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Nonlinear bi-skew Jordan-type higher derivations on $*$-algebras

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Abstract. This article studies the structure of nonlinear bi-skew Jordan-type higher derivations of unital $*$-algebras and proves that each nonlinear bi-skew Jordan-type higher derivation is an additive higher $*$-derivation. As applications, nonlinear bi-skew Jordan-type higher derivations on some classical unital $*$-algebras are characterized.

1. Introduction

Let $\mathfrak{B}$ be an unital $*$-algebra, where $*$ satisfies the relation $(xy)^* = y^*x^*$ and $((x)^*)^* = x$ for all $x, y \in \mathfrak{B}$. The symbol $y_1 \triangleleft y_2 = y_1^* y_2 + y_2^* y_1$ is called a bi-skew Jordan product for arbitrary $y_1, y_2 \in \mathfrak{B}$. A mapping $\delta_1 : \mathfrak{B} \to \mathfrak{B}$ (not necessarily linear) is called a nonlinear $*$-derivation if $\delta_1(y_1y_2) = \delta_1(y_1)y_2 + y_1\delta_1(y_2)$ and $\delta_1(x^*) = \delta_1(x)^*$; A mapping $\delta_1 : \mathfrak{B} \to \mathfrak{B}$ (not necessarily linear) is called a nonlinear bi-skew Jordan derivation if

$$\delta_1(U_2(y_1, y_2)) = U_2(\delta_1(y_1), y_2) + U_2(y_1, \delta_1(y_2))$$

for all $y_1, y_2 \in \mathfrak{B}$. Furthermore, for integer $n \geq 2$ and $y_1, \cdots, y_n \in \mathfrak{B}$, define $U_1(y_1) = y_1, U_2(y_1, y_2) = y_1 \triangleleft y_2$ and $U_n(y_1, \cdots, y_n) = U_{n-1}(y_1, \cdots, y_{n-1}) \triangleleft y_n$. Then, according to [5], we introduce the concept of nonlinear bi-skew Jordan n-derivation. If a nonlinear mapping $\Psi_1 : \mathfrak{B} \to \mathfrak{B}$ satisfies an equation

$$\Psi_1(U_n(x_1, \cdots, x_n)) = \sum_{k=1}^{n} U_n(x_1, x_2, \cdots, \Psi_1(x_k) \cdots, x_n), \tag{1.1}$$

it's called a nonlinear bi-skew Jordan n-derivation. We now introduce a more general mapping which contains the above mappings such as bi-skew Jordan derivations, bi-skew Jordan n-derivations, etc., as its special form. Let $\mathcal{N}$ be the set of all non-negative integers and $\Delta = \{\Psi_m\}_{m\in\mathcal{N}}$ be a family of mapping $\Psi_m : \mathfrak{B} \to \mathfrak{B}$ on $\mathfrak{B}$ such that $\Psi_0 = id_{\mathfrak{B}}$. Δ is called:

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(a) an additive higher $*$-derivation if

$$\Psi_m(xy) = \sum_{i+j=m} \Psi_i(x)\Psi_j(y), \ \ \Psi_m(x+y) = \Psi_m(x) + \Psi_m(y) \ \text{ and } \ \Psi_m(y)^* = \Psi_m(y^*) \tag{1.2}$$

for all $x, y \in \mathfrak{B}$ and for each $m \in \mathcal{N}$;

(b) a nonlinear bi-skew Jordan higher n-derivation if

$$\Psi_m(U_n(x_1, \cdots, x_n)) = \sum_{i_1+\cdots+i_n=m} U_n(\Psi_{i_1}(x_1), \Psi_{i_2}(x_2), \cdots, \Psi_{i_n}(x_n)) \tag{1.3}$$

for all $x_1, \cdots, x_n \in \mathfrak{B}$ and for each $n, m \in \mathcal{N}$.

This notion makes the best use of the definition of nonlinear bi-skew Jordan-type derivation, It's named similarly to [1]. The main statement is as follows: when $m = 1$ in (1.2) and (1.3), the map $\Psi_1 : \mathfrak{B} \to \mathfrak{B}$ is an additive $*$-derivation and a nonlinear bi-skew Jordan n-derivation, respectively. Many mappings associated with Jordan n-derivations have been studied by scholars, see [2–4].

In the scope of the author's research, the structure of bi-skew Jordan derivations and their associated maps on $*$-algebra $\mathfrak{B}$ has attracted the attention of many scholars. In 2022, the problem of the structure of nonlinear bi-skew Jordan derivations on prime $*$-algebras has attracted the attention of Darvish and his collaborators[7], who characterized every nonlinear bi-skew Jordan derivation evolved into an additive $*$-derivation. In 2023, Ashraf and co-authors[6] shown that every nonlinear bi-skew Jordan-type derivation on factor von Neumann algebra is an additive $*$-derivation. This result[6] generalizes the main conclusion of [7]. Meanwhile, Zhao and co-authors[5] generalized the results of [7] and [6] to $*$-algebra $\mathfrak{B}$, that is, they proved that every nonlinear bi-skew Jordan-type derivation on unital $*$-algebra $\mathfrak{B}$ is an additive $*$-derivation. It was noted that Wani and his collaborators[1] have studied the structure of multiplicative $*$-Jordan-type higher derivations on von Neumann algebras without nonzero central abelian projections, and proved that every multiplicative $*$-Jordan-type higher derivations on von Neumann algebras is an additive higher $*$-derivation. Based on these facts[1, 5, 7], there is a natural question:

Problem 1.1. *Is a nonlinear bi-skew Jordan-type higher derivation on an unital $*$-algebra an additive higher $*$-derivation?*

This is an interesting problem, and its solution not only extends existing results, but also solves some problems of the same type in algebra, such as standard operator algebras, factor von Neumann algebras, von Neumann algebras of type I_1 and prime $*$-algebras, and so on.

This paper takes the above questions as the main research topic, and gives the positive answers to the above Questions 1.1, that is, it proves that every nonlinear bi-skew Jordan-type higher derivations on unital $*$-algebra is an additive higher $*$-derivation, and gives a series of conclusions as its corollaries.

2. Nonlinear bi-skew Jordan-type higher derivations

Let us begin this section with the following concept of unital $*$-algebras $\mathfrak{B}$.

Suppose that the symbol $\mathfrak{B}$ represents unital $*$-algebra with idempotents F_1 and F_2 satisfying the condition $F_2 = I - F_1$, where I is the identity of algebra $\mathfrak{B}$. We introduce the following notation to illustrate the proof process. Define $\mathfrak{B}_b = \{B \in \mathfrak{B} : B = B^*\}$, $\mathfrak{B}_{11} = F_1\mathfrak{B}_bF_1$, $\mathfrak{B}_{12} = \{F_1BF_2 + F_2BF_1 : B \in \mathfrak{B}_b\}$ and $\mathfrak{B}_{22} = F_2\mathfrak{B}_bF_2$. For each $B \in \mathfrak{B}_b$, we may write $B = B_{11} + B_{12} + B_{22}$, where $B_{11} \in \mathfrak{B}_{11}$, $B_{12} \in \mathfrak{B}_{12}$ and $B_{22} \in \mathfrak{B}_{22}$. In this article, we assume that the unital $*$-algebra $\mathfrak{B}$ meet the conditions:

$$\mathfrak{C} = \begin{cases} Y\mathfrak{B}F_1 = 0 \ \text{ implies } Y = 0 \\ Y\mathfrak{B}F_2 = 0 \ \text{ implies } Y = 0. \end{cases}$$

Under the condition of $\mathfrak{C}$, it contains many important algebras as its classic examples[5]: such as standard operator algebras, factor von Neumann algebras, von Neumann algebras of type I_1 and prime $*$-algebras, and so on.

Theorem 2.1. *Let $\mathfrak{B}$ be an unital $*$-algebra with identity element I that satisfies condition $\mathfrak{C}$. Then every nonlinear bi-skew Jordan higher n-derivation as defined in* (1.3) *is an additive higher $*$-derivation.*

To prove this theorem, we use mathematical induction for m, which appears in equation (1.3). When $m = 1$ in Eq (1.3), every nonlinear bi-skew Jordan higher n-derivation will evolve into a nonlinear bi-skew Jordan n-derivation, which provides the results underlying the use of mathematical induction in this paper. It can be seen from [5] that every nonlinear bi-skew Jordan n-derivation $\mathfrak{B}$ is an additive $*$-derivation and satisfies the following conditions:

$$\mathfrak{H}_1 = \begin{cases} \Psi_1(0) = 0;\ \Psi_1(x) \in \mathfrak{B}_b \text{ for all } x \in \mathfrak{B}_b;\ \Psi_1(I) = \Psi_1(iI) = 0; \\ \Psi_1(C_{11} + C_{12} + C_{22}) = \Psi_1(C_{11}) + \Psi_1(C_{12}) + \Psi_1(C_{22}) \text{ for all } C_{11} \in \mathfrak{B}_{11}, C_{12} \in \mathfrak{B}_{12}, C_{22} \in \mathfrak{B}_{22}; \\ \Psi_1(C_{12} + D_{12}) = \Psi_1(C_{12}) + \Psi_1(D_{12}) \text{ for all } C_{12}, D_{12} \in \mathfrak{B}_{12}; \\ \Psi_1(C_{ii} + D_{ii}) = \Psi_1(C_{ii}) + \Psi_1(D_{ii}) \text{ for all } C_{ii}, D_{ii} \in \mathfrak{B}_{ii}, i \in \{1,2\}; \\ \Psi_1(M)^* = \Psi_1(M^*);\ \Psi_1(iM) = i\Psi_1(M) \text{ for all } M \in \mathfrak{B}. \end{cases}$$

We assume that the mapping Ψ_s holds for all $1 < s < m$, $m \in \mathcal{N}$ on an unital $*$-algebra $\mathfrak{B}$ satisfies the following:

$$\mathfrak{H}_s = \begin{cases} \Psi_s(0) = 0;\ \Psi_s(x) \in \mathfrak{B}_b \text{ for all } x \in \mathfrak{B}_b;\ \Psi_s(I) = \Psi_s(iI) = 0; \\ \Psi_s(C_{11} + C_{12} + C_{22}) = \Psi_s(C_{11}) + \Psi_s(C_{12}) + \Psi_s(C_{22}) \text{ for all } C_{11} \in \mathfrak{B}_{11}, C_{12} \in \mathfrak{B}_{12}, C_{22} \in \mathfrak{B}_{22}; \\ \Psi_s(C_{12} + D_{12}) = \Psi_s(C_{12}) + \Psi_s(D_{12}) \text{ for all } C_{12}, D_{12} \in \mathfrak{B}_{12}; \\ \Psi_s(C_{ii} + D_{ii}) = \Psi_s(C_{ii}) + \Psi_s(D_{ii}) \text{ for all } C_{ii}, D_{ii} \in \mathfrak{B}_{ii}, i \in \{1,2\}; \\ \Psi_s(M)^* = \Psi_s(M^*);\ \Psi_s(iM) = i\Psi_s(M) \text{ for all } M \in \mathfrak{B}. \end{cases}$$

In the remaining part we will prove that the above condition $\mathfrak{H}_s$ holds for $s = m$. Finally, it is proved that a nonlinear bi-skew Jordan higher n-derivation Ψ_m on algebra $\mathfrak{B}$ is also an additive higher $*$-derivation. We prove the main conclusion through a series of lemmas.

Lemma 2.2. $\Psi_m(0) = 0$.

Proof. By the hypothesis $\mathfrak{H}_s$ $(1 \le s \le m-1)$, i.e., $\Psi_s(0) = 0$, we have

$$\begin{aligned} \Psi_m(0) &= \Psi_m(U_n(0, 0, \cdots, 0)) \\ &= \sum_{i_1+\cdots+i_n=m} U_n(\Psi_{i_1}(0), \Psi_{i_2}(0), \cdots, \Psi_{i_n}(0)) \\ &= U_n(\Psi_m(0), 0, \cdots, 0) + \cdots + U_n(0, \cdots, 0, \Psi_m(0)) \\ &\quad + \sum_{\substack{i_1+\cdots+i_n=m, \\ i_1,\cdots,i_n<m}} U_n(\Psi_{i_1}(0), \Psi_{i_2}(0), \cdots, \Psi_{i_n}(0)) \\ &= 0. \end{aligned}$$

□

Lemma 2.3. $\Psi_m(B) \in \mathfrak{B}_b$ *for every* $B \in \mathfrak{B}_b$.

Proof. For every $B \in \mathfrak{B}_b$, by the fact $B = U_n(B, \frac{I}{2}, \cdots, \frac{I}{2})$, $U_2(B_1, B_2) = B_1^*B_2 + B_2^*B_1 \in \mathfrak{B}_b$ for any $B_1, B_2 \in \mathfrak{B}$ and inductive hypothesis $\mathfrak{H}_s$ $(1 \le s \le m-1)$, i.e., $\Psi_s(B) \in \mathfrak{B}_b$ for every $B \in \mathfrak{B}_b$, we have

$$\begin{aligned}
\Psi_m(B) &= \Psi_m(U_n(B, \frac{I}{2}, \cdots, \frac{I}{2})) \\
&= \sum_{i_1+\cdots+i_n=m} U_n(\Psi_{i_1}(B), \Psi_{i_2}(\frac{I}{2}), \cdots, \Psi_{i_n}(\frac{I}{2})) \\
&= U_n(\Psi_m(B), \frac{I}{2}, \cdots, \frac{I}{2}) + \sum_{j=2}^{n} U_n(B, \frac{I}{2}, \cdots, \underbrace{\Psi_m(\frac{I}{2})}_{\text{j-th component}}, \cdots, \frac{I}{2}) \\
&\quad + \sum_{\substack{i_1+\cdots+i_n=m,\\ i_1,\cdots,i_n\in\{0,\cdots,m-1\}}} U_n(\Psi_{i_1}(B), \Psi_{i_2}(\frac{I}{2}), \cdots, \Psi_{i_n}(\frac{I}{2})) \\
&= \frac{I}{2}\{\Psi_m(B) + \Psi_m(B)^*\} + (n-1)\{B\Psi_m(\frac{I}{2}) + \Psi_m(\frac{I}{2})^* B\} \\
&\quad + \sum_{\substack{i_1+\cdots+i_n=m,\\ i_1,\cdots,i_n\in\{0,\cdots,m-1\}}} U_n(\Psi_{i_1}(B), \Psi_{i_2}(\frac{I}{2}), \cdots, \Psi_{i_n}(\frac{I}{2})).
\end{aligned}$$

Hence $\Psi_m(B)^* = \Psi_m(B)$ for all $B \in \mathfrak{B}_b$.
□

Lemma 2.4. *With notations as above, we obtain*

$$\Psi_m(B_{11} + C_{12}) = \Psi_m(B_{11}) + \Psi_m(C_{12}) \ \text{ and } \ \Psi_m(D_{22} + C_{12}) = \Psi_m(D_{22}) + \Psi_m(C_{12})$$

for all $B_{11} \in \mathfrak{B}_{11}$, $D_{22} \in \mathfrak{B}_{22}$ *and* $C_{12} \in \mathfrak{B}_{12}$.

Proof. To prove this lemma, we introduce symbol $V^m = \Psi_m(B_{11}+C_{12}) - \Psi_m(B_{11}) - \Psi_m(C_{12})$. In agreement with Lemma 2.3, we have $(V^m)^* = V^m$. Since $U_n(F_2, B_{11}, I, \cdots, I) = 0$, and inductive hypothesis $\mathfrak{H}_s$ $(1 \le s \le m-1)$, i.e., $\Psi_s(B_{11} + C_{12}) = \Psi_s(B_{11}) + \Psi_s(C_{12})$ for all $B_{11} \in \mathfrak{B}_{11}$ and $C_{12} \in \mathfrak{B}_{12}$, we get

$$\begin{aligned}
& U_n(F_2, \Psi_m(B_{11} + C_{12}), I, \cdots, I) \\
& + \sum_{\substack{i_1+\cdots+i_n=m,\\ i_2\in\{0,\cdots,m-1\}}} U_n(\Psi_{i_1}(F_2), \Psi_{i_2}(B_{11}) + \Psi_{i_2}(C_{12}), \Psi_{i_3}(I), \cdots, \Psi_{i_n}(I)) \\
&= U_n(F_2, \Psi_m(B_{11} + C_{12}), I, \cdots, I) \\
&\quad + \sum_{\substack{i_1+\cdots+i_n=m,\\ i_2\in\{0,\cdots,m-1\}}} U_n(\Psi_{i_1}(F_2), \Psi_{i_2}(B_{11} + C_{12}), \Psi_{i_3}(I), \cdots, \Psi_{i_n}(I)) \\
&= \Psi_m(U_n(F_2, B_{11} + C_{12}, I, \cdots, I)) \\
&= \Psi_m(U_n(F_2, B_{11}, I, \cdots, I)) + \Psi_m(U_n(F_2, C_{12}, I, \cdots, I)) \\
&= U_n(F_2, \Psi_m(B_{11}), I, \cdots, I) + \sum_{\substack{i_1+\cdots+i_n=m,\\ i_2\in\{0,\cdots,m-1\}}} U_n(\Psi_{i_1}(F_2), \Psi_{i_2}(B_{11}), \Psi_{i_3}(I), \cdots, \Psi_{i_n}(I)) \\
&\quad + U_n(F_2, \Psi_m(C_{12}), I, \cdots, I) + \sum_{\substack{i_1+\cdots+i_n=m,\\ i_2\in\{0,\cdots,m-1\}}} U_n(\Psi_{i_1}(F_2), \Psi_{i_2}(C_{12}), \Psi_{i_3}(I), \cdots, \Psi_{i_n}(I)) \\
&= U_n(F_2, \Psi_m(B_{11}) + \Psi_m(C_{12}), I, \cdots, I) \\
&\quad + \sum_{\substack{i_1+\cdots+i_n=m,\\ i_2\in\{0,\cdots,m-1\}}} U_n(\Psi_{i_1}(F_2), \Psi_{i_2}(B_{11}) + \Psi_{i_2}(C_{12}), \Psi_{i_3}(I), \cdots, \Psi_{i_n}(I)).
\end{aligned}$$

It follows from the above equation that $U_n(F_2, V^m, I, \cdots, I) = 0$, which implies that $V^m_{22} = V^m_{12} = 0$. It follows from $U_n(F_1 - F_2, C_{12}, I, \cdots, I) = 0$ and inductive hypothesis $\mathfrak{H}_s$ $(1 \le s \le m-1)$ that

$$\begin{aligned}
& U_n(F_1 - F_2, \Psi_m(B_{11} + C_{12}), I, \cdots, I) \\
& + \sum_{\substack{i_1+\cdots+i_n=m,\\ i_2\in\{0,\cdots,m-1\}}} U_n(\Psi_{i_1}(F_1 - F_2), \Psi_{i_2}(B_{11} + C_{12}), \Psi_{i_3}(I), \cdots, \Psi_{i_n}(I)) \\
& = \Psi_m(U_n(F_1 - F_2, B_{11} + C_{12}, I, \cdots, I)) \\
& = \Psi_m(U_n(F_1 - F_2, B_{11}, I, \cdots, I)) + \Psi_m(U_n(F_1 - F_2, C_{12}, I, \cdots, I)) \\
& = U_n(F_1 - F_2, \Psi_m(B_{11}), I, \cdots, I) + \sum_{\substack{i_1+\cdots+i_n=m,\\ i_2\in\{0,\cdots,m-1\}}} U_n(\Psi_{i_1}(F_1 - F_2), \Psi_{i_2}(B_{11}), \Psi_{i_3}(I), \cdots, \Psi_{i_n}(I)) \\
& \quad + U_n(F_1 - F_2, \Psi_m(C_{12}), I, \cdots, I) + \sum_{\substack{i_1+\cdots+i_n=m,\\ i_2\in\{0,\cdots,m-1\}}} U_n(\Psi_{i_1}(F_1 - F_2), \Psi_{i_2}(C_{12}), \Psi_{i_3}(I), \cdots, \Psi_{i_n}(I)) \\
& = U_n(F_1 - F_2, \Psi_m(B_{11}) + \Psi_m(C_{12}), I, \cdots, I) \\
& \quad + \sum_{\substack{i_1+\cdots+i_n=m,\\ i_2\in\{0,\cdots,m-1\}}} U_n(\Psi_{i_1}(F_1 - F_2), \Psi_{i_2}(B_{11}) + \Psi_{i_2}(C_{12}), \Psi_{i_3}(I), \cdots, \Psi_{i_n}(I)) \\
& = U_n(F_1 - F_2, \Psi_m(B_{11}) + \Psi_m(C_{12}), I, \cdots, I) \\
& \quad + \sum_{\substack{i_1+\cdots+i_n=m,\\ i_2\in\{0,\cdots,m-1\}}} U_n(\Psi_{i_1}(F_1 - F_2), \Psi_{i_2}(B_{11} + C_{12}), \Psi_{i_3}(I), \cdots, \Psi_{i_n}(I)).
\end{aligned}$$

As maintained by the first equation and the last equation, we arrive at $U_n(F_1 - F_2, V^m, I, \cdots, I) = 0$, and then we get $V^m_{11} = 0$. Combining the equations $V^m_{ij} = 0$ $(1 \le i \le j \le 2)$ can be obtained $V^m = 0$, i.e., $\Psi_m(B_{11} + C_{12}) = \Psi_m(B_{11}) + \Psi_m(C_{12})$ for all $B_{11} \in \mathfrak{B}_{11}$ and $C_{12} \in \mathfrak{B}_{12}$.

Using similar computational techniques, we can obtain $\Psi_m(D_{22} + C_{12}) = \Psi_m(D_{22}) + \Psi_m(C_{12})$ for all $D_{22} \in \mathfrak{B}_{22}$ and $C_{12} \in \mathfrak{B}_{12}$. □

Lemma 2.5. *With notations as above, we have*

$$\Psi_m(B_{11} + C_{12} + D_{22}) = \Psi_m(B_{11}) + \Psi_m(C_{12}) + \Psi_m(D_{22})$$

for all $B_{11} \in \mathfrak{B}_{11}$, $C_{12} \in \mathfrak{B}_{12}$ *and* $D_{22} \in \mathfrak{B}_{22}$.

Proof. To prove this, we introduce notation $V^m = \Psi_m(B_{11} + C_{12} + D_{22}) - \Psi_m(B_{11}) - \Psi_m(C_{12}) - \Psi_m(D_{22})$. In accordance with Lemma 2.3, we have $(V^m)^* = V^m$. By combining Eq. $U_n(F_1, D_{22}, I, \cdots, I) = 0$ and Lemma 2.4 with the induction hypothesis $\mathfrak{H}_s$ $(1 \le s \le m-1)$, i.e.,$\Psi_s(B_{11} + C_{12} + D_{22}) = \Psi_s(B_{11}) + \Psi_s(C_{12}) + \Psi_s(D_{22})$ for all $B_{11} \in \mathfrak{B}_{11}$, $C_{12} \in \mathfrak{B}_{12}$ and $D_{22} \in \mathfrak{B}_{22}$, we know

$$\begin{aligned}
& U_n(F_1, \Psi_m(B_{11} + C_{12} + D_{22}), I, \cdots, I) \\
& + \sum_{\substack{i_1+\cdots+i_n=m,\\ i_2\in\{0,\cdots,m-1\}}} U_n(\Psi_{i_1}(F_1), \Psi_{i_2}(B_{11}) + \Psi_{i_2}(C_{12}) + \Psi_{i_2}(D_{22}), \Psi_{i_3}(I), \cdots, \Psi_{i_n}(I)) \\
& = U_n(F_1, \Psi_m(B_{11} + C_{12} + D_{22}), I, \cdots, I) \\
& \quad + \sum_{\substack{i_1+\cdots+i_n=m,\\ i_2\in\{0,\cdots,m-1\}}} U_n(\Psi_{i_1}(F_1), \Psi_{i_2}(B_{11} + C_{12} + D_{22}), \Psi_{i_3}(I), \cdots, \Psi_{i_n}(I)) \\
& = \Psi_m(U_n(F_1, B_{11} + C_{12} + D_{22}, I, \cdots, I))
\end{aligned}$$

$$
\begin{aligned}
&= \Psi_m(U_n(F_1, B_{11} + C_{12}, I, \cdots, I)) + \Psi_m(U_n(F_1, D_{22}, I, \cdots, I)) \\
&= U_n(F_1, \Psi_m(B_{11} + C_{12}), I, \cdots, I) + \sum_{\substack{i_1+\cdots+i_n=m,\\ i_2\in\{0,\cdots,m-1\}}} U_n(\Psi_{i_1}(F_1), \Psi_{i_2}(B_{11} + C_{12}), \Psi_{i_3}(I), \cdots, \Psi_{i_n}(I)) \\
&\quad + U_n(F_1, \Psi_m(D_{22}), I, \cdots, I) + \sum_{\substack{i_1+\cdots+i_n=m,\\ i_2\in\{0,\cdots,m-1\}}} U_n(\Psi_{i_1}(F_1), \Psi_{i_2}(D_{22}), \Psi_{i_3}(I), \cdots, \Psi_{i_n}(I)) \\
&= U_n(F_1, \Psi_m(B_{11}) + \Psi_m(C_{12}), I, \cdots, I) + \sum_{\substack{i_1+\cdots+i_n=m,\\ i_2\in\{0,\cdots,m-1\}}} U_n(\Psi_{i_1}(F_1), \Psi_{i_2}(B_{11}) + \Psi_{i_2}(C_{12}), \Psi_{i_3}(I), \cdots, \Psi_{i_n}(I)) \\
&\quad + U_n(F_1, \Psi_m(D_{22}), I, \cdots, I) + \sum_{\substack{i_1+\cdots+i_n=m,\\ i_2\in\{0,\cdots,m-1\}}} U_n(\Psi_{i_1}(F_1), \Psi_{i_2}(D_{22}), \Psi_{i_3}(I), \cdots, \Psi_{i_n}(I)) \\
&= U_n(F_1, \Psi_m(B_{11}) + \Psi_m(C_{12}) + \Psi_m(D_{22}), I, \cdots, I) \\
&\quad + \sum_{\substack{i_1+\cdots+i_n=m,\\ i_2\in\{0,\cdots,m-1\}}} U_n(\Psi_{i_1}(F_1), \Psi_{i_2}(B_{11}) + \Psi_{i_2}(C_{12}) + \Psi_{i_2}(D_{22}), \Psi_{i_3}(I), \cdots, \Psi_{i_n}(I)).
\end{aligned}
$$

Then, we have $U_n(F_1, V^m, I, \cdots, I) = 0$, which implies that $V^m_{11} = V^m_{12} = 0$.

On the other hand, we also combine Eq. $U_n(F_2, B_{11}, I, \cdots, I) = 0$ and Lemma 2.4 and the induction hypothesis $\mathfrak{H}_s$ $(1 \le s \le m-1)$ can be obtained

$$
\begin{aligned}
&U_n(F_2, \Psi_m(B_{11} + C_{12} + D_{22}), I, \cdots, I) \\
&+ \sum_{\substack{i_1+\cdots+i_n=m,\\ i_2\in\{0,\cdots,m-1\}}} U_n(\Psi_{i_1}(F_2), \Psi_{i_2}(B_{11} + C_{12} + D_{22}), \Psi_{i_3}(I), \cdots, \Psi_{i_n}(I)) \\
&= \Psi_m(U_n(F_2, B_{11} + C_{12} + D_{22}, I, \cdots, I))
\end{aligned}
$$

$$
\begin{aligned}
&= \Psi_m(U_n(F_2, D_{22} + C_{12}, I, \cdots, I)) + \Psi_m(U_n(F_2, B_{11}, I, \cdots, I)) \\
&= U_n(F_2, \Psi_m(D_{22} + C_{12}), I, \cdots, I) + \sum_{\substack{i_1+\cdots+i_n=m,\\ i_2\in\{0,\cdots,m-1\}}} U_n(\Psi_{i_1}(F_2), \Psi_{i_2}(D_{22} + C_{12}), \Psi_{i_3}(I), \cdots, \Psi_{i_n}(I)) \\
&\quad + U_n(F_2, \Psi_m(B_{11}), I, \cdots, I) + \sum_{\substack{i_1+\cdots+i_n=m,\\ i_2\in\{0,\cdots,m-1\}}} U_n(\Psi_{i_1}(F_2), \Psi_{i_2}(B_{11}), \Psi_{i_3}(I), \cdots, \Psi_{i_n}(I)) \\
&= U_n(F_2, \Psi_m(B_{11}) + \Psi_m(C_{12}) + \Psi_m(D_{22}), I, \cdots, I) \\
&\quad + \sum_{\substack{i_1+\cdots+i_n=m,\\ i_2\in\{0,\cdots,m-1\}}} U_n(\Psi_{i_1}(F_2), \Psi_{i_2}(B_{11} + C_{12} + D_{22}), \Psi_{i_3}(I), \cdots, \Psi_{i_n}(I)).
\end{aligned}
$$

Then, we have $U_n(F_2, V^m, I, \cdots, I) = 0$, which implies that $V^m_{22} = 0$. Thus $V^m = 0$.
□

Lemma 2.6. *With notations as above, we have*

$$\Psi_m(B_{12} + C_{12}) = \Psi_m(B_{12}) + \Psi_m(C_{12})$$

for all $B_{12}, C_{12} \in \mathfrak{B}_{12}$.

Proof. Set $B_{12}, C_{12} \in \mathfrak{B}_{12}$, according to the construction method of the elements in set $\mathfrak{B}_{12}$, for $B, C \in \mathfrak{B}_b$, we know $B_{12} = F_1BF_2 + F_2BF_1$ and $C_{12} = F_1CF_2 + F_2CF_1$, which implies that $B_{12} + C_{12} = F_1(C+B)F_2 + F_2(C+B)F_1 \in \mathfrak{B}_{12}$.

According to Lemma 2.4 and the following two relations $U_n(F_1 + B_{12}, F_2 + C_{12}, \frac{I}{2}, \cdots, \frac{I}{2}) = B_{12} + C_{12} + B_{12}C_{12} + C_{12}B_{12}$ and $B_{12}C_{12} + C_{12}B_{12} = F_1(BF_2C + CF_2B)F_1 + F_2(BF_1C + CF_1B)F_2 \in \mathfrak{B}_{11} + \mathfrak{B}_{22}$ and the induction hypothesis $\mathfrak{H}_s$ $(1 \leq s \leq m-1)$, i.e., $\Psi_s(B_{12} + C_{12}) = \Psi_s(B_{12}) + \Psi_s(C_{12})$ for all $B_{12}, C_{12} \in \mathfrak{B}_{12}$, we know that

$$\begin{aligned}
&\Psi_m(B_{12} + C_{12}) + \Psi_m(B_{12}C_{12} + C_{12}B_{12}) \\
&= \Psi_m(B_{12} + C_{12} + B_{12}C_{12} + C_{12}B_{12}) \\
&= \Psi_m(U_n(F_1 + B_{12}, F_2 + C_{12}, \frac{I}{2}, \cdots, \frac{I}{2})) \\
&= \sum_{i_1+\cdots+i_n=m} U_n(\Psi_{i_1}(F_1 + B_{12}), \Psi_{i_2}(F_2 + C_{12}), \Psi_{i_3}(\frac{I}{2}), \cdots, \Psi_{i_n}(\frac{I}{2})) \\
&= \sum_{i_1+\cdots+i_n=m} U_n(\Psi_{i_1}(F_1), \Psi_{i_2}(F_2), \Psi_{i_3}(\frac{I}{2}), \cdots, \Psi_{i_n}(\frac{I}{2})) \\
&\quad + \sum_{i_1+\cdots+i_n=m} U_n(\Psi_{i_1}(F_1), \Psi_{i_2}(C_{12}), \Psi_{i_3}(\frac{I}{2}), \cdots, \Psi_{i_n}(\frac{I}{2})) \\
&\quad + \sum_{i_1+\cdots+i_n=m} U_n(\Psi_{i_1}(B_{12}), \Psi_{i_2}(F_2), \Psi_{i_3}(\frac{I}{2}), \cdots, \Psi_{i_n}(\frac{I}{2})) \\
&\quad + \sum_{i_1+\cdots+i_n=m} U_n(\Psi_{i_1}(B_{12}), \Psi_{i_2}(C_{12}), \Psi_{i_3}(\frac{I}{2}), \cdots, \Psi_{i_n}(\frac{I}{2})) \\
&= \Psi_m(U_n(F_1, F_2, \frac{I}{2}, \cdots, \frac{I}{2})) + \Psi_m(U_n(F_1, C_{12}, \frac{I}{2}, \cdots, \frac{I}{2})) \\
&\quad + \Psi_m(U_n(B_{12}, F_2, \frac{I}{2}, \cdots, \frac{I}{2})) + \Psi_m(U_n(B_{12}, C_{12}, \frac{I}{2}, \cdots, \frac{I}{2})) \\
&= \Psi_m(B_{12}) + \Psi_m(C_{12}) + \Psi_m(B_{12}C_{12} + C_{12}B_{12}).
\end{aligned}$$

And then it yields that $\Psi_m(B_{12} + C_{12}) = \Psi_m(B_{12}) + \Psi_m(C_{12})$.
□

Lemma 2.7. *With notations as above, we have*

$$\Psi_m(C_{ii} + D_{ii}) = \Psi_m(C_{ii}) + \Psi_m(D_{ii})$$

for all $C_{ii}, D_{ii} \in \mathfrak{B}_{ii}, i \in \{1, 2\}$.

Proof. The main purpose of this lemma is to prove that the nonlinear map Ψ_m agrees with additivity on $\mathfrak{B}_{ii}$ $(i \in \{1, 2\})$, i.e., $\Psi_m(C_{ii} + D_{ii}) = \Psi_m(C_{ii}) + \Psi_m(D_{ii})$ for all $C_{ii}, D_{ii} \in \mathfrak{B}_{ii}, i \in \{1, 2\}$. We just need to prove the case where $i = 1$, and the case where $i = 2$ can be calculated in a similar way. So as to prove the conclusion, we make $V^m = \Psi_m(C_{11} + D_{11}) - \Psi_m(C_{11}) - \Psi_m(D_{11})$.

It follows from Eq. $U_2(F_2, C_{11}) = U_2(F_2, D_{11}) = 0$ and the induction hypothesis $\mathfrak{H}_s$ $(1 \leq s \leq m-1)$, i.e., $\Psi_s(C_{11} + D_{11}) = \Psi_s(C_{11}) + \Psi_s(D_{11})$ for all $C_{11}, D_{11} \in \mathfrak{B}_{11}$ that

$$\begin{aligned}
U_n(F_2, V^m, I, \cdots, I) &= U_n(F_2, \Psi_m(C_{11} + D_{11}) - \Psi_m(C_{11}) - \Psi_m(D_{11}), I, \cdots, I) \\
&= U_n(F_2, \Psi_m(C_{11} + D_{11}), I, \cdots, I) \\
&\quad - U_n(F_2, \Psi_m(C_{11}), I, \cdots, I) - U_n(F_2, \Psi_m(D_{11}), I, \cdots, I) \\
&= \sum_{i_1+\cdots+i_n=m} U_n(\Psi_{i_1}(F_2), \Psi_{i_2}(C_{11} + D_{11}), \Psi_{i_3}(I), \cdots, \Psi_{i_n}(I)) \\
&\quad - \sum_{i_1+\cdots+i_n=m} U_n(\Psi_{i_1}(F_2), \Psi_{i_2}(C_{11}), \Psi_{i_3}(I), \cdots, \Psi_{i_n}(I)) \\
&\quad - \sum_{i_1+\cdots+i_n=m} U_n(\Psi_{i_1}(F_2), \Psi_{i_2}(D_{11}), \Psi_{i_3}(I), \cdots, \Psi_{i_n}(I))
\end{aligned}$$

$$
\begin{aligned}
=&\Psi_m(U_n(F_2, C_{11}+D_{11}, I, \cdots, I)) - \Psi_m(U_n(F_2, C_{11}, I, \cdots, I)) \\
&- \Psi_m(U_n(F_2, D_{11}, I, \cdots, I)) \\
=&0.
\end{aligned}
$$

And then, we have $U_n(F_2, V^m, I, \cdots, I) = 0$, which implies that $V^m_{22} = V^m_{12} = 0$.

For arbitrary element $E \in \mathfrak{B}$, let's say $G_{12} = F_1EF_2+F_2E^*F_1$, it is obvious that $G_{12}, U_2(C_{11}, G_{12}), U_2(D_{11}, G_{12}) \in \mathfrak{B}_b$.

In agreement with Lemma 2.6, and the induction hypothesis $\mathfrak{H}_s$ $(1 \le s \le m-1)$, we have

$$
\begin{aligned}
U_n(V^m, G_{12}, I, \cdots, I) =&U_n(\Psi_m(C_{11}+D_{11}) - \Psi_m(C_{11}) - \Psi_m(D_{11}), G_{12}, I, \cdots, I) \\
=&U_n(\Psi_m(C_{11}+D_{11}), G_{12}, I, \cdots, I) \\
&- U_n(\Psi_m(C_{11}), G_{12}, I, \cdots, I) - U_n(\Psi_m(D_{11}), G_{12}, I, \cdots, I) \\
=&\sum_{i_1+\cdots+i_n=m} U_n(\Psi_{i_1}(C_{11}+D_{11}), \Psi_{i_2}(G_{12}), \Psi_{i_3}(I), \cdots, \Psi_{i_n}(I)) \\
&- \sum_{i_1+\cdots+i_n=m} U_n(\Psi_{i_1}(C_{11}), \Psi_{i_2}(G_{12}), \Psi_{i_3}(I), \cdots, \Psi_{i_n}(I)) \\
&- \sum_{i_1+\cdots+i_n=m} U_n(\Psi_{i_1}(D_{11}), \Psi_{i_2}(G_{12}), \Psi_{i_3}(I), \cdots, \Psi_{i_n}(I)) \\
=&\Psi_m(U_n(C_{11}+D_{11}, G_{12}, I, \cdots, I)) - \Psi_m(U_n(C_{11}, G_{12}, I, \cdots, I)) \\
&- \Psi_m(U_n(D_{11}, G_{12}, I, \cdots, I)) \\
=&0.
\end{aligned}
$$

It follows from $V^m_{22} = V^m_{12} = 0$ that $0 = U_n(V^m, G_{12}, I, \cdots, I) = U_n(V^m_{11}, G_{12}, I, \cdots, I)$ for all $G_{12} \in \mathfrak{B}_{12}$. In other words, $V^m_{11}(F_1EF_2) + (F_2E^*F_1)V^m_{11} = 0$. In the above equation, multipling the left by F_1 and the right by F_2, we pick up $V^m_{11}(F_1EF_2) = 0$. Consistent with $\mathfrak{C}$, we get your hands on $V^m_{11} = 0$. To sum up, it can be seen that $V^m = 0$.

Likewise, we get hold of $\Psi_m(C_{22} + D_{22}) = \Psi_m(C_{22}) + \Psi_m(D_{22})$ for all $C_{22}, D_{22} \in \mathfrak{B}_{22}$. □

Lemma 2.8. *Ψ_m is additive on $\mathfrak{B}_b$.*

Proof. Considering Lemma 2.5-Lemma 2.7 uniformly, we obtain that this lemma holds.
□

Based on the additivity of mapping Ψ_m, we introduce the imaginary number i in the remaining part to prove that the mapping Ψ_m is an additive higher $*$-derivation. Therefore, we give the new decomposition form of the element.

For arbitrary $X \in \mathfrak{B}$, we obtain that $X = \frac{X-X^*}{2} + \mathrm{i}(-\mathrm{i}\frac{X+X^*}{2})$. It follows from Eq $(\mathrm{i})^* = -\mathrm{i}$ that $(-\mathrm{i}\frac{X+X^*}{2})^* = \mathrm{i}\frac{X+X^*}{2}$ and $(\frac{X-X^*}{2})^* = -(\frac{X-X^*}{2})$. Based on this, for any element $X \in \mathfrak{B}$, we have the following decomposition

$$X = X_1 + \mathrm{i}X_2 \;\text{ for some } X_1{}^* = -X_1, X_2{}^* = -X_2.$$

Lemma 2.9. *Following the notation above, we realize*

(1) $\Psi_m(I) = \Psi_m(iI) = 0$;

(2) $\Psi_m(L)^* = -\Psi_m(L)$ and $\Psi_m(iL) = i\Psi_m(L)$ for arbitrary $L^* = -L$.

Proof. In the proof process, we carry out the proof process of conclusion (1) and conclusion (2) together.

In harmony with Lemma 2.3, Lemma 2.8 and the induction hypothesis $\mathfrak{H}_s$ $(1 \le s \le m-1)$, i.e., $\Psi_s(I) = 0$, we have

$$\begin{aligned} 2^{n-1}\Psi_m(I) &= \Psi_m(2^{n-1}I) = \Psi_m(U_n(I, I, \cdots, I)) \\ &= \sum_{k=1}^{n} U_n(I, I, \cdots, \underbrace{\Psi_m(I)}_{\text{k-th component}}, \cdots, I) \\ &\quad + \sum_{\substack{i_1+\cdots+i_n=m,\\ i_1,\cdots,i_n\in\{0,1,\cdots,m-1\}}} U_n(\Psi_{i_1}(I), \Psi_{i_2}(I), \cdots, \Psi_{i_n}(I)) \\ &= \sum_{k=1}^{n} U_n(I, I, \cdots, \underbrace{\Psi_m(I)}_{\text{k-th component}}, \cdots, I) \\ &= 2^{n-1}n\Psi_m(I). \end{aligned}$$

And then $\Psi_m(I) = 0$. Thanks to $\Psi_m(I) = 0$, $L^* = -L$ and Lemma 2.8, we attain

$$0 = \Psi_m(U_n(L, I, \cdots, I)) = U_n(\Psi_m(L), I, \cdots, I) = 2^{n-2}(\Psi_m(L)^* + \Psi_m(L)).$$

Furthermore, we have $\Psi_m(L)^* = -\Psi_m(L)$.

Now let's prove that conclusions $\Psi_m(\mathrm{i}I) = 0$ and $\Psi_m(\mathrm{i}L) = \mathrm{i}\Psi_m(L)$ are true.

Because of $(\mathrm{i})^* = -\mathrm{i}$, $\Psi_m(I) = 0$, $\Psi_m(L)^* = -\Psi_m(L)$ for arbitrary $L^* = -L$, Lemma 2.8, and inductive hypothesis $\mathfrak{H}_s$ $(1 \le s \le m-1)$, i.e., $\Psi_s(\mathrm{i}I) = 0$, we arrive at

$$\begin{aligned} 0 = 2^{n-1}\Psi_m(I) &= \Psi_m(2^{n-1}I) = \Psi_m(U_n(\mathrm{i}I, \mathrm{i}I, I, \cdots, I)) \\ &= U_n(\Psi_m(\mathrm{i}I), \mathrm{i}I, I, \cdots, I) + U_n(\mathrm{i}I, \Psi_m(\mathrm{i}I), I, \cdots, I) \\ &\quad + \sum_{\substack{i_1+\cdots+i_n=m,\\ i_1,i_2\in\{0,1,\cdots,m-1\}}} U_n(\Psi_{i_1}(\mathrm{i}I), \Psi_{i_2}(\mathrm{i}I), \cdots, \Psi_{i_n}(I)) \\ &= U_n(\Psi_m(\mathrm{i}I), \mathrm{i}I, I, \cdots, I) + U_n(iI, \Psi_m(\mathrm{i}I), I, \cdots, I) \\ &= -2^n i\Psi_m(\mathrm{i}I). \end{aligned}$$

And then we can get $\Psi_m(\mathrm{i}I) = 0$. For any element L that satisfies Eq $L^* = -L$, Because of $\Psi_m(L)^* = -\Psi_m(L)$, $\Psi_m(I) = 0$, and $\Psi_m(\mathrm{i}I) = 0$. we have

$$\begin{aligned} -2^{n-1}\Psi_m(\mathrm{i}L) &= \Psi_m(-2^{n-1}\mathrm{i}L) = \Psi_m(U_n(\mathrm{i}I, L, I, \cdots, I)) \\ &= U_n(\mathrm{i}I, \Psi_m(L), I, \cdots, I) \\ &= -2^{n-1}\mathrm{i}\Psi_m(L). \end{aligned}$$

So we have $\Psi_m(\mathrm{i}L) = \mathrm{i}\Psi_m(L)$.

□

Lemma 2.10. *According to the above registered symbols, we obtain*

$$\Psi_m(T_1 + T_2) = \Psi_m(T_1) + \Psi_m(T_2) \ \textit{and} \ \Psi_m(T_1 + iT_2) = \Psi_m(T_1) + i\Psi_m(T_2)$$

for arbitrary $T_1 = -T_1^*$ *and* $T_2 = -T_2^*$.

Proof. As a result of Lemma 2.8 and Lemma 2.9, we obtain

$$\mathrm{i}(\Psi_m(T_1 + T_2)) = \Psi_m(\mathrm{i}(T_1 + T_2)) = \mathrm{i}\Psi_m(T_1) + \mathrm{i}\Psi_m(T_2)$$

for all $T_1 = -T_1^*$ and $T_2 = -T_2^*$. In addition, We can see that the first conclusion is true. Now let's prove the second conclusion.

Because the following two equations are true, which means

$$\begin{aligned}2^{n-1}\mathrm{i}\Psi_m(T_1) &= \Psi_m(U_n(T_1+\mathrm{i}T_2, I, \cdots, I))\\ &= \sum_{i_1+\cdots+i_n=m} U_n(\Psi_{i_1}(T_1+\mathrm{i}T_2), \Psi_{i_2}(I), \cdots, \Psi_{i_n}(I))\\ &= U_n(\Psi_m(T_1+\mathrm{i}T_2), I, \cdots, I)\\ &= 2^{n-2}(\Psi_m(T_1+\mathrm{i}T_2)^* + \Psi_m(T_1+\mathrm{i}T_2))\end{aligned}$$

and

$$\begin{aligned}-2^{n-1}\mathrm{i}\Psi_m(T_2) &= \Psi_m(U_n(T_1+\mathrm{i}T_2, \mathrm{i}I, I, \cdots, I))\\ &= \sum_{i_1+\cdots+i_n=m} U_n(\Psi_{i_1}(T_1+\mathrm{i}T_2), \Psi_{i_2}(\mathrm{i}I), \Psi_{i_3}(I), \cdots, \Psi_{i_n}(I))\\ &= U_n(\Psi_m(T_1+\mathrm{i}T_2), \mathrm{i}I, I, \cdots, I)\\ &= 2^{n-1}\mathrm{i}(\Psi_m(T_1+T_2)^* - \Psi_m(T_1+\mathrm{i}T_2)).\end{aligned}$$

Combining the above two expressions, we obtain $\Psi_m(T_1+\mathrm{i}T_2) = \Psi_m(T_1) + \mathrm{i}\Psi_m(T_2)$. □

Lemma 2.11. *According to the above registered symbols, for arbitrary $B \in \mathfrak{B}$, we obtain*

(1) $\Psi_m(iB) = i\Psi_m(B)$ and $\Psi_m(B^*) = \Psi_m(B)^*$;

(2) the nonlinear mapping Ψ_m is additive on $\mathfrak{B}$.

Proof. **(1).** For arbitrary $B \in \mathfrak{B}$, we arrive at $B = B_1 + \mathrm{i}B_2$, where $B_j^* = -B_j$, $j \in \{1, 2\}$. By reasons of Lemma 2.9 and Lemma 2.10, it is obvious that $\Psi_m(\mathrm{i}B) = \mathrm{i}\Psi_m(B)$ and $\Psi_m(B^*) = \Psi_m(B)^*$.

(2). For arbitrary $B \in \mathfrak{B}$ and $C \in \mathfrak{B}$, we arrive at $B = B_1 + \mathrm{i}B_2$ and $C = C_1 + \mathrm{i}C_2$, where $B_j^* = -B_j$ and $C_j^* = -C_j$, $j \in \{1, 2\}$. In conformity with Lemma 2.10, we prevail $\Psi_m(B + C) = \Psi_m(B) + \Psi_m(C)$. □

Lemma 2.12. *According to the above marked symbols, we obtain that the nonlinear mapping Ψ_m is an additive higher $*$-derivation on $\mathfrak{B}$.*

Proof. For arbitrary $B \in \mathfrak{B}$ and $C \in \mathfrak{B}$, in concert with Lemma 2.11, we have

$$\begin{aligned}2^{n-2}\mathrm{i}\Psi_m(B^*C - C^*B) &= \Psi_m(2^{n-2}\mathrm{i}(B^*C - C^*B))\\ &= \Psi_m(U_n(B, \mathrm{i}C, I, \cdots, I))\\ &= \sum_{j_1+\cdots+j_n=m} U_n(\Psi_{j_1}(B), \Psi_{j_2}(\mathrm{i}C), \Psi_{j_3}(I), \cdots, \Psi_{j_n}(I))\\ &= \sum_{j_1+j_2=m} U_n(\Psi_{j_1}(B), \mathrm{i}\Psi_{j_2}(C), I, \cdots, I)\\ &= 2^{n-2}\mathrm{i}\sum_{j_1+j_2=m}(\Psi_{j_1}(B)^*\Psi_{j_2}(C) - \Psi_{j_2}(C)^*\Psi_{j_1}(B)),\end{aligned}$$

which implies that

$$\Psi_m(B^*C - C^*B) = \sum_{j_1+j_2=m}(\Psi_{j_1}(B)^*\Psi_{j_2}(C) - \Psi_{j_2}(C)^*\Psi_{j_1}(B)).$$

In addition, we also need to consider the following equation:

$$\begin{aligned}
2^{n-2}\Psi_m(B^*C + C^*B) &= \Psi_m(2^{n-2}(B^*C + C^*B)) \\
&= \Psi_m(U_n(B, C, I, \cdots, I)) \\
&= \sum_{j_1+\cdots+j_n=m} U_n(\Psi_{j_1}(B), \Psi_{j_2}(C), \Psi_{j_3}(I), \cdots, \Psi_{j_n}(I)) \\
&= \sum_{j_1+j_2=m} U_n(\Psi_{j_1}(B), \Psi_{j_2}(C), I, \cdots, I) \\
&= 2^{n-2} \sum_{j_1+j_2=m} (\Psi_{j_1}(B)^*\Psi_{j_2}(C) + \Psi_{j_2}(C)^*\Psi_{j_1}(B)),
\end{aligned}$$

which implies that

$$\Psi_m(B^*C + C^*B) = \sum_{j_1+j_2=m} (\Psi_{j_1}(B)^*\Psi_{j_2}(C) + \Psi_{j_2}(C)^*\Psi_{j_1}(B)).$$

According to the above equation and additivity of map Ψ_m, it can be seen that

$$\Psi_m(B^*C) = \sum_{j_1+j_2=m} \Psi_{j_1}(B)^*\Psi_{j_2}(C).$$

Furthermore, according to conclusion (2) in Lemma 2.11, we have

$$\Psi_m(BC) = \sum_{j_1+j_2=m} \Psi_{j_1}(B)\Psi_{j_2}(C).$$

In summary, combining Lemma 2.11, it can be concluded that mapping Ψ_m is an additive higher $*$-derivation on $\mathfrak{B}$.

□

It immediately follows from Theorem 2.1 and [5] that the following corollary holds.

Corollary 2.13. *[5, Theorem 2.1] Let $\mathfrak{B}$ be an unital $*$-algebra with identity element I that satisfies condition $\mathfrak{C}$. Then every nonlinear bi-skew Jordan-type derivation is an additive $*$-derivation.*

According to Theorem 2.1, we can immediately deduce the following from the typical example of unital $*$-algebra, prime $*$-algebras, factor von Neumann algebra, von Neumann algebra of type I_1 and standard operator algebra.

We know from the concept of prime $*$-algebras[7, 3.Corollaries] that prime $*$-algebras satisfy condition $\mathfrak{C}$, and then we obtain that the following corollary holds.

Corollary 2.14. *Let $\mathfrak{B}$ be a prime $*$-algebra. Then every nonlinear bi-skew Jordan-type higher derivation on $\mathfrak{B}$ is an additive higher $*$-derivation.*

At the same time, the above inference improves on the existing results[7, Theorem 2.3].

Corollary 2.15. *[7, Theorem 2.3] Let $\mathfrak{B}$ be a prime $*$-algebra. Then every nonlinear bi-skew Jordan derivation is an additive $*$-derivation.*

An algebra $\mathfrak{B}(\mathcal{S})$ composed of all bounded operators is defined on a complex Hilbert space $\mathcal{S}$. If the center $Z(\mathbb{A})$ of an von Neumann algebra $\mathbb{A} \subseteq \mathfrak{B}(\mathcal{S})$ satisfies condition $Z(\mathbb{A}) = CI$, then $\mathbb{A}$ is called a factor von Neumann algebra. It is clear that the algebra $\mathbb{A}$ is a prime algebra, and the following corollary holds.

Corollary 2.16. *Let $\mathbb{A}$ be a factor von Neumann algebra acting on complex Hilbert space with $dim(\mathbb{A}) \geq 2$. Then every nonlinear bi-skew Jordan-type higher derivation on $\mathbb{A}$ is an additive higher $*$-derivation.*

At the same time, the above Corollary 2.16 improves on the existing results[6, Theorem 2.1].

Corollary 2.17. *[6, Theorem 2.1] Let $\mathbb{A}$ be a factor von Neumann algebra acting on complex Hilbert space with dim*$(\mathbb{A}) \geq 2$*. Then every nonlinear bi-skew Jordan n-derivation is an additive* $*$*-derivation.*

We use the symbol $\mathfrak{G}(\mathcal{S}) \subseteq \mathfrak{B}(\mathcal{S})$ to denote the subalgebras of the algebra $\mathfrak{B}(\mathcal{S})$ formed by all finite rank operators. If a subalgebra $\mathfrak{R}$ contains $\mathfrak{G}(\mathcal{S})$, it is called a standard operator algebra.

Corollary 2.18. *Let $\mathfrak{S}$ be an infinite dimensional complex Hilbert space and $\mathfrak{R}$ be a standard operator algebra on $\mathfrak{S}$ containing the identity operator I. Suppose that $\mathfrak{R}$ is closed under the adjoint operation. Then every nonlinear bi-skew Jordan-type higher derivation on $\mathfrak{R}$ is an additive higher* $*$*-derivation.*

Corollary 2.19. *[5, Corollary 3.3] Let $\mathfrak{S}$ be an infinite dimensional complex Hilbert space and $\mathfrak{R}$ be a standard operator algebra on $\mathfrak{S}$ containing the identity operator I. Suppose that $\mathfrak{R}$ is closed under the adjoint operation. Then every nonlinear bi-skew Jordan-type derivation Ψ_1 on $\mathfrak{R}$ is an additive* $*$*-derivation. Moreover, there exists an operator $T \in \mathfrak{B}(\mathcal{S})$ satisfying $T + T^* = 0$ such that $\Psi_1(A) = AT - TA$ for all $A \in \mathfrak{R}$. This is Ψ_1 is inner.*

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