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The atomic characterization of weighted local Hardy spaces and its applications

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Abstract. The purpose of this paper is to obtain atomic decomposition characterization of the weighted local Hardy space $h^p_\omega(\mathbb{R}^n)$ with $\omega \in A_\infty$. We apply the discrete version of Calderón's identity and the weighted Littlewood–Paley–Stein theory to prove that $h^p_\omega(\mathbb{R}^n)$ coincides with the weighted-(p,q,s) atomic local Hardy space $h^{p,q,s}_{\omega,atom}(\mathbb{R}^n)$ for $0 < p < \infty$. The atomic decomposition theorems in our paper improve the previous atomic decomposition results of local weighted Hardy spaces in the literature. As applications, we derive the boundedness of inhomogeneous Calderón–Zygmund singular integrals and local fractional integrals on weighted local Hardy spaces.

1. Introduction

The real-variable theory of global Hardy spaces on $\mathbb{R}^n$ was essentially developed by Stein and Weiss [25] and systematically studied by Fefferman and Stein [10]. Hardy spaces $H^p(\mathbb{R}^n)$ serve as a substitute for $L^p(\mathbb{R}^n)$ when $p \leq 1$. However, the principle of $H^p(\mathbb{R}^n)$ breaks down at some key points, for example, pseudo-differential operators are not bounded on $H^p(\mathbb{R}^n)$. Hence, Goldberg in [13] introduced the class of local Hardy spaces $h^p(\mathbb{R}^n)$ with $p \in (0,1]$. Moreover, Goldberg [13] established the maximal function characterization of $h^p(\mathbb{R}^n)$ for $p \in ((n-1)/n, 1]$. From then on, local Hardy spaces have become an indispensable part in terms of harmonic analysis and partial differential equations. Then Peloso and Secco [21] obtained local Riesz transforms of local Hardy spaces and extended some characterizations of Hardy spaces $H^p(\mathbb{R}^n)$ to the local Hardy spaces $h^p(\mathbb{R}^n)$ for $0 < p \leq 1$. In 1983, Triebel [33] first established the Littlewood–Paley characterization of $h^p(\mathbb{R}^n)$ which is a tool to prove that $h^p(\mathbb{R}^n)$ coincides with the Triebel-Lizorkin space $F^0_{p,2}(\mathbb{R}^n)$. In 1981, the weighted version $h^p_\omega(\mathbb{R}^n)$ of $h^p(\mathbb{R}^n)$ with $\omega \in A_\infty$ was developed by Bui [2]. Later, Rychkov [22] extended a part of the theory of weighted local Hardy spaces to A^{loc}_∞ weights and obtained the Littlewood-Paley function characterization of $h^p_\omega(\mathbb{R}^n)$. In 2012, Tang [32] established the weighted atomic characterization of $h^p_\omega(\mathbb{R}^n)$ with $\omega \in A^{loc}_\infty$ via the local grand maximal function. For more information on various Hardy-type spaces, we refer to [20, 23, 29, 36].

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As is well-known to us, the atomic decomposition plays an important role in the study of the boundedness of operators on Hardy-type spaces and many theories of it have been established. In 1974, Coifman [3] first introduced an atomic decomposition characterization of Hardy spaces on $\mathbb{R}$. Later, the extension to higher dimensions was obtained by Latter [17]. In fact, the marked difference between the atomic characterization of $H^p(\mathbb{R}^n)$ and $h^p(\mathbb{R}^n)$ is the cancellation property of atoms. To be precise, the vanishing moment is needed only for the atoms with small supports in $h^p(\mathbb{R}^n)$ while the vanishing moment is needed for all atoms in $H^p(\mathbb{R}^n)$. In [8], Y. Ding et al. established the atomic decomposition characterization of the weighted Hardy spaces $H^p_\omega(\mathbb{R}^n)$ for $p \in (0,1]$ and obtained the $(H^p_\omega(\mathbb{R}^n), L^p_\omega(\mathbb{R}^n))$-boundedness for singular integrals via the discrete Calderón's identity and the weighted Littlewood–Paley–Stein theory. In [7], W. Ding et al. obtained the $L^2(\mathbb{R}^n)$ atomic decomposition of local Hardy spaces $h^p(\mathbb{R}^n)$ for $0 < p \leq 1$. Motivated by these results, we give the atomic decomposition characterization of the weighted local Hardy spaces $h^p_\omega(\mathbb{R}^n)$ and a proof of the convergence of the atomic decomposition in both $h^p_\omega(\mathbb{R}^n)$ and $L^q(\mathbb{R}^n)$ norms for any $f \in h^p_\omega(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$. The atomic decomposition characterization in our paper provides extensions of the results in [18] by w-(p,q,s)-atom and w-(p,q,s)-block. Moreover, the results have a wide applicability to more general settings in that we avoid the maximal function characterization and the Calderón–Zymund decomposition. Very recently, Izuki et al. [16] defined and studied the theory of local weighted Hardy spaces with variable exponents by applying the local grand maximal function characterization.

The class of weighted local Hardy spaces $h^p_\omega(\mathbb{R}^n)$ can be defined by the finiteness of the quasi-norm [22]. To be precise, let $\Phi \in \mathcal{S}(\mathbb{R}^n)$ with $\int \Phi \neq 0$ and $\Phi_t(x) = t^{-n}\Phi(\frac{x}{t})$, then

$$M_\Phi(f)(x) = \sup_{0<t<1} |\Phi_t * f(x)|.$$

Then the weighted local Hardy space $h^p_\omega(\mathbb{R}^n)$ for $0 < p < \infty$ and $\omega \in A_\infty$ is defined by

$$h^p_\omega(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) \colon M_\Phi(f) \in L^p_\omega(\mathbb{R}^n)\},$$

where

$$\left\|f\right\|_{h^p_\omega} = \left\|M_\Phi(f)\right\|_{L^p_\omega}.$$

In fact, we can also define the weighted local Hardy space via the discrete Littlewood–Paley–Stein theory. Thus, we firstly recall some definitions as follows. For more details, see [15].

Definition 1.1. *Let ϕ_0, $\phi \in \mathcal{S}(\mathbb{R}^n)$ with*

$$\mathrm{supp}\widehat{\phi_0} \subseteq \{\xi \in \mathbb{R}^n \colon |\xi| \leq 2\};\ \widehat{\phi_0}(\xi) = 1,\ \textit{if}\ |\xi| \leq 1, \tag{1}$$

and

$$\mathrm{supp}\widehat{\phi} \subseteq \{\xi \in \mathbb{R}^n \colon \frac{1}{2} \leq |\xi| \leq 2\}, \tag{2}$$

and for all $\xi \in \mathbb{R}^n$

$$|\widehat{\phi_0}(\xi)|^2 + \sum_{j=1}^{\infty} |\widehat{\phi}(2^{-j}\xi)|^2 = 1. \tag{3}$$

Additionally, define $\phi_j(x) = 2^{jn}\phi(2^jx)$ for $j \in \mathbb{N}^+$. For any $j \in \mathbb{Z}$, denote Π_j={Q: Q are dyadic cubes in $\mathbb{R}^n$ with $l(Q) = 2^{-j}$ and the left lower corners of Q are $x_Q = 2^{-j}l$, $l \in \mathbb{Z}^n$}. By applying Fourier transform and equation (3), we can obtain the continuous Calderón's identity [7]:

$$f(x) = \sum_{j=0}^{\infty} \phi_j * \phi_j * f(x), \tag{4}$$

where the series converges in $L^q(\mathbb{R}^n)$, $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$. Furthermore, we can discretize the above identity:

$$f(x)=\sum_{j=0}^{\infty}\sum_{Q\in\Pi_j}|Q|(\phi_j*f)(x_Q)\phi_j(x-x_Q).$$

where the series converges in $L^q(\mathbb{R}^n)$, $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$.

Suppose that $\phi_0,\phi\in\mathcal{S}(\mathbb{R}^n)$ satisfies (1.1)-(1.3). Based on the above reproducing formula, we give the definition of inhomogeneous Littlewood–Paley–Stein square function

$$g(f)(x)=\left\{\sum_{i\in\mathbb{N}}|\phi_i*f(x)|^2\right\}^{\frac{1}{2}}$$

and the definition of the discrete Littlewood–Paley–Stein square function

$$g_d(f)(x)=\left\{\sum_{j\in\mathbb{N}}\sum_{Q\in\Pi_j}|\phi_j*f(x_Q)|^2\chi_Q(x)\right\}^{\frac{1}{2}}.$$

Now we can give the definition of the weighted local Hardy space.

Definition 1.2. *Let $0<p<\infty$, $\omega\in A_\infty$. Then the weighted local Hardy space $h^p_\omega(\mathbb{R}^n)$ is defined by*

$$h^p_\omega(\mathbb{R}^n)=\{f\in\mathcal{S}'(\mathbb{R}^n):\ \|f\|_{h^p_\omega}<\infty\},$$

where

$$\|f\|_{h^p_\omega}=\|g_d(f)\|_{L^p_\omega}.$$

The definitions of the atom a and the block b are as follows. Details are referred to [29].

Definition 1.3. *Let $0<p<\infty$, $1\le q<\infty$, $\omega\in A_q$ with critical index q_ω and $s\in\mathbb{Z}$ fulfilling $s\ge\max\{[n(\frac{q_\omega}{p}-1)],-1\}$. Fix a constant $C\ge1$. Then define a ω-(p,q,s)-atom of $h^p_\omega(\mathbb{R}^n)$ to be a function a which is supported in a cube $Q\subseteq\mathbb{R}^n$ with $|Q|\le C$ and satisfies*

$$\|a\|_{L^q}\le|Q|^{\frac{1}{q}}\omega(Q)^{-\frac{1}{p}}\quad\text{and}\quad\int_Q a(x)x^\alpha dx=0,\ \text{for all } |\alpha|\le s.$$

Definition 1.4. *Let $0<p<\infty$, $1\le q<\infty$, $\omega\in A_q$ with critical index q_ω and $s\in\mathbb{Z}$ fulfilling $s\ge\max\{[n(\frac{q_\omega}{p}-1)],-1\}$. Fix a constant $C\ge1$. Then define a ω-(p,q,s)-block of $h^p_\omega(\mathbb{R}^n)$ to be a function b which is supported in a cube $P\subseteq\mathbb{R}^n$ with $|P|>C$ and satisfies $\|b\|_{L^q}\le|P|^{\frac{1}{q}}\omega(P)^{-\frac{1}{p}}$.*

Naturally, we can give the definition of the weighted-(p,q,s) atomic local Hardy space $h^{p,q,s}_{\omega,atom}(\mathbb{R}^n)$.

Definition 1.5. *Let $0<p<\infty$, $q_\omega<q<\infty$, $\omega\in A_\infty$ with the critical index q_ω and $s\in\mathbb{Z}$ fulfilling $s\ge\max\{[n(\frac{q_\omega}{p}-1)],-1\}$. Then the weighted-$(p,q,s)$ atomic local Hardy space $h^{p,q,s}_{\omega,atom}(\mathbb{R}^n)$ is defined by*

$$h^{p,q,s}_{\omega,atom}(\mathbb{R}^n)=\left\{f\in\mathcal{S}'(\mathbb{R}^n):\ f=\sum_j\lambda_ja_j+\sum_j\mu_jb_j\right\},$$

where each a_j is a ω-(p,q,s)-atom and each b_j is a ω-(p,q,s)-block sastifying

$$\left\|\sum_{j=1}^{\infty}\frac{\lambda_j\chi_{Q_j}}{\omega(Q_j)^{\frac{1}{p}}}\right\|_{L^p_\omega}+\left\|\sum_{j=1}^{\infty}\frac{\mu_j\chi_{P_j}}{\omega(P_j)^{\frac{1}{p}}}\right\|_{L^p_\omega}<\infty.$$

Furthermore, we have

$$\|f\|_{h^{p,q,s}_{\omega,atom}} = \inf\left\{\left\|\sum_{j=1}^{\infty}\frac{\lambda_j\chi_{Q_j}}{\omega(Q_j)^{\frac{1}{p}}}\right\|_{L^p_\omega} + \left\|\sum_{j=1}^{\infty}\frac{\mu_j\chi_{P_j}}{\omega(P_j)^{\frac{1}{p}}}\right\|_{L^p_\omega}\right\},$$

where the infimum is taken over all decompositions $f = \sum_j \lambda_j a_j + \sum_j \mu_j b_j$.

If $\omega \in A_\infty$, there exists $r > 1$ such that $\omega \in RH_r$. Fix a constant q_r such that $q_r > \max\{p, 1\}$ and $(\frac{q_r}{p})' \leq r$.

Theorem 1.6. *If* $0 < p < \infty$ *and* $\omega \in A_\infty$, *then for any* $\max\{q_\omega, q_r\} < q < \infty$ *and any* $s \in \mathbb{Z}$ *fulfilling* $s \geq \max\{[n(\frac{q_\omega}{p} - 1)], -1\}$,

$$h^p_\omega(\mathbb{R}^n) = h^{p,q,s}_{\omega,atom}(\mathbb{R}^n)$$

with the equivalent norms.

In fact, Theorem 1.6 can be split into two parts as follows.

Theorem 1.7. *Let* $0 < p < \infty$, $\omega \in A_\infty$, $q_\omega = \inf\{q\colon \omega \in A_q\}$, $q_\omega < q < \infty$ *and* $s \in \mathbb{Z}$ *fulfilling* $s \geq \max\{[n(\frac{q_\omega}{p} - 1)], -1\}$. *If* $f \in h^p_\omega(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$, *there exist a sequence of* ω-(p, q, s)-*atoms* $\{a_j\}_{j=1}^\infty$ *with a corresponding sequence of non-negative numbers* $\{\lambda_j\}_{j=1}^\infty$ *and a sequence of* ω-(p, q, s)-*blocks* $\{b_j\}_{j=1}^\infty$ *with a corresponding sequence of non-negative numbers* $\{\mu_j\}_{j=1}^\infty$ *such that*

$$f = \sum_j \lambda_j a_j + \sum_j \mu_j b_j$$

and

$$\left\|\left(\sum_{j=1}^{\infty}\left(\frac{\lambda_j\chi_{Q_j}}{\omega(Q_j)^{\frac{1}{p}}}\right)^\eta\right)^{\frac{1}{\eta}}\right\|_{L^p_\omega} + \left\|\left(\sum_{j=1}^{\infty}\left(\frac{\mu_j\chi_{P_j}}{\omega(P_j)^{\frac{1}{p}}}\right)^\eta\right)^{\frac{1}{\eta}}\right\|_{L^p_\omega} \leq C_\eta \|f\|_{h^p_\omega}$$

for any $0 < \eta < \infty$. *Furthermore, the series converges to* f *in both* $h^p_\omega(\mathbb{R}^n)$ *and* $L^q(\mathbb{R}^n)$ *norms.*

Theorem 1.8. *Given* $0 < p < \infty$, $\omega \in A_\infty$, $q_\omega = \inf\{q\colon \omega \in A_q\}$, $q_r < q < \infty$ *and* $s \in \mathbb{Z}$ *fulfilling* $s \geq \max\{[n(\frac{q_\omega}{p} - 1)], -1\}$. *Suppose that* $\{a_j\}_{j=1}^\infty$ *is a sequence of* ω-(p, q, s)-*atoms with a corresponding sequence of non-negative numbers* $\{\lambda_j\}_{j=1}^\infty$ *and* $\{b_j\}_{j=1}^\infty$ *is a sequence of* ω-(p, q, s)-*blocks with a corresponding sequence of non-negative numbers* $\{\mu_j\}_{j=1}^\infty$ *satisfying*

$$\left\|\sum_{j=1}^{\infty}\frac{\lambda_j\chi_{Q_j}}{\omega(Q_j)^{\frac{1}{p}}}\right\|_{L^p_\omega} + \left\|\sum_{j=1}^{\infty}\frac{\mu_j\chi_{P_j}}{\omega(P_j)^{\frac{1}{p}}}\right\|_{L^p_\omega} < \infty.$$

Then the series $f = \sum_j \lambda_j a_j + \sum_j \mu_j b_j$ *converges in* $h^p_\omega(\mathbb{R}^n)$ *and satisfies*

$$\left\|\sum_j \lambda_j a_j\right\|_{h^p_\omega} \leq C\left\|\sum_{j=1}^{\infty}\frac{\lambda_j\chi_{Q_j}}{\omega(Q_j)^{\frac{1}{p}}}\right\|_{L^p_\omega}$$

and

$$\left\|\sum_j \mu_j b_j\right\|_{h^p_\omega} \leq C\left\|\sum_{j=1}^{\infty}\frac{\mu_j\chi_{P_j}}{\omega(P_j)^{\frac{1}{p}}}\right\|_{L^p_\omega}.$$

Theorem 1.6 follows from Theorem 1.7 and Theorem 1.8 together with the fact $h^p_\omega(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$ is dense in $h^p_\omega(\mathbb{R}^n)$. As applications of the above atomic decomposition results, we shall prove the boundedness of the inhomogenous Calderón-Zygmund singular integrals and the local fractional integrals on weighted local Hardy spaces. The groundbreaking work of Hardy estimates for Calderón–Zygmund operators is completed by Stein and Weiss [25], Stein [24], and Fefferman and Stein [10]. In particular, weighted Hardy spaces estimates for singular integrals were proved by Strömberg and Torchinsky [26]. We remark that the proof of Theorem 1.9 and 1.10 is an adaption from the ones for local variable Hardy spaces in [30]. Moreover, fractional integrals have been investigated extensively by several authors in recent years. Weighted Hardy space estimates for fractional integrals were first proved by Strömberg and Wheeden [27]; see also Gatto et al. [12] and Tan [31]. Theorem 1.12 extends these results to weighted local Hardy spaces. We remark that the proof of this theorem is similar to the proof of [5, Theorem 1.5], but we need to concentrate on the differences.

Now we recall the inhomogeneous Calderón–Zygmund singular integrals in [6]. Define $\mathcal{D}(\mathbb{R}^n)$ to be the space of all smooth functions with compact support. The operator T is said to be an inhomogeneous Calderón–Zygmund integral if T is a continuous linear operator from $\mathcal{D}(\mathbb{R}^n)$ to $\mathcal{D}'(\mathbb{R}^n)$ defined by

$$\langle T(f), g\rangle = \int_{\mathbb{R}^n} \mathcal{K}(x,y) f(y) g(x) dx dy$$

for all $f, g \in \mathcal{D}(\mathbb{R}^n)$ with disjoint supports, where $\mathcal{K}(x,y)$, the kernel of T, satisfies the following conditions

$$|\mathcal{K}(x,y)| \le C \min\left\{ \frac{1}{|x-y|^n}, \frac{1}{|x-y|^{n+\delta}} \right\} \text{ for some } \delta > 0 \text{ and } x \neq y$$

and for $\epsilon \in (0,1)$

$$|\mathcal{K}(x,y) - \mathcal{K}(x,y')| + |\mathcal{K}(y,x) - \mathcal{K}(y',x)| \le C \frac{|y-y'|^\epsilon}{|x-y|^{n+\epsilon}},$$

where $|y - y'| \le \frac{1}{2}|x-y|$.

Theorem 1.9. *Let $\frac{n}{n+\eta} < p < \infty$ and $\omega \in A_{(\frac{n+\eta}{n})p}$ where $\eta = \epsilon \wedge \delta$. Suppose that T is an inhomogeneous Calderón–Zygmund singular integral. If T is a bounded operator on $L^2(\mathbb{R}^n)$, then T can be extended to an $(h^p_\omega(\mathbb{R}^n), L^p_\omega(\mathbb{R}^n))$-bounded operator. To be precise, there exists a constant C such that*

$$\left\|T(f)\right\|_{L^p_\omega} \le C \left\|f\right\|_{h^p_\omega}.$$

To state the $(h^p_\omega(\mathbb{R}^n), h^p_\omega(\mathbb{R}^n))$-boundedness of T, we assume one additional condition on T, $\int_{\mathbb{R}^n} T(a)(x)dx = 0$ for the ω-(p,q,s)-atom a. Then if T satisfies the above moment condition, we write $T^{loc}_*(1) = 0$.

Theorem 1.10. *Let $\frac{n}{n+\eta} < p < \infty$ and $\omega \in A_{(\frac{n+\eta}{n})p}$ where $\eta = \epsilon \wedge \delta$. Suppose that T is an inhomogeneous Calderón-Zygmund singular integral. If T is a bounded operator on $L^2(\mathbb{R}^n)$ and $T^{loc}_*(1) = 0$, then T has a unique extension on $h^p_\omega(\mathbb{R}^n)$ and, moreover, there exists a constant C such that*

$$\left\|T(f)\right\|_{h^p_\omega} \le C \left\|f\right\|_{h^p_\omega}$$

for all $f \in h^p_\omega(\mathbb{R}^n)$.

We also recall the following local fractional integral which is introduced by D. Yang and S. Yang [35].

Definition 1.11. *Let $\alpha \in [0, n)$ and let $\varphi_0 \in \mathcal{D}(\mathbb{R}^n)$ be such $\varphi_0 \equiv 1$ on $Q(0,1)$ and $\operatorname{supp}(\varphi_0) \subset Q(0,2)$. The local fractional integral $I^{loc}_\alpha(f)$ of f is defined by*

$$I^{loc}_\alpha(f)(x) \equiv \int_{\mathbb{R}^n} \frac{\varphi_0(y)}{|y|^{n-\alpha}} f(x-y) dy.$$

Now we show that the local fractional integrals are bounded from $h^p_{\omega^p}(\mathbb{R}^n)$ to $L^q_{\omega^q}(\mathbb{R}^n)$ when $1 < q < \infty$ and from $h^p_{\omega^p}(\mathbb{R}^n)$ to $h^q_{\omega^q}(\mathbb{R}^n)$ when $0 < q \le 1$.

Theorem 1.12. *Let $0 < \alpha < n$ and $0 < p < \frac{n}{\alpha}$. Define q by $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. If a weight ω is such that $\omega^p \in RH_{\frac{q}{p}}$, then I^{loc}_α admits a bounded extension from $h^p_{\omega^p}(\mathbb{R}^n)$ to $L^q_{\omega^q}(\mathbb{R}^n)$ when $1 < q < \infty$ and I^{loc}_α admits a bounded extension from $h^p_{\omega^p}(\mathbb{R}^n)$ to $h^q_{\omega^q}(\mathbb{R}^n)$ when $0 < q \le 1$.*

Throughout this paper, C or c denotes a positive constant that is independent of the main parameters involved but may vary at each occurrence. To denote the dependence of the constants on some parameter s, we will write C_s. We denote $f \le Cg$ by $f \lesssim g$. If $f \lesssim g \lesssim f$, we write $f \sim g$ or $f \approx g$. Denote $Q(x, l(Q))$ the closed cube centered at x and of side-length $l(Q)$. Similarly, given $Q = Q(x, l(Q))$ and $\lambda > 0$, λQ means the cube with the same center x and with side-length $\lambda l(Q)$. We denote $Q^* = 2\sqrt{n}Q$. Moreover, we use the notation $j \wedge k = \min\{j, k\}$. We write $\mathbb{N} = \{0, 1, 2, \dots\}$.

2. Preliminaries

In this section, we present some known results that will be used in the next sections and establish a new reproducing formula.

Firstly, we recall some known results about weights. For more details, see [4, 9, 11]. Suppose that a weight ω is a non-negative, locally integrable function such that $0 < \omega(x) < \infty$ for almost every $x \in \mathbb{R}^n$. It is said that ω is in the Muckenhoupt class A_p for $1 < p < \infty$ if

$$[\omega]_{A_p} = \sup_Q \left(\frac{1}{Q}\int_Q \omega(x)dx\right)\left(\frac{1}{Q}\int_Q \omega(x)^{-\frac{1}{p-1}}dx\right)^{p-1} < \infty,$$

where Q is any cube in $\mathbb{R}^n$ and when $p = 1$, a weight $\omega \in A_1$ if for almost every $x \in \mathbb{R}^n$,

$$M\omega(x) \le C\omega(x),$$

where M is the Hardy–Littlewood maximal operator defined by

$$M(f)(x) = \sup_{x\in Q} \frac{1}{|Q|}\int_Q f(u)du.$$

Therefore, define the set

$$A_\infty = \bigcup_{1\le p<\infty} A_p.$$

Given a weight $\omega \in A_\infty$, define

$$q_\omega = \inf\{q \ge 1 \colon \omega \in A_q\}.$$

Given a weight $\omega \in A_\infty$ and $0 < p < \infty$. Then the weighted Lebesgue space is defined by

$$L^p_\omega(\mathbb{R}^n) = \left\{f \colon \int_{\mathbb{R}^n} |f(x)|^p \omega(x)dx < \infty\right\},$$

where f are measurable functions on $\mathbb{R}^n$. A weight $\omega \in A_\infty$ if and only if $\omega \in RH_r$ for some $r > 1$: that is, for every cube Q,

$$\left(\frac{1}{|Q|}\int_Q \omega(x)^r dx\right)^{\frac{1}{r}} \le \frac{C}{|Q|}\int_Q \omega(x)dx.$$

Furthermore, we can obtain the property that $\omega \in RH_r$ if and only if $\omega^r \in A_\infty$. Given $1 < p,\ q < \infty$, a weight satisfies the $A_{p,q}$ condition of Muckenhoupt and Wheeden if for every cube Q,

$$\left(\frac{1}{|Q|}\int_Q \omega^q dx\right)^{\frac{1}{q}}\left(\frac{1}{|Q|}\int_Q \omega^{-p'} dx\right)^{\frac{1}{p'}} \leq C.$$

It follows from the defintion that $\omega \in A_{p,q}$ if and only if $\omega^q \in A_{1+\frac{q}{p'}}$. When $p = 1$ and $q > 1$, it is said that $\omega \in A_{1,q}$ if for every cube Q and almost every $x \in Q$,

$$\frac{1}{|Q|}\int_Q \omega(x)^q dx \leq C\omega(x)^q,$$

which is clearly equivalent to $\omega^q \in A_1$.
Given $0 \leq \alpha < n$ and $1 < p < \frac{n}{\alpha}$, define q by $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$. If $\omega \in A_{p,q}$, the fractional maximal operator

$$M_\alpha(f)(x) = \sup_Q |Q|^{\frac{\alpha}{n}}\left(\frac{1}{|Q|}\int_Q |f(y)|dy\right)\chi_Q(x)$$

is bounded from $L^p_{\omega^p}(\mathbb{R}^n)$ to $L^q_{\omega^q}(\mathbb{R}^n)$.

Now we recall two lemmas which will be applied to the proofs in Section 3. First we need the weighted Fefferman–Stein vector-valued maximal inequality [1] as follows.

Lemma 2.1. *Let $1 < p,\ q < \infty$, $\omega \in A_p$, $f = \{f_i\}_{i\in\mathbb{Z}}$, $f_i \in L_{loc}(\mathbb{R}^n)$,*

$$\left\| \|\mathbb{M}(f)\|_{l^q} \right\|_{L^p_\omega} \leq C \left\| \|f\|_{l^q} \right\|_{L^p_\omega}$$

where $\mathbb{M}(f) = \{M(f_i)\}_{i\in\mathbb{Z}}$.

Remark 2.2. *If we let $f_i = \chi_{Q_i}$, for some collection of cubes Q_i, then given $0 < p < \infty$, $\tau > 1$ and $\omega \in A_\infty$, there exists $r > 1$ such that $\omega \in A_{rp}$. Thus we have that*

$$\left\|\sum_i \chi_{\tau Q_i}\right\|_{L^p_\omega} \lesssim \left\|\left\{\sum_i \left(M\chi_{Q_i}\right)^r\right\}^{\frac{1}{r}}\right\|^r_{L^{rp}_\omega} \lesssim \left\|\sum_i \chi_{Q_i}\right\|_{L^p_\omega}.$$

Lemma 2.3 ([5]). *Fix $q > 1$. Suppose that $0 < p < q$ and $\omega \in RH_{(\frac{q}{p})'}$. We are given countable collections of cubes $\{Q_j\}_{j=1}^\infty$, of non-negative numbers $\{\lambda_j\}_{j=1}^\infty$ and of non-negative measurable functions $\{a_j\}_{j=1}^\infty$ such that $\operatorname{supp}(a_j) \subset Q_j$, $\|a_j\|_{L^q} \leq |Q_j|^{\frac{1}{q}}\omega(Q_j)^{-\frac{1}{p}}$. Then*

$$\left\|\sum_{j=1}^\infty \lambda_j a_j\right\|_{L^p_\omega} \leq C\left\|\sum_{j=1}^\infty \frac{\lambda_j\chi_{Q_j}}{\omega(Q_j)^{\frac{1}{p}}}\right\|_{L^p_\omega}.$$

In order to obtain the atomic decomposition, we need a new reproducing formula. Thus, we introduce test functions as follows.

Definition 2.4. *Let $\psi_0,\ \psi \in \mathcal{S}(\mathbb{R}^n)$ satisfies*

$$\operatorname{supp}\psi_0 \subseteq \{x \in \mathbb{R}^n : |x| \leq 1\};\ \int \psi_0 = 1, \tag{5}$$

$$\operatorname{supp}\psi \subseteq \{x \in \mathbb{R}^n : |x| \leq 1\};\ \int \psi(x)x^\alpha dx = 0,\ \textit{for all } |\alpha| \leq M, \tag{6}$$

and

$$|\widehat{\psi_0}(\xi)|^2 + \sum_{j=1}^{\infty} |\widehat{\psi}(2^{-j}\xi)|^2 = 1, \text{ for all } \xi \in \mathbb{R}^n, \tag{7}$$

where a constant $M = M_{p,n}$ *is large enough.*

Lemma 2.5. *Let* $0 < p < \infty$, $\omega \in A_\infty$, $q_\omega = \inf\{q\colon \omega \in A_q\}$ *and* $q_\omega < q < \infty$. *Suppose that* ψ_0, $\psi \in \mathcal{S}(\mathbb{R}^n)$ *satisfies (2.1)-(2.3). Then there exists a positive integer N such that for any* $f \in h^p_\omega(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$

$$f(x) = \sum_{j\in\mathbb{N}} \sum_{Q\in\Pi_{j+N}} |Q|\psi_j(x-u_Q)(\psi_j * h)(u_Q),$$

where u_Q *is any point in Q and* $h \in h^p_\omega(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$ *satisfies*

$$\|h\|_{L^q} \sim \|f\|_{L^q}, \ \|h\|_{h^p_\omega} \sim \|f\|_{h^p_\omega}.$$

Moreover, the series converges in $L^q(\mathbb{R}^n)$.

Proof. Applying the Calderón reproducing formula on $L^2(\mathbb{R}^n)$ and the Coifman's decomposition, we have that

$$\begin{aligned} f(x) &= \sum_{j\in\mathbb{N}} \psi_j * \psi_j * f(x) \\ &= \sum_{j\in\mathbb{N}} \sum_{Q\in\Pi_{j+N}} \int_Q \psi_j(x-u)(\psi_j * f)(u)du \\ &=: T_N(f)(x) + R_n(f)(x), \end{aligned}$$

where

$$T_N(f)(x) = \sum_{j\in\mathbb{N}} \sum_{Q\in\Pi_{j+N}} |Q|\psi_j(x-u_Q)(\psi_j * f)(u_Q),$$

$$R_N(f)(x) = \sum_{j\in\mathbb{N}} \sum_{Q\in\Pi_{j+N}} \int_Q [\psi_j(x-u)(\psi_j * f)(u) - \psi_j(x-u_Q)(\psi_j * f)(u_Q)]du,$$

the positive integer N will be chosen later and u_Q is any point in Q.

Details are similar to those in [14, 18, 34]. By a standard almost orthogonality estimation, we can prove that

$$\left\|R_N(f)\right\|_{h^p_\omega} \le C2^{-N}\left\|f\right\|_{h^p_\omega} \quad \text{and} \quad \left\|R_N(f)\right\|_{L^q} \le C2^{-N}\left\|f\right\|_{L^q}.$$

We can choose N large enough so that $C2^{-N} < 1$. Since $I = T_N + R_N$ and R_N is bounded on $h^p_\omega(\mathbb{R}^n)$ and $L^q(\mathbb{R}^n)$, then T_N and T_N^{-1} are bounded on $h^p_\omega(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n)$. Moreover, $T_N^{-1} = \sum_{n=0}^{\infty}(R_N)^n$. Let $h(x) = T_N^{-1}(f)(x)$ and then

$$\|h\|_{h^p_\omega} \sim \|f\|_{h^p_\omega}, \ \|h\|_{L^q} \sim \|f\|_{L^q}.$$

Furthermore,

$$f(x) = T_N(T_N^{-1}(f))(x) = \sum_{j\in\mathbb{N}} \sum_{Q\in\Pi_{j+N}} |Q|\psi_j(x-u_Q)(\psi_j * h)(u_Q).$$

where the series converges in $L^2(\mathbb{R}^n)$.

Next we will prove that the series above converges in $L^q(\mathbb{R}^n)$ for any $1 < q < \infty$. Since $L^q(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ is dense in $L^q(\mathbb{R}^n)$, it suffices to show that the series converges in $L^q(\mathbb{R}^n)$ for any function $f \in L^q(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Let

$$B_l = \{Q\colon l(Q) = 2^{-j-N},\ Q \subset B(0,l),\ |j| \le l\},$$

where $B(0,l)$ are balls centered at origin with radii l in $\mathbb{R}^n$. Write $\psi_Q = \psi_j$. We claim that for each function $f \in L^q(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$

$$\left\| \sum_{l>L} \sum_{Q\in B_l} |Q|\psi_Q(x-u_Q)(\psi_Q * h)(u_Q) \right\|_{L^q} \to 0,\ as\ L \to +\infty.$$

In fact, by the duality argument and the fact that ψ_Q is radial, we have that

$$\begin{aligned}
&\left\| \sum_{l>L} \sum_{Q\in B_l} |Q|\psi_Q(x-u_Q)(\psi_Q * h)(u_Q) \right\|_{L^q} \\
&= \sup_{\|g\|_{L^{q'}}\le 1} \left\langle \sum_{l>L} \sum_{Q\in B_l} |Q|\psi_Q(x-u_Q)(\psi_Q * h)(u_Q), g \right\rangle \\
&= \sup_{\|g\|_{L^{q'}}\le 1} \sum_{l>L} \sum_{Q\in B_l} |Q|(\psi_Q * h)(u_Q) \int_{\mathbb{R}^n} \psi_Q(x-u_Q) g(x)dx \\
&= \sup_{\|g\|_{L^{q'}}\le 1} \left| \sum_{l>L} \sum_{Q\in B_l} |Q|(\psi_Q * h)(u_Q)(\psi_Q * g)(u_Q) \right| \\
&\le \sup_{\|g\|_{L^{q'}}\le 1} \left| \int_{\mathbb{R}^n} \sum_{l>L} \sum_{Q\in B_l} (\psi_Q * h)(u_Q)(\psi_Q * g)(u_Q)\chi_Q(y)dy \right| \\
&\le \sup_{\|g\|_{L^{q'}}\le 1} \int_{\mathbb{R}^n} \left\{ \sum_{l>L} \sum_{Q\in B_l} |(\psi_Q * h)(u_Q)|^2 \chi_Q(y) \right\}^{\frac{1}{2}} \\
&\quad \times \left\{ \sum_{l>L} \sum_{Q\in B_l} |(\psi_Q * g)(u_Q)|^2 \chi_Q(y) \right\}^{\frac{1}{2}} dy \\
&\le \sup_{\|g\|_{L^{q'}}\le 1} \left\| \left\{ \sum_{l>L} \sum_{Q\in B_l} |(\psi_Q * g)(u_Q)|^2 \chi_Q(y) \right\}^{\frac{1}{2}} \right\|_{L^{q'}} \\
&\quad \times \left\| \left\{ \sum_{l>L} \sum_{Q\in B_l} |(\psi_Q * h)(u_Q)|^2 \chi_Q(y) \right\}^{\frac{1}{2}} \right\|_{L^q} \\
&\le C \left\| \left\{ \sum_{l>L} \sum_{Q\in B_l} |(\psi_Q * h)(u_Q)|^2 \chi_Q(y) \right\}^{\frac{1}{2}} \right\|_{L^q},
\end{aligned}$$

which tends to zero as L goes to infinity. Then by a standard density argument, we can obtain the desired result. □

By Lemma 2.5, we can obtain the following corollary.

Corollary 2.6. *Let $0 < p < \infty$, $\omega \in A_\infty$, $q_\omega < q < \infty$. Suppose that ψ_0, $\psi \in \mathcal{S}(\mathbb{R}^n)$ satisfies (2.1)-(2.3). Then for any $f \in h^p_\omega(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$,*

$$\|f\|_{h^p_\omega} \sim \left\| \left\{ \sum_{j\in\mathbb{N}} \sum_{Q\in\Pi_{j+N}} \sup_{u\in Q} |\psi_j * f(u)|^2 \chi_Q(x) \right\}^{\frac{1}{2}} \right\|_{L^p_\omega}.$$

Proof. From the above proof, we know that

$$\begin{aligned}\|T_N(f)\|_{h^p_\omega} &= \left\| \sum_{j\in\mathbb{N}} \sum_{Q\in\Pi_{j+N}} |Q|\psi_j(x-u_Q)(\psi_j * f)(u_Q) \right\|_{h^p_\omega} \\ &\le C \left\| \left\{ \sum_{j\in\mathbb{N}} \sum_{Q\in\Pi_{j+N}} |(\psi_j * f)(u_Q)|^2 \chi_Q \right\}^{\frac{1}{2}} \right\|_{L^p_\omega}.\end{aligned}$$

Hence, for any $f \in L^q_\omega(\mathbb{R}^n) \cap h^p_\omega(\mathbb{R}^n)$, we can obtain that

$$\begin{aligned}\|f\|_{h^p_\omega} &= \|T_N^{-1} \circ T_N(f)\|_{h^p_\omega} \\ &\le C \|T_N(f)\|_{h^p_\omega} \\ &\le C \left\| \left\{ \sum_{j\in\mathbb{N}} \sum_{Q\in\Pi_{j+N}} |(\psi_j * f)(u_Q)|^2 \chi_Q \right\}^{\frac{1}{2}} \right\|_{L^p_\omega},\end{aligned}$$

which implies that

$$\|f\|_{h^p_\omega} \le C \left\| \left\{ \sum_{j\in\mathbb{N}} \sum_{Q\in\Pi_{j+N}} \inf_{u\in Q} |(\psi_j * f)(u)|^2 \chi_Q \right\}^{\frac{1}{2}} \right\|_{L^p_\omega}.$$

Then, repeating the same process, we can obtain that

$$\left\| \left\{ \sum_{j\in\mathbb{N}} \sum_{Q\in\Pi_{j+N}} \sup_{u\in Q} |(\psi_j * f)(u)|^2 \chi_Q \right\}^{\frac{1}{2}} \right\|_{L^p_\omega} \le C \|f\|_{h^p_\omega}.$$

Details are similar to those in [7]. Furthermore, we have that

$$\|f\|_{h^p_\omega} \approx \left\| \left\{ \sum_{j\in\mathbb{N}} \sum_{Q\in\Pi_{j+N}} \sup_{u\in Q} |(\psi_j * f)(u)|^2 \chi_Q \right\}^{\frac{1}{2}} \right\|_{L^p_\omega}.$$

Therefore, we complete the proof. □

Then we give the following lemma which is need for the proof of Theorem 1.8. The proof of the lemma is similar to but easier than those in [8, 29].

Lemma 2.7. *Let $0 < p < \infty$, $\omega \in A_\infty$. Then for any $f \in \mathcal{S}'(\mathbb{R}^n)$,*

$$\|f\|_{h^p_\omega} \sim \|g(f)\|_{L^p_\omega}.$$

We also recall the following key lemmas which are need for the proof of Section 4. For more details, see [5].

Lemma 2.8. *Fix $q > 1$. If $0 < p < q$ and $\omega \in RH_{(\frac{q}{p})'}$, then for all sequences of cubes $\{Q_k\}$ and non-negative functions $\{g_k\}$ such that* $\operatorname{supp}(g_k) \subset Q_k$,

$$\left\|\sum_k g_k\right\|_{L^p_\omega} \lesssim \left\|\sum_k \left(\frac{1}{|Q_k|}\int_{Q_k} g_k^q dy\right)^{\frac{1}{q}} \chi_{Q_k}\right\|_{L^p_\omega}.$$

Lemma 2.9. *Suppose $0 < \alpha < n$, $0 < p < \frac{n}{\alpha}$, and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. If $\omega^p \in RH_{\frac{q}{p}}$, then for any countable collection of cubes $\{Q_k\}$ and $\lambda_k > 0$,*

$$\left\|\sum_k \lambda_k |Q_k|^{\frac{\alpha}{n}} \chi_{Q_k}\right\|_{L^q_{\omega^q}} \lesssim \left\|\sum_k \lambda_k \chi_{Q_k}\right\|_{L^p_{\omega^p}}.$$

Here and throughout this paper, we assume that $\Phi \in \mathcal{S}(\mathbb{R}^n)$ with $\int \Phi \neq 0$ and $\operatorname{supp}(\Phi) \subset B(0, R)$, where R is a given constant in $(0, \infty)$.

Lemma 2.10. *Fix $N \geq 0$ and $0 \leq \alpha < n$. Let $\mathcal{K}$ be a distribution such that $|\widehat{\mathcal{K}}(\xi)| \lesssim |\xi|^{-\alpha}$. Suppose further that away from the origin $\mathcal{K}$ agrees with a function in C^{N+1}, and for all multi-indices β such that $|\beta| \leq N + 1$,*

$$|\partial^\beta \mathcal{K}(x)| \lesssim |x|^{-n+\alpha-|\beta|}.$$

Define the operator T by $Tf = \mathcal{K} * f$. Let a be any ω-(p, q, s)-atom with $\operatorname{supp}(a) \subset Q$ *for $0 < p < \infty$ and $1 \leq q < \infty$. Then for all $x \in (Q^*)^c$,*

$$M_\Phi(Ta)(x) \lesssim \frac{M_{\alpha_\tau}(\chi_Q)(x)^\tau}{\omega(Q)^{\frac{1}{p}}},$$

where $\tau = \frac{n+N+1}{n}$ and $\alpha_\tau = \alpha/\tau$.

Lemma 2.11. *Given $0 < \alpha < n$, $1 < r < \infty$, and $1 < p < \frac{n}{\alpha}$, define q by $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$. If $\omega \in A_{p,q}$, then*

$$\left\|\left(\sum_k (M_\alpha g_k)^r\right)^{\frac{1}{r}}\right\|_{L^q_{\omega^q}} \lesssim \left\|\left(\sum_k |g_k|^r\right)^{\frac{1}{r}}\right\|_{L^p_{\omega^p}}.$$

3. Proofs of Theorems 1.7 and 1.8

In this section, we will establish the atomic decomposition characterization of $h^p_\omega(\mathbb{R}^n)$ for $0 < p < \infty$ and $\omega \in A_\infty$. Now we give the proof of the atom decomposition.

Proof of Theorem 1.7. Suppose that $f \in h^p_\omega(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$, $0 < p < \infty$, $q_\omega < q < \infty$. By Lemma 2.5, we can obtain

$$\begin{aligned} f(x) &= \sum_{j\in\mathbb{N}} \sum_{Q\in\Pi_{j+N}} |Q|\psi_j(x-u_Q)(\psi_j * h)(u_Q) \\ &= \sum_{Q\in\Pi_N} |Q|\psi_0(x-u_Q)(\psi_0 * h)(u_Q) + \sum_{j\geq1} \sum_{Q\in\Pi_{j+N}} |Q|\psi_j(x-u_Q)(\psi_j * h)(u_Q) \\ &= I + II. \end{aligned}$$

Define

$$S^0(h)(x) = \left\{ \sum_{P\in\Pi_N} \sup_{u\in P} |\psi_0 * h(u)|^2 \chi_P(x) \right\}^{\frac{1}{2}}$$

and

$$S^1(h)(x) = \left\{ \sum_{j\geq 1} \sum_{Q\in\Pi_{j+N}} \sup_{u\in Q} |\psi_j * h(u)|^2 \chi_Q(x) \right\}^{\frac{1}{2}}.$$

For any $i \in \mathbb{Z}$ and $k = 0, 1$, set

$$\Omega_{i,k} = \left\{ x \in \mathbb{R}^n : S^k(h)(x) > 2^i \right\}$$

and

$$\widetilde{\Omega}_{i,k} = \left\{ x \in \mathbb{R}^n : M(\chi_{\Omega_{i,k}})(x) > \frac{1}{1000} \right\}.$$

Denote

$$B_{i,0} = \left\{ P : P \in \Pi_N,\ |P \cap \Omega_{i,0}| > \frac{1}{2}|P|,\ |P \cap \Omega_{i+1,0}| \leq \frac{1}{2}|P| \right\}$$

and

$$B_{i,1} = \left\{ Q : Q \in \bigcup_{j\geq 1} \Pi_{j+N},\ |Q \cap \Omega_{i,1}| > \frac{1}{2}|Q|,\ |Q \cap \Omega_{i+1,1}| \leq \frac{1}{2}|Q| \right\}.$$

Denote that $\widetilde{Q} \in B_{i,1}$ are maximal dyadic cubes in $B_{i,1}$. If $l(Q) = 2^{-j-N}$, use ψ_Q to denote ψ_j.

Now we estimate II. We can rewrite

$$\begin{aligned} II &= \sum_{i=-\infty}^{+\infty} \sum_{\widetilde{Q}\in B_{i,1}} \sum_{Q\subset\widetilde{Q}, Q\in B_{i,1}} |Q|(\psi_Q * h)(u_Q)\psi_Q(x - u_Q) \\ &=: \sum_{i=-\infty}^{+\infty} \sum_{\widetilde{Q}\in B_{i,1}} \lambda^i_{\widetilde{Q}} a^i_{\widetilde{Q}}(x), \end{aligned}$$

where

$$a^i_{\widetilde{Q}}(x) := \frac{1}{\lambda^i_{\widetilde{Q}}} \sum_{Q\subset\widetilde{Q}} |Q|(\psi_Q * h)(u_Q)\psi_Q(x - u_Q)$$

and

$$\lambda^i_{\widetilde{Q}} := \widetilde{C} \frac{\omega(\widetilde{Q})^{\frac{1}{p}}}{|\widetilde{Q}|^{\frac{1}{q}}} \left\| \left\{ \sum_{Q\subset\widetilde{Q}} |\psi_Q * h(u_Q)|^2 \chi_Q \right\}^{\frac{1}{2}} \right\|_{L^q}.$$

By the definition of ψ_Q, we find that $a^i_{\widetilde{Q}}$ is supported in $c_1\widetilde{Q}$ where $c_1 = 2^{N+2}$ and the vanishing moment condition of $a^i_{\widetilde{Q}}$ follows from the vanishing moment condition of ψ_Q. There exists a constant $C \geq 1$ such that $|c_1\widetilde{Q}| \leq C$. Then we try to obtain the size condition of $a^i_{\widetilde{Q}}$. By the duality argument,

$$\begin{aligned}
&\left\|\sum_{Q\subset\widetilde{Q}} |Q|(\psi_Q * h)(u_Q)\psi_Q(x-u_Q)\right\|_{L^q} \\
&= \sup_{\|g\|_{L^{q'}}\leq 1} \left\langle \sum_{Q\subset\widetilde{Q}} |Q|(\psi_Q * h)(u_Q)\psi_Q(x-u_Q), g\right\rangle \\
&= \sup_{\|g\|_{L^{q'}}\leq 1} \left|\int_{\mathbb{R}^n} \sum_{Q\subset\widetilde{Q}} (\psi_Q * h)(u_Q)(\psi_Q * g)(u_Q)\chi_Q(y)dy\right| \\
&\leq \sup_{\|g\|_{L^{q'}}\leq 1} \int_{\mathbb{R}^n} \left(\sum_{Q\subset\widetilde{Q}} |(\psi_Q * h)(u_Q)|^2\chi_Q(y)\right)^{\frac{1}{2}} \\
&\quad \times \left(\sum_{Q\subset\widetilde{Q}} |(\psi_Q * g)(u_Q)|^2\chi_Q(y)\right)^{\frac{1}{2}} dy \\
&\leq \sup_{\|g\|_{L^{q'}}\leq 1} \left\{\int_{\mathbb{R}^n} \left(\sum_{Q\subset\widetilde{Q}} |(\psi_Q * h)(u_Q)|^2\chi_Q(y)\right)^{\frac{q}{2}} dy\right\}^{\frac{1}{q}} \\
&\quad \times \left\{\int_{\mathbb{R}^n} \left(\sum_{Q\subset\widetilde{Q}} |(\psi_Q * g)(u_Q)|^2\chi_Q(y)\right)^{\frac{q'}{2}} dy\right\}^{\frac{1}{q'}} \\
&\leq \sup_{\|g\|_{L^{q'}}\leq 1} \left\|S^1(g)\right\|_{L^{q'}} \left\|\left\{\sum_{Q\subset\widetilde{Q}} |(\psi_Q * h)(u_Q)|^2\chi_Q(y)\right\}^{\frac{1}{2}}\right\|_{L^q}.
\end{aligned}$$

Therefore, we can choose an appropriate constant $\widetilde{C}$ such that

$$\left\|a^i_{\widetilde{Q}}(x)\right\|_{L^q} \leq \frac{|\widetilde{Q}|^{\frac{1}{q}}}{\omega(\widetilde{Q})^{\frac{1}{p}}}.$$

In conclusion, each $a^i_{\widetilde{Q}}(x)$ is a ω-(p,q,s)-atom of $h^p_\omega(\mathbb{R}^n)$.

Then, we try to prove that for any $0 < \eta < \infty$, we have

$$\left\|\left(\sum_i \sum_{\widetilde{Q}\in B_{i,1}} \left(\frac{\lambda^i_{\widetilde{Q}}\chi_{c_1\widetilde{Q}}}{\omega(\widetilde{Q})^{\frac{1}{p}}}\right)^\eta\right)^{\frac{1}{\eta}}\right\|_{L^p_\omega} \leq C_\eta \left\|f\right\|_{h^p_\omega}.$$

We claim that

$$\left\|\left\{\sum_{Q\subset\widetilde{Q}} |\psi_Q * h(u_Q)|^2\chi_Q\right\}^{\frac{1}{2}}\right\|_{L^q} \leq C2^i \left|\widetilde{Q}\right|^{\frac{1}{q}}. \tag{8}$$

When $x \in Q$ and $Q \in B_{i,1}$, $M(\chi_{Q\cap\widetilde{\Omega}_{i,1}\backslash\Omega_{i+1,1}})(x) > \frac{1}{2}$. Moreover, since

$$\chi_Q(x) \le 2M(\chi_{Q\cap\widetilde{\Omega}_{i,1}\backslash\Omega_{i+1,1}})(x),$$

then

$$\chi_Q(x) \le 4M^2(\chi_{Q\cap\widetilde{\Omega}_{i,1}\backslash\Omega_{i+1,1}})(x).$$

By Lemma 2.1, for any $1 < q < \infty$,

$$\begin{aligned}
&\left\|\left\{\sum_{Q\subset\widetilde{Q}} |\psi_Q * h(u_Q)|^2\chi_Q\right\}^{\frac{1}{2}}\right\|_{L^q}^q \\
&= \int_{\mathbb{R}^n}\left(\sum_{Q\subset\widetilde{Q}} |\psi_Q * h(u_Q)|^2\chi_Q(x)\right)^{\frac{q}{2}} dx \\
&\le C\int_{\mathbb{R}^n}\left(\sum_{Q\subset\widetilde{Q}} |\psi_Q * h(u_Q)|^2 M^2(\chi_{Q\cap\widetilde{\Omega}_{i,1}\backslash\Omega_{i+1,1}})(x)\right)^{\frac{q}{2}} dx \\
&\le C\int_{\mathbb{R}^n}\left(\sum_{Q\subset\widetilde{Q}} |\psi_Q * h(u_Q)|^2 \chi_{Q\cap\widetilde{\Omega}_{i,1}\backslash\Omega_{i+1,1}}(x)\right)^{\frac{q}{2}} dx \\
&\le C\int_{\widetilde{Q}\cap\widetilde{\Omega}_{i,1}\backslash\Omega_{i+1,1}}\left(\sum_{Q\subset\widetilde{Q}} |\psi_Q * h(u_Q)|^2\chi_Q(x)\right)^{\frac{q}{2}} dx \\
&\le C\int_{\widetilde{Q}\cap\widetilde{\Omega}_{i}\backslash\Omega_{i+1}}\left(S^1(h)\right)^q dx \le C2^{iq}\left|\widetilde{Q}\right|.
\end{aligned}$$

Hence, we finished the proof of the claim (8). Now we can obtain

$$\left\|\left(\sum_i\sum_{\widetilde{Q}\in B_{i,1}}\left(\frac{\lambda^i_{\widetilde{Q}}\chi_{c_1\widetilde{Q}}}{\omega(\widetilde{Q})^{\frac{1}{p}}}\right)^{\eta}\right)^{\frac{1}{\eta}}\right\|_{L^p_\omega} \le C\left\|\left(\sum_i\sum_{\widetilde{Q}\in B_{i,1}}\left(2^i\chi_{c_1\widetilde{Q}}\right)^{\eta}\right)^{\frac{1}{\eta}}\right\|_{L^p_\omega}.$$

Since $\Omega_{i,1} \subset \widetilde{\Omega}_{i,1}$ for any $i \in \mathbb{Z}$ and $|\widetilde{\Omega}_{i,1}| \le C|\Omega_{i,1}|$ for any $x \in \mathbb{R}^n$, we have

$$\chi_{\widetilde{\Omega}_{i,1}}(x) \le CM^{\gamma}(\chi_{\Omega_{i,1}})(x),$$

where γ is large enough such that $\gamma p > q_\omega$ and $\gamma\eta > 1$. Applying Lemma 2.1 with $\omega \in A_{\gamma p}$, we can obtain

$$\begin{aligned}
&\left\|\left(\sum_i\sum_{\widetilde{Q}\in B_{i,1}}\left(2^i\chi_{c_1\widetilde{Q}}\right)^{\eta}\right)^{\frac{1}{\eta}}\right\|_{L^p_\omega} \le C_\eta\left\|\left(\sum_i\left(2^i\chi_{\widetilde{\Omega}_{i,1}}\right)^{\eta}\right)^{\frac{1}{\eta}}\right\|_{L^p_\omega} \\
&\le C_\eta\left\|\left(\sum_i\left(2^{\frac{i}{\gamma}}M(\chi_{\Omega_{i,1}})\right)^{\gamma\eta}\right)^{\frac{1}{\gamma\eta}}\right\|_{L^{\gamma p}_\omega}^{\gamma} \le C_\eta\left\|\left(\sum_i 2^{i\eta}\chi_{\Omega_{i,1}}\right)^{\frac{1}{\eta}}\right\|_{L^p_\omega}.
\end{aligned}$$

It is easy to know that $\Omega_{i+1,1} \subset \Omega_{i,1}$ and $|\bigcap_{i=1}^{\infty} \Omega_{i,1}| = 0$. Then for almost every $x \in \mathbb{R}^n$, we have

$$\left(\sum_i 2^{i\eta}\chi_{\Omega_{i,1}}(x)\right)^{\frac{1}{\eta}} \sim \left(\sum_i 2^{i\eta}\chi_{\Omega_{i,1}\setminus\Omega_{i+1,1}}(x)\right)^{\frac{1}{\eta}}.$$

Hence, together with Corollary 2.6, we conclude that

$$\begin{aligned}
&\left\|\left(\sum_i 2^{i\eta}\chi_{\Omega_{i,1}}\right)^{\frac{1}{\eta}}\right\|_{L^p_\omega}^p \leq C\left\|\left(\sum_i 2^{i\eta}\chi_{\Omega_{i,1}\setminus\Omega_{i+1,1}}\right)^{\frac{1}{\eta}}\right\|_{L^p_\omega}^p \\
&= C\int_{\mathbb{R}^n}\left(\sum_i 2^i\chi_{\Omega_{i,1}\setminus\Omega_{i+1,1}}\right)^p\omega(x)dx = C\sum_i\int_{\Omega_{i,1}\setminus\Omega_{i+1,1}} 2^{ip}\omega(x)dx \\
&\leq C\int_{\mathbb{R}^n}\left(S^1(f)\right)^p\omega(x)dx \leq C\left\|f\right\|_{h^p_\omega}^p.
\end{aligned}$$

Next we estimate I. We can use P to denote Q if $Q \in \Pi_N$ and rewrite

$$I =: \sum_i\sum_{P\in B_{i,0}} \mu_P^i b_P^i(x)$$

where $b_P^i(x) = \frac{1}{\mu_P^i}|P|(\psi_0 * h)(u_P)\psi_0(x-u_P)$ and $\mu_P^i = \widetilde{C}|(\psi_0 * h)(u_P)|$. Let $\widetilde{C} = 2^{-Nn}\omega(P)^{\frac{1}{p}}|P|^{-\frac{1}{q}}\left\|\psi_0\right\|_{L^q}$. Similarly, by the definition of ψ_0, we find that b_P^i is supported in c_0P where $c_0 = 3$. Moreover, there exist a constant $C \geq 1$ such that $|c_0P| > C$. It is easy to prove that $\left\|b_P^i\right\|_{L^q} = \frac{1}{\mu_P^i}|P||\psi_0 * h(u_P)|(\int_{\mathbb{R}^n}|\psi_0(x-u_P)|^q dx)^{\frac{1}{q}} \leq |P|^{\frac{1}{q}}\omega(P)^{-\frac{1}{p}}$. In conclusion, each $b_P^i(x)$ is a ω-(p,q,s)-block of $h^p_\omega(\mathbb{R}^n)$. Repeating the similar but easier argument, we can obtain

$$\left\|\left(\sum_i\sum_{P\in B_{i,0}}\left(\frac{\mu_P^i\chi_{c_0P}}{\omega(P)^{\frac{1}{p}}}\right)^\eta\right)^{\frac{1}{\eta}}\right\|_{L^p_\omega} \leq C_\eta\left\|f\right\|_{h^p_\omega}.$$

Consequently, we can know that

$$\left\|\left(\sum_i\sum_{\widetilde{Q}\in B_{i,1}}\left(\frac{\lambda_{\widetilde{Q}}^i\chi_{c_1\widetilde{Q}}}{\omega(\widetilde{Q})^{\frac{1}{p}}}\right)^\eta\right)^{\frac{1}{\eta}}\right\|_{L^p_\omega} + \left\|\left(\sum_i\sum_{P\in B_{i,0}}\left(\frac{\mu_P^i\chi_{c_0P}}{\omega(P)^{\frac{1}{p}}}\right)^\eta\right)^{\frac{1}{\eta}}\right\|_{L^p_\omega} \leq C_\eta\left\|f\right\|_{h^p_\omega}.$$

Therefore, we complete the proof. □

Next we will prove the reconstruction theorem for the atomic decomposition.

Proof of Theorem 1.8. Notice that for almost every $x \in \mathbb{R}^n$

$$|g(f)(x)| \leq \sum_{j=1}^\infty \lambda_j|g(a_j)(x)| + \sum_{j=1}^\infty \mu_j|g(b_j)(x)| = I + II.$$

For II,

$$\begin{aligned}
II &= \sum_{j=1}^\infty \mu_j|g(b_j)(x)|\chi_{4P_j}(x) + \sum_{j=1}^\infty \mu_j|g(b_j)(x)|\chi_{(4P_j)^c}(x) \\
&= II_1 + II_2.
\end{aligned}$$

Now we estimate the term II_1. Denote $h_j(x) = g(b_j)(x)\chi_{4P_j}$. By the size condition of atoms and $q > q_r$, we obtain

$$\left\|h_j\right\|_{L^q} \leq \left\|g(b_j)\right\|_{L^q} \leq C\left\|b_j\right\|_{L^q} \leq C|P_j|^{\frac{1}{q}}\omega(P_j)^{-\frac{1}{p}}.$$

Together with the fact $\mathrm{supp}(h_j) \subset 4P_j$, Lemma 2.3 and Remark 2.2, we obtain

$$\|II_1\|_{L^p_\omega} = \left\|\sum_{j=1}^{\infty}\mu_j h_j(x)\right\|_{L^p_\omega} \leq C\left\|\sum_{j=1}^{\infty}\frac{\mu_j\chi_{4P_j}}{\omega(P_j)^{\frac{1}{p}}}\right\|_{L^p_\omega} \leq C\left\|\sum_{j=1}^{\infty}\frac{\mu_j\chi_{P_j}}{\omega(P_j)^{\frac{1}{p}}}\right\|_{L^p_\omega}.$$

Next we estimate II_2. For all $x \in (4P_j)^c$, we have

$$\begin{aligned}
|(\phi_i * b_j)(x)| &\leq \int_{P_j}|\phi_i(x-y)b_j(y)|dy \\
&\leq \sup_{z\in P_j}|\phi_i(x-z)|\int_{P_j}|b_j(y)|dy \\
&\leq C\frac{2^{in}}{(1+2^i|x-x_{P_j}|)^M}\left\|b_j\right\|_{L^q}|P_j|^{\frac{1}{q'}} \\
&\leq C\frac{2^{in}}{(1+2^i|x-x_{P_j}|)^M}\frac{|P_j|^{\frac{1}{q}}|P_j|^{\frac{1}{q'}}}{\omega(P_j)^{\frac{1}{p}}} \\
&\leq C\frac{2^{in}}{(1+2^i|x-x_{P_j}|)^M}\frac{l(P_j)^M}{\omega(P_j)^{\frac{1}{p}}}
\end{aligned}$$

for some sufficient large $M > n > 0$. Observe that $|P_j| > C \geq 1$ and if $M > n$,

$$\sum_{i=0}^{\infty}C\frac{2^{in}}{(1+2^i|x-x_{P_j}|)^M} \leq \frac{C}{|x-x_{P_j}|^M}.$$

Therefore, we obtain

$$\begin{aligned}
II_2 &= \sum_{j=1}^{\infty}\mu_j\left\{\sum_{i\in\mathbb{N}}|\phi_i * b_j(x)|^2\right\}^{\frac{1}{2}}\chi_{(4P_j)^c}(x) \\
&\leq \sum_{j=1}^{\infty}\mu_j\left(\sum_{i\in\mathbb{N}}|\phi_i * b_j(x)|\right)\chi_{(4P_j)^c}(x) \\
&\leq C\sum_{j=1}^{\infty}\mu_j\frac{\omega(P_j)^{-\frac{1}{p}}(l(P_j))^M}{|x-x_{P_j}|^M}\chi_{(4P_j)^c}(x).
\end{aligned}$$

Let $M = n + s + 1$ and $\gamma = \frac{M}{n}$. We have

$$\begin{aligned}
II_2 &\leq C\sum_{j=1}^{\infty}\mu_j\omega(P_j)^{-\frac{1}{p}}\left(\left(\frac{l(P_j)}{|x-x_{P_j}|}\right)^n\right)^\gamma\chi_{(4P_j)^c}(x) \\
&\leq C\sum_{j=1}^{\infty}\mu_j\omega(P_j)^{-\frac{1}{p}}(M\chi_{P_j})^\gamma(x).
\end{aligned}$$

Since $\omega \in A_\infty$ with the critical index q_ω and $s \geq \max\{[n(\frac{q_\omega}{p}-1)], -1\}$, we know that $\gamma p > q_\omega$ and then $\omega \in A_{\gamma p}$. Applying Lemma 2.1 yields that

$$\begin{aligned}\|II_2\|_{L^p_\omega} &\leq C\left\|\sum_{j=1}^{\infty}\frac{\mu_j(M\chi_{P_j})^{\gamma}}{\omega(P_j)^{\frac{1}{p}}}\right\|_{L^p_\omega} = C\left\|\left(\sum_{j=1}^{\infty}\frac{\mu_j(M\chi_{P_j})^{\gamma}}{\omega(P_j)^{\frac{1}{p}}}\right)^{\frac{1}{\gamma}}\right\|^{\gamma}_{L^{\gamma p}_\omega}\\ &\leq C\left\|\left(\sum_{j=1}^{\infty}\frac{\mu_j\chi_{P_j}}{\omega(P_j)^{\frac{1}{p}}}\right)^{\frac{1}{\gamma}}\right\|^{\gamma}_{L^{\gamma p}_\omega} = C\left\|\sum_{j=1}^{\infty}\frac{\mu_j\chi_{P_j}}{\omega(P_j)^{\frac{1}{p}}}\right\|_{L^p_\omega}.\end{aligned}$$

By Lemma 2.7 and the estimates of II_1 and II_2, we obtain

$$\left\|\sum_{j=1}^{\infty}\mu_j b_j\right\|_{h^p_\omega} \leq C\left\|\sum_{j=1}^{\infty}\mu_j g(b_j)\right\|_{L^p_\omega} \leq C\left\|\sum_{j=1}^{\infty}\frac{\mu_j\chi_{P_j}}{\omega(P_j)^{\frac{1}{p}}}\right\|_{L^p_\omega}.$$

Similarly, for I, we can find that

$$\begin{aligned}I &= \sum_{j=1}^{\infty}\lambda_j|g(a_j)(x)|\chi_{2Q_j}(x) + \sum_{j=1}^{\infty}\lambda_j|g(a_j)(x)|\chi_{(2Q_j)^c}(x)\\ &:= I_1 + I_2,\end{aligned}$$

and

$$\|I_1\|_{L^p_\omega} \leq C\left\|\sum_{j=1}^{\infty}\frac{\lambda_j\chi_{Q_j}}{\omega(Q_j)^{\frac{1}{p}}}\right\|_{L^p_\omega}.$$

Note that $Q_j \equiv Q_j(x_j, l(Q_j))$. Denote by p_i^s the sum of first $s+1$ terms in the Taylor expansion of $\phi_i(y-z)$ at $y - x_j$. Details are similar to those in [28]. Applying the vanishing moment and size condition of a_j and the smoothness conditions on ϕ_j, we can obtain

$$\begin{aligned}g(a_j)^2(y) &= \sum_i\left|\int_{\mathbb{R}^n}a_j(z)[\phi_i(y-z) - p_i^s(y,z,x_j)]dz\right|^2\\ &\leq C\frac{\omega(Q_j)^{-\frac{2}{p}}l(Q_j)^{2(n+s+1)}}{|y-x_j|^{2(n+s+1)}}.\end{aligned}$$

Let $\gamma = \frac{n+s+1}{n}$. By repeating the similar analysis as in the estimate of II, we can obtain

$$\|I_2\|_{L^p_\omega} \leq C\left\|\sum_{j=1}^{\infty}\frac{\lambda_j\chi_{Q_j}}{\omega(Q_j)^{\frac{1}{p}}}\right\|_{L^p_\omega}.$$

Therefore, it concludes that

$$\left\|\sum_{j=1}^{\infty}\lambda_j a_j\right\|_{h^p_\omega} \leq C\left\|\sum_{j=1}^{\infty}\frac{\lambda_j\chi_{Q_j}}{\omega(Q_j)^{\frac{1}{p}}}\right\|_{L^p_\omega}.$$

Observe that

$$\left\|\sum_{j=1}^{\infty}\frac{\lambda_j\chi_{Q_j}}{\omega(Q_j)^{\frac{1}{p}}}\right\|_{L^p_\omega} + \left\|\sum_{j=1}^{\infty}\frac{\mu_j\chi_{P_j}}{\omega(P_j)^{\frac{1}{p}}}\right\|_{L^p_\omega} < \infty,$$

which implies that

$$\left\|\sum_{j=N}^{\infty}\frac{\lambda_j\chi_{Q_j}}{\omega(Q_j)^{\frac{1}{p}}}\right\|_{L^p_\omega}+\left\|\sum_{j=N}^{\infty}\frac{\mu_j\chi_{P_j}}{\omega(P_j)^{\frac{1}{p}}}\right\|_{L^p_\omega}\to 0,\ as\ N\to\infty.$$

Thus,

$$\lim_{N\to\infty}\left\|\sum_{j=N}^{\infty}\frac{\lambda_j\chi_{Q_j}}{\omega(Q_j)^{\frac{1}{p}}}\right\|_{L^p_\omega}=0,\ \lim_{N\to\infty}\left\|\sum_{j=N}^{\infty}\frac{\mu_j\chi_{P_j}}{\omega(P_j)^{\frac{1}{p}}}\right\|_{L^p_\omega}=0.$$

Notice that

$$\left\|\sum_{j=N}^{\infty}\lambda_j a_j\right\|_{h^p_\omega}\le C\left\|\sum_{j=N}^{\infty}\frac{\lambda_j\chi_{Q_j}}{\omega(Q_j)^{\frac{1}{p}}}\right\|_{L^p_\omega};$$

$$\left\|\sum_{j=N}^{\infty}\mu_j b_j\right\|_{h^p_\omega}\le C\left\|\sum_{j=N}^{\infty}\frac{\mu_j\chi_{P_j}}{\omega(P_j)^{\frac{1}{p}}}\right\|_{L^p_\omega}.$$

Therefore, we can obtain

$$\lim_{N\to\infty}\left\|\sum_{j=N}^{\infty}\lambda_j a_j\right\|_{h^p_\omega}=0,\ \lim_{N\to\infty}\left\|\sum_{j=N}^{\infty}\mu_j b_j\right\|_{h^p_\omega}=0,$$

which implies that the series $\sum_j \lambda_j a_j+\sum_j \mu_j b_j$ converges in $h^p_\omega(\mathbb{R}^n)$. □

4. Proofs of Theorems 1.9, 1.10 and 1.12

This section is devoted to proving the boundedness results given in Theorems 1.9 and 1.10 for the inhomogenous Calderón–Zygmund singular integrals and Theorem 1.12 for the local fractional integrals.

Proof of Theorem 1.9. Recalling the atomic decomposition of weighted local Hardy spaces in Theorem 1.7, we know that if $f\in h^p_\omega(\mathbb{R}^n)\cap L^q(\mathbb{R}^n)$, there exist a sequence of ω-(p,q,s)-atoms $\{a_j\}_{j=1}^{\infty}$ with a corresponding sequence of non-negative numbers $\{\lambda_j\}_{j=1}^{\infty}$ and a sequence of ω-(p,q,s)-blocks $\{b_j\}_{j=1}^{\infty}$ with a corresponding sequence of non-negative numbers $\{\mu_j\}_{j=1}^{\infty}$ such that $f=\sum_j \lambda_j a_j+\sum_j \mu_j b_j$ in $h^p_\omega(\mathbb{R}^n)\cap L^q(\mathbb{R}^n)$ with $(\frac{n+\eta}{n})p<q<\infty$, and that

$$\left\|\sum_{j=1}^{\infty}\frac{\lambda_j\chi_{Q_j}}{\omega(Q_j)^{\frac{1}{p}}}\right\|_{L^p_\omega}+\left\|\sum_{j=1}^{\infty}\frac{\mu_j\chi_{P_j}}{\omega(P_j)^{\frac{1}{p}}}\right\|_{L^p_\omega}\le C\left\|f\right\|_{h^p_\omega}.$$

To prove the theorem, it will suffice to prove that

$$\left\|T(f)\right\|_{L^p_\omega}\le C_1\left\|\sum_{j=1}^{\infty}\frac{\lambda_j\chi_{Q_j}}{\omega(Q_j)^{\frac{1}{p}}}\right\|_{L^p_\omega}+C_2\left\|\sum_{j=1}^{\infty}\frac{\mu_j\chi_{P_j}}{\omega(P_j)^{\frac{1}{p}}}\right\|_{L^p_\omega}.$$

In fact, for $x \in \mathbb{R}^n$, we have

$$\left|T(f)(x)\right| \leq \sum_j |\lambda_j||T(a_j)(x)| + \sum_j |\mu_j||T(b_j)(x)| =: I + II.$$

First we can prove that

$$\|I\|_{L^p_\omega} \leq C\left\|\sum_{j=1}^{\infty} \frac{\lambda_j \chi_{Q_j}}{\omega(Q_j)^{\frac{1}{p}}}\right\|_{L^p_\omega}.$$

For $x \in \mathbb{R}^n$,

$$I \leq \sum_{j=1}^{\infty} |\lambda_j||T(a_j)(x)|\chi_{Q_j^*}(x) + \sum_{j=1}^{\infty} |\lambda_j||T(a_j)(x)|\chi_{(Q_j^*)^c}(x) =: I_1 + I_2.$$

Since T is a bounded operator on $L^2(\mathbb{R}^n)$, from the Calderón-Zygmund real method in [19, Section 7.3], we know that T is bounded on $L^q(\mathbb{R}^n)$ for any $1 < q < \infty$. Together with the size condition of a_j, we obtain that for any $(\frac{n+\eta}{n})p < q < \infty$

$$\left(\frac{1}{|Q_j^*|}\int_{Q_j^*} |T(a_j)\chi_{Q_j^*}|^q dx\right)^{\frac{1}{q}} \leq \frac{\left\|a_j\right\|_{L^q}}{|Q_j^*|^{\frac{1}{q}}} \leq \frac{|Q_j|^{\frac{1}{q}}}{\omega(Q_j)^{\frac{1}{p}}|Q_j^*|^{\frac{1}{q}}} \leq \frac{C}{\omega(Q_j)^{\frac{1}{p}}}.$$

Since $\omega \in A_{(\frac{n+\eta}{n})p}$, then there exists $r > 1$ such that $\omega \in RH_r$. Fix $q_0 > \max\{p, 1\}$ such that $(\frac{q_0}{p})' < r$. For I_1, by Lemma 2.8 and Remark 2.2, we can get that

$$\begin{aligned}
\|I_1\|_{L^p_\omega} &\leq \left\|\sum_j |\lambda_j||T(a_j)|\chi_{Q_j^*}\right\|_{L^p_\omega} \\
&\leq C\left\|\sum_j \lambda_j \left(\frac{1}{|Q_j^*|}\int_{Q_j^*} |T(a_j)\chi_{Q_j^*}|^{q_0} dx\right)^{\frac{1}{q_0}} \chi_{Q_j^*}\right\|_{L^p_\omega} \\
&\leq C\left\|\sum_j \frac{\lambda_j \chi_{Q_j^*}}{\omega(Q_j)^{\frac{1}{p}}}\right\|_{L^p_\omega} \leq C\left\|\sum_j \frac{\lambda_j \chi_{Q_j}}{\omega(Q_j)^{\frac{1}{p}}}\right\|_{L^p_\omega}.
\end{aligned}$$

For I_2, note that $x \in (Q_j^*)^c$ and c_{Q_j} is the center of Q_j. We can know that $|x - c_{Q_j}| \geq 2|y - c_{Q_j}|$ and $|y - c_{Q_j}| \leq l(Q_j)$. Applying the smooth condition of the kernel $\mathcal{K}$, we obtain that

$$\begin{aligned}
|T(a_j)(x)| &= \left|\int_{Q_j} \mathcal{K}(x, y)a_j(y)dy\right| \leq \int_{Q_j} |\mathcal{K}(x, y) - \mathcal{K}(x, c_{Q_j})||a_j(y)|dy \\
&\leq C\int_{Q_j} \frac{|y - c_{Q_j}|^\epsilon}{|x - c_{Q_j}|^{n+\epsilon}}|a_j(y)|dy \leq C\frac{l(Q_j)^\epsilon}{|x - c_{Q_j}|^{n+\epsilon}}\int_{Q_j} |a_j(y)|dy \\
&\leq C\frac{l(Q_j)^{n+\epsilon}}{\omega(Q_j)^{\frac{1}{p}}|x - c_{Q_j}|^{n+\epsilon}} \leq C\frac{(M(\chi_{Q_j})(x))^{\frac{n+\eta}{n}}}{\omega(Q_j)^{\frac{1}{p}}}.
\end{aligned}$$

Denote that $\gamma = \frac{n+\eta}{n}$. Notice that $\gamma p > 1$ and $\omega \in A_{\gamma p}$. Applying Fefferman–Stein vector-valued maximal inequality yields that

$$\|I_2\|_{L^p_\omega} \le C\left\|\sum_j \frac{|\lambda_j|M^\gamma(\chi_{Q_j})}{\omega(Q_j)^{\frac{1}{p}}}\right\|_{L^p_\omega} \le C\left\|\left(\sum_j \frac{\lambda_j M^\gamma(\chi_{Q_j})}{\omega(Q_j)^{\frac{1}{p}}}\right)^{\frac{1}{\gamma}}\right\|^{\gamma}_{L^{\gamma p}_\omega} \le C\left\|\sum_j \frac{\lambda_j \chi_{Q_j}}{\omega(Q_j)^{1/p}}\right\|_{L^p_\omega}.$$

Combining the estimates of I_1 and I_2, we can obtain the desired result.

Then we can prove that

$$\|II\|_{L^p_\omega} \le C\left\|\sum_{j=1}^{\infty} \frac{\mu_j \chi_{P_j}}{\omega(P_j)^{\frac{1}{p}}}\right\|_{L^p_\omega}.$$

By repeating the similar argument, we can know that for $x \in \mathbb{R}^n$,

$$II \le \sum_{j=1}^{\infty} |\mu_j||T(b_j)(x)|\chi_{P_j^*}(x) + \sum_{j=1}^{\infty} |\mu_j||T(b_j)(x)|\chi_{(P_j^*)^c}(x) =: II_1 + II_2$$

and

$$\|II_1\|_{L^p_\omega} \le C\left\|\sum_{j=1}^{\infty} \frac{\mu_j \chi_{P_j}}{\omega(P_j)^{\frac{1}{p}}}\right\|_{L^p_\omega}.$$

For I_2, when $x \in (P_j^*)^c$ and $y \in P_j$, we have $|x-y| \sim |x - c_{P_j}|$ and $|x-y| > 1/2$. By using the size condition of $\mathcal{K}$ and the fact that $|P_j| > C$, we can get that for any $x \in (P_j^*)^c$,

$$\begin{aligned} |T(b_j)(x)| &= \left|\int_{P_j} \mathcal{K}(x,y) b_j(y)dy\right| \le \int_{P_j} |\mathcal{K}(x,y)||b_j(y)|dy \\ &\le C\frac{1}{|x-c_{P_j}|^{n+\delta}} \int_{P_j} |b_j(y)|dy \le C\frac{1}{|x-c_{P_j}|^{n+\delta}} \left\|b_j\right\|_{L^q} |P_j|^{\frac{1}{q'}} \\ &\le C\frac{l(P_j)^{n+\delta}}{\omega(P_j)^{\frac{1}{p}} |x-c_{P_j}|^{n+\delta}} \le C\frac{(M(\chi_{P_j})(x))^{\frac{n+\eta}{n}}}{\omega(P_j)^{\frac{1}{p}}}. \end{aligned}$$

Then, it concludes that

$$\|II_2\|_{L^p_\omega} \le C\left\|\sum_j \frac{\mu_j \chi_{P_j}}{\omega(P_j)^{1/p}}\right\|_{L^p_\omega}.$$

Therefore, by a density argument, we finish the proof of the theorem. □

Proof of Theorem 1.10. By the argument similar to that used in the above proof, it will suffice to prove that for $f \in h^p_\omega(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$ with $(\frac{n+\eta}{n})p < q < \infty$,

$$\left\|T(f)\right\|_{h^p_\omega} = \left\|M_\Phi(T(f))\right\|_{L^p_\omega} \le C_1\left\|\sum_{j=1}^{\infty} \frac{\lambda_j \chi_{Q_j}}{\omega(Q_j)^{\frac{1}{p}}}\right\|_{L^p_\omega} + C_2\left\|\sum_{j=1}^{\infty} \frac{\mu_j \chi_{P_j}}{\omega(P_j)^{\frac{1}{p}}}\right\|_{L^p_\omega}.$$

We claim that for $x \in \mathbb{R}^n$, we have

$$\sup_{0<t<1} |\Phi_t * T(f)(x)| \le I + II,$$

where

$$I=\sum_j \frac{\lambda_j}{\omega(Q_j)^{\frac{1}{p}}}(M(T(a_j))(x)\chi_{2\sqrt{n}Q_j^*}(x)+(M(\chi_{Q_j})(x))^{\gamma}\chi_{(2\sqrt{n}Q_j^*)^c}(x))$$

and

$$II=\sum_j \frac{\mu_j}{\omega(P_j)^{\frac{1}{p}}}(M(T(b_j))(x)\chi_{2\sqrt{n}P_j^*}(x)+(M(\chi_{P_j})(x))^{\gamma}\chi_{(2\sqrt{n}P_j^*)^c}(x))$$

with $\gamma=\frac{n+\eta}{n}$. Applying the claim and repeating the nearly identical argument to the proof of Theorem 1.9, we can obtain the desired result. In fact, when $x\in 2\sqrt{n}Q_j^*$, we just need the pointwise estimate

$$M_\Phi(T(\sum_j \lambda_j a_j))(x)\le C\sum_j \lambda_j M(T(a_j))(x).$$

When $x\in(2\sqrt{n}Q_j^*)^c$, we have

$$|\Phi_t*T(a_j)(x)|=\left|\int_{\mathbb{R}^n}\Phi_t(x-y)T(a_j)(y)dy\right|\le t^{-n}\int_{B(x,t)}|T(a_j)(y)|dy\le \sup_{y\in B(x,t)}|T(a_j)(y)|.$$

Notice that $|Q_j|\le C$ and $x\in(2\sqrt{n}Q_j^*)^c$. If $0<t\le|x-c_{Q_j}|/2$, we can get that $y\in(Q_j^*)^c$. Therefore, from the proof of Theorem 1.9, we conclude that

$$\sup_{y\in B(x,t)}|T(a_j)(y)|\le C\frac{(M(\chi_{Q_j})(x))^{\gamma}}{\omega(Q_j)^{\frac{1}{p}}}.$$

Then we consider the case that $t>|x-c_{Q_j}|/2$. Observe that a_j satisfies $\int_{\mathbb{R}^n}T(a_j)(x)dx=0$. For any $x\in(2\sqrt{n}Q_j^*)^c$, applying the mean value theorem and Hölder's inequality yields that

$$\begin{aligned}
|\Phi_t*T(a_j)(x)|&=\left|\int_{\mathbb{R}^n}(\Phi_t(x-y)-\Phi_t(x-c_{Q_j}))T(a_j)(y)dy\right|\\
&\le t^{-n}\int_{\mathbb{R}^n}\left|\frac{y-c_{Q_j}}{t}\right||\Phi'((x-c_{Q_j}+\theta(c_{Q_j}-y))/t)||T(a_j)(y)|dy\\
&\le C|x-c_{Q_j}|^{-n-1}\left(\int_{Q_j^*}|y-c_{Q_j}||T(a_j)(y)|dy+\int_{(Q_j^*)^c}|y-c_{Q_j}||T(a_j)(y)|dy\right)\\
&\le C|x-c_{Q_j}|^{-n-1}\left(l(Q_j)^{\frac{n}{q'}+1}\left\|T(a_j)\right\|_{L^q}+\int_{(Q_j^*)^c}\frac{l(Q_j)^{n+\eta}}{\omega(Q_j)^{\frac{1}{p}}|y-c_{Q_j}|^{n+\eta-1}}dy\right)\\
&\le C|x-c_{Q_j}|^{-n-1}l(Q_j)^{n+1}\omega(Q_j)^{-\frac{1}{p}}\le C\frac{(M(\chi_{Q_j})(x))^{\gamma}}{\omega(Q_j)^{\frac{1}{p}}},
\end{aligned}$$

where $\theta\in(0,1)$.

Similarly, when $x\in 2\sqrt{n}P_j^*$, we can get that

$$M_\Phi(T(\sum_j \mu_j b_j)(x))\le C\sum_j \mu_j M(T(b_j))(x)$$

and when $x\in(2\sqrt{n}P_j^*)^c$,

$$|\Phi_t*T(b_j)(x)|\le \sup_{y\in B(x,t)}|T(b_j)(y)|.$$

Notice that $|P_j| > C$. Then we can get that $y \in (P_j^*)^c$. Then, repeating the same argument as used above, we can obtain that

$$\sup_{y\in B(x,t)} |T(b_j)(y)| \le C\frac{(M(\chi_{P_j})(x))^\gamma}{\omega(P_j)^{\frac{1}{p}}}.$$

Therefore, we complete the proof of the theorem. □

Proof of Theorem 1.12. To prove the first part of this theorem, we apply the argument similar to that used in the proof of Theorem 1.9 and Theorem 1.10 and so we only need to concentrate on the differences. Now we consider the case when $1 < q < \infty$. By the atomic decomposition of $h^p_\omega(\mathbb{R}^n)$ and a dense argument, in order to show that I^{loc}_α admits a bounded extension from $h^p_{\omega^p}(\mathbb{R}^n)$ to $L^q_{\omega^q}(\mathbb{R}^n)$, we only need to prove that

$$\left\|\sum_j \lambda_j I^{loc}_\alpha(a_j)\right\|_{L^q_{\omega^q}} \le C_1\left\|\sum_j \frac{\lambda_j\chi_{Q_j}}{\omega(Q_j)^{\frac{1}{p}}}\right\|_{L^p_{\omega^p}} \tag{9}$$

and

$$\left\|\sum_j \mu_j I^{loc}_\alpha(b_j)\right\|_{L^q_{\omega^q}} \le C_2\left\|\sum_j \frac{\mu_j\chi_{P_j}}{\omega(P_j)^{\frac{1}{p}}}\right\|_{L^p_{\omega^p}}, \tag{10}$$

where each a_j is ω-(p,t,s)-atom, each b_j is ω-(p,t,s)-block and the exact value of t will be chosen below.

Then we prove (9). We first estimate the condition that $|x - c_{Q_j}| \le l(Q_j^*)$. Since $\omega^p \in RH_{\frac{q}{p}}$, $\omega^q \in A_\infty$, there exists $r > 1$ such that $\omega^q \in RH_r$. Fix $q_0 > \max\{q, 1\}$ such that $(\frac{q_0}{q})' < r$. Then $\omega^q \in RH_{(\frac{q_0}{q})'}$. Define $p_0 > 1$ by $\frac{1}{p_0} - \frac{1}{q_0} = \frac{\alpha}{n}$. Moreover, we choose $t = p_0$. Thus, by Lemma 2.8, we have that

$$\begin{aligned}
\left\|\sum_j \lambda_j I^{loc}_\alpha(a_j)\chi_{Q_j^*}\right\|_{L^q_{\omega^q}} &\le \left\|\sum_j \lambda_j |I^{loc}_\alpha(a_j)|\chi_{Q_j^*}\right\|_{L^q_{\omega^q}}\\
&\le C\left\|\sum_j \lambda_j\left(\frac{1}{|Q_j^*|}\int_{Q_j^*}|I^{loc}_\alpha(a_j)\chi_{Q_j^*}|^{q_0}dx\right)^{\frac{1}{q_0}}\chi_{Q_j^*}\right\|_{L^q_{\omega^q}}\\
&\le C\left\|\sum_j \lambda_j|Q_j^*|^{-\frac{1}{q_0}}\left(\int_{Q_j^*}|a_j|^{p_0}dx\right)^{\frac{1}{p_0}}\chi_{Q_j^*}\right\|_{L^q_{\omega^q}}\\
&\le C\left\|\sum_j \lambda_j\frac{|Q_j^*|^{-\frac{1}{q_0}}|Q_j|^{\frac{1}{p_0}}\chi_{Q_j^*}}{\omega(Q_j)^{\frac{1}{p}}}\right\|_{L^q_{\omega^q}}\\
&\le C\left\|\sum_j \lambda_j\frac{l(Q_j^*)^\alpha\chi_{Q_j^*}}{\omega(Q_j)^{\frac{1}{p}}}\right\|_{L^q_{\omega^q}}\\
&\le C\left\|\sum_j \frac{\lambda_j\chi_{Q_j}}{\omega(Q_j)^{\frac{1}{p}}}\right\|_{L^p_{\omega^p}},
\end{aligned}$$

where the third inequality follows from the boundedness of I^{loc}_α on classical Lebesgue spaces ([35, Lemma 8.9]) and the last inequality follows from the Lemma 2.9, Remark 2.2.

Let $P_N(y)$ be the Taylor polynomial of degree d of the kernel of I_α^{loc} centered at c_{Q_j} where the exact value of d will be chosen below. When $|x - c_{Q_j}| > l(Q_j^*)$, by the moment condition of a_j and the Taylor expansion theorem, we obtain that

$$|I_\alpha^{loc}(a_j)(x)| \leq C\frac{l(Q_j)^\alpha (M\chi_{Q_j}(x))^\gamma}{\omega(Q_j)^{\frac{1}{p}}},$$

where $\gamma = \frac{n+d+1-\alpha}{n+1}$. Since $\omega^p \in RH_{\frac{q}{p}}$, then $\omega^q \in A_\infty$. We choose d such that $\gamma q > q_{\omega^q}$. Therefore, by Fefferman-Stein vector-valued maximal inequality and Lemma 2.9, we have

$$\begin{aligned}
&\left\|\sum_j \lambda_j I_\alpha^{loc}(a_j)\chi_{(Q_j^*)^c}\right\|_{L^q_{\omega^q}} \leq C\left\|\sum_j \lambda_j \frac{l(Q_j)^\alpha (M\chi_{Q_j}(x))^\gamma}{\omega(Q_j)^{\frac{1}{p}}}\right\|_{L^q_{\omega^q}} \\
&\leq C\left\|\sum_j \lambda_j \frac{l(Q_j)^\alpha \chi_{Q_j}}{\omega(Q_j)^{\frac{1}{p}}}\right\|_{L^q_{\omega^q}} \leq C\left\|\sum_j \frac{\lambda_j \chi_{Q_j}}{\omega(Q_j)^{\frac{1}{p}}}\right\|_{L^p_{\omega^p}}.
\end{aligned}$$

Now we prove (10). Notice that

$$\operatorname{supp}(I_\alpha^{loc}(b_j)) \subset P_j(c_{P_j}, l(P_j)+4) \subset 10P_j.$$

Repeating the similar argument, we can obtain that

$$\left\|\sum_j \mu_j I_\alpha^{loc}(b_j)\right\|_{L^q_{\omega^q}} \leq C\left\|\sum_j \frac{\mu_j \chi_{P_j}}{\omega(P_j)^{\frac{1}{p}}}\right\|_{L^p_{\omega^p}}.$$

Therefore, we have proved the first part of the theorem.

Now we consider the boundedness of I_α^{loc} from $h^p_{\omega^p}(\mathbb{R}^n)$ to $h^q_{\omega^q}(\mathbb{R}^n)$. To end this, we need to prove that

$$\left\|\sum_j \lambda_j M_\Phi(I_\alpha^{loc}(a_j))\right\|_{L^q_{\omega^q}} \leq C\left\|\sum_j \frac{\lambda_j \chi_{Q_j}}{\omega(Q_j)^{\frac{1}{p}}}\right\|_{L^p_{\omega^p}} \tag{11}$$

and

$$\left\|\sum_j \mu_j M_\Phi(I_\alpha^{loc}(b_j))\right\|_{L^q_{\omega^q}} \leq C\left\|\sum_j \frac{\mu_j \chi_{P_j}}{\omega(P_j)^{\frac{1}{p}}}\right\|_{L^p_{\omega^p}}, \tag{12}$$

where each a_j is ω-(p,t,N)-atom, each b_j is ω-(p,t,N)-block and the exact values of t and N will be chosen below.

First we prove (11). For $x \in \mathbb{R}^n$,

$$\begin{aligned}
&\left|\sum_j \lambda_j M_\Phi(I_\alpha^{loc}(a_j))(x)\right| \\
&\leq \sum_j \lambda_j |M_\Phi(I_\alpha^{loc}(a_j))(x)|\chi_{Q_j^*}(x) + \sum_j \lambda_j |M_\Phi(I_\alpha^{loc}(a_j))(x)|\chi_{(Q_j^*)^c}(x) \\
&=: I + II.
\end{aligned}$$

To estimate I, arguing as before we may assume that $q_0 > \max\{q, 1\}$. Define $p_0 > 1$ by $\frac{1}{p_0} - \frac{1}{q_0} = \frac{\alpha}{n}$. Together with the fact that M_Φ is bounded on $L^{q_0}(\mathbb{R}^n)$, we can obtain that

$$\begin{aligned}\|I\|_{L^q_{\omega^q}} &\le C \left\| \sum_j \lambda_j \left(\frac{1}{|Q_j^*|} \int_{Q_j^*} \left(M_\Phi I_\alpha^{loc}(a_j) \right)^{q_0} dx \right)^{\frac{1}{q_0}} \chi_{Q_j^*} \right\|_{L^q_{\omega^q}} \\ &\le C \left\| \sum_j \lambda_j |Q_j^*|^{-\frac{1}{q_0}} \left(\int_{Q_j^*} |I_\alpha^{loc}(a_j)|^{q_0} dx \right)^{\frac{1}{q_0}} \chi_{Q_j^*} \right\|_{L^q_{\omega^q}} \\ &\le C \left\| \sum_j \frac{\lambda_j |Q_j^*|^{\frac{\alpha}{n}}}{\omega(Q_j)^{\frac{1}{p}}} \chi_{Q_j^*} \right\|_{L^q_{\omega^q}} \\ &\le C \left\| \sum_j \frac{\lambda_j \chi_{Q_j}}{\omega(Q_j)^{\frac{1}{p}}} \right\|_{L^p_{\omega^p}}.\end{aligned}$$

To estimate II, we choose N so that

$$\left(\frac{n - \alpha + N + 1}{n} \right) q > q_{\omega^q}.$$

Let $\tau = \frac{n+N+1}{n}$. Then, since $\frac{1}{\tau p} - \frac{1}{\tau q} = \frac{\alpha}{\tau n}$, we have that

$$1 + \frac{\tau q}{(\tau p)'} = \tau p \left(1 - \frac{\alpha}{\tau n} \right) = \left(\frac{n - \alpha + N + 1}{n} \right) q.$$

Let $v = \omega^{\frac{1}{\tau}}$. Then we have that $v^{\tau q} = \omega^q \in A_{1+\frac{\tau q}{(\tau p)'}}$. Equivalently, we have that $v \in A_{\tau p, \tau q}$. Therefore, by Lemma 2.10 and Lemma 2.11 applied to the fractional maximal operator M_{α_τ},

$$\|II\|_{L^q_{\omega^q}} \le C \left\| \left(\sum_j \frac{\lambda_j (M_{\alpha_\tau}(\chi_{Q_j}))^\tau}{\omega(Q_j)^{\frac{1}{p}}} \right)^{\frac{1}{\tau}} \right\|^\tau_{L^{q\tau}_{v^{q\tau}}} \le C \left\| \left(\sum_j \frac{\lambda_j \chi_{Q_j}}{\omega(Q_j)^{\frac{1}{p}}} \right)^{\frac{1}{\tau}} \right\|^\tau_{L^{p\tau}_{v^{p\tau}}} = C \left\| \sum_j \frac{\lambda_j \chi_{Q_j}}{\omega(Q_j)^{\frac{1}{p}}} \right\|_{L^p_{\omega^p}}.$$

Then we prove (12). From the definition of $M_\Phi(I_\alpha^{loc}(b_j))$, we can obtain that

$$\mathrm{supp}(M_\Phi(I_\alpha^{loc}(b_j))) \subset P_j(c_{P_j}, l(P_j) + 8) \subset 20P_j.$$

Applying the similar argument in the above proof, we can obtain the desired results. □

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