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Extension of the notion of P-symmetric operators using the Aluthge transform I

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Abstract. The class of $\tilde{P}$-symmetric operators is introduced. Certain properties of this class of operators are obtained. Among other things, it is shown that (1) This class includes the quasinormal operators, isometries, co-isometries, partial isometries with a square normal, the cyclic subnormal operators and all P-symmetric operators (2) If A is an iw-hyponormal operator then A is P-symmetric if and only if $\tilde{A}$ is P-symmetric. Conditions under which a $\tilde{P}$-symmetric operator becomes normal are given.

Introduction

Let $\mathcal{L}(H)$ denote the algebra of all bounded linear operators on a separable infinite dimensional complex Hilbert space H. Given $A, B \in \mathcal{L}(H)$, the generalized derivation δ_{AB} as an operator on $\mathcal{L}(H)$ is defined by

$$\delta_{A,B}(X) = AX - XB.$$

The operator δ_{AB} was initially systematically studied in [15]. The properties of such operators have been studied extensively ([16], [18]). When $A = B$, we simply write δ_A for $\delta_{A,A}$, and is called the inner derivation induced by A.
Let $T = U|T|$ be the polar decomposition of T, where U is partial isometry and $|T|$ is a positive square root of T^*T, that is $|T| = (T^*T)^{\frac{1}{2}}$, with the condition $\ker(T) = \ker(|T|) = \ker(U)$, where $\ker(T)$ denotes the kernel of T. Aluthge [1] introduced the the operator $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ which is called the Aluthge transform of T.
Recently some investigation in the operator theory have been related to relationship between operators and their Aluthge transform, see for example ([2],[12]).
An operator $A \in \mathcal{L}(H)$ is called P-symmetric if $AT = TA$ implies $A^*T = TA^*$ for every $T \in C_1(H)$, where $C_1(H)$ is the ideal of trace class operators and S^* denote the adjoint of $S \in \mathcal{L}(H)$. S. Bouali and J. Charles ([6], [7]) introduced the class of P-symmetric operators, and they gave some basic properties of those operators. In this paper, we would like to explore this class of operators. We initiate the study of a more general class

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of operators A that have the following property: $AT = TA$ implies $\tilde{A}T = T\tilde{A}$ for all $T \in C_1(H)$, where $\tilde{S}$ denote the Aluthge transform of $S \in \mathcal{L}(H)$. We call such operators $\tilde{P}$-symmetric. In the first part We use different arguments to establish a characterization and some basic properties of $\tilde{P}$-symmetric operators. In the second part we give some properties concerning this class, of the same type as those established for the P-symmetric operators. We conclude this section with some notations.
Let $\mathcal{K}(\mathcal{H})$ and $C_1(\mathcal{H})$ be respectively the ideal of compact operators and the ideal of trace class operators on $\mathcal{H}$. The trace function is defined on $C_1(\mathcal{H})$ by

$$tr(T) = \sum_n < Te_n, e_n >,$$

where (e_n) is any complete orthonormal system in $\mathcal{H}$. For X a linear operator acting on Banach space E, we denote by X^*, $\ker(X)$, $\ker^{\perp}(X)$, $R(X)$ and $X|M$ respectively the adjoint, the kernel, the orthogonal complement of the kernel, the range of X and the restriction of X to an invariant subspace M. Also we denote by $\sigma(X)$, $\overline{R(X)}$ and $\overline{R(X)}^{w^*}$ respectively The spectrum of X, the closure of the range of X respect to the norm topology and the ultra-weak topology. Given $\mathcal{B}$ be a Banach and $\mathcal{S}$ be a subspace of $\mathcal{B}$. By $\mathcal{B}'$ we denote the dual of $\mathcal{S}$, the set

$$\mathcal{S}^{\circ} = \{\Phi \in \mathcal{L}'(\mathcal{H}) : \Phi(x) = 0 \quad \forall x \in \mathcal{S}\}.$$

denotes the annihilator of $\mathcal{S}$. For g and ω two vectors in H, we define $g \otimes \omega \in \mathcal{L}(H)$ as follows :

$$g \otimes \omega(x) = \langle x, \omega \rangle g \text{ for all } x \in H.$$

Recall that an operator $T \in \mathcal{L}(H)$ is said to be hyponormal if $T^*T \geq TT^*$. Hyponormal operators have been studied by many authors and it is known that hyponormal operators have many interesting properties similar to those of normal operators [11]. An operator T is said to be p-hyponormal if $(T^*T)^p \geq (TT^*)^p$ for $p \in]0, 1]$ and an operator T is said to be log-hyponormal if T is invertible and $\log|T| \geq \log|T^*|$. p-hyponormal and log-hyponormal operators are defined as extension of hyponormal operator. An operator T is called w-hyponormal if $|\tilde{T}| \geq |T| \geq |\tilde{T}^*|$. The classes of log- and w-hyponormal operators were introduced, and their properties were studied in [2]. In particular, it was shown in [2] that the class of w-hyponormal operators contains both p-and log-hyponormal operators.

1. $\tilde{P}$-symmetric operators

Definition 1.1 ([6]). *Let $A \in L(H)$. The operator A is said to be P-symmetric if it satisfies the following property: $AT = TA$ implies $A^*T = TA^*$ for every $T \in C_1(H)$.*

Theorem 1.2. *([6]) An operator A in $\mathcal{L}(H)$ is P-symmetric if and only if $\overline{\mathcal{R}(\delta_A)}^{w^*} = \overline{\mathcal{R}(\delta_{A^*})}^{w^*}$, i.e. $\overline{\mathcal{R}(\delta_A)}^{w^*}$ is a self adjoint subspace of $\mathcal{L}(H)$.*

Definition 1.3. *Let $A \in \mathcal{L}(H)$. We say that A is $\tilde{P}$-symmetric if $AT = TA$ implies $\tilde{A}T = T\tilde{A}$ for all $T \in C_1(H)$. We denote the class of $\tilde{P}$-symmetric operators by $\tilde{\mathcal{P}}(H)$.*

Example 1.4.
1. *Recall that an operator A is called quasinormal if $A(A^*A) = (A^*A)A$. It was shown in [4, Proposition 1.10] that $\tilde{A} = A$ if and only if A is quasinormal. It follows that if A is quasinormal then A is $\tilde{P}$-symmetric.*
2. *If A is normal then A is $\tilde{P}$-symmetric.*

Next we wish to extend the $\tilde{P}$-symmetry for a large class of operators.

Definition 1.5. *A vector $e_{\circ} \in H$ is cyclic for $A \in \mathcal{L}(H)$, if H is the smallest invariant subspace for A that contains e_0. The operator A is said to be cyclic if it has a cyclic vector.*

Definition 1.6. *An operator $A \in \mathcal{L}(H)$ is called subnormal, if there exists a Hilbert space K and a normal operator $N \in \mathcal{L}(K)$, such that H is a subspace of K and $A = N|H$. The Operator N is called a normal extension of A.*

Theorem 1.7. *Let $A \in \mathcal{L}(H)$ be a cyclic subnormal operator. Then A is $\tilde{P}$-symmetric.*

Proof. Let $A = U|A|$ and $T = V|T| \in C_1(H)$ be the polar decompositions of A and T respectively. Since $AT = TA$ and A is cyclic subnormal, then T is subnormal by [19, Theorem 3]. Also, since T is compact, it follows that T is normal, and so Fuglede's theorem [9, Theorem I] ensures that $AT^* = T^*A$. Hence we obtain from [10, Corollary 3, p64] that

$$AT = UV|AT| = (UV)(|A||T|) \quad \text{and} \quad TA = VU|TA| = (VU)|T||A|$$

are the polar decompositions of AT and TA respectively. Then we have

$$\widetilde{AT} = (|A||T|)^{\frac{1}{2}} UV(|A||T|)^{\frac{1}{2}} \quad \text{and} \quad \widetilde{TA} = (|T||A|)^{\frac{1}{2}} VU(|T||A|)^{\frac{1}{2}}$$

are the Aluthge transform of AT and TA respectively. On the other hand, by [10, Theorem 2, p64] we get

$$(1)\ \ |A||T| = |T||A|\,, \quad (2)\ \ U|T| = |T|U, \quad (3)\ \ |A|V = V|A|.$$

Let $\{P_n\}$ be a sequence of polynomials with no constant term such that $P_n(t) \to t^{\frac{1}{2}}$ uniformly on a certain compact set as $n \to \infty$. Thus, it results from (2) and (3) that

$$UP_n(|T|) = P_n(|T|)\,U \quad \text{and} \quad VP_n(|A|) = P_n(|A|)\,V.$$

Hence $U|T|^{\frac{1}{2}} = |T|^{\frac{1}{2}}U$ and $V|A|^{\frac{1}{2}} = |A|^{\frac{1}{2}}V$. Furthermore, by using the assumption $AT = TA$ with T is normal and the result (1) there holds

$$\begin{aligned}
\widetilde{AT} &= \widetilde{TA} \\
|A|^{\frac{1}{2}}|T|^{\frac{1}{2}}UV|A|^{\frac{1}{2}}|T|^{\frac{1}{2}} &= |T|^{\frac{1}{2}}|A|^{\frac{1}{2}}VU|T|^{\frac{1}{2}}|A|^{\frac{1}{2}} \\
|A|^{\frac{1}{2}}U|T|^{\frac{1}{2}}|A|^{\frac{1}{2}}V|T|^{\frac{1}{2}} &= |T|^{\frac{1}{2}}V|A|^{\frac{1}{2}}|T|^{\frac{1}{2}}U|A|^{\frac{1}{2}} \\
|A|^{\frac{1}{2}}U|A|^{\frac{1}{2}}|T|^{\frac{1}{2}}V|T|^{\frac{1}{2}} &= |T|^{\frac{1}{2}}V|T|^{\frac{1}{2}}|A|^{\frac{1}{2}}U|A|^{\frac{1}{2}} \\
\tilde{A}\tilde{T} &= \tilde{T}\tilde{A} \\
\tilde{A}T &= T\tilde{A}
\end{aligned}$$

This completes the proof. □

Corollary 1.8. *Let $A \in \mathcal{L}(H)$. Suppose that $f(A)$ is a cyclic subnormal operator, where f is a nonconstant analytic function on an open set containing $\sigma(A)$. Then A is $\tilde{P}$-symmetric.*

Proof. Since $Af(A) = f(A)A$, the result follows form [19, Theorem 3] and Theorem 1.7. □

Lemma 1.9. *If $A \in \mathcal{L}(H)$ is a partial isometry, then $A = A|A|$ is the polar decomposition of A and $\tilde{A} = A^*A^2$ is the Aluthge transform of A.*

Proof. If A is a partial isometry, then we have A^*A is a projection. It follows that $(A^*A)^2 = A^*A$, and so $|A| = A^*A$. We get $A = A|A|$ is the polar decomposition of A, then the Aluthge transform of A is given by

$$\tilde{A} = |A|^{\frac{1}{2}}A|A|^{\frac{1}{2}} = (A^*A)A(A^*A) = A^*A^2.$$

□

Theorem 1.10. *If $A \in \mathcal{L}(H)$ is an isometry or co-isometry, then A is $\tilde{P}$-symmetric.*

Proof. If A is isometric, then A is quasinormal and so A is trivially $\tilde{P}$-symmetric. If A is a co-isometry, it is well known that A is partial isometry, then it follows from Lemma 1.9 that the Aluthge transform of A is given by $\tilde{A} = A^*A^2$. So, if $T \in C_1(H)$ such that $AT = TA$, then we have $\tilde{A}T = A^*TA^2$ and $T\tilde{A} = TA^*A^2$. On the other hand, A is a contraction and T is a compact with $ATA^* = T$, hence by [17, Theorem 2.2] we obtain $A^*TA = T$ and $A^*T = TA^*$. This implies that $\tilde{A}T = T\tilde{A}$. □

Lemma 1.11. *[18] If $A \in \mathcal{L}(H)$, then*

$$\mathcal{R}(\delta_A)^0 \simeq \mathcal{R}(\delta_A)^0 \cap \mathcal{K}(H)^0 \oplus \{A\}' \cap C_1(H)$$

Theorem 1.12. *Let $A \in \mathcal{L}(H)$ such that* $\ker A \neq \{0\}$ *and* $\{0\} \neq \ker A^* \not\subset \ker \tilde{A}^*$ *where $\tilde{A}^*$ is the adjoint operator of the Aluthge transform of A, then A is not $\tilde{P}$-symmetric.*

Proof. From the hypothesis, there exists two nonzero elements f and g in H such that $A(f) = 0$, $A^*(g) = 0$ and $\tilde{A}^*(g) \neq 0$ and since $\ker(A) = \ker(|A|) = \ker(|A|^{\frac{1}{2}})$ we get $\tilde{A}(f) = 0$. Note that $\tilde{A}^*(g) = \omega \neq 0$. If $X = \|f\|^{-2}(\omega \otimes f)$ and $Y \in \mathcal{L}(H)$, then

$$\begin{aligned}\left\langle \left(\tilde{A}X - X\tilde{A}\right) f, g\right\rangle &= \left\langle \tilde{A}X(f), g\right\rangle - \left\langle X\tilde{A}(f), g\right\rangle \\ &= \langle X(f), w\rangle - \langle 0, g\rangle \\ &= \|\omega\|^2\end{aligned}$$

and

$$< (AY - YA)f, g >=< Yf, A^*g > - < 0, g >= 0$$

Suppose that A $\tilde{P}$-symmetric, it follows from Lemma 1.11 that $\overline{\mathcal{R}(\delta_{\tilde{A}})}^{w^*} \subset \overline{\mathcal{R}(\delta_A)}^{w^*}$. Then $\tilde{A}X - X\tilde{A} \in \overline{\mathcal{R}(\delta_A)}^{w^*}$ and there exists a net $(Y_\alpha)_\alpha$ in $\mathcal{L}(H)$ such that for all x and y in H, we have :

$$< (AY_\alpha - Y_\alpha A) x, y > \longrightarrow < \left(\tilde{A}X - X\tilde{A}\right) x, y >$$

So that

$$0 =< (AY_\alpha - Y_\alpha A) f, g > \longrightarrow \left\langle \left(\tilde{A}X - X\tilde{A}\right) f, g\right\rangle = \|\omega\|^2$$

It follows that $\omega = 0$. □

Example 1.13. *Let $(e_n)_{n\geq 1}$ be an orthonormal basis of H. Let us consider $H_0 = Vect\{e_1, e_2, e_3\}$ and define*

$$A_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \in \mathcal{L}(H_0).$$

Then an easy calculation shows that A_0 is a partial isometry. It follows from Lemma 1.9 that the Aluthge transform of A_0 is given by

$$\tilde{A_0} = A_0^* A_0^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let $A = A_0 \oplus I$ with respect the decomposition $H = H_0 \oplus H_0^\perp$, since $\tilde{A} = \tilde{A_0} \oplus I$ then it is easy to see that

$$Ae_3 = 0, \quad A^*(e_2 - \sqrt{3}e_3) = 0 \;\; and \;\; \tilde{A}^*(e_2 - \sqrt{3}e_3) \neq 0.$$

So by Theorem 1.12 the operator A is not $\tilde{P}$-symmetric.

Theorem 1.14. *$\mathcal{P}(H)$ is strictly included in $\tilde{\mathcal{P}}(H)$.*

Proof. Let $A = U|A|$ be a P-symmetric operator, and let $T = V|T| \in C_1(H)$ be such that $AT = TA$. Then we have $A^*T = TA^*$, it follows from [10, Theorem 2, p64] that

$$(1)\ \ |A||T| = |T||A|\,,\quad (2)\ \ U|T| = |T|U\,,\quad (3)\ \ |A|V = V|A|\,,\quad (4)\ \ UV = VU.$$

Let $\{P_n\}$ be a sequence of polynomials with no constant term such that $P_n(t) \to t^{\frac{1}{2}}$ uniformly on a certain compact set as $n \to \infty$, and so by (1) and (3) we get

$$P_n(|A|)\,|T| = P_n(|A|)\,|T| \text{ and } P_n(|A|)\,V = VP_n(|A|)\,,$$

then $|A|^{\frac{1}{2}}|T| = |T||A|^{\frac{1}{2}}$ and $|A|^{\frac{1}{2}}V = V|A|^{\frac{1}{2}}$. Hence we have

$$V|A|^{\frac{1}{2}}|T| = V|T||A|^{\frac{1}{2}} \implies |A|^{\frac{1}{2}}T = T|A|^{\frac{1}{2}}.$$

On the other hand by (2) and (4), one obtains

$$UT = UV|T| = VU|T| = TU,$$

which gives

$$U|A|^{\frac{1}{2}}T = UT|A|^{\frac{1}{2}} = TU|A|^{\frac{1}{2}}.$$

Therefore

$$\tilde{A}T = |A|^{\frac{1}{2}}U|A|^{\frac{1}{2}}T = |A|^{\frac{1}{2}}TU|A|^{\frac{1}{2}} = T|A|^{\frac{1}{2}}U|A|^{\frac{1}{2}} = T\tilde{A}.$$

Consequently, the operator A is $\tilde{P}$-symmetric.
We Now show that the inclusion is proper. Let $(e_n)_{n\geq 1}$ be an orthonormal basis of H, we define the operator $S \in \mathcal{L}(H)$ as follows

$$Se_k = \begin{cases} 0 & \text{if } k = 1 \\ e_{k+1} & \text{if } k \geq 2 \end{cases}$$

A simple calculation shows that S is quasinormal operator, then S is trivially $\tilde{P}$-symmetric. However, it results from [6, Theorem 1.6] that S is not P-symmetric. □

Theorem 1.15. *Let $A \in \mathcal{L}(H)$ be a partial isometry. If A^2 is normal, then A is $\tilde{P}$-symmetric.*

Proof. It follows from Lemma 1.9 that $\tilde{A} = A^*A^2$ is the Aluthge transform of A. So if $T \in C_1(H)$ such that $AT = TA$, then we get $A^2T = TA^2$, that is $A^2A^*AT = TA^2A^*A$. Since $A^2A = AA^2$, it follows from Fuglede's theorem that $A^2A^* = A^*A^2 = \tilde{A}$, hence we get $(\tilde{A}T - T\tilde{A})A = 0$, thus $\tilde{A}T - T\tilde{A}$ vanish on $\overline{R(A)}$. Furthermore, if $x \in \ker(A^*) \subset \ker(A^2)$, then by using $AA^2A^* = AA^*A^2 = A^2$ and $A^2T = TA^2$ we obtain $Tx \in \ker(A^2) = \ker(\tilde{A})$. Consequently, $\tilde{A}T - T\tilde{A}$ vanish also on $\ker(A^*)$. We conclude $\tilde{A}T = T\tilde{A}$, then A is $\tilde{P}$-symmetric. □

Example 1.16. *Let $H_0 = H \oplus H$ and define the operator $A = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} \in \mathcal{L}(H_0)$. Then a straightforward computation shows that A is a partial isometry and A^2 is normal. So by the Theorem 1.15, A is $\tilde{P}$-symmetric but not P-symmetric* (See Example 3.1 in [8]).

Remark 1.17. *S. Bouali and all proved in* [8, Proposition 3.1] *that every nonzero nilpotent operator is not P-symmetric. Then if A is nilpotent of order 2, it results by Theorem 4 in* [12] *that $\tilde{A} = 0$ and hence A is trivially $\tilde{P}$-symmetric. But the following example proves that if $A \in \mathcal{L}(H)$ is a nilpotent operator of order $n \geq 3$, then A is not $\tilde{P}$-symmetric.*

Example 1.18. *Let $H_0 = H \oplus H \oplus H$, and define the operator $A = \begin{pmatrix} 0 & B & 0 \\ 0 & 0 & B \\ 0 & 0 & 0 \end{pmatrix}$ such that $B^2 \neq 0$. If we consider*

$$T = \begin{pmatrix} 0 & C & 0 \\ 0 & 0 & C \\ 0 & 0 & 0 \end{pmatrix} \in C_1(H_0)\,,\quad C \neq 0 \quad \text{and} \quad BC = CB.$$

A simple calculation shows that $A^3 = 0$ and $AT = TA$. On the other hand if $B = V|B|$ is the polar decomposition of B then a computation shows that $A = U|A|$ is the polar decomposition of A where

$$U = \begin{pmatrix} 0 & V & 0 \\ 0 & 0 & V \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad |A| = \begin{pmatrix} 0 & 0 & 0 \\ 0 & |B| & 0 \\ 0 & 0 & |B| \end{pmatrix}.$$

Then the Aluthge transform of A is given by

$$\tilde{A} = |A|^{\frac{1}{2}} U |A|^{\frac{1}{2}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \tilde{B} \\ 0 & 0 & 0 \end{pmatrix},$$

with $\tilde{B} \neq 0$ since $A^2 \neq 0$ and by using [12, Theorem 4] *again. Therefore if we take $B = I$, it follows that $\tilde{B} = I$, $\tilde{A}T - T\tilde{A} = \begin{pmatrix} 0 & 0 & -C \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq 0$ and so A is not $\tilde{P}$-symmetric.*

Proposition 1.19. *Let $A \in C_1(H)$ be a partial isometry. If A is nilpotent of order $n \geq 3$ then A is not $\tilde{P}$-symmetric.*

Proof. Suppose A is $\tilde{P}$-symmetric such that $A \in C_1(H)$, then we have $\tilde{A}A = A\tilde{A}$. Since A is a partial isometry, it follows from Lemma 1.9 that $A^*A^3 = AA^*A^2$. Hence, we obtain $A^*A^3A^{n-3} = A^2A^{n-3}$, from this we get $A^{n-1} = 0$, which is absurd. □

Proposition 1.20. *Let $A \in \mathcal{L}(H)$ be such that $AT = TA$ implies $|A|T = T|A|$ for every $T \in C_1(H)$. Then A is $\tilde{P}$-symmetric.*

Proof. Let $T \in C_1(H)$, such that $AT = TA$. So by hypothesis $T|A|^{\frac{1}{2}} = |A|^{\frac{1}{2}}T$ and since $|A|^{\frac{1}{2}}A = \tilde{A}|A|^{\frac{1}{2}}$, we have $|A|^{\frac{1}{2}}AT = |A|^{\frac{1}{2}}TA$ which implies that $\left(\tilde{A}T - T\tilde{A}\right)|A|^{\frac{1}{2}} = 0$. It follows that $\tilde{A}T - T\tilde{A}$ vanish on $\overline{R(|A|)}$. On the other hand, if $x \in \ker(|A|) = \ker(|A|^{\frac{1}{2}})$ we get $|A|^{\frac{1}{2}}Tx = T|A|^{\frac{1}{2}}x = 0$, hence $\tilde{A}Tx = 0$. It results that $\tilde{A}T - T\tilde{A}$ vanish on $\ker(|A|)$. Consequently

$$\tilde{A}T - T\tilde{A} = 0 \text{ on } H = \overline{R(|A|)} \oplus \ker(|A|).$$

Thus, A is $\tilde{P}$-symmetric. □

Corollary 1.21. *If $A = U|A| \in \mathcal{L}(H)$ such that U is a normal operator. If $UT = TU$ for every $T \in \{A\}' \cap C_1(H)$ then A is $\tilde{P}$-symmetric.*

Proof. Let $T \in C_1(H)$ such that $AT = TA$, then by hypothesis we get $U(|A|T - T|A|) = 0$, and by taking adjoints $|A|T^* - T^*|A|$ vanish on $\overline{R(U^*)}$. Let $x \in \ker(U) = \ker(|A|) = \ker(U^*)$, then from $UT = TU$ we see that $T^*x \in \ker(U^*) = \ker(|A|)$. Hence $|A|T^* - T^*|A|$ vanish also on $\ker(U)$ which means

$$|A|T - T|A| = -\left(|A|T^* - T^*|A|\right)^* = 0 \text{ on } H = \overline{R(U^*)} \oplus \ker(U).$$

which is equivalent to $|A|^{\frac{1}{2}}T - T|A|^{\frac{1}{2}} = 0$ and A is $\tilde{P}$-symmetric by proposition 1.20. □

Lemma 1.22. *Let $A, B \in \mathcal{L}(H)$ and $S = A \oplus B$. Then the Aluthge transform of S is given by $\tilde{S} = \tilde{A} \oplus \tilde{B}$.*

Proof. Let $A = U|A|$, $B = V|B|$ and $S = P|S|$ are the polar decompositions of A, B and S respectively, where P and $|S|$ are defined on $H \oplus H$ by $P = U \oplus V$ and $|S| = |A| \oplus |B|$. It follows that

$$\tilde{S} = |S|^{\frac{1}{2}} P |S|^{\frac{1}{2}} = (|A|^{\frac{1}{2}} \oplus |B|^{\frac{1}{2}})(U \oplus V)(|A|^{\frac{1}{2}} \oplus |B|^{\frac{1}{2}}) = \tilde{A} \oplus \tilde{B}.$$

□

Theorem 1.23. *Let $A \in \mathcal{L}(H)$. If A is $\tilde{P}$-symmetric and $H_0 \subset H$ is a reducing subspace for A, then $A_0 = A|H_0$ is $\tilde{P}$-symmetric.*

Proof. We have $A = A_0 \oplus A_1$ with respect the decomposition $H = H_0 \oplus H_0^{\perp}$. Suppose that $A_0T_0 = T_0A_0$ for $T_0 \in C_1(H_0)$. If $T = \begin{pmatrix} T_0 & 0 \\ 0 & 0 \end{pmatrix}$ then $AT = TA$ and $T \in C_1(H)$. Since A is $\tilde{P}$-symmetric we get $\tilde{A}T = T\tilde{A}$, and $(\tilde{A_0} \oplus \tilde{A_1})T = T(\tilde{A_0} \oplus \tilde{A_1})$, hence $\tilde{A_0}T = T\tilde{A_0}$. □

Theorem 1.24. *Let $A, B \in \mathcal{L}(H)$. If A and B are $\tilde{P}$-symmetric operators with disjoint spectra, then $A \oplus B$ is $\tilde{P}$-symmetric.*

Proof. Let $T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} \in C_1(H \oplus H)$. Then $(A \oplus B)T = T(A \oplus B)$ implies that

$$AT_1 = T_1A, BT_4 = T_4B, AT_2 = T_2B \text{ and } BT_3 = T_3A.$$

Since $\sigma(A) \cap \sigma(B) = \emptyset$, then $\delta_{A,B}$ and $\delta_{B,A}$ are invertible [15, Corollary 3.3]; consequently we have $T_2 = T_3 = 0$. Or A and B are $\tilde{P}$-symmetric then we get $\tilde{A}T_1 = T_1\tilde{A}$ and $\tilde{B}T_4 = T_4\tilde{B}$. This implies that $(\tilde{A} \oplus \tilde{B})T = T(\tilde{A} \oplus \tilde{B})$ and $\tilde{S}T = T\tilde{S}$ by Lemma 1.22. □

Theorem 1.25. *Let $A = \int \lambda dE(\lambda)$ be a normal operator and B a $\tilde{P}$-symmetric operator. If $E(\sigma(A) \cap \sigma(B)) = 0$, then $A \oplus B$ is $\tilde{P}$-symmetric.*

Proof. Let $T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} \in C_1(H \oplus H)$. Then $(A \oplus B)T = T(A \oplus B)$ implies that

$$AT_1 = T_1A, BT_4 = T_4B, AT_2 = T_2B \text{ and } BT_3 = T_3A.$$

It follows from [16, Lemma 5] that $T_2 = T_3 = 0$. Since A is normal ($\tilde{A} = A$) and B is $\tilde{P}$-symmetric, we deduce that $\tilde{A}T_1 = T_1\tilde{A}$ and $\tilde{B}T_4 = T_4\tilde{B}$, hence $(\tilde{A} \oplus \tilde{B})T = T(\tilde{A} \oplus \tilde{B})$. □

Remark 1.26. *By virtue of the Theorem 1.14 our results generalize Bouali and Charles's [6] results to $\tilde{P}-$symmetric operators.*

Definition 1.27 ([2]). *Let $T \in \mathcal{L}(H)$, we say that T is w-hyponormal, if*

$$|\tilde{T}| \geq |T| \geq \left|\tilde{T}^*\right|.$$

T is said iw-hyponormal if T is invertible and w-hyponormal. Recall that an operator T is called w_*-hyponormal, if T is w-hyponormal and satisfying the condition $\ker(T) \subseteq \ker(T^*)$. So Clearly, every iw-hyponormal operator is w_*-hyponormal.

Lemma 1.28 ([13]). *Let $A \in \mathcal{L}(H)$. Then the following assertions are equivalent:*

1. *$AT = TA$ implies $A^*T = TA^*$ for all $T \in C_1(H)$. i.e. A is P-symmetric.*
2. *If $AT = TA$, then $\overline{R(T)}$ and $(\ker T)^{\perp}$ are reducing subspaces for A, and $A|\overline{R(T)}$, $A|(kerT)^{\perp}$ are unitarily equivalent normal operators.*

Theorem 1.29. *Let $A \in \mathcal{L}(H)$ be a iw-hyponormal operator. If $\tilde{A}$ is P-symmetric then A is P-symmetric.*

Proof. Let $T \in C_1(H)$ such that $AT = TA$. Since A invertible, the Lemma 2.1 and the Theorem 2.2 from [2] ensures that $|A|$ is invertible. So since $|A|^{\frac{1}{2}}A = \tilde{A}|A|^{\frac{1}{2}}$ we have $A|A|^{\frac{-1}{2}} = |A|^{\frac{-1}{2}}\tilde{A}$. Then from $AT = TA$ we get $|A|^{\frac{1}{2}}AT|A|^{\frac{-1}{2}} = |A|^{\frac{1}{2}}TA|A|^{\frac{-1}{2}}$ which equivalent to $\tilde{A}X = X\tilde{A}$ with $X = |A|^{\frac{1}{2}}T|A|^{\frac{-1}{2}} \in C_1(H)$. So if $\tilde{A}$ is

P-symmetric, we get from Lemma 1.28 that $\overline{R(X)}$ and $(\ker(X))^{\perp}$ are reducing spaces for $\tilde{A}$, and $\tilde{A}|\overline{R(X)}$ and $\tilde{A}\Big|(\ker(X))^{\perp}$ are unitarily equivalent normal operators. Therefore

$$\tilde{A} = M \oplus R \text{ on } H_1 = H = (\ker(X))^{\perp} \oplus \ker(X)$$

and

$$\tilde{A} = N \oplus S \text{ on } H_2 = H = \overline{R(X)} \oplus R(X)^{\perp}$$

where N and M are normal operators. So since A is w_*-hyponormal, it follows by [14, Lemma 4.5] that

$$A = M \oplus R' \text{ on } H_1 \text{ and } A = N \oplus S' \text{ on } H_2$$

The operator A is invertible and so are N, S', M and R'. Then we can write T and X on H_1 into H_2 as $X = \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$. Clearly $|A|^{-1} = |M|^{-1} \oplus |R'|^{-1}$ on H_1 and $|A| = |N| \oplus |S'|$ on H_2. It follows from $X = |A|^{\frac{1}{2}} T |A|^{\frac{-1}{2}}$ that

$$\begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} |N|^{\frac{1}{2}} T_1 |M|^{\frac{-1}{2}} & |N|^{\frac{1}{2}} T_2 |R'|^{\frac{-1}{2}} \\ |S'|^{\frac{1}{2}} T_3 |M|^{\frac{-1}{2}} & |S'|^{\frac{1}{2}} T_4 |R'|^{\frac{-1}{2}} \end{pmatrix}$$

Hence $T_2 = T_3 = T_4 = 0$, so $T = T_1 \oplus 0$. Since $AT = TA$, then $NT_1 = T_1 M$, and by applying Fuglede-Putnam's theorem we obtain $N^* T_1 = T_1 M^*$, which gives $A^* T = T A^*$. This completes the proof. □

Theorem 1.30. *Let $A \in \mathcal{L}(H)$. If one of the following assertions*

1. *A w_*-hyponormal operator such that $\tilde{A}$ is P-symmetric.*
2. *$f(\tilde{A})$ is cyclic subnormal for some nonconstant analytic function f on an open set containing $\sigma(A)$.*

is verified, then A is $\tilde{P}$-symmetric if and only if A is P-symmetric.

Proof. By Theorem 1.14, it suffices to show the property: A is $\tilde{P}$-symmetric implies that A is P-symmetric.

1. Suppose that A is $\tilde{P}$-symmetric and let $T \in C_1(H)$ such that $AT = TA$, so we have also $\tilde{A}T = T\tilde{A}$, and since $\tilde{A}$ is P-symmetric, the Lemma 1.28 ensures that $\overline{R(T)}$ and $(kerT)^{\perp}$ are reducing spaces for $\tilde{A}$, and $\tilde{A}|\overline{R(T)}$ and $\tilde{A}\Big|(kerT)^{\perp}$ are unitarily equivalent normal operators.
   Therefore
   $$\tilde{A} = M \oplus R \text{ on } H_1 = H = (\ker T)^{\perp} \oplus \ker T$$
   and
   $$\tilde{A} = N \oplus S \text{ on } H_2 = H = \overline{R(T)} \oplus R(T)^{\perp}$$
   where N and M are normal operators. It follows by hypothesis and [14, Lemma 4.5] that
   $$A = M \oplus R' \text{ on } H_1 \text{ and } A = N \oplus S' \text{ on } H_2$$
   So we can write T on H_1 into H_2 as $T = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix}$ and since $AT = TA$ then $NT_1 = T_1 M$, so by Fuglede-Putnam's theorem $N^* T_1 = T_1 M^*$ which give $A^* T = T A^*$.
2. Assume that A is $\tilde{P}$-symmetric and let $T \in C_1(H)$ such that $AT = TA$, so we have $\tilde{A}T = T\tilde{A}$, then $f(\tilde{A})T = Tf(\tilde{A})$. Hence T is subnormal by [19, Theorem 3] then hyponormal. Since T is compact, it follows that T is normal. So from $AT = TA$ and Fuglede's theorem [9, Theorem I] we deduce that $A^* T = T A^*$. Consequently, A is P-symmetric.

□

Corollary 1.31. *Let $A \in C_1(H)$ such that $f(\tilde{A})$ is cyclic subnormal for some nonconstant analytic function f on an open set containing $\sigma(A)$. Then A is $\tilde{P}$-symmetric if and only if A is normal.*

Proof. If A is normal, A is trivially $\tilde{P}$-symmetric. But if A is $\tilde{P}$-symmetric, we get that A is normal by replacing T by A in the proof of part 2 in the Theorem 1.30. □

2. Ultraweak closures of derivation ranges

Theorem 2.1. *If $A \in \mathcal{L}(H)$, then the following statements are equivalent:*

1. $\overline{\mathcal{R}(\delta_A)}^{w^*} = \overline{\mathcal{R}(\delta_{\tilde{A}})}^{w^*}$;
2. (a) *A is $\tilde{P}$-symmetric and*
   (b) *$\tilde{A}T = T\tilde{A}$ implies $AT = TA$ for all $T \in C_1(H)$.*

Proof. Note that $\overline{\mathcal{R}(\delta_A)}^{w^*} = \overline{\mathcal{R}(\delta_{\tilde{A}})}^{w^*}$ if and only if

$$\mathcal{R}(\delta_A)^0 \cap \mathcal{L}(H)'^{w^*} \simeq \mathcal{R}(\delta_{\tilde{A}})^0 \cap \mathcal{L}(H)'^{w^*}.$$

Using Lemma 1.11, we have

$$\mathcal{R}(\delta_A)^0 \cap \mathcal{L}(H)'^{w^*} \simeq \{A\}' \cap C_1(H).$$

It follows that $\overline{\mathcal{R}(\delta_A)}^{w^*} = \overline{\mathcal{R}(\delta_{\tilde{A}})}^{w^*}$ if and only if $\{A\}' \cap C_1(H) = \{\tilde{A}\}' \cap C_1(H)$. This gives the result. □

Corollary 2.2. *Let $A \in \mathcal{L}(H)$. If A satisfy the following conditions*

1. *A is P-symmetric and*
2. *$\tilde{A}T = T\tilde{A}$ implies $AT = TA$ for all $T \in C_1(H)$,*

then $\overline{\mathcal{R}(\delta_A)}^{w^*} = \overline{\mathcal{R}(\delta_{A^*})}^{w^*} = \overline{\mathcal{R}(\delta_{\tilde{A}})}^{w^*}$ and $\tilde{A}$ is P-symmetric.

Proof. it is an immediate consequence of Theorem 1.2, Theorem 1.14 and Theorem 2.1. □

Proposition 2.3. *Let $A = U|A| \in \mathcal{L}(H)$ such that $\ker(A) \subset \ker(A^*)$ and $\tilde{A}T = T\tilde{A}$ implies $|A|T = T|A|$ for every $T \in C_1(H)$. Then A is $\tilde{P}$-symmetric if and only if $\overline{\mathcal{R}(\delta_A)}^{w^*} = \overline{\mathcal{R}(\delta_{\tilde{A}})}^{w^*}$.*

Proof. By Theorem 2.1, it suffices to show the property that $\tilde{A}T = T\tilde{A}$ implies $AT = TA$ for all $T \in C_1(H)$. So if $\tilde{A}T = T\tilde{A}$ for $T \in C_1(H)$ then by virtue of hypothesis and since $\tilde{A}|A|^{\frac{1}{2}} = |A|^{\frac{1}{2}}A$ we have $\tilde{A}T|A|^{\frac{1}{2}} = T\tilde{A}|A|^{\frac{1}{2}}$, which implies that $|A|^{\frac{1}{2}}(AT - TA) = 0$ and hence $(T^*A^* - A^*T^*)|A|^{\frac{1}{2}} = 0$. Therefore $T^*A^* - A^*T^*$ vanish on $\overline{R(|A|)}$. On the other hand, if $x \in \ker(|A|) = \ker(A) \subset \ker(A^*)$, we get by hypothesis $|A|T^*x = T^*|A|x = 0$ hence $A^*T^*x = 0$ and as result $T^*A^* - A^*T^*$ vanish on $\ker(|A|)$. Then we obtain

$$AT - TA = (T^*A^* - A^*T^*)^* = 0 \text{ on } H = \overline{R(|A|)} \oplus \ker(|A|).$$

□

Proposition 2.4. *Let $A \in \mathcal{L}(H)$ such that $\overline{\mathcal{R}(\delta_A)}^{w^*} = \overline{\mathcal{R}(\delta_{\tilde{A}})}^{w^*}$, $H_0 \subset H$ is a reducing subspace for A and $A_0 = A_{|H_0}$. Then $\overline{\mathcal{R}(\delta_{A_0})}^{w^*} = \overline{\mathcal{R}\left(\delta_{\tilde{A}_0}\right)}^{w^*}$ and A_0 is $\tilde{P}$-symmetric.*

Proof. It is an consequence of Theorem 2.1 and Theorem 1.23. □

Proposition 2.5. *Let $A, B \in \mathcal{L}(H)$ and $S = A \oplus B$. If one of the following conditions*

1. $\overline{\mathcal{R}(\delta_A)}^{w^*} = \overline{\mathcal{R}(\delta_{\tilde{A}})}^{w^*}$ *and* $\overline{\mathcal{R}(\delta_B)}^{w^*} = \overline{\mathcal{R}(\delta_{\tilde{B}})}^{w^*}$ *with $\sigma(A) \cap \sigma(B) = \emptyset$.*
2. *$A = \int \lambda dE(\lambda)$ is normal and $\overline{\mathcal{R}(\delta_B)}^{w^*} = \overline{\mathcal{R}(\delta_{\tilde{B}})}^{w^*}$ such that $E(\sigma(A) \cap \sigma(B)) = 0$.*

is verified, then $\overline{\mathcal{R}(\delta_S)}^{w^*} = \overline{\mathcal{R}(\delta_{\tilde{S}})}^{w^*}$ and S is $\tilde{P}$-symmetric.

Proof. 1. It is an consequence of Theorem 2.1 and Theorem 1.24.

2. It is an consequence of Theorem 2.1 and Theorem 1.25.
□

Theorem 2.6. *Let $A \in \mathcal{L}(H)$ be an invertible P-symmetric operator. Then $\overline{\mathcal{R}(\delta_A)}^{w^*} = \overline{\mathcal{R}(\delta_{A^*})}^{w^*} = \overline{\mathcal{R}(\delta_{\tilde{A}})}^{w^*}$ and $\tilde{A}$ is P-symmetric.*

Proof. On the light of the Corollary 2.2, it suffices to show the property that $\tilde{A}T = T\tilde{A}$ implies $AT = TA$ for all $T \in C_1(H)$. Let $A = U|A|$ be the polar decomposition of A, since A is invertible, the Lemma 2.1 and the Theorem 2.2 from [2] ensures that $|A|$ is invertible. So from $\tilde{A}|A|^{\frac{1}{2}} = |A|^{\frac{1}{2}}A$ we get $|A|^{\frac{-1}{2}}\tilde{A} = A|A|^{\frac{-1}{2}}$. Hence if $T \in C_1(H)$ such that $\tilde{A}T = T\tilde{A}$, we have :

$$|A|^{\frac{-1}{2}}\tilde{A}T|A|^{\frac{1}{2}} = |A|^{\frac{-1}{2}}T\tilde{A}|A|^{\frac{1}{2}} \implies A|A|^{\frac{-1}{2}}T|A|^{\frac{1}{2}} = |A|^{\frac{-1}{2}}T|A|^{\frac{1}{2}}A.$$

Let $X = |A|^{\frac{-1}{2}}T|A|^{\frac{1}{2}} \in C_1(H)$. Then $AX = XA$, hence by hypothesis and Lemma 1.28, $\overline{R(X)}$ and $(\ker X)^{\perp}$ are reducing spaces for A, and $A|_{\overline{R(X)}}$ and $A|_{(\ker X)^{\perp}}$ are normal operators. Therefore

$$A = M \oplus R \text{ on } H_1 = H = (\ker X)^{\perp} \oplus \ker X$$

and

$$A = N \oplus S \text{ on } H_2 = H = \overline{R(X)} \oplus R(X)^{\perp}$$

where N and M are normal operators, and note that $\tilde{N} = N$ and $\tilde{M} = M$. The operator A is invertible and so are N, S, M and R. Also, we can write X and T on H_1 into H_2 as

$$X = \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$$

Clearly $|A| = |M| \oplus |R|$ on H_1 and $|A|^{-1} = |N|^{-1} \oplus |S|^{-1}$ on H_2 .It follows from $X = |A|^{\frac{-1}{2}}T|A|^{\frac{1}{2}}$ that

$$\begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} |N|^{\frac{-1}{2}}T_1|M|^{\frac{1}{2}} & |N|^{\frac{-1}{2}}T_2|R|^{\frac{1}{2}} \\ |S|^{\frac{-1}{2}}T_3|M|^{\frac{1}{2}} & |S|^{\frac{-1}{2}}T_4|R|^{\frac{1}{2}} \end{pmatrix}$$

Hence $T_2 = T_3 = T_4 = 0$ so $T = T_1 \oplus 0$. On the other hand, since $\tilde{A} = N \oplus \tilde{S}$ on H_2 and $\tilde{A} = M \oplus \tilde{R}$ on H_1 by Lemma 1.22, then from $\tilde{A}T = T\tilde{A}$ we get $NT_1 = T_1M$, therefore $AT = TA$. So the proof is complete. □

Remark 2.7. *The invertibility condition of Theorem 2.6 is essential. We confirm this in the following example:*

Example 2.8. *Let $(e_k)_{k\geq 1}$ be an orthonormal basis of H, and $S \in \mathcal{L}(H)$ the unilateral shift operator, that is $Se_k = e_{k+1}$ for all $k \geq 1$. Put $A = S^*$, So A is a co-isometry since S is an isometry. Hence A is P-symmetric operator (see [8]), however A is not invertible. It results from Lemma 1.9 that $\tilde{A} = A^*A^2 = |A|A$. A simple calculation shows that:*

$$|A|e_k = \begin{cases} 0 & \text{if } k = 1 \\ e_k & \text{if } k \geq 2 \end{cases} \implies \tilde{A}e_k = \begin{cases} 0 & \text{if } 1 \leq k \leq 2 \\ e_{k-1} & \text{if } k \geq 3 \end{cases} \implies \left(\tilde{A}\right)^* e_k = \begin{cases} 0 & \text{if } k = 1 \\ e_{k+1} & \text{if } k \geq 2 \end{cases}.$$

It follows from [6, Theorem 1.6] that $\tilde{A}$ is not P-symmetric.

Corollary 2.9. *Let $A \in \mathcal{L}(H)$ be iw-hyponormal operator, then A is P-symmetric if and only if $\tilde{A}$ is P-symmetric.*

Proof. it is an immediate consequence of Theorem 1.29 and Theorem 2.6. □

Proposition 2.10. *Let $A \in \mathcal{L}(H)$ be invertible such that $\|A^{-1}\|\|A\| = 1$. Then $\overline{\mathcal{R}(\delta_A)}^{w^*} = \overline{\mathcal{R}(\delta_{A^*})}^{w^*} = \overline{\mathcal{R}(\delta_{\tilde{A}})}^{w^*}$.*

Proof. It follows from [3] that A is P-symmetric, and $\overline{\mathcal{R}(\delta_A)}^{w^*} = \overline{\mathcal{R}(\delta_{A^*})}^{w^*} = \overline{\mathcal{R}(\delta_{\tilde{A}})}^{w^*}$ by Theorem 2.6. □

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