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An iterative algorithm for split equality of variational inequality problem of a finite family of pseudomonotone mappings

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Abstract. This paper presents an iterative algorithm for approximating a solution to a split equality variational problem involving a finite family of pseudomonotone mappings in Hilbert spaces. We demonstrate the strong convergence of the sequence produced by the algorithm to a solution of the problem in Hilbert spaces, under the assumption that the mappings are uniformly continuous. Additionally, we apply our main findings to solve split variational inequality and split equality zero point problems for a finite family of pseudomonotone mappings in Hilbert spaces, expanding on existing literature.

1. Introduction

The split equality problem was first introduced by Moudafi [7] and has garnered significant attention due to its applications in various fields such as decomposition methods for partial differential equations, game theory, medical image reconstruction, and radiation therapy treatment planning. The concept of variational inequalities has been utilized as an analytical tool in a wide range of disciplines including engineering, physics, optimization theory, and economics. Stampacchia [11] and Fichera [2] introduced the variational inequality in 1964, in potential theory and mechanics, respectively, as a means to study differential equations in infinite-dimensional spaces with practical applications. The variational inequality problem combines key concepts in applied mathematics such as systems of nonlinear equations, necessary optimality conditions for optimization problems, complementarity problems, obstacle problems, and network equilibrium problems.

Pseudomonotone mappings, introduced by Karamardian [4], generalize the concept of monotone operators and have been extensively studied for over 40 years. They have found numerous applications in variational inequalities and economics. Various authors have explored pseudomonotone variational inequality and split equality variational inequality problems in Hilbert space using different iterative algorithms and classes of mappings. For instance, Shehu, Dong, and Jiang [10] introduced a single projection method for pseudomonotone variational inequalities in Hilbert space in 2019. Reich, Thong, Dong, Li, and Dong [9] proposed new algorithms and convergence theorems for solving variational inequalities with non-Lipschitz mapping in 2021.

Several authors have also studied the split equality problem for variational inequality problems, known as the split equality variational inequality problem. For example, Wega and Zegeye [12] developed an

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algorithm for approximating solutions to split equality monotone inclusion problems and obtained strong convergence results in 2020. Izuchukwu, Ezeora, and Martinez-Moreno [3] proposed a new modified contraction method for solving a certain class of split monotone variational inclusion problems in real Hilbert spaces in 2020. More recently, Kwelegano, Zegeye, and Boikanyo [5] introduced an iterative method for split equality variational inequality problems for non-Lipschitz pseudomonotone mappings in 2021.

Motivated by the research works of [3, 5, 12], the goal of this paper is to study iterative algorithms for approximating a common solution to split equality monotone inclusion problems for a finite family of pseudomonotone mappings in Hilbert spaces. To facilitate this study, we provide some necessary notions and definitions. Throughout this research, we use (VIP) to denote variational inequality problems, (SEP) for split equality problems, (SEVIP) for split equality variational inequality problems, H for a Hilbert space, and C for a closed, non-empty, and convex subset of a Hilbert space.

Definition 1.1. *Let $T : C \to H$ be a mapping.*

i) T is called an L-Lipschitz mapping with Lipschitz constant $L > 0$ if $\|Tx - Ty\| \leq L\|x - y\|$ for all $x, y \in C$. If $0 \leq L < 1$, then T is a contraction. If $L = 1$, then T is nonexpansive.
ii) T is called a monotone mapping if $\langle Tx - Ty, y - x\rangle \geq 0$ for all $x, y \in C$.
iii) T is called a pseudomonotone mapping if $\langle Tx, y - x\rangle \geq 0$ implies $\langle Ty, y - x\rangle \geq 0$ for all $x, y \in C$.

We note that pseudomonotone mappings are more general than monotone mappings.

Definition 1.2. *Let $A : C \to H$ be a mapping.*
The variational inequality problem is formulated to find a point x^* in C such that for all $x \in C$,

$$\langle Ax^*, x - x^*\rangle \geq 0. \tag{1}$$

The solution set of (1) is denoted by $VI(C, A)$.

2. Preliminaries

In this section we recall some known results which are used in our subsequent analysis. The projection mapping $P_c : H \to C$ is defined by

$$\|P_c x - x\| = inf_{y\in c}\|x - y\|, \tag{2}$$

and hence, P_c satisfies: $\|P_c x - P_c y\|^2 \leq \langle P_c x - P_c y, x - y\rangle$, for all $x, y \in H$.

Definition 2.1. *The mapping $T : C :\to H$ is called sequentially weakly continuous if for each sequence $\{x_n\}$, we have $\{x_n\}$ converges weakly to p implies $\{Tx_n\}$ converges to Tp.*

Lemma 2.2. *For all $x, y \in H$, it is known that the following inequalities hold.*

i) $2\langle x, y\rangle = \|x\|^2 + \|y\|^2 - \|x - y\|^2$.
ii) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$.

Lemma 2.3. *Let $x \in H$. Then*

$$P_c x \in C \text{ if and only if } \langle y - P_c x, x - P_c x\rangle \leq 0, \text{ for every } y \in C. \tag{3}$$

This result implies that for all $x \in H$

$$\|P_C x - z\|^2 \leq \|x - z\|^2 - \|x - P_c x\|^2 z \in C. \tag{4}$$

Lemma 2.4 ([6]). *Let $\{a_k\}$ be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence a_{k_j} of $\{a_k\}$ such that $a_{k_j} < a_{k_j+1}$ for all $j \geq 0$. Define an integer sequence $\{m_k\}_{k\geq k_0}$ as*

$$m_k = \max\{k_0 \leq l \leq k : a_l < a_{l+1}\}.$$

Then, $m_k \to \infty$ as $k \to \infty$ and for all $k \geq k_0$

$$\max\{a_{m_k}, a_k\} \leq a_{m_k+1}.$$

Lemma 2.5 ([13]). *Let $\{\alpha_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\gamma_n \text{ for } n \geq n_0 \text{ where } \{\alpha_n\} \subseteq (0,1) \text{ and } \{\gamma_n\} \subseteq R, \text{ satisfies}$$

$$\sum_{n=1}^{\infty} \alpha_n = \infty, \text{ and } \limsup_{n\to\infty} \gamma_n \leq 0. \text{ Then } \lim_{n\to\infty} a_n = 0.$$

Lemma 2.6 ([8]). *Let H be a real Hilbert space, for all $x_i \in H$ and $\alpha_i \in [0,1]$ for $i = 1,2,3,...n$, such that $\alpha_1 + \alpha_2 + \alpha_3 + ... + \alpha_n = 1$, the following holds:*

$$\|\alpha_0 x_0 + \alpha_1 x_1 + ... + \alpha_n x_n\|^2 = \sum_{i=0}^{n} \alpha_i \|x_i\|^2 - \sum_{0\leq i,j\leq n} \alpha_i\alpha_j \|x_i - x_j\|^2.$$

Lemma 2.7. *Let $r(x)$, be a real valued function on H and defined $K := \{x \in C : r(x) \leq 0\}$. If K, is nonempty and r is L-Lipshitz continuous with $L > 0$, then*

$$\|P_K x - x\| \geq \frac{1}{L}\max\{r(x), 0\}, \text{ for } x \in C.$$

3. Main results

In this section, we shall make use of the following assumptions:

Assumption 1:

- A1: Let $T_1, T_2 : H_1 \to H_1$ and $S_2, S_2 : H_2 \to H_2$ be sequentially weakly continuous and uniformly continuous pseudomonotone mappings on bounded subset of H_1 and H_2, respectively.
- A2: Let $\Omega := \{(p,q) \in H_1 \times H_2 : p \in VI(C,T_1) \cap VI(C,T_2), q \in VI(D,S_1) \cap VI(D,S_2) \text{ and } Ap = Bq\} \neq \emptyset$, where $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are bounded linear mappings with adjoints A^* and B^*, respectively.
- A3: Let $\iota \in (0,1)$, $\mu > 0$ and $\delta \in [\underline{\delta}, \bar{\delta}] \subset (0, \frac{1}{\mu})$
- A4: Let $\{\alpha_n\} \subset (0,\epsilon)$ for some constant real number $\epsilon > 0$ be a real sequence such that,

$$lim_{n\to\infty}\alpha_n = 0, \text{ and } \sum_{n=1}^{\infty} \alpha_n = \infty.$$

- A5: Let $g_1 : H_1 \to H_1$ and $g_2 : H_2 \to H_2$ be contraction mappings with constants $\alpha_1, \alpha_2 \in (0, \frac{1}{\sqrt{2}})$, respectively and we denote $\alpha = \max\{\alpha_1, \alpha_2\}$.
- A6 : Let the sequence γ_n satisfies

$$0 < \xi \leq \gamma_n \leq \frac{\|Ax_n - Bt_n\|^2}{\|A^*(Ax_n - \mathcal{B}t_n)\|^2 + \|B^*(Bt_n - Ax_n)\|^2}, \; for\; n \in \Upsilon$$

otherwise, $\gamma_n = \gamma > 0$, such that the indexes

$$\Upsilon = \{n \in \mathbb{N} : Ax_n - Bt_n \neq 0\}.$$

Algorithm 1. For arbitrary $(x_0, t_0) \in H_1 \times H_2$, define an iterative algorithm by

Step 1. Compute

$$\begin{cases} z_{i,n} = P_C(x_n - \delta T_i x_n) \text{ and } d_i(x_n) = x_n - z_{i,n}, \quad \text{for } \; i = 1, 2, \\ u_{i,n} = P_D(t_n - \delta S_i t_n) \text{ and } d_i(t_n) = t_n - u_{i,n}, \quad \text{for } \; i = 1, 2. \end{cases} \tag{5}$$

Step 2. Compute

$$\begin{cases} y_{i,n} = x_n - \Upsilon_{i,n} d(x_n), \quad \text{for } \; i = 1, 2. \\ v_{i,n} = t_n - \Upsilon'_{i,n} d(t_n), \quad \text{for } \; i = 1, 2., \end{cases} \tag{6}$$

where, $\Upsilon_{i,n} = \iota^{j_{i,n}}$ such that $j_{i,n}$ is the smallest nonnegative integer j_i satisfying

$$\langle T_i x_n - T_i(x_n - \iota^{j_i} d_i(x_n)), d_i(x_n) \rangle \leq \mu \|d_i(x_n)\|^2,$$

and $\Upsilon'_n = \iota^{j'_{i,n}}$ such that $j'_{i,n}$ is the smallest nonnegative integer j'_i satisfying

$$\langle S_i t_n - S_i(t_n - \iota^{j'_i} d_i(t_n)), d_i(t_n) \rangle \leq \mu \|d_i(t_n)\|^2.$$

Step 3. Compute

$$\begin{cases} a_n = P_C(x_n - \gamma_n A^*(Ax_n - Bt_n), \\ b_n = P_C(t_n - \gamma_n B^*(Bt_n - Ax_n), \\ w_n = \theta_n a_n + \beta_n p_{1,n} + \eta_n p_{2,n}, \\ r_n = \theta_n b_n + \beta_n q_{1,n} + \eta_n q_{2,n} \end{cases} \tag{7}$$

where $C_{i,n} = \{x \in H : h_{i,n} = \langle y_{i,n} - T_i y_{i,n}, x - y_{i,n} \rangle \leq 0\}$,
$D_{i,n} = \{x \in H : e_{i,n} = \langle v_{i,n} - S_i v_{i,n}, x - v_{i,n} \rangle \leq 0\}$ and $\{\theta_n\}, \{\beta_n\}, \{\eta_n\} \subset [\rho, 1)$ for $\rho > 0$ such that $\beta_n + \theta_n + \eta_n = 1$ for all $n \geq 0$ and $p_{i,n} = P_{C_{i,n}} x_n$, $q_{i,n} = P_{D_{i,n}} t_n$ for $i = 1, 2$.

Step 4. Compute

$$\begin{cases} x_{n+1} = \alpha_n g_1(x_n) + (1 - \alpha_n) w_n, \\ t_{n+1} = \alpha_n g_2(t_n) + (1 - \alpha_n) r_n. \end{cases}$$

Step 5. Set $n := n + 1$ and go to **Step 1**.

Lemma 3.1. *Suppose that the assumption A_1–A_2 hold, and $\{x_n\}, \{t_n\}, \{y_{i,n}\}, \{z_{i,n}\}, \{u_{i,n}\}, \{v_{i,n}\}$ are sequences, generated by Algorithm 1 for $i = 1, 2$. Then, the search rules in step 2 are well defined.*

Proof. Since $\iota \in (0, 1)$, T_i and S_i are uniformly continuous on H_1 and H_2, respectively, we have

$$\langle T_i x_n - T_i(x_n - \iota^{j_i} d_i(x_n)), d_i(x_n) \rangle \to 0 \text{ as } j_i \to \infty,$$

and

$$\langle S_i t_n - S_i(t_n - \iota^{j'_i} d_i(t_n)), d_i(t_n) \rangle \to 0 \text{ as } j'_i \to \infty.$$

Moreover, since $\|d_i(x_n)\| > 0$ and $\|d_i(t_n)\| > 0$ there exist a non-negative integers $j_{i,n}$ and $j'_{i,n}$, satisfying the inequalities in Step 2. □

Lemma 3.2. *Suppose that the assumption $A_1 - A_3$ hold. If $\{x_n\}$, $\{t_n\}$, $\{y_{i,n}\}$, $\{z_{i,n}\}$, $\{u_{i,n}\}$, $\{v_{i,n}\}$ are sequences generated by Algorithm 1, then*

$$\langle T_i x_n, d_i(x_n)\rangle \geq \frac{1}{\delta}\|d_i(x_n)\|^2$$

and

$$\langle S_i t_n, d_i(t_n)\rangle \geq \frac{1}{\delta}\|d_i(t_n)\|^2.$$

Proof. From equations (5), for $n \geq 0$ and $i = 1, 2$, we have,

$$\begin{aligned}\|x_n - P_C(x_n - \delta T_i x_n)\|^2 &\leq \langle x_n - (x_n - \delta T_i x_n), x_n - P_C(x_n - \delta T_i x_n)\rangle \\ &= \delta\langle T_i x_n), x_n - P_C(x_n - \delta T_i x_n)\rangle,\end{aligned}$$

which implies $\langle T_i x_n, d_i(x_n)\rangle \geq \frac{1}{\delta}\|d_i(x_n)\|^2$.

Similarly, we get $\langle S_i t_n, d_i(t_n)\rangle \geq \frac{1}{\delta}\|d_i(t_n)\|^2$. □

Lemma 3.3. *Suppose the assumptions $A_1 - A_3$ holds. Let $(p, q) \in \Omega$, let $h_{i,n}(x_n) = \langle T_i y_{i,n}, x_n - y_{i,n}\rangle$, and let $e_{i,n}(t_n) = \langle S_i v_{i,n}, x_n - v_{i,n}\rangle$. Then,*

$$h_{i,n}(p) \leq 0,\ e_{i,n}(q) \leq 0,\ h_{i,n}(x_n) \geq \Upsilon_n(\frac{1}{\delta} - \mu)\|d_i(x_n)\|^2,$$

$$\text{and } e_{i,n}(t_n) \geq \Upsilon_n'(\frac{1}{\delta} - \mu)\|d_i(t_n)\|^2.$$

In particular, if $d_i(x_n) \neq 0$ and $d_i(t_n) \neq 0$, then $h_{i,n}(x_n) > 0$ and $e_{i,n}(t_n) > 0$.

Proof. For the fact that $(p, q) \in \Omega$, we have

$$\langle T_i p, y_{i,n} - p\rangle \geq 0.$$

This inequality and the fact that T_i is pseudomonotone mapping, we obtain

$$h_n(p) = \langle T_i y_n, y_{i,n} - p\rangle \geq 0,$$

which gives us,

$$h_{i,n}(p) = \langle T_i y_{i,n}, p - y_{i,n}\rangle \leq 0.$$

Similarly, we obtain $e_{i,n}(q) \leq 0$. In addition, from Step 2, of Algorithm 1, we have,

$$h_{i,n}(x_n) = \langle T_i y_{i,n}, x_n - y_{i,n}\rangle = \langle T_i y_{i,n}, x_n - (x_n - \Upsilon_n d_i(x_n)\rangle = \langle T_i y_{i,n}, d_i(x_n)\rangle.$$

Furthermore, from the inequalities in Step 2, we have,

$$\langle T_i x_n - T_i y_{i,n}, d(x_n)\rangle \leq \mu\|d(x_n)\|^2,$$

which implies

$$\langle T_i y_{i,n}, d_i(x_n)\rangle \geq \langle T_i x_n, d_i(x_n)\rangle - \mu\|d(x_n)\|^2 \tag{8}$$

From Lemma 3.2 and inequality above, we obtain

$$\langle T_i y_{i,n}, d(x_n)\rangle \geq (\frac{1}{\delta} - \mu)\|d(x_n)\|^2 \tag{9}$$

By combining (8) and (9), we obtain,

$$h_{i,n}(x_n) \geq \Upsilon_n(\frac{1}{\delta} - \mu)\|d_i(x_n)\|^2.$$

Similarly, we obtain,

$$e_{i,n}(x_n) \geq \Upsilon'_n(\frac{1}{\delta} - \mu)\|d_i(t_n)\|^2,$$

for $i = 1, 2$. □

Lemma 3.4. *Suppose that the assumption A_1–A_2 hold, and $\{x_n\}, \{t_n\}, \{y_{i,n}\}, \{z_{i,n}\}, \{u_{i,n}\}, \{v_{i,n}\}$ are sequences, generated by Algorithm 1 for $i = 1, 2$. Let $\{(x_{n_k}, t_{n_k})\}$ be a subsequence of $\{(x_n, t_n)\}$ such that*
$(x_{n_k}, y_{n_k}) \rightharpoonup (\overline{p}, \overline{q})$, $\lim_{k\to\infty} \|x_{n_k} - z_{i,n_k}\| = 0$ and $\lim_{k\to\infty} \|t_{n_k} - u_{i,n_k}\| = 0$.
Then $(\overline{p}, \overline{q}) \in [VI(C, T_1) \cap VI(C, T_2)] \times [VI(D, S_1) \cap VI(D, S_2)]$.

Proof. For the fact that $z_{i,n_k} = P_C(x_{n_k} - \delta T_i x_{n_k})$, from (3), we get

$$\langle x_{n_k} - \delta T_i x_{n_k} - z_{i,n_k}, x - x_{n_k}\rangle \leq 0 \ \forall x \in C,$$

which implies

$$\langle x_{n_k} - z_{i,n_k}, x - z_{i,n_k}\rangle \leq \delta\langle T_i x_{n_k}, x - z_{i,n_k}\rangle \ \forall x \in C,$$

and hence

$$\langle x_{n_k} - z_{i,n_k}, x - z_{i,n_k}\rangle + \langle T_i x_{n_k}, z_{i,n_k} - x_{n_k}\rangle \leq \delta\langle T_i x_{n_k}, x - x_{n_k}\rangle \ \forall x \in C.$$

Since $\lim_{k\to\infty} \|x_{n_k} - z_{i,n_k}\| = 0$ and the fact that T_i is bounded, we obtain

$$\liminf_{k\to\infty} \langle T_i x_{n_k}, x - x_{n_k}\rangle \geq 0, \forall x \in C. \tag{10}$$

Moreover, let $\{\xi_k\}$ be a sequence of decreasing numbers such that $\xi_k \to 0$ as $k \to \infty$ and w be an arbitrary element of C. Using inequality (10), we can find a large enough N_k such that

$$\langle T_i x_{n_k}, w - x_{n_k}\rangle + \xi_k \geq 0, \forall k \geq N_k. \tag{11}$$

From (11) and the fact that $T_i x_{n_k} \neq 0$, we get

$$\langle T_i x_{n_k}, \xi_k d_k + w - x_{n_k}\rangle \geq 0, \forall k \geq N_k, \tag{12}$$

for some $d_k \in C$ satisfying $\langle T_i x_{n_k}, d_k\rangle = 1$. In addition, from definition of T_i and inequality (12), we have

$$\langle T_i(w + \xi_n d_k w), w + \xi_k d_k w - x_{n_k}\rangle \geq 0, \forall k \geq N_k,$$

which implies that

$$\begin{aligned}\langle T_i w, w - x_{n_k}\rangle \geq\ & \langle T_i w - A_1(w + \xi_k d_k w), w + \xi_k d_k w - x_{n_k}\rangle \\ & -\xi_k\langle T_i w, d_k\rangle, \forall k \geq N_k.\end{aligned} \tag{13}$$

Since $\xi_k \to 0$ as $k \to \infty$ and T_i is continuous, then from inequality (13), we obtain

$$\langle T_i w, w - \overline{p}\rangle = \liminf_{k\to\infty} \langle T_i w, w - x_{n_k}\rangle \geq 0, \forall w \in C,$$

and for $i = 1, 2$. Thus, $\overline{p} \in VI(C, T_1) \cap VI(C, T_2)$. Similarly, we get

$$\langle S_i z, z - \overline{q}\rangle = \liminf_{k\to\infty} \langle S_i z, z - t_{n_k}\rangle \geq 0, \forall z \in D,$$

and for $i = 1, 2$. Thus, $\overline{q} \in VI(D, S_1) \cap VI(D, S_2)$. □

Theorem 3.5. *Suppose the assumptions $A_1 - A_4$ hold. Then, the sequence $\{(x_n, t_n)\}$, generated by the Algorithm 1 is bounded in Hilbert space, $C \times D$.*

Proof. Now, from Lemma 2.6, and (4), we get

$$\begin{aligned} ||w_n - p||^2 &= ||\theta_n a_n + \beta_n p_{1,n} + \eta_n p_{2,n} - p||^2 \\ &= ||\theta_n(a_n - p) + \beta_n(P_{C_{1,n}} x_n - p) + \beta_n(P_{C_{2,n}} x_n - p)||^2 \\ &\leq \theta_n ||a_n - p||^2 + \beta_n ||P_{C_{1,n}} x_n - p||^2 + \eta_n ||P_{C_{2,n}} x_n - p||^2 \\ &\leq \theta_n ||a_n - p||^2 + \beta_n [||x_n - p||^2 - ||P_{C_{1,n}} x_n - x_n||^2] \\ &\quad + \eta_n [||x_n - p||^2 - ||P_{C_{2,n}} x_n - x_n||^2]. \end{aligned} \tag{14}$$

Similarly, we obtain

$$\begin{aligned} ||r_n - q||^2 &\leq \theta_n ||b_n - q||^2 + \beta_n [||t_n - q||^2 - ||P_{D_{1,n}} t_n - t_n||^2] \\ &\quad + \eta_n [||t_n - q||^2 - ||P_{D_{2,n}} t_n - t_n||^2]. \end{aligned} \tag{15}$$

Thus, by adding inequalities (14) and (15), we get

$$\begin{aligned} ||w_n - p||^2 + ||r_n - q||^2 &\leq \theta_n [||a_n - p||^2 + ||b_n - q||^2] \\ &\quad + \beta_n [||x_n - p||^2 - ||P_{C_{1,n}} x_n - x_n||^2] \\ &\quad + \eta_n [||x_n - p||^2 - ||P_{C_{2,n}} x_n - x_n||^2] \\ &\quad + \beta_n [||t_n - q||^2 - ||P_{D_{1,n}} t_n - t_n||^2] \\ &\quad + \eta_n [||t_n - q||^2 - ||P_{D_{2,n}} t_n - t_n||^2]. \end{aligned} \tag{16}$$

In addition from (7) and (4), we obtain

$$\begin{aligned} ||a_n - p||^2 &= ||P_C(x_n - \gamma_n A^*(Ax_n - Bt_n) - p||^2 \\ &\leq ||x_n - \gamma_n A^*(Ax_n - Bt_n||^2 - ||a_n - (x_n - \gamma_n A^*(Ax_n - Bt_n)||^2 \\ &\leq ||x_n - p||^2 + \gamma_n^2 ||A^*(Ax_n - Bt_n)||^2 - \gamma_n ||Ax_n - Bt_n||^2 \\ &\quad - ||x_n - a_n - \gamma_n A^*(Ax_n - Bt_n)||^2. \end{aligned} \tag{17}$$

Similarly, we get

$$\begin{aligned} ||b_n - q||^2 &\leq ||t_n - q||^2 + \gamma_n^2 ||B^*(Bt_n - Ax_n)||^2 - \gamma_n ||Ax_n - Bt_n||^2 \\ &\quad - ||t_n - b_n - \gamma_n B^*(Bt_n - Ax_n)||^2. \end{aligned} \tag{18}$$

By adding inequalities (17) and (18), we get

$$\begin{aligned} ||a_n - p||^2 + ||b_n - q||^2 &\leq ||x_n - p||^2 + ||t_n - q||^2 \\ &\quad + \gamma_n^2 [||A^*(Ax_n - Bt_n)||^2 + ||B^*(Bt_n - Ax_n)||^2] \\ &\quad - 2\gamma_n ||Ax_n - Bt_n||^2 - ||x_n - a_n - \gamma_n A^*(Ax_n - Bt_n)||^2 \\ &\quad - ||t_n - b_n - \gamma_n B^*(Bt_n - Ax_n)||^2. \end{aligned} \tag{19}$$

Moreover, (19) and (A6), we obtain

$$\begin{aligned} ||a_n - p||^2 + ||b_n - q||^2 &\leq ||x_n - p||^2 + ||t_n - q||^2 - \gamma_n ||Ax_n - Bt_n||^2 \\ &\quad - ||x_n - a_n - \gamma_n A^*(Ax_n - Bt_n)||^2 \\ &\quad - ||t_n - b_n - \gamma_n B^*(Bt_n - Ax_n)||^2. \end{aligned} \tag{20}$$

Now, by substituting (20) in (16), we get

$$\begin{aligned}||w_n - p||^2 + ||r_n - q||^2 \leq\ & ||x_n - p||^2 + ||t_n - q||^2 \\ & -\beta_n[||P_{C_{1,n}}x_n - x_n||^2 + ||P_{D_{1,n}}t_n - t_n||^2] \\ & -\eta_n[||P_{C_{2,n}}x_n - x_n||^2 + ||P_{D_{2,n}}t_n - t_n||^2] \\ & -\theta_n\gamma_n||Ax_n - Bt_n||^2 - \theta_n||x_n - a_n - \gamma_n A^*(Ax_n - Bt_n)||^2 \\ & -\theta_n||t_n - b_n - \gamma_n B^*(Bt_n - Ax_n)||^2. \qquad (21)\end{aligned}$$

From (21), Lemma 2.6 and (A5), we get

$$\begin{aligned}||x_{n+1} - p||^2 + ||t_{n+1} - q||^2 =\ & ||\alpha_n g_1(x_n) + (1 - \alpha_n)w_n - p||^2 \\ & +||\alpha_n g_2(t_n) + (1 - \alpha_n)r_n - q||^2 \\ \leq\ & \alpha_n||g_1(x_n) - p||^2 + (1 - \alpha_n)||w_n - p||^2 \\ & +\alpha_n||g_2(t_n) - q||^2 + (1 - \alpha_n)|r_n - p||^2 \\ \leq\ & \alpha_n(||g_1(x_n) - g_1(p)|| + ||g_1(p) - p||)^2 + (1 - \alpha_n)||x_n - p||^2 \\ & +\alpha_n(||g_2(t_n) - g_2(q)|| + ||g_2(q) - q||)^2 + (1 - \alpha_n)||t_n - q||^2 \\ \leq\ & \alpha_n[\alpha^2||x_n - p||^2 + ||g_1(x_n) - p||^2] + (1 - \alpha_n)||x_n - p||^2 \\ & +\alpha_n[\alpha^2||t_n - q||^2 + ||g_2(t_n) - q||^2] + (1 - \alpha_n)||t_n - q||^2 \\ \leq\ & 2\alpha_n(\alpha^2||x_n - p||^2 + ||g_1(p) - p||^2) + (1 - \alpha_n)||x_n - p||^2 \\ & +2\alpha_n(\alpha^2||t_n - q||^2 + ||g_2(q) - q||^2) \\ & +(1 - \alpha_n)||t_n - q||^2. \qquad (22)\end{aligned}$$

By setting $R_n(p, q) = ||x_n - p||^2 + ||t_n - q||^2$, from inequality (22), we get

$$\begin{aligned}R_{n+1}(p, q) \leq\ & (1 - \alpha_n(1 - 2\alpha^2))R_n(p, q) \\ & +2\alpha_n(||g_1(p) - p||^2 + ||g_2(q) - q||^2) \\ \leq\ & \max\Big\{R_n(p, q), \frac{2}{1 - 2\alpha^2}(||g_1(p) - p||^2 + ||g_2(q) - q||^2)\Big\},\end{aligned}$$

and hence by induction

$$R_n(p, q) \leq \max\Big\{R_0(p, q), \frac{2(||g_1(p) - p||^2 + ||g_2(q) - q||^2)}{1 - 2\alpha^2}\Big\},$$

which implies that $\{x_n\}$, $\{t_n\}$ and hence $\{y_{i,n}\}$, $\{v_{i,n}\}$, $\{T_i x_n\}$ and $\{S_i v_n\}$ for $i = 1, 2$ are bounded. □

Theorem 3.6. *Suppose the assumption* $(A1) - (A6)$ *hold. Then, the sequence* $\{(x_n, t_n)\}$*, generated by Algorithm 1 converges strongly to* $(p, q) = P_\Omega(g_1(p), g_2(q))$.

Proof. Now, let $(p, q) = P_\Omega(g_1(p), g_2(q))$. Then, from equation 2, for $i = 1, 2$, we have,

$$||p - p_{i,n}||^2 \leq ||p - x_n||^2 - ||x_n - p_{i,n}||^2.$$

Similarly, we get

$$||q - q_{i,n}||^2 \leq ||q - t_n||^2 - ||t_n - q_{i,n}||^2, \qquad (23)$$

Since for $i = 1, 2$, T_i is bounded on bounded subset of H_1 ,Then their exists $L_i > 0$, such that $||T_i y_n|| \leq L_i$, for all $n \geq 0$ and $i = 1, 2$. Thus,

$$\begin{aligned} |h_{i,n}(z) - h_{i,n}(w)| &= |\langle T_i y_{i,n}, z - y_{i,n}\rangle - \langle T_i y_{i,n}, w - y_{i,n}\rangle| \\ &= |\langle T_i y_{i,n}, z - w\rangle \\ &\leq \|T_i y_{i,n}\|\|z - w\| \\ &\leq L_i\|z - w\|, \end{aligned}$$

which gives us that $h_{i,n}$ is L_i- Lipschitz continuous on H_1. Thus, from Lemma 2.7 and Lemma 3.3, we obtain

$$\|x_n - p_{i,n}\|^2 \geq \frac{h_{i,n}x_n}{2L_i^2} \geq \Upsilon_{i,n}^2(\frac{1}{\delta} - \mu)^2\|d_i(x_n)\|^4 \tag{24}$$

Thus, from (23) and (24), for $i = 1, 2$, we get

$$\|p - p_{i,n}\|^2 \leq \|p - x_n\|^2 - \Upsilon_{i,n}^2(\frac{1}{\delta} - \mu)^2\|d_i(x_n)\|^4. \tag{25}$$

Similarly, for $i = 1, 2$, we get

$$\|q - q_{i,n}\|^2 \leq \|q - t_n\|^2 - \Upsilon_{i,n}'^2(\frac{1}{\delta} - \mu)^2\|d_i(t_n)\|^4.$$

Now, from Lemma 2.6, (25), (21) and (17), we get

$$\begin{aligned} \|w_n - p\|^2 + \|r_n - q\|^2 &= \|\theta_n a_n + \beta_n p_{1,n} + \eta_n p_{2,n} - p\|^2 \\ &\quad +\|\theta_n b_n + \beta_n q_{1,n} + \eta_n q_{2,n} - q\|^2 \\ &\leq \theta_n\|a_n - p\|^2 + \beta_n\|p_{1,n} - p\|^2 + \eta_n\|p_{2,n} - p\|^2 \\ &\quad +\theta_n\|b_n - q\|^2 + \beta_n\|q_{1,n} - q\|^2 + \eta_n\|q_{2,n} - q\|^2 \\ &\leq \theta_n\|x_n - p\|^2 + \beta_n\|x_n - p\|^2\eta_n\|x_n - p\|^2 + \eta_n\|x_n - p\|^2 \\ &\quad +\theta_n\|t_n - q\|^2 + \beta_n\|t_n - q\|^2\eta_n\|t_n - q\|^2 + \eta_n\|t_n - q\|^2 \\ &\quad -\Big(\Upsilon_{1,n}^2(\frac{1}{\delta} - \mu)^2\|d_1(x_n)\|^4 + \Upsilon_{2,n}^2(\frac{1}{\delta} - \mu)^2\|d_2(x_n)\|^4\Big) \\ &\quad -\Big(\Upsilon_{1,n}'^2(\frac{1}{\delta} - \mu)^2\|d_1(t_n)\|^4 + \Upsilon_{2,n}'^2(\frac{1}{\delta} - \mu)^2\|d_2(t_n)\|^4\Big) \\ &\leq \|x_n - p\|^2 + \|t_n - q\|^2 \\ &\quad -\Big(\Upsilon_{1,n}^2(\frac{1}{\delta} - \mu)^2\|d_1(x_n)\|^4 + \Upsilon_{2,n}^2(\frac{1}{\delta} - \mu)^2\|d_2(x_n)\|^4\Big) \\ &\quad -\Big(\Upsilon_{1,n}'^2(\frac{1}{\delta} - \mu)^2\|d_1(t_n)\|^4 \\ &\quad +\Upsilon_{2,n}'^2(\frac{1}{\delta} - \mu)^2\|d_2(t_n)\|^4\Big). \end{aligned} \tag{26}$$

By Lemma 2.2, Lemma 2.6 and (26), we obtain

$$\begin{aligned}
R_{n+1}(p,q) &= \|\alpha_n g_1(x_n) + (1-\alpha_n)w(n) - p\|^2 + \|\alpha_n g_2(t_n) + (1-\alpha_n)r(n) - q\|^2 \\
&\leq \|\alpha_n(g_1(x_n) - g_1(p)) + (1-\alpha_n)(w_n - p) + \alpha_n(g_1(p) - p)\|^2 \\
&\quad + \|\alpha_n(g_2(t_n) - g_2(q)) + (1-\alpha_n)(r_n - q) + \alpha_n(g_2(q) - q)\|^2 \\
&\quad + 2\alpha_n[\langle g_1(p) - p, x_{n+1} - p\rangle + \langle g_2(q) - q, t_{n+1} - q\rangle] \\
&\leq \alpha\alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|w_n - p\|^2 + \alpha\alpha_n\|t_n - q\|^2 + (1-\alpha_n)\|r_n - q\|^2 \\
&\quad + 2\alpha_n\|g_1(x_n) - p\|\|x_{n+1} - x_n\| + 2\alpha_n\|g_2(t_n) - q\|\|t_{n+1} - t_n\| \\
&\quad + 2\alpha_n[\langle g_1(p) - p, x_n - p\rangle + \langle g_2(q) - q, t_n - q\rangle] \\
&\leq \Big(1 - (1-\alpha)\alpha_n\Big)R_n(p,q) \\
&\quad + 2\alpha_n\|g_1(x_n) - p\|\|x_{n+1} - x_n\| + 2\alpha_n\|g_2(t_n) - q\|\|t_{n+1} - t_n\| \\
&\quad + 2\alpha_n[\langle g_1(p) - p, x_n - p\rangle + \langle g_2(q) - q, t_n - q\rangle] \\
&\quad - (1-\alpha_n)\Big(\Upsilon_{1,n}^2(\frac{1}{\delta} - \mu)^2\|d_1(x_n)\|^4 + \Upsilon_{2,n}^2(\frac{1}{\delta} - \mu)^2\|d_2(x_n)\|^4\Big) \\
&\quad - (1-\alpha_n)\Big(\Upsilon_{1,n}'^2(\frac{1}{\delta} - \mu)^2\|d_1(t_n)\|^4 + \Upsilon_{2,n}'^2(\frac{1}{\delta} - \mu)^2\|d_2(t_n)\|^4\Big),
\end{aligned}$$

which gives us

$$\begin{aligned}
&(1-\alpha_n)\Big(\Upsilon_{1,n}^2(\frac{1}{\delta} - \mu)^2\|d_1(x_n)\|^4 + \Upsilon_{2,n}^2(\frac{1}{\delta} - \mu)^2\|d_2(x_n)\|^4\Big) \\
&+ (1-\alpha_n)\Big(\Upsilon_{1,n}'^2(\frac{1}{\delta} - \mu)^2\|d_1(t_n)\|^4 + \Upsilon_{2,n}'^2(\frac{1}{\delta} - \mu)^2\|d_2(t_n)\|^4\Big) \\
&\leq R_n(p,q) - R_{n+1}(p,q) \\
&+ 2\alpha_n\|g_1(x_n) - p\|\|x_{n+1} - x_n\| + 2\alpha_n\|g_2(t_n) - q\|\|t_{n+1} - t_n\| \\
&+ 2\alpha_n[\langle g_1(p) - p, x_n - p\rangle + \langle g_2(q) - q, t_n - q\rangle]. \qquad (27)
\end{aligned}$$

In addition, from (21), we get

$$\begin{aligned}
R_{n+1}(p,q) &\leq R_n(p,q) + 2\alpha_n[\langle g_1(p) - p, x_{n+1} - p\rangle + \langle g_2(q) - q, t_{n+1} - q\rangle] \\
&\quad - \beta_n[\|P_{C_{1,n}}x_n - x_n\|^2 + \|P_{D_{1,n}}t_n - t_n\|^2] \\
&\quad - \eta_n[\|P_{C_{2,n}}x_n - x_n\|^2 + \|P_{D_{2,n}}t_n - t_n\|^2] \\
&\quad - \theta_n\gamma_n\|Ax_n - Bt_n\|^2 - \theta_n\|x_n - a_n - \gamma_n A^*(Ax_n - Bt_n)\|^2 \\
&\quad - \theta_n\|t_n - b_n - \gamma_n B^*(Bt_n - Ax_n)\|^2. \qquad (28)
\end{aligned}$$

Next, we show that the sequence $\{R_n(p,q)\}$ converges strongly to zero. For this we consider two cases as follows:

Case 1: Assume that there exist $n_0 \in N$, such that the sequence of real numbers $\{R_n(p,q)\}$ is decreasing for all $n \geq n_0$. Thus, the sequence $\{R_n(p,q)\}$ convergent and hence from (28) and the fact that $\alpha_n \to 0$, we obtain

$$\lim_{n\to\infty} \|P_{C_{1,n}}x_n - x_n\|^2 = \lim_{n\to\infty} \|P_{C_{2,n}}x_n - x_n\|^2 = 0,$$

$$\lim_{n\to\infty} \|P_{D_{1,n}}t_n - t_n\| = \lim_{n\to\infty} \|P_{D_{2,n}}t_n - t_n\| = 0,$$

$$\lim_{n\to\infty} \|Ax_n - Bt_n\| = 0,$$

and

$$\lim_{n\to\infty} \|x_n - a_n - \gamma_n A^*(Ax_n - Bt_n)\| = \lim_{n\to\infty} \|t_n - b_n - \gamma_n B^*(Bt_n - Ax_n)\| = 0,$$

which implies that

$$\lim_{n\to\infty} \|x_n - a_n\| \le \|x_n - a_n - \gamma_n A^*(Ax_n - Bt_n)\|^2 + \|\gamma_n A^*(Ax_n - Bt_n)\| = 0, \tag{29}$$

and

$$\lim_{n\to\infty} \|t_n - b_n\| \le \|t_n - b_n - \gamma_n B^*(Bt_n - Ax_n)\|^2 + \|\gamma_n B^*(Bt_n - Ax_n)\| = 0,$$

In addition, from (27), we have

$$\lim_{n\to\infty} \Upsilon_{1,n}^2 \|d_1(x_n)\|^4 = \lim_{n\to\infty} \Upsilon_{i,n}^2 \|d_2(x_n)\|^4 = 0,$$

and

$$\lim_{n\to\infty} \Upsilon_{1,n}'^2 \|d_1(t_n)\|^4 = \lim_{n\to\infty} \Upsilon_{2,n}'^2 \|d_2(t_n)\|^4 = 0.$$

Then, from this we obtain that

$$\lim_{n\to\infty} \Upsilon_{1,n} \|d_1(x_n)\|^2 = \lim_{n\to\infty} \Upsilon_{2,n} \|d_2(x_n)\|^2 = 0,$$

and

$$\lim_{n\to\infty} \Upsilon_{1,n}' \|d_1(t_n)\|^2 = \lim_{n\to\infty} \Upsilon_{2,n}' \|d_2(t_n)\|^2 = 0. \tag{30}$$

Since the sequence $\{(x_n, t_n)\}$ is bounded, there exists a subsequence $\{(x_{n_k}, (x_{n_k})\}$, of $\{(x_n, t_n)\}$ which converges weakly to $(\overline{p}, \overline{q}) \in H_1 \times H_2$ and

$$\begin{aligned} &\limsup_{n\to\infty} [\langle g_1(p) - p, x_n - p\rangle + \langle g_2(q) - p, t_n - q\rangle] \\ &= \lim_{k\to\infty} [\langle g(p) - p, x_{n_k} - p\rangle + g_2(q) - q, t_{n_k} - q\rangle]. \end{aligned} \tag{31}$$

Now, we prove that for $i = 1, 2$

$$\lim_{k\to\infty} \|x_{n_k} - z_{i,n_k}\| = 0, \lim_{k\to\infty} \|t_{n_k} - u_{i,n_k}\| = 0 \tag{32}$$

First consider the case, when $\liminf_{k\to\infty} \Upsilon_{n_k} > 0$ In this case there is $\Upsilon > 0$ such that $\Upsilon_{n_k} > \Upsilon > 0$, for all $k \in N$. Thus, we have

$$\|x_{n_k} - z_{n_k}\|^2 = \frac{1}{\Upsilon_{n_k}} \Upsilon_{n_k} \|x_{n_k} - z_{n_k}\|^2 \le \frac{1}{\Upsilon} \Upsilon_{n_k} \|x_{n_k} - z_{n_k}\|^2.$$

From this inequality and (30), we obtain

$$\lim_{k\to\infty} \|x_{n_k} - z_{n_k}\|^2 = 0$$

and hence

$$\lim_{k\to\infty} \|x_{n_k} - z_{n_k}\|$$

Second consider, when $\lim inf_{k\to\infty} \Upsilon_{i,n_k} = 0$. In this case

$$\lim_{k\to\infty} \Upsilon_{i,n_k} = 0 \text{ and } \lim_{k\to\infty} \|x_{n_k} - z_{i,n_k}\|^2 = c > 0 \tag{33}$$

Consider, $y'_{i,n_k} = \frac{1}{\iota} \Upsilon_{i,n_k} z_{i,n_k} + (1 - \frac{1}{\iota}) \Upsilon_{i,n_k} x_{i,n_k}$

Thus, from (33), we have

$$\lim_{k\to\infty} \|y'_{i,n_k} - z_{i,n_k}\| = \lim_{k\to\infty} \frac{1}{\iota}\Upsilon_{i,n_k}\|.x_{n_k} - z_{i,n_k}\| = 0 \tag{34}$$

From inequality in Step 2 and definition of y'_{i,n_k}, we obtain

$$\begin{aligned} \mu\|x_{n_k} - z_{i,n_k}\|^2 &< \langle x_{n_k} - y'_{i,n_k} + T_i y'_{i,n_k} - T_i x_{n_k}, x_{n_k} - z_{i,n_k}\rangle \\ &\leq \langle x_{n_k} - y'_{i,n_k}, x_{n_k} - z_{i,n_k}\rangle + \langle T_i y'_{i,n_k} - T_i x_{n_k}, x_{n_k} - z_{i,n_k}\rangle \\ &\leq \|x_{n_k} - y'_{i,n_k}\|\|x_{n_k} - z_{i,n_k}\| \\ &+ \| + T_i y'_{i,n_k} - T_i x_{n_k}\|\|x_{n_k} - z_{i,n_k}\|. \end{aligned} \tag{35}$$

From (34), (35) and the fact that T_i is uniformly continuous, we get $\lim_{n\to\infty}\|x_{n_k} - z_{i,n_k}\| = 0$, which contradict (33). In a similar way we can show that $\lim_{n\to\infty}\|t_{n_k} - u_{i,n_k}\| = 0$ Thus, from this fact the equations (32) hold. Moreover, since $\{(x_{n_k}, t_{n_k})\}$, which converges weakly to $(\bar{p},\bar{q})$, then $x_{n_k} \rightharpoonup \bar{p}$ and $t_{n_k} \rightharpoonup \bar{q}$. Thus, from (33) and Lemma 3.4, we get $\bar{p} \in VI(C,T_1)\cap VI(C,T_2)$ and $\bar{q} \in VI(D,S_1)\cap VI(D,S_2)$.
Next we show that $A\bar{p} = B\bar{q}$. But, observe that from Lemma 2.2 (ii) we get

$$\begin{aligned} \|A\bar{p} - B\bar{q}\|^2 &= \|A\bar{p} - Ax_{n_k} + Bt_{n_k} - B\bar{q} + Ax_{n_k} - Bt_{n_k}\|^2 \\ &\leq \|Ax_{n_k} - Bt_{n_k}\|^2 + 2\langle A\bar{p} - B\bar{q}, A\bar{p} - Ax_{n_k} + Bt_{n_k} - B\bar{q}\rangle, \\ &\to 0 \text{ as } k\to\infty, \end{aligned}$$

and this implies $A\bar{p} = B\bar{q}$. That is $(\bar{p},\bar{q}) \in \Omega$. From the definition of x_{n+1} and t_{n+1}, we have $\|x_{n+1} - w_n\| = \alpha_n\|g_1(x_n) - w_n\| \to 0$, as $n\to\infty$, and $\|t_{n+1} - r_n\| = \alpha_n\|g_2(t_n) - r_n\| \to 0$, as $n\to\infty$, since $\alpha_n\to\infty$, as $n\to\infty$. From (27) and (29), we get

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq \|x_{n+1} - w_n\| + \|w_n - x_n\| \\ &\leq \|x_{n+1} - w_n\| + \|\theta_n a_n + \beta_n p_{1,n} + \eta_n p_{2,n} - x_n\| \\ &\leq \|x_{n+1} - w_n\| + \theta_n\|a_n - x_n\| \\ &\quad +\beta_n\|p_{1,n} - x_n\| + \eta_n\|p_{2,n} - x_n\| \to 0 \text{ as } n\to 0. \end{aligned} \tag{36}$$

Moreover,

$$\|x_n - w_n\| \leq \theta_n\|a_n - x_n\| + \beta_n\|p_{1,n} - x_n\| + \eta_n\|p_{2,n} - x_n\| \to 0, \text{ as } n\to 0. \tag{37}$$

Thus, from (36) and (37), we obtain

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|x_{n+1} - w_n + w_n - x_n\| \\ &\leq \|x_{n+1} - w_n\| + \|w_n - x_n\| \to 0, \text{ as } n\to 0. \end{aligned} \tag{38}$$

Similarly we can show that

$$\|t_{n+1} - t_n\| \quad \to 0, \text{ as } n\to 0.$$

From (31) and Lemma 2.3, we have

$$\begin{aligned} \limsup_{n\to\infty}[\langle g_1(p) - p, x_n - p\rangle + \langle g_2(q) - q, t_n - q\rangle] &\leq \lim_{k\to\infty}[\langle g_1(p) - q, x_{n_k} - p\rangle \\ &\quad +\langle g_2(q) - q, t_{n_k} - q\rangle] \\ &= \langle g_1(p) - p, \bar{p} - p\rangle + \langle g_2(q) - q, \bar{q} - q\rangle \\ &\leq 0. \end{aligned} \tag{39}$$

Now, we show that the sequence $\{R_n(p,q)\}$ converges strongly to 0. Indeed, from Lemma 2.2 and 26, we obtain

$$\begin{aligned} R_{n+1}(p,q) &\le (1-(1-\alpha)2\alpha_n)R_n(p,q) \\ &+\alpha_n(1-\alpha)\frac{2}{1-\alpha}[\langle g_1(p)-p, x_{n+1}-p\rangle \\ &+\langle g_2(q)-q, t_{n+1}-q\rangle] \end{aligned} \tag{40}$$

Finally, from (40), (38), (39) and Lemma 2.5, we get $R_n(p,q) \to 0$, as $n \to \infty$ and hence $x_n \to p$ and $t_n \to q$ as $n \to \infty$.

Case 2: Suppose that there exists a subsequence $\{R_{n_j}(p,q)\}$ of $\{R_n(p,q)\}$ such that

$$R_{n_j}(p,q) < R_{n_j+1}(p,q), \text{ for } j \ge 0. \tag{41}$$

Thus by Lemma 2.4 there exists a non-decreasing sequence $\{m_k\}$, of the set of positive integer of numbers such that $m_k \to 0$, as $k \to \infty$,
$\|x_{m_k} - p\|^2 \le \|x_{m_k+1} - p\|^2$ and

$$\max\{R_{m_k}(p,q), R_k(p,q)\} \le R_{m_k+1}(p,q) \text{ for all } k \ge 1.$$

Following the method of Case 1, we obtain

$$\lim_{k\to\infty} \|P_{C_{1,m_k}} x_{m_k} - x_{m_k}\|^2 = \lim_{k\to\infty} \|P_{C_{2,m_k}} x_{m_k} - x_{m_k}\|^2 = 0,$$

$$\lim_{k\to\infty} \|P_{D_{1,m_k}} t_{m_k} - t_{m_k}\| = \lim_{k\to\infty} \|P_{D_{2,m_k}} x_{m_k} - t_{m_k}\| = 0,$$

$$\lim_{k\to\infty} \|Ax_{m_k} - Bt_{m_k}\| = 0,$$

and

$$\lim_{k\to\infty} \|x_{m_k} - a_{m_k} - \gamma_{m_k} A^*(Ax_{m_k} - Bt_{m_k})\| = \lim_{k\to\infty} \|t_{m_k} - b_{m_k} - \gamma_{m_k} B^*(Bt_{m_k} - Ax_{m_k})\| = 0,$$

In addition, by following the method of Case 1, from the inequality (31), for $i = 1, 2$, we obtain

$$\lim_{k\to\infty} \|x_{m_k} - p_{i,m_k}\| = \lim_{k\to\infty} \|t_{n_k} - q_{i,m_k}\| = 0.$$

In addition, for $i = 1, 2$

$$\lim_{k\to\infty} \|x_{m_k} - z_{i,m_k}\| = 0 = \lim_{k\to\infty} \|t_{m_k} - u_{i,m_k}\| = 0,$$

$$\lim_{k\to\infty} \|x_{m_k} - x_{m_k+1}\| = 0 = \lim_{k\to\infty} \|t_{m_k} - t_{m_k+1}\| = 0,$$

and

$$\limsup_{k\to\infty} [\langle g_1(p) - p, x_{m_k+1} - p\rangle + \langle g_2(q) - q, t_{m_k+1} - q\rangle] \le 0. \tag{42}$$

Now, from (40), we get

$$\begin{aligned} R_{m_k+1}(p,q) &\le (1-(1-\alpha)\alpha_{m_k})R_{n_k}(p,q) \\ &+\alpha_{m_k}(1-\alpha)\frac{2}{1-\alpha}[\langle g_1(p)-p, x_{m_k+1}-p\rangle \\ &+\langle g_2(q)-q, t_{m_k+1}-q\rangle] \\ &\le (1-(1-\alpha)2\alpha_n)R_{m_k+1}(p,q) \\ &+\alpha_{m_k}(1-\alpha)\frac{2}{1-\alpha}[\langle g_1(p)-p, x_{m_k+1}-p\rangle \\ &+\langle g_2(q)-q, t_{m_k+1}-q\rangle] \end{aligned}$$

which implies that

$$\begin{aligned}\alpha_{m_k}(1-\alpha)R_{m_k+1}(p,q) &\leq \alpha_{m_k}(1-\alpha)\frac{2}{1-\alpha}[\langle g_1(p)-p, x_{m_k+1}-p\rangle \\ &+\langle g_2(q)-q, t_{m_k+1}-q\rangle]. \end{aligned} \tag{43}$$

Thus, from (41) and (43), we have

$$\begin{aligned}R_k(p,q) \leq R_{m_k+1}(p,q) &\leq \frac{2}{1-\alpha}[\langle g_1(p)-p, x_{m_k+1}-p\rangle \\ &+\langle g_2(q)-q, t_{m_k+1}-q\rangle].\end{aligned}$$

Hence using (42), we get

$$\begin{aligned}\limsup_{k\to\infty} R_k(p,q) &\leq \limsup_{k\to\infty} \frac{2}{1-\alpha}[\langle g_1(p)-p, x_{m_k+1}-p\rangle \\ &+\langle g_2(q)-q, t_{m_k+1}-q\rangle] \leq 0,\end{aligned}$$

which implies

$$\limsup_{k\to\infty} R_k(p,q) = 0,$$

and hence $x_k \to p$ and $t_k \to q$ as, $k \to \infty$. □

We note that the method of proof of Theorem 3.6 provides the following result for split equality variational inequality problems of a finite family of pseudomonotone mappings in Hilbert spaces.

Algorithm 2. For arbitrary $(x_0, t_0) \in H_1 \times H_2$, define an iterative algorithm by

Step 1. Compute

$$\begin{cases} z_{i,n} = P_C(x_n - \delta T_i x_n) \text{ and } d_i(x_n) = x_n - z_{i,n}, & \text{for } i = 1,2,\cdots,m \\ u_{i,n} = P_D(t_n - \delta S_i t_n) \text{ and } d_i(t_n) = t_n - u_{i,n}, & \text{for } i = 1,2,\cdots,m. \end{cases}$$

Step 2. Compute

$$\begin{cases} y_{i,n} = x_n - \Upsilon_{i,n} d(x_n), & \text{for } i = 1,2,\cdots,m \\ v_{i,n} = t_n - \Upsilon'_{i,n} d(t_n), & \text{for } i = 1,2,\cdots,m \end{cases}$$

where, $\Upsilon_{i,n} = \iota^{j_{i,n}}$ such that $j_{i,n}$ is the smallest nonnegative integer j_i satisfying

$$\langle T_i x_n - T_i(x_n - \iota^{j_i} d_i(x_n)), d_i(x_n)\rangle \leq \mu \|d_i(x_n)\|^2,$$

and $\Upsilon'_n = \iota^{j'_{i,n}}$ such that $j'_{i,n}$ is the smallest nonnegative integer j'_i satisfying

$$\langle S_i t_n - S_i(t_n - \iota^{j'_i} d_i(t_n)), d_i(t_n)\rangle \leq \mu \|d_i(t_n)\|^2.$$

Step 3. Compute

$$\begin{cases} a_n = P_C(x_n - \gamma_n A^*(Ax_n - Bt_n)), \\ b_n = P_C(t_n - \gamma_n B^*(Bt_n - Ax_n)), \\ w_n = \theta_n a_n + \beta_{1,n} p_{1,n} + \beta_{2,n} p_{2,n} + \cdots + \beta_{m,n} p_{m,n}, \\ r_n = \theta_n b_n + \beta_{1,n} q_{1,n} + \beta_{2,n} q_{2,n} + \cdots + \beta_{m,n} q_{m,n}, \end{cases}$$

where $C_{i,n} = \{x \in H : h_{i,n} = \langle y_{i,n} - T_i y_{i,n}, x - y_{i,n}\rangle \leq 0\}$,

$D_{i,n} = \{x \in H : e_{i,n} = \langle v_{i,n} - S_i v_{i,n}, x - v_{i,n} \rangle \leq 0\}$ and $\{\theta_n\}, \{\beta_n\}, \{\eta_n\} \subset [\rho, 1)$ for $\rho > 0$ such that $\beta_n + \theta_n + \eta_n = 1$ for all $n \geq 0$ and $p_{i,n} = P_{C_{i,n}} x_n$, $q_{i,n} = P_{D_{i,n}} t_n$ for $i = 1, 2, \cdots, m$.

Step 4. Compute

$$\begin{cases} x_{n+1} = \alpha_n g_1(x_n) + (1 - \alpha_n) w_n, \\ t_{n+1} = \alpha_n g_2(t_n) + (1 - \alpha_n) r_n. \end{cases}$$

Step 5. Set $n := n + 1$ and go to **Step 1**.

Theorem 3.7. *Suppose Assumption* $(A3) - (A6)$ *hold. Let* $T_i : H_1 \to H_1$ *and* $S_i : H_2 \to H_2$ *sequentially weakly continuous and uniformly continuous pseudomonotone mappings on bounded subset of* H_1 *and* H_2*, respectively for* $i = 1, 2, ..., m$ *such that*

$$\Omega := \{(p, q) \in H_1 \times H_2 : p \in \cap_{i=1}^{m} VI(C, T_i), q \in \cap_{i=1}^{m} VI(D, S_i) \text{ and } Ap = Bq\} \neq \emptyset.$$

Then, the sequence $\{(x_n, t_n)\}$ *generated by Algorithm 2 converges strongly to an element* $(p, q) = P_\Omega(g_1(p), g_2(q))$.

Corollary 3.8. *Suppose Assumption* $(A3) - (A6)$ *hold. Let* $T_i : H_1 \to H_1$ *and* $S_i : H_2 \to H_2$ *be uniformly continuous monotone mappings on bounded subset of* H_1 *and* H_2*, respectively for* $i = 1, 2, ..., m$ *such that*

$$\Omega := \{(p, q) \in H_1 \times H_2 : p \in \cap_{i=1}^{m} VI(C, T_i), q \in \cap_{i=1}^{m} VI(D, S_i) \text{ and } Ap = Bq\} \neq \emptyset.$$

Then, the sequence $\{(x_n, t_n)\}$ *generated by Algorithm 2 converges strongly to an element* $(p, q) = P_\Omega(g_1(p), g_2(q))$.

If in Theorem 3.7, we assume $g_1(x) = u$ for all $x \in C$ and $g_2(t) = v$ for all $t \in D$, we get the following result.

Corollary 3.9. *Suppose Assumption* $(A3), (A4)$ *and* $(A6)$ *hold. Let* $T_i : H_1 \to H_1$ *and* $S_i : H_2 \to H_2$ *sequentially weakly continuous and uniformly continuous pseudomonotone mappings on bounded subset of* H_1 *and* H_2*, respectively for* $i = 1, 2, ..., m$ *such that*

$$\Omega := \{(p, q) \in H_1 \times H_2 : p \in \cap_{i=1}^{m} VI(C, T_i), q \in \cap_{i=1}^{m} VI(D, S_i) \text{ and } Ap = Bq\} \neq \emptyset.$$

Then, the sequence $\{(x_n, t_n)\}$ *generated by Algorithm 2* $g_1(x) = u$ *for all* $x \in C$ *and* $g_2(t) = v$ *for all* $t \in D$ *converges strongly to an element* $(p, q) = P_\Omega(u, v)$.

4. Application

In this section we present some applications of Theorem 3.7.

4.1. Split Variational Inequality Problem

Let H_1 and H_2 be real Hilbert spaces. Let $C \subset H_1$ and $D \subset H_2$ be two nonempty, closed and convex sets; let $T : H_1 \to H_1$ and $S : H_2 \to H_2$ be two given mappings and $A : H_1 \to H_2$ be a bounded linear mapping. The split variational inequality problem (SVIP) introduced by Censor, Gibali and Reich [1] can mathematically be formulated as the problem of finding:

$$\begin{cases} x^* \in C \text{ such that } \langle T(x^*), x - x^* \rangle \geq 0 \text{ for all } x \in C, \\ \qquad\qquad \text{and} \\ y^* = Ax^* \in D \text{ solves } \langle S(y^*), y - y^* \rangle \geq 0 \text{ for all } y \in D. \end{cases}$$

Thus, in Algorithm 2, we assume $H_2 = H_3$ and $B = I$, then split equality variational inequality problem reduces to split variational inequality problem for pseudomonotone mappings and the method of proof of Theorem 3.7 provides the following corollary for approximating a solution of split variational inequality problem for a finite family of pseudomonotone mappings in Hilbert spaces.

Corollary 4.1. *Suppose Assumption $(A3)-(A6)$ hold. Let $T_i : H_1 \to H_1$ and $S_i : H_2 \to H_2$ sequentially weakly continuous and uniformly continuous pseudomonotone mappings on bounded subset of H_1 and H_2, respectively for $i = 1, 2, ..., m$ such that*

$$\Omega := \{(p, q) \in H_1 \times H_2 : p \in \cap_{i=1}^{m} VI(C, T_i), q \in \cap_{i=1}^{m} VI(D, S_i) \text{ and } Ap = q\} \neq \emptyset.$$

Then, the sequence $\{(x_n, t_n)\}$ generated by Algorithm 2 converges strongly to an element $(p, q) = P_{\Omega}(g_1(p), g_2(q))$.

4.2. Split Equality Zero Point Problem

If in Algorithm 2, we assume $C = H_1$ and $D = H_2$, then $P_C = I_1$, $P_D = I_2$ and hence $VI(C, T_i) = T_i^{-1}(0)$ and $VI(C, S_i) = S_i^{-1}(0)$ where I_1 and I_2 are identity mappings in H_1 and H_2, respectively. Thus, split equality variational inequality problem reduces to split equality zero point problem and the method of proof of Theorem 3.7 provides the following corollary for approximating a solution of split equality zero point problem for pseudomonotone mappings in Hilbert spaces.

Corollary 4.2. *Suppose Assumption $(A3)-(A6)$ hold. Let $T_i : H_1 \to H_1$ and $S_i : H_2 \to H_2$ sequentially weakly continuous and uniformly continuous pseudomonotone mappings on bounded subset of H_1 and H_2, respectively for $i = 1, 2, ..., m$ such that*

$$\Omega := \{(p, q) \in H_1 \times H_2 : p \in \cap_{i=1}^{m} T_i^{-1}(0), q \in \cap_{i=1}^{m} S_i^{-1}(0) \text{ and } Ap = Bq\} \neq \emptyset.$$

Then, the sequence $\{(x_n, t_n)\}$ generated by Algorithm 2 converges strongly to an element $(p, q) = P_{\Omega}(g_1(p), g_2(q))$.

5. Numerical Example

In this section, we provide a numerical example to explain the conclusion of our main result. The following numerical example verifies the conclusion of Theorem 3.6.

Example 5.1. *Let $H_1 = H_2 = H_3 = \mathbb{R}^3$ be with the standard topology. Let $C = \{x \in \mathbb{R}^3 : \|x\| \leq 1\}$ and $D = \{x \in \mathbb{R}^3 : \|x\| \leq 2\}$. Let $T_1, T_2 : C \to \mathbb{R}^3$ be defined by $T_1(x) = \frac{3}{2}x - \|x\|x$ and $T_2(x) = x$, were $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. Let $S_1, S_2 : D \to \mathbb{R}^3$ be defined by $S_1(x) = (x_1 + 1, x_2 - 1, 2x_3)$ and $S_2(x) = (\frac{x_1+1}{2}, \frac{2x_2-2}{3}, \frac{x_3}{4})$, then T_1 and T_2 are continuous pseudomonotone and hence they are sequentially weakly continuous and uniformly continuous pseudomonotone mappings on C with $VI(C, T_1) \cap VI(C, T_2) = \{(0, 0, 0)\}$. In addition, one can observe that S_1 and S_2 are monotone and hence they are sequentially weakly continuous and uniformly continuous pseudomonotone mappings on D with $VI(C, S_1) \cap VI(D, S_2) = \{(-1, 1, 0)\}$. Let $A, B : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $A(x) = (2x_1, x_2, 3x_3)$ and $B(x) = (0, 0, 2x_3)$, were $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. Thus, $A(0, 0, 0) = (0, 0, 0) = B(-1, 1, 0)$ and hence $\Omega \neq \emptyset$. Let $g_1 : H_1 \to H_1$ and $g_2 : H_2 \to H_2$ be defined by $g_1(x) = \frac{x}{5}$ and $g_2(x) = \frac{x}{2}$, respectively, were $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. Now, if we assume $\alpha_n = \frac{1}{10n+10^5}$, $l = 0.5$, $\mu = 0.9$, $\theta = 1$, $\beta_n = \frac{1}{n+10000} + 0.02 = \theta_n$ and $\eta_n = 0.96 - \frac{2}{n+10000}$ for all $n \geq 0$, and take different initial points $(x_0, t_0) = ((0.1, 0.2, 0.3), (1.0, 1.0, 0.0))$, $(x_0^{'}, t_0^{'}) = ((0.4, 0.3, 0.1), (1.2, 0.6, 0.1))$ and $(x_0^{''}, t_0^{''}) = ((0.3, 0.1, 0.2), (-1.0, 0.5, 0.3)$, then in all cases, the numerical experiment results using MATLAB provide that the sequence $\{(x_n, t_n)\}$ generated by Algorithm 1 converges strongly to $(p, q) = ((0, 0, 0), (-1, 1, 0))$. (see, Figure 5.1, below).*

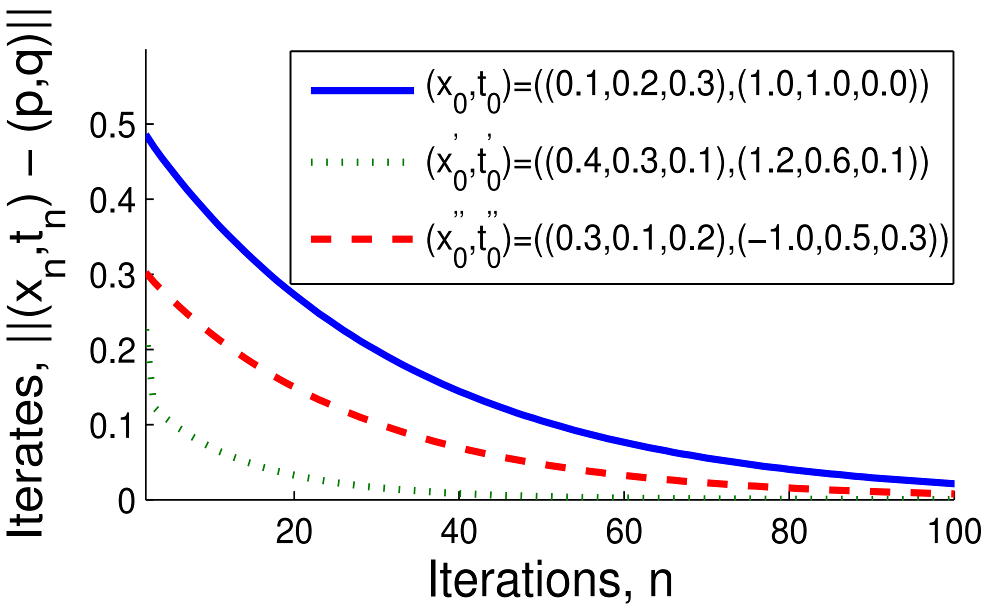

Figure 1: The graph of $\|(x_n, t_n) - (p, q)\|$ versus number of iterations with different choices of (x_0, t_0)

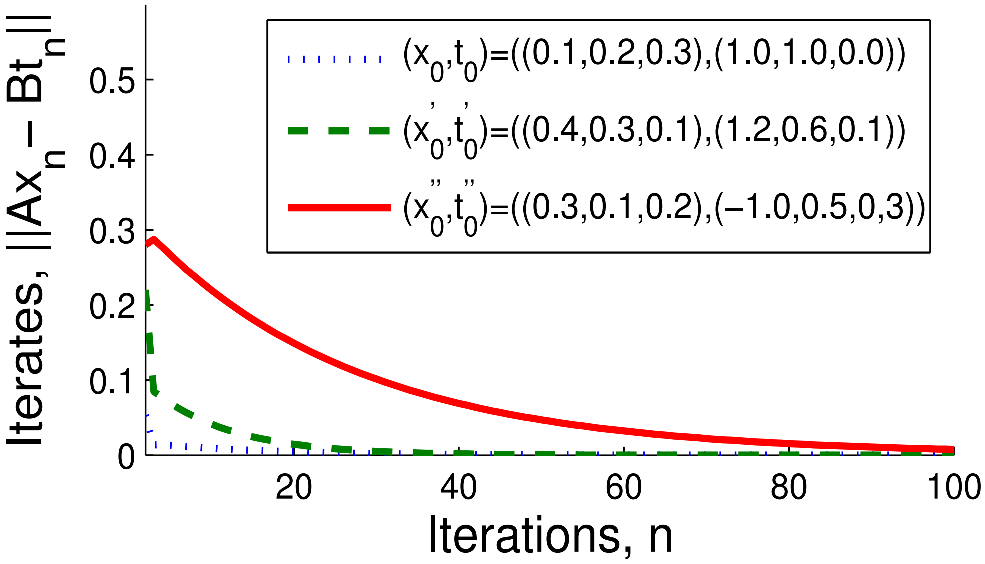

Figure 2: The graph of $\|Ax_n - Bt_n\|$ versus number of iterations with different choices of (x_0, t_0)

In addition, we have sketched the difference term $\|Ax_n - Bt_n\|$ for each initial point. From the sketch we observe that $\|Ax_n - Bt_n\| \to 0$ as $n \to \infty$ (see, Figure 5.1, below).

6. Conclusions

In conclusion, this research article has focused on studying iterative algorithms for approximating a common solution to split equality monotone inclusion problems for a finite family of pseudomonotone mappings in Hilbert spaces. The significance of the split equality problem and variational inequalities in various fields such as decomposition methods for partial differential equations, game theory, and medical image reconstruction has been highlighted. Building on the works of previous researchers, this study contributes to the understanding and development of algorithms for solving split equality variational inequality problems. By introducing necessary notions and definitions, the paper lays the foundation for further exploration and advancement in this area of applied mathematics. The findings and methods presented in this article provide valuable insights for future research in the field of variational inequalities and optimization theory.

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