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# More on the weakly 2-prime ideals of commutative rings

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**Abstract.** Let $R$ be a commutative ring with a nonzero identity. In this paper, we introduce the concept of weakly 2-prime ideal which is a generalization of 2-prime ideal and both are generalizations of prime ideals. A proper ideal $I$ of $R$ is called *weakly 2-prime ideal* if whenever $a, b \in R$ with $0 \neq ab \in I$, then $a^2$ or $b^2$ lies in $I$. A number results concerning weakly 2-prime ideals are given. Furthermore, we characterize the valuation domain and the rings over which every weakly 2-prime ideal is 2-prime and rings over which every weakly 2-prime ideal is semi-primary (i.e $\sqrt{I}$ is a prime ideal). We study the transfer the notion of weakly 2-prime ideal to amalgamted algebras along an ideal $A \bowtie^f J$.

## 1. Introduction

We assume throughout this paper that all rings are commutative with nonzero identity. Let $R$ be a ring, we recall that $\mathrm{Nil}(R)$ is the set of all nilpotent elements of $R$ called the nilradical of $R$ and defined by $\mathrm{Nil}(R) := \sqrt{0} = \{a \in R : a^n = 0 \text{ for some positive integer } n\}$. Also, a ring $R$ is called reduced if it has no nonzero nilpotent elements, (i.e., $\mathrm{Nil}(R) = 0$). Finally, we define $Z_I(R) = \{r \in R : rs \in I \text{ for some } s \in R \setminus I\}$.

Prime ideals play a central role in ring theory and so this notion has been generalized and studied in several directions. The importance of some these generalizations is same as the prime, like primary ideals which determine how an ideal is far to be prime. In 1978, Hedstrom and Houston in [12], defined the strongly prime ideal, to be a proper ideal $P$ of $R$ such that for any $a, b \in K$, if $ab \in P$, then either $a \in P$ or $b \in P$ where $K$ is the quotient field of $R$. In 2003, Anderson and Smith [2], introduced the notion of weakly prime ideal, to be a proper ideal $P$ of $R$ such that if $a, b \in R$ and $0 \neq ab \in P$, then $a \in P$ or $b \in P$. So it's easy to see that prime ideals are weakly prime. However, the converse is not true in general. For instance, for a prime number $p$, $(\overline{0})$ is a weakly prime ideal of $\mathbb{Z}_{p^2}$ which is not prime.

In [5], Beddani and Messirdi introduced and studied 2-prime ideals which are generalization of prime ideals. A proper ideal $I$ of a ring $R$ is said to be *2-prime* if whenever $a, b \in R$ and $ab \in I$, then either $a^2 \in I$ or $b^2 \in I$. This concept is also studied by Nikandish, Nikmehr and Yassine in [20].

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In [17], Suat Koç introduced and studied weakly 2-prime ideal which are generalization of 2-prime ideals. A proper ideal $I$ of a ring $R$ is said to be *weakly 2-prime* if whenever $a, b \in R$ and $0 \neq ab \in I$, then either $a^2 \in I$ or $b^2 \in I$.

In Section 2, we give some properties of the concept of weakly 2-prime ideal as well as some new characterizations of weakly 2-prime ideals.

In Section 3, we study the rings over which every weakly 2-prime ideal is 2-prime ideal and the rings, and gives a new characterization of a valuation domain.

In Section 4, we study the rings over which every weakly 2-prime ideal is a semi-primary ideal ( $\sqrt{I}$ is a prime ideal).

Let $f : A \to B$ be a ring homomorphism and $J$ be an ideal of $B$. We define the following subring of $A \times B$ :

$$A \bowtie^f J = \{(a, f(a) + j)/a \in A, j \in J\}$$

called *the amalgamation of A with B along J with respect to f*. This construction is a generalization of the amalgamated duplication of a ring along an ideal introduced and studied in [7, 15]. If $A$ is a commutative ring with unity, and $I$ be a ideal of $A$, the amalgamated duplication of $A$ along the ideal, coincides with $A \bowtie^{id} I$, we have

$$A \bowtie I = \{(a, a + i)/a \in A, i \in I\}.$$

The interest of amalgamation resides, partly, in its ability to cover several constructions in commutative rings, including specially trivial extension (also called Nagata's idealizations)([19]). Moreover, other constructions ($A + XB[X], A + XB[[X]]$, and the $D + M$ construction) can be studied as particular cases amalgamation [6](Example 2.5 and 2.6).

Recall from [8] that a prime ideal $P$ of a ring $R$ is said to be *divided prime ideal* if $P \subset (x)$ for every $x \in R \setminus P$; thus a divided prime ideal is comparable to every ideal of $R$. A ring $R$ is said to be a *divided ring* if every prime ideal of $R$ is a divided prime ideal. For more informations on divided rings, the reader may consult [4, 18, 21]. Let $A$ be a commutative ring and $E$ be an $A$-module. The trivial ring extension of $A$ by $E$ (also called the idealization of $E$ over $A$) is the ring $R := A \propto E$ whose underlying group is $A \times E$ with multiplication given by $(a, e)(a', e') = (aa', ae' + a'e)$ [3].

We recall that if $I$ is a proper ideal of $A$, then $I \propto E$ is an ideal of $A \propto E$. And if $F$ is a submodule of $E$ such that $IE \subseteq F$, then $I \propto F$ is an ideal of $A \propto E$. The ideals of $A \propto E$ are not all of the form $I \propto E$ or $I \propto F$, but if $A$ is an integral domain, and $E$ is a divisible $A$-module (that is, $aE = E$ for all $a \neq 0$), the ideals of $A \propto E$ are the form $I \propto E$ or $0 \propto F$, where $I$ is an ideal of $A$ and $F$ is a submodule of $E$. If $E$ is a $K$-vector space, then $A \propto E$ is local with maximal ideal $0 \propto E$.

## 2. Properties of weakly 2-prime ideals

In this section, we examine weakly 2-prime ideals and present their new properties.

**Definition 2.1.** *[17] Let I be a proper ideal of a ring R. We say that I is weakly 2-prime ideal if for all $x, y \in R$ such that $0 \neq xy \in I$, then either $x^2$ or $y^2$ lies in I.*

**Remark 2.2.**
1. *It is clear that every 2-prime ideal is a weakly 2-prime, but the converse is not true in general, for example $(\overline{0})$ is weakly 2-prime ideal in $\mathbb{Z}_{pq}$ which is not 2-prime, where $p \neq q$ are prime numbers.*
2. *Every weakly prime ideal is also a weakly 2-prime. However, the converse is not true in general. For instance, consider a local ring $(R, \mathfrak{m})$, where $0 \neq \mathfrak{m}^2 \subsetneq \mathfrak{m}$. Then $\mathfrak{m}^2$ is a weakly 2-prime ideal which is not weakly prime.*

The following result give many examples of weakly 2-prime ideals which are not 2-prime ideals.

**Theorem 2.3.** *Let I be an ideal of a ring R. Then:*

1. *If $I$ is a weakly 2-prime and not 2-prime, then $I \subseteq Nil(R)$.*
2. *If $I$ is a weakly 2-prime and $Nil(R) \not\supseteq I$, then $I$ is 2-prime ideal.*
3. *If $R$ is reduced, an ideal $I$ is weakly 2-prime if and only if either $I = 0$ or $I$ is 2-prime.*

*Proof.* Let $I$ be a weakly 2-prime ideal, if $I$ is not 2-prime, then from the [17, Theorem 1] we have $I^2 = 0$, then $I \subseteq \text{Nil}(R)$. Now, if $\text{Nil}(R) \not\subseteq I$, we conclude from [17, Theorem 1] that $I$ is a 2-prime ideal. The rest of result is clear. □

Now we give an example of an ideal which is not weakly 2-prime.

**Example 2.4.** *We consider the ring $\mathbb{Z}_{12}$ and let $I = \{\overline{0}, \overline{6}\}$ be an ideal of $\mathbb{Z}_{12}$. We have $I^2 = \overline{0}$, but $\overline{2}.\overline{3} = \overline{6} \in I$ and $\overline{2}^2 = \overline{4} \notin I$ and $\overline{3}^2 = \overline{9} \notin I$. Thus $I$ is not a weakly 2-prime ideal.*

In the following result, we will give some characterizations of a weakly 2-prime ideal.

**Theorem 2.5.** *Let $I$ be a proper ideal of a ring $R$. The following are equivalent:*

1. *$I$ is a weakly 2-prime ideal .*
2. *For every $x \in R$, if $x^2 \notin I$, then $(I : x) \subseteq (0 : x) \cup \{x \in R : x^2 \in I\}$.*

*Proof.* (1) ⇒ (2). Let $x \in R$ such that $x^2 \notin I$ and let $y \in (I : x)$, so $xy \in I$. If $0 \neq xy$ and since $I$ is a weakly 2-prime ideal we have $y^2 \in I$, then $y \in \{x \in R : x^2 \in I\}$. Now if $xy = 0$, then $y \in (0 : x)$, so $(I : x) \subseteq (0 : x) \cup \{x \in R : x^2 \in I\}$ as desired.

(2) ⇒ (1). Let $a, b \in R$ such that $0 \neq ab \in I$. Assume that $a^2 \notin I$, since $0 \neq ab$, so $b \notin (0 : a)$ this gives $b \in \{x \in R : x^2 \in I\}$, thus $b^2 \in I$ and $I$ is a weakly 2-prime ideal. □

In the following result we will give another characterization of weakly 2-prime. First, we need the following definitions.

**Definition 2.6.** 1. *Let $I$ be a weakly 2-prime ideal of a ring $R$ and $a, b \in R$. We call $(a, b)$ a double-zero of $I$ if $ab = 0, a^2 \notin I, b^2 \notin I$.*

2. *Let $I$ be a weakly 2-prime ideal of a ring $R$, and $AB \subseteq I$ for some ideals $A, B$ of $R$. If $(a, b)$ is not a double-zero of $I$ for every $a \in A$ and $b \in B$, then we call $I$ is double-zero free with respect to $AB$.*

Observe that if $I$ is a weakly 2-prime ideal of a ring $R$ without double-zeros, then $I$ is a 2-prime ideal of $R$. So if $I$ is a weakly 2-prime ideal which is not 2-prime ideal, then there exists a double-zero of $I$.

Let $I$ be a proper ideal of $R$. The ideal generated by $n^{th}$ powers of elements of $I$ is denoted by $I_{[n]} := \langle\{a^n : a \in I\}\rangle$ (See [1]). It is easy to note that $I_{[n]} \subseteq I^n \subseteq I$ and also the equality holds if $n = 1$. Further if $n!.1_R$ is a unit of $R$, then $I_{[n]} = I^n$ by [1, Theorem 5].

**Theorem 2.7.** *Let $I$ be a weakly 2-prime ideal and let $J$ be a proper ideal of $R$ with $aJ \subseteq I$ for some $a \in R$. If $(a, j)$ is not a double-zero of $I$ for all $j \in J$ and $a^2 \notin I$, then $J_{[2]} \subseteq I$. Furthermore if $2.1_R$ is a unit, then $J^2 \subseteq I$.*

*Proof.* Suppose that $I$ is a weakly 2-prime ideal and assume that $J_{[2]} \not\subseteq I$, then there exists $j \in J$ such that $j^2 \notin I$, so $aj \in I$. If $aj \neq 0$, then it contradicts our assumption that $a^2 \notin I$ and $j^2 \notin I$, thus $aj = 0$. Since $(a, j)$ is not a double-zero of $I$ and $a^2 \notin I$, we conclude that $j^2 \in I$, a contradiction. Then $J_{[2]} \subseteq I$. Furthermore is clear since $J_{[2]} = J^2$ □

**Theorem 2.8.** *Let $I$ be a weakly 2-prime ideal and $0 \neq AB \subseteq I$ for some ideals $A$ and $B$ of $R$. If $I$ is double-zero free with respect to $AB$, then either $A_{[2]} \subseteq I$ or $B_{[2]} \subseteq I$. Furthermore if $2.1_R$ is unit, then either $A^2 \subseteq I$ or $B^2 \subseteq I$.*

*Proof.* Suppose that $I$ is double-zero free with respect to $AB$, and $0 \neq AB \subseteq I$. If $A_{[2]} \nsubseteq I$, so there exists $a \in A$ such that $a^2 \notin I$. Since $I$ is double-zero free with respect to $AB$, we conclude that $(a, x)$ is not a double-zero for all $x \in B$. Thus $B_{[2]} \subseteq I$ by theorem 2.7. The furthermore is clear. □

The next theorem gives a characterization of weakly 2-prime ideals of direct product of rings.

**Theorem 2.9.** *Let $R = R_1 \times R_2$ be a direct product of rings with identity and let $I$ be a proper ideal of $R_1$. The following statements are equivalent:*

1. *$I \times R_2$ is a weakly 2-prime ideal of $R$,*
2. *$I \times R_2$ is a 2-prime ideal of $R$,*
3. *$I$ is a 2-prime ideal of $R_1$.*

*Proof.* (1) ⇒ (2). Follows from Theorem 2.3 and the fact that $I \times R_2 \nsubseteq \mathrm{Nil}(R)$.

(2) ⇒ (3). Clear [20, Proposition 2.4].

(3) ⇒ (1). From [20, Proposition 2.4], $I \times R_2$ is a 2-prime ideal of $R$, thus a weakly 2-prime ideal of $R$. □

Let $I$ be an ideal of $R$. Then we denote the set of all elements $a \in R$ whose square is in $I$ by $\sqrt[2]{I}$, that is, $\sqrt[2]{I} = \{x \in R : x^2 \in I\}$. Note that $I \subseteq \sqrt[2]{I} \subseteq \sqrt{I}$ and also $\sqrt[2]{I}$ may not be an ideal of $R$. See the following example.

**Example 2.10.** *Let $k$ be a field of characteristic $\neq 2$ and $R = k[X, Y]$, where $X, Y$ are indeterminates over $k$. Consider the ideal $I = (X^2, Y^2)$ of $R$. Then note that $\sqrt{I} = (X, Y)$ and also $X, Y \in \sqrt[2]{I}$. Since $(X + Y)^2 = X^2 + 2XY + Y^2 \notin I$, it follows that $\sqrt[2]{I}$ is not an ideal of $R$.*

From the above example, one can naturally asks when $\sqrt[2]{I}$ is an ideal of $R$. Now, we give an answer to this question with the following result.

**Proposition 2.11.** *(i) Let $R$ be a ring and $I$ be an ideal of $R$. Then $\sqrt[2]{I}$ is an ideal of $R$ if and only if $2\left(\sqrt[2]{I}\right)^2 \subseteq I$.*

*(ii) Let $R$ be a ring of characteristic 2. Then $\sqrt[2]{I}$ is an ideal of $R$ for every ideal $I$ of $R$.*

*(iii) Let $R$ be a ring and $I, J$ be two ideals of $R$ such that $2\left(\sqrt[2]{I}\right)^2 \subseteq I$. Then $J \subseteq \sqrt[2]{I}$ if and only if $J_{[2]} \subseteq I$.*

*Proof.* (i) Suppose that $\sqrt[2]{I}$ is an ideal of $R$. Let $x, y \in \sqrt[2]{I}$. Then we have $x^2, y^2 \in I$. Since $\sqrt[2]{I}$ is an ideal of $R$, $x+y \in \sqrt[2]{I}$ which implies that $(x+y) = x^2+2xy+y^2 \in I$. Then we conclude that $2xy \in I$, that is, $2\left(\sqrt[2]{I}\right)^2 \subseteq I$. For the converse, assume that $2\left(\sqrt[2]{I}\right)^2 \subseteq I$. Let $x \in \sqrt[2]{I}$, that is, $x^2 \in I$. Then for each $R$, $(rx)^2 \in I$ which implies that $rx \in \sqrt[2]{I}$. Now, choose $x, y \in \sqrt[2]{I}$. Then $x^2, y^2 \in I$. Also, by assumption, we have $2xy \in 2\left(\sqrt[2]{I}\right)^2 \subseteq I$. This gives $(x + y)^2 = x^2 + 2xy + y^2 \in I$, that is, $x + y \in \sqrt[2]{I}$. Thus, $\sqrt[2]{I}$ is an ideal of $R$.

(ii) Since characteristic of $R$ is 2, we have $2\left(\sqrt[2]{I}\right)^2 = (0) \subseteq I$. Thus the result follows from (i).

(iii) Suppose that $2\left(\sqrt[2]{I}\right)^2 \subseteq I$. Then by (i), $\sqrt[2]{I}$ is an ideal of $R$. The rest is clear by definition. □

Now, we are ready to give a new characterization of weakly 2-prime ideals and also we will use it to investigate the weakly 2-prime ideals in polynomial rings and formal power series ring.

**Theorem 2.12.** *Let $P$ be an ideal of $R$ such that $2\left(\sqrt[2]{P}\right)^2 \subseteq P$. The following statements are equivalent:*

*(i) $P$ is a weakly 2-prime ideal of $R$.*

*(ii) For every $x \in R - \sqrt[2]{P}$, either $(P : x) \subseteq ann(x)$ or $(P : x) \subseteq \sqrt[2]{P}$.*

*(iii) For every $x \in R$ with $x^2 \notin P$, either $(P : x) \subseteq ann(x)$ or $(P : x)_{[2]} \subseteq P$.*

*(iv) If $0 \neq xJ \subseteq P$ for some $x \in R$ and some ideal $J$ of $R$, then either $x^2 \in P$ or $J_{[2]} \subseteq P$.*

*(v) If $0 \neq IJ \subseteq P$ for some ideals $I, J$ of $R$, then either $I_{[2]} \subseteq P$ or $J_{[2]} \subseteq P$.*

*Proof.* $(i) \Rightarrow (ii)$ : Let $P$ be a weakly 2-prime ideal of $R$ and choose $x \in R - \sqrt[2]{P}$. Then $x^2 \notin P$. By Theorem 2.5, we have $(P : x) \subseteq ann(x) \cup \sqrt[2]{P}$. As $2\left(\sqrt[2]{P}\right)^2 \subseteq P$, $\sqrt[2]{P}$ is an ideal of $R$. Thus we have either $(P : x) \subseteq ann(x)$ or $(P : x) \subseteq \sqrt[2]{P}$.

$(ii) \Leftrightarrow (iii)$ : Follows from previous proposition (iii).

$(iii) \Rightarrow (iv)$ : Suppose that $0 \neq xJ \subseteq P$ for some $x \in R$ and some ideal $J$ of $R$. Let $x^2 \notin P$. Then by (iii), either $J \subseteq (P : x) \subseteq ann(x)$ or $J_{[2]} \subseteq (P : x)_{[2]} \subseteq P$. Since $xJ \neq 0$, the first case is impossible. So we have $J_{[2]} \subseteq P$.

$(iv) \Rightarrow (i), (v) \Rightarrow (ii)$ : are clear.

$(ii) \Rightarrow (v)$ : Suppose that $0 \neq IJ \subseteq P$ for some ideals $I, J$ of $R$. Now, assume that $I_{[2]} \nsubseteq P$. Then there exists $x \in I$ such that $x^2 \notin P$, that is $x \in R - \sqrt[2]{P}$. If $xJ \neq 0$, then by (ii), $J \subseteq (P : x) \subseteq \sqrt[2]{P}$ which implies that $J_{[2]} \subseteq P$. So assume that $xJ = 0$. Since $IJ \neq 0$, there exists $a \in I$ such that $aJ \neq 0$. If $a \in R - \sqrt[2]{P}$, again by (ii), we have $J_{[2]} \subseteq P$. Thus, we may assume that $a \in \sqrt[2]{P}$. As $2\left(\sqrt[2]{P}\right)^2 \subseteq P$, $\sqrt[2]{P}$ is an ideal so $a + x \in R - \sqrt[2]{P}$. Also note that $0 \neq (a + x)J = aJ \subseteq P$. Then by $(ii)$, we conclude that $J \subseteq (P : x) \subseteq \sqrt[2]{P}$ which implies that $J_{[2]} \subseteq P$. This completes the proof. □

As an immediate consequences of the previous theorem, we give the following result.

**Corollary 2.13.** *Let $R$ be a ring of characteristic 2 and $P$ be a proper ideal of $R$. The following statements are equivalent:*

*(i) $P$ is a weakly 2-prime ideal of $R$.*

**Theorem 2.14.** *(ii) For every $x \in R - \sqrt[2]{P}$, either $(P : x) \subseteq ann(x)$ or $(P : x) \subseteq \sqrt[2]{P}$.*

*(iii) For every $x \in R$ with $x^2 \notin P$, either $(P : x) \subseteq ann(x)$ or $(P : x)_{[2]} \subseteq P$.*

*(iv) If $0 \neq xJ \subseteq P$ for some $x \in R$ and some ideal $J$ of $R$, then either $x^2 \in P$ or $J_{[2]} \subseteq P$.*

*(v) If $0 \neq IJ \subseteq P$ for some ideals $I, J$ of $R$, then either $I_{[2]} \subseteq P$ or $J_{[2]} \subseteq P$.*

Let $R$ be a ring and $R[X]$ be the polynomial ring, where $X$ is an indeterminate over $R$. For any $f(X) = \sum_{k=0}^{n} a_k X^k$, the content of $f$ is defined by $c(f) = (a_0, a_1, \ldots, a_n)$ [9]. If $I$ is an ideal of $R$, then $I[X] = \{f \in R[X] : c(f) \subseteq I\}$ is an ideal of $R[X]$.

**Theorem 2.15.** *Let $R$ be a ring and $2.1_R$ be a unit of $R$. Suppose that $P$ is a radical ideal of $R$. Then $P$ is a weakly 2-prime ideal of $R$ if and only if $P[X]$ is a weakly 2-prime ideal of $R[X]$.*

*Proof.* $(\Leftarrow)$ : is easy.

$(\Rightarrow)$ : Let $P$ be a weakly 2-prime ideal of $R$ and $\sqrt{P} = P$. Since $\sqrt[2]{P} \subseteq \sqrt{P} = P$, we have $2\left(\sqrt[2]{P}\right)^2 \subseteq P$. Now, choose $f, g \in R[X]$ such that $0 \neq fg \in P[X]$. This gives $c(fg) \subseteq P$. Let $\deg(f) = n$. Then by Dedekin-Mertens Theorem ([9, Theorem 28.1]), $c(f)c(g)^{n+1} = c(fg)c(g)^n \subseteq P$. Since $P$ is a radical ideal, we have $0 \neq c(f)c(g) \subseteq P$. Then by Theoerem 2.12, $c(f)_{[2]} \subseteq P$ or $c(g)_{[2]} \subseteq P$. As $2.1_R$ is a unit of $R$, we conclude that $c(f^2) \subseteq c(f)^2 = c(f)_{[2]} \subseteq P$ or $c(g^2) \subseteq c(g)_{[2]} \subseteq P$. Then we have either $f^2 \in P[X]$ or $g^2 \in P[X]$. Hence, $P[X]$ is a weakly 2-prime ideal of $R[X]$. □

Recall from [10] that a ring $R$ is said to be a Gaussian ring if $c(fg) = c(f)c(g)$ for every $f, g \in R[X]$. If $R$ is a Gaussian ring, then we can remove the condition "$P$ is a radical ideal of $R$" in the previous theorem. Since its proof is similar to previous one, we omit the proof.

**Theorem 2.16.** *Let $R$ be a Gaussian ring and $2.1_R$ be a unit of $R$. Then $P$ is a weakly 2-prime ideal of $R$ if and only if $P[X]$ is a weakly 2-prime ideal of $R[X]$.*

Let $R$ be a ring and $R[[X]]$ be ring of formal power series, where $X$ is an indeterminate over $R$. For any $f = \sum_{k=0}^{\infty} a_k X^k \in R[[X]]$, the content of $f$ is denoted by $c(f) = \langle\{a_k : k \in \mathbb{N} \cup \{0\}\}\rangle$. If $I$ is an ideal of $R$, then $I[[X]] = \{f \in R[[X]] : c(f) \subseteq I\}$ is an ideal of $R[[X]]$. In [11], the authors proved that a similar version of Dedekin-Mertens Theorem for Noetherian formal power series ring. Now, we will investigate the weakly 2-prime ideals of Noetherian formal power series ring.

**Theorem 2.17.** *Let $R$ be a Noetherian ring and $2.1_R$ be a unit of $R$. Suppose that $P$ is a radical ideal of $R$. Then $P$ is a weakly 2-prime ideal of $R$ if and only if $P[[X]]$ is a weakly 2-prime ideal of $R[[X]]$.*

*Proof.* $(\Leftarrow)$ : is clear.

$(\Rightarrow)$ : Let $P$ be a radical ideal and $P[[X]]$ be a weakly 2-prime ideal of $R[[X]]$. Choose $f, g \in R[[X]]$ such that $0 \neq fg \in P[[X]]$. Then $c(fg) \subseteq P$. Let $\mu(c(f))$ denote the minimal number of generators of $f$. Since $R$ is Noetherian, we can choose $n$ as maximum of the numbers $\mu(c(f)_{\mathfrak{m}})$, taken over all maximal ideals $\mathfrak{m}$ of $R$. Then by [11, Theorem 2.6], $c(f)c(g)^n = c(fg)c(g)^{n-1} \subseteq P$. As $P$ is a radical ideal, we have $0 \neq c(f)c(g) \subseteq P$. Since 2 is unit and $2\left(\sqrt[2]{P}\right)^2 \subseteq P$, by a similar argument in the previous theorem, we have $c(f^2) \subseteq P$ or $c(g^2) \subseteq P$. Which implies that $f^2 \in P[[X]]$ or $g^2 \in P[[X]]$. Thus $P[[X]]$ is a weakly 2-prime ideal of $R[[X]]$. □

Now, we study the weakly 2-prime property over $R(+)M$ constructions.

**Theorem 2.18.** *[17, Proposition 5] Let $R$ be a ring and let $M$ be an $R$-module. For a proper ideal $I$ of $R$, the following statement are equivalent.*

1. *$I(+)M$ is a weakly 2-prime ideal of $R(+)M$.*
2. *$I$ is a weakly 2-prime ideal of $R$ and for every double zero $(a, b)$ of $I$ we have $aM = bM = 0$.*

*Proof.* See [17, Proposition 5]. □

**Example 2.19.** *Let $R$ be a reduced ring and $M$ be an $R$-module. The unique ideal of $R(+)M$ which has the form $I(+)M$ and which is weakly 2-prime and not 2-prime is $0(+)M$.*

*Indeed, from Theorem 2.9, if $I(+)M$ is weakly 2-prime not a 2-prime ideal of $R(+)M$, we have $I$ is weakly 2-prime not 2-prime and by Theorem 2.3 $I^2 = 0$. Hence $I = 0$ since $R$ is reduced as desired.*

Let $f : A \to B$ be a ring homomorphism and $J$ be an ideal of $B$. We define the following subring of $A \times B$ :

$$A \bowtie^f J = \{(a, f(a) + j)/a \in A, j \in J\}$$

called *the amalgamation of $A$ with $B$ along $J$ with respect to $f$*.This construction is a generalization of the amalgamated duplication of a ring along an ideal introduced and studied in [7, 15]. If $A$ is a commutative ring with unity, and $I$ is an ideal of $A$, the amalgamated duplication of $A$ along the ideal, coincides with $A \bowtie^{id} I$, we have

$$A \bowtie I = \{(a, a + i)/a \in A, i \in I\}.$$

We next show how to construct examples of a weakly 2-prime ideal using the method of amalgamated algebras along an ideal.

**Theorem 2.20.** *Let $f : A \to B$ be a homomorphism of rings and $J$ be an ideal of $B$. Let $I$ be an ideal of $A$.*

1. *If $I \bowtie^f J$ is a weakly 2-prime ideal of $A \bowtie^f J$, then $I$ is a weakly 2-prime ideal of $A$.*
2. *If $I$ is a weakly 2-prime ideal which is not a 2-prime ideal of $A$. Then the following statements are equivalent*
   - *$I \bowtie^f J$ is a weakly 2-prime ideal of $A \bowtie^f J$,*
   - *For every double zero $(a, b) \in A \times A$ of $I$, we have $f(a)J = f(b)J = 0$ and $J^2 = 0$.*

We begin by demonstrating the following lemma.

**Lemma 2.21.** *Let $f : A \to B$ be a homomorphism of rings, $J$ be an ideal of $B$ and $I$ be an ideal of $A$. Then*

$$(I \bowtie^f J)^2 = I^2 \bowtie^f \left(f(I)J + J^2\right).$$

*Proof.* [14, Lemma 3.4]. □

**Lemma 2.22.** *Let $f : A \to B$ be a homomorphism of rings, $J$ be an ideal of $B$ and $I$ be an ideal of $A$. The following are equivalent:*

1. *$I$ is 2-prime ideal of $A$.*
2. *$I \bowtie^f J$ is 2-prime ideal of $A \bowtie^f J$.*

*Proof.* By [20, Corollary 2.5], an ideal $I$ of $A$ is 2-prime if and only if $\{0\}$ is a 2-prime ideal of $A/I$. On the other hand, by the [6, Proposition 5.1], we have $A \bowtie^f J/I \bowtie^f J \simeq A/I$, so the result follows. □

*Proof.* **(Proof of Theorem 2.20):**

1. Suppose that $I \bowtie^f J$ is a weakly 2-prime ideal of $A \bowtie^f J$, and let $0 \neq ab \in I$ for some $a, b \in A$. Then $0 \neq (a, f(a))(b, f(b)) \in I \bowtie^f J$. Since $I \bowtie^f J$ is a weakly 2-prime ideal of $A \bowtie^f J$, we have $(a, f(a))^2 \in I \bowtie^f J$ or $(b, f(b))^2 \in I \bowtie^f J$ which implies that $a^2 \in I$ or $b^2 \in I$. Thus $I$ is a weakly 2-prime ideal.
   - Assume that $I$ is weakly 2-prime which is not a 2-prime ideal of $A$. Suppose that $I \bowtie^f J$ is a weakly 2-prime ideal of $A \bowtie^f J$. Let $(a, b) \in A \times A$ be a double zero of $I$, and assume that $f(a) \notin \text{ann}(J)$, so there exists $j \in J$ such that $f(a)j \neq 0$, consequently $(0, 0) \neq (a, f(a))(b, f(b) + j) = (0, f(a)f(b)j) \in I \bowtie^f J$, but neither $(a, f(a))^2 \in I \bowtie^f J$ nor $(b, f(b) + j)^2 \in I \bowtie^f J$. This is a contradiction. So $f(a)J = 0$. Likewise, $f(b)J = 0$. On the other hand $I \bowtie^f J$ is not a 2-prime ideal of $A \bowtie^f J$ from the Lemma2.22. Then by the Lemma2.21 and the [17, Theorem 1] we have $(I \bowtie^f J)^2 = I^2 \bowtie^f (f(I)J + J^2) = 0$, thus $J^2 = 0$.
   - Conversely, let $(a, f(a) + i), (b, f(b) + j) \in A \bowtie^f J$ such that $(0, 0) \neq (a, f(a) + i)(b, f(b) + j) = (ab, f(ab) + m) \in I \bowtie^f J$ where $m \in J$.

     **Case one:** $ab \neq 0$.

     Since $I$ is a weakly 2-prime ideal of $A$, suppose for example that $a^2 \in I$, then $(a, f(a) + i)^2 \in I \bowtie^f J$.

     **Case two:** $ab = 0$. Without loss the generality, we may assume neither $a^2 \in I$ nor $b^2 \in I$. Hence $(a, b)$ is a double zero of $I$, by the hypothesis, we have $f(a)J = f(b)J = 0$, then $(a, f(a) + i)(b, f(b) + j) = (ab, f(ab) + f(a)j + f(b)i + ij) = (0, ij) = (0, 0)$ because $J^2 = 0$.

□

**Corollary 2.23.** *Let $f : A \longrightarrow B$ be a ring homomorphism and let $J$ be an ideal of $B$ with $J^2 = 0$ and $I$ be a weakly 2-prime ideal of $A$ which is not 2-prime ideal such that for every $(a, b) \in A \times A$ a double zero of $I$, $(f(a), f(b)) \in ann(J) \times ann(J)$. Then $I \bowtie^f J$ is weakly 2-prime ideal which is not a 2-prime ideal of $A \bowtie^f J$.*

*Proof.* Follows from the Theorem 2.20. □

**Corollary 2.24.** *Let $(A, M)$ be a local ring with a maximal ideal $M$, $f : A \longrightarrow B$ be a ring homomorphism, $J$ be an ideal of $B$ such that $f(M)J = 0$. Then the following are equivalent:*

1. *$I$ is a weakly 2-prime ideal which is not 2-prime of $A$ and $J^2 = 0$.*
2. *$I \bowtie^f J$ is a weakly 2-prime ideal which is not 2-prime of $A \bowtie^f J$.*

*Proof.* (1) $\Longrightarrow$ (2). Let $I$ be a weakly 2-prime ideal which is not 2-prime, and take $(a, b) \in A \times A$ a double zero of $I$. We claim that $a, b \in M$. Deny assume that $a \notin M$, hence $a$ is invertible and so $b = 0$, a contradiction. Hence $(a, b) \in M \times M$ and so $f(a)J = f(b)J = 0$. The result follows from Theorem 2.20.

(2) $\Longrightarrow$ (1). Assume that $I \bowtie^f J$ is a weakly 2-prime ideal which is not 2-prime of $A \bowtie^f J$, then $I$ is a weakly 2-prime ideal which is not 2-prime of $A$, by the Theorem 2.20, we conclude that $J^2 = 0$. □

**Corollary 2.25.** *Let $(A, M)$ be a local ring with a maximal ideal $M$, $f : A \longrightarrow B$ be a ring homomorphism, $J$ be an ideal of $B$ such that $f(M)J = 0$ and $M^2 = 0$. If $J^2 = 0$, then every ideal of $A \bowtie^f J$ is weakly 2-prime.*

*Proof.* It is well known that $A \bowtie^f J$ is a local ring with a maximal ideal $M \bowtie^f J$. Now $(M \bowtie^f J)^2 = 0$, by the Lemma 2.21. Now the result follows from [17, Lemma 1]. □

**Example 2.26.** *Let $(A, M)$ be a local ring which is not a field such that $M^2 = 0$(for instance $A = \mathbb{Z}/4\mathbb{Z}, M = 2\mathbb{Z}/4\mathbb{Z}$), $E$ be an $A/M$ vector-space, $B = A(+)E$ be the trivial extension of $A$ by $E$. Consider $f : A \hookrightarrow B$ defined by $f(a) = (a, 0)$ for all $a \in A$. Then $f$ is an injective ring homomorphism and $J := I(+)E$ is a proper ideal of $B$ where $I$ is a proper ideal of $A$. We have $f(M)J = (M(+)0)(I(+)E) = 0$ and $J^2 = 0$. Following the Corollary 2.24 we get $I \bowtie^f J$ is a weakly 2-prime ideal which is not a 2-prime ideal of $A \bowtie^f J$ if and only if $I$ is a weakly 2-prime ideal which is not a 2-prime ideal of $A$. Since $M^2 = 0$, by the Corollary 2.25 every ideal of $A \bowtie^f J$ is weakly 2-prime ideal.*

**Proposition 2.27.** *Let $f : A \to B$ be a homomorphism of rings, $J$ be an ideal of $B$ and $I$ be an ideal of $A$ such that $Nil(A) \not\supseteq I$. Then the following are equivalent:*

1. *$I \bowtie^f J$ is a 2-prime ideal of $A \bowtie^f J$.*
2. *$I \bowtie^f J$ is a weakly 2-prime ideal of $A \bowtie^f J$.*

*Proof.* (1) ⇒ (2) : Clear.

(2) ⇒ (1) : Follows from Theorem 2.20, Theorem 2.3 and Lemma 2.22. □

## 3. Rings over which every weakly 2-prime ideal is a 2-prime ideal

In this section we study the rings over which every weakly 2-prime ideal is a 2-prime ideal.

**Theorem 3.1.** *Let $R$ be a ring. Then, every weakly 2-prime ideal is a 2-prime ideal if and only if $(0)$ is 2-prime.*

*Proof.* Assume that every weakly 2-prime ideal is 2-prime. Then (0) is a 2-prime ideal since it is weakly 2-prime. Conversely, assume that (0) is a 2-prime ideal and let $I$ be a weakly 2-prime ideal. Let $ab \in I$ for some $a, b \in R$ such that $ab \neq 0$, then either $a^2 \in I$ or $b^2 \in I$. Now, suppose that $ab = 0$, since (0) is 2-prime ideal, we conclude that $a^2 = 0 \in I$ or $b^2 = 0 \in I$. Finally $I$ is a 2-prime ideal as desired. □

**Example 3.2.** *Let $R$ be an integral domain and $M$ be a divisible $R$-module. Then every ideal $J$ of $R(+)M$ is weakly 2-prime if and only if $J$ is a 2-prime ideal of $R(+)M$. Indeed, we claim that the ideal $(0(+)0)$ is 2-prime. Deny, let $(a, e)(b, f) = (ab, af + be) \in (0(+)0)$ for some $(a, e), (b, f) \in R(+)M$. Since $R$ is an integral domain, we get either $a = 0$ or $b = 0$, in both cases we obtain either $(a, e)^2 = (0, 0) \in (0(+)0)$ or $(b, f)^2 = (0, 0) \in (0(+)0)$. Hence $(0(+)0)$ is a 2-prime ideal. Now the result follows from Theorem 3.1.*

**Corollary 3.3.** *Let $R$ be a reduced ring. Then every weakly 2-prime ideal is a 2-prime ideal of $R$ if and only if $R$ is an integral domain.*

*Proof.* If every weakly 2-prime ideal is 2-prime, then (0) is a 2-prime ideal, hence $\sqrt{0} = (0)$ is a prime ideal and $R$ is an integral domain. The converse is clear since every prime ideal is a 2-prime ideal. □

In the following theorem see that the integral domain over which every ideal is weakly 2-prime is exactly the valuation domain.

**Theorem 3.4.** *Let $R$ be an integral domain, then the following statements are equivalent:*

1. *Every proper ideal is a weakly 2-prime.*
2. *Every proper ideal is a 2-prime.*
3. *$R$ is a valuation domain.*

*Proof.* (2) ⇒ (1). Clear.

(1) ⇒ (2). Let $ab \in I$ for some $a, b \in R$, if $0 \neq ab$, since $I$ is weakly 2-prime ideal of $R$, then either $a^2 \in I$ or $b^2 \in I$. Now if $ab = 0$, we conclude that $a^2 = 0 \in I$ or $b^2 = 0 \in I$ since $R$ is a domain. Hence $I$ is a 2-prime ideal.

(2) ⇔ (3). Follows from the [5, Theorem 2.4]. □

## 4. Rings over which every weakly 2-prime ideal is semi-primary

An ideal $I$ of a ring $R$ is said to be semi-primary ideal if $\sqrt{I}$ is a prime ideal of $R$. It is proved in [20] that a 2-prime ideal is semi-primary ideal. However, this is not the case for weakly 2-prime ideal. For instance, $I = (\bar{0})$ is a weakly 2-prime ideal of $\mathbb{Z}_6$. But $\sqrt{I} = (\bar{0})$ is not a semi-primary ideal of $\mathbb{Z}_6$. The next result characterizes a rings over which every weakly 2-prime ideal is semi-primary.

**Theorem 4.1.** *Let $R$ be a ring. Then, every weakly 2-prime ideal is a semi-primary ideal if and only if $Nil(R)$ is a prime ideal.*

*Proof.* ($\Rightarrow$) Trivial since (0) is a weakly 2-prime ideal.

($\Leftarrow$) Let $I$ be a weakly 2-prime ideal and let $ab \in \sqrt{I}$ for some $a, b \in R$ and $a \notin \sqrt{I}$, we will show that $b \in \sqrt{I}$. So there exists an integer $n \geq 1$ such that $a^n b^n \in I$, if $a^n b^n \neq 0$ then $a^{2n} \in I$ or $b^{2n} \in I$, so $b \in \sqrt{I}$ since $a \notin \sqrt{I}$. So, we suppose that $a^n b^n = 0$, if $a^n I \neq 0$ there exists $x \in I$ such that $a^n x \neq 0$, and so $0 \neq a^n(x + b^n) \in I$. And since $a^{2n} \notin I$, we get $(b^n + x)^2 \in I$ and thus $x + b^n \in \sqrt{I}$, hence $b \in \sqrt{I}$. If $a^n I = 0 \subseteq \mathrm{Nil}(R)$, since $a \notin \mathrm{Nil}(R)$ and $\mathrm{Nil}(R)$ is prime, we conclude that $I \subseteq \mathrm{Nil}(R)$. Thus $\sqrt{I} = \mathrm{Nil}(R)$ is a prime ideal, as desired. □

Recall that a ring $R$ is called *divided* if for every prime ideal $P$ or $R$ and for every $x \in R \setminus P$, we have $x$ divides $p$ for every $p \in P$. We have the following result.

**Corollary 4.2.** *Let $R$ be a divided ring and $I$ be a proper ideal of $R$. Then $I$ is a weakly 2-prime ideal if and only if $I$ is a semi-primary ideal.*

*Proof.* Since $R$ is divided ring, then the prime ideals are comparable, we conclude that $\mathrm{Nil}(R)$ is a prime ideal. Now the result follows from the Theorem 4.1. □

**Corollary 4.3.** *Let $R$ a reduced ring. Then every weakly 2-prime ideal is a semi-primary ideal if and only if $R$ is an integral domain.*

*Proof.* Assume that $R$ is a reduced ring, then $\mathrm{Nil}(R) = (0)$. Now the result follows from the Theorem 4.1. □

**Theorem 4.4.** *Let $R$ be a reduced ring. If $I$ is a nonzero weakly 2-prime ideal of $R$, then $\sqrt{I}$ is a prime ideal. In particular if $\sqrt{I}$ is a maximal ideal, then it is a primary ideal.*

*Proof.* Suppose that $0 \neq ab \in \sqrt{I}$ for some $a, b \in R$, then there exists a positive integer $n \geq 1$ such that $(ab)^n \in I$. Since $\mathrm{Nil}(R) = 0$, we have $(ab)^n \neq 0$, hence either $a^{2n} \in I$ or $b^{2n} \in I$ and therefore $\sqrt{I}$ is a weakly prime ideal. Since $R$ is reduced and $I \neq 0$, we conclude that $\sqrt{I}$ is a prime ideal from [2, Corollary 2]. The proof of the "in particular" statement is now clear. □

**Corollary 4.5.** *Let $R$ be a von-Neumann regular ring. Then every nonzero weakly 2-prime ideal is a maximal ideal.*

*Proof.* Suppose that $0 \neq I$ is a weakly 2-prime ideal of $R$. Since $R$ is a von-Neumann regular, we know that $R$ is reduced. Hence by Theorem 4.4, $\sqrt{I}$ is a prime ideal and since every prime ideal of a von-Neumann regular ring is maximal, by [16, Theorem 2], we conclude that $\sqrt{I} = I = M$ is a maximal ideal of $R$. Then $I$ is a $M$-primary ideal of $R$ by Theorem 4.4. □

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