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Bernstein type gradient estimation for weighted local heat equation

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Abstract. In this article we derive Bernstein type gradient estimation for local weighted heat equation on static weighted Riemannian manifold and evolving weighted Riemannian manifold along local Ricci flow and extended local Ricci flow. We showed that along local Ricci flow and extended local Ricci flow we can derive Bernstein type estimation for weighted heat equation without any assumption on the bound of Bakry-Émery Ricci curvature.

1. Introduction

Gradient estimation is method in analysis on partial differntial equations where one can derive bounds for $|\nabla u|$, where u is a solution to some partial differntial equation, ∇ being the gradient operator. Depending on the type of the partial differntial equation, different types of gradient estimations can be derived. After Li and Yau's [15] work this method becomes popular. In recent days, there are several studies going on finding gradient estimation for positive solution of nonlinear heat type equations over Riemannian manifolds. One can follow works of Abolarinwa et al. [1] where they derived gradient estimates for nonlinear weighted parabolic equation. In [7], Chen and Zhao studied Li-Yau and Souplet-Zhang type gradient estimation for parabolic equations involving V-Laplacian. Dung and Khanh [9] derived gradient estimation for semilinear parabolic equations. Abolarinwa [2, 3] also studied gradient estimation for elliptic equations, Harnack estimation and derived Liouville type theorem. Results related to first eigenvalue of weighted p-Laplacian under cotton flow was studied by Saha et al. [17]. Some recent developments can also be found in Hui et al. [11–14], these includes Hamilton and Souplet-Zhang type estimation, gradient estimation on system of equations, nonlinear elliptic equations, Li-Yau type gradient estimation along geometric flow etc. Results related to Hessian estimates can be found in the work of Wang et al. [19]. It should be mentioned here that majority of those studies are for heat equations with nonlinear potential terms. Our study is different than the classical study of gradient estimation. We study gradient estimation for local heat equation along local Ricci flow and extended local Ricci flow. The idea of 'local heat equation' was originated after the development of local Ricci flow. Let (M^n, g) be an n-dimensional Riemannian manifold with g being the Riemannian metric. The local Ricci flow is defined by

$$\frac{\partial g}{\partial t} = -2\chi^2 Ric, \tag{1}$$

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where *Ric* denotes the Ricci tensor. Here $\chi : M \to \mathbb{R}$ is a function with compact support in a smooth bounded domain $\Omega \subset M$. Following this idea a new type of heat equation is formed viz. local heat equation which is defined as

$$\begin{cases} \frac{\partial u}{\partial t} = \chi^2 \Delta u \\ u(x,0) = u_0, \ \forall x \in M, \end{cases} \tag{2}$$

where χ is defined as earlier. Next observe that $u(x,t) = u_0(x)$ for all $x \in M \setminus \operatorname{supp}(\chi)$. One can see [8, Chapter 14] for a complete study. Next we mention some interesting areas in which this type of heat equations plays a significant role.

1.1. Thermal diffusivity

Heat equation of the form (2) are important in various different field of science, especially in Thermography. In this field one can find thermal diffusivity constant α for different metals. Thermal diffusivity constant is the ration of *conducted heat* and *stored heat* in a material, mostly metals. In mathematical notation

$$\alpha = \frac{\kappa}{\rho c_p},$$

where κ is the material conductivity, ρ is material density and c_p is the specific heat of the material. The function χ^2 is treated as the thermal diffusivity constant α. To make a quick review of this constant one can follow [18] and the references therein. There are different types of methods available to determine α in practical scenario.

1.2. Cable theory

In Neuroscience, the propagation of action potential in nerve cells is an effective area of modern research. In Cable theory one can make quantitative studies of flow of currents in axons. One can see [16] and the references therein for a detailed study. The partial differential equation that is used in this field is

$$(\tau \partial_t - \lambda^2 \partial^2_{xx})V = V,$$

we just refer the reader to [16] for the notations and implications of the above equation. Just to mention in our case λ^2 is nothing but χ^2.

Thus the local heat equation has a great significance in modern day science. Hence motivated by the above mentioned work, in this article we derive the Bernstein type estimate for weighted local heat type equation

$$\begin{cases} \frac{\partial u}{\partial t} = \chi^2 \Delta_f u \\ u(x,0) = u_0, \ \forall x \in M, \end{cases} \tag{3}$$

on a weighted Riemannian manifold M equipped with the weighted volume measure $e^{-f} d\mu$. Here $f \in C^\infty(M)$ and Δ_f denotes the weighted Laplacian defined by

$$\Delta_f u = \Delta u - \langle \nabla f, \nabla u \rangle, \tag{4}$$

where Δ is the Laplace-Beltrami operator. Some recent developments on gradient estimation related to weighted Laplacian can be found in [11–14]. Next we have some fundamental results and definitions that will be required to initiate the estimation.

Definition 1.1 ([6] Bakry-Émery Ricci tensor). *For any $m \geq n > 0$ and $m, n \in \mathbb{Z}$ and any $f \in C^\infty(M)$, the $(m-n)$-Bakry-Émery Ricci tensor is defined by*

$$Ric_f^{m-n} := Ric + Hess\ f - \frac{\nabla f \otimes \nabla f}{m-n}.$$

Remark 1.2. *The case $m = n$ happens $\iff$ f is a constant map. The case when $m \to \infty$ will produce the ∞-Bakry-Émery Ricci tensor is defined by*

$$Ric_f^{\infty} = Ric + Hess\ f.$$

Alternatively written as Ric_f.

Lemma 1.3 (Weighted Bochner formula). *For any $u \in C^3(M)$ we have*

$$\frac{1}{2}\Delta_f|\nabla u|^2 = |Hess\ u|^2 + \langle \nabla u, \nabla \Delta_f u\rangle + Ric_f(\nabla u, \nabla u).$$

We are giving the proof of the above lemma just to enrich the reader.

Proof. From Bochner's formula we have

$$\begin{aligned} \frac{1}{2}\Delta|\nabla u|^2 &= |\text{Hess } u|^2 + \langle \nabla u, \nabla \Delta u\rangle + Ric(\nabla u, \nabla u) \\ \implies \frac{1}{2}\Delta_f|\nabla u|^2 &= |\text{Hess } u|^2 + \langle \nabla u, \nabla \Delta_f u\rangle + Ric(\nabla u, \nabla u) - \frac{1}{2}\langle \nabla f, \nabla|\nabla u|^2\rangle + \langle \nabla u, \nabla\langle \nabla f, \nabla u\rangle\rangle \\ &= |\text{Hess } u|^2 + \langle \nabla u, \nabla \Delta_f u\rangle + Ric(\nabla u, \nabla u) - \text{Hess } u(\nabla f, \nabla u) \\ &\quad + \text{Hess } u(\nabla f, \nabla u) + \text{Hess } f(\nabla u, \nabla u). \end{aligned}$$

The second and the last term gives the ∞-Bakry-Émery Ricci tensor. This completes the proof. □

2. Main Results

2.1. Bernstein type estimate on static manifold

In this subsection we derive Bernstein type estimate on a static weighted Riemannian manifold $(M^n, g, e^{-f}d\mu)$ with $Ric_f^{m-n} \geq -K(m-1)g$, where $K \geq 0$ is a real number.

Theorem 2.1 (Bernstein type estimate for weighted local heat equation). *If u is a solution to the heat equation* (3) *on $(M^n, g, e^{-f}d\mu)$ with $Ric_f^{m-n} \geq -K(m-1)g$ then*

$$|\nabla u|^2 \leq \frac{B}{\chi^2 t},\ on\ \Omega \times [0, T], \tag{5}$$

where Ω is a bounded domain of M, $T > 0$ is a real number and $B < \infty$ is a constant depends only on K, m, T, $\max_{\Omega} u_0$, $\max_{\Omega} \chi^2$, $\max_{\Omega} (-\chi\Delta_f\chi)$ and $\max_{\Omega} |\nabla\chi|^2$.

Proof. It is east to check that

$$\frac{\partial}{\partial t}(u^2) = \chi^2\Delta_f(u^2) - 2\chi^2|\nabla u|^2. \tag{6}$$

From weighted Bochner formula (Lemma 1.3) we get

$$\begin{aligned} \partial_t|\nabla u|^2 &= 2\langle \nabla u, \nabla(\chi^2\Delta_f u)\rangle \\ &= \chi^2\Delta_f|\nabla u|^2 - 2\chi^2|\text{Hess } u|^2 - 2\chi^2 Ric_f(\nabla u, \nabla u) + 4\chi\langle \nabla u, \nabla\chi\rangle\Delta_f u. \end{aligned}$$

Using the above equation we derive

$$\partial_t(\chi^2|\nabla u|^2) = \chi^4\left(\Delta_f|\nabla u|^2 - 2|\text{Hess } u|^2 - 2Ric_f(\nabla u, \nabla u)\right) + 4\chi^3\langle \nabla u, \nabla\chi\rangle\Delta_f u. \tag{7}$$

Similarly we infer

$$\begin{aligned}\Delta_f(\chi^2|\nabla u|^2) &= \chi^2\Delta_f|\nabla u|^2 + 2|\nabla u|^2(\chi\Delta_f\chi + |\nabla\chi|^2) + 4\chi\langle\nabla\chi, \nabla|\nabla u|^2\rangle \\ &= \chi^2\Delta_f|\nabla u|^2 + 2|\nabla u|^2(\chi\Delta_f\chi + |\nabla\chi|^2) + 8\chi\langle\nabla\chi\,\nabla u, \text{Hess } u\rangle. \end{aligned} \tag{8}$$

Combining (7) and (8) gives

$$\begin{aligned}(\partial_t - \chi^2\Delta_f)(\chi^2|\nabla u|^2) &= \chi^4\left(\Delta_f|\nabla u|^2 - 2|\text{Hess } u|^2 - 2Ric_f(\nabla u, \nabla u)\right) + 4\chi^3\langle\nabla u, \nabla\chi\rangle\Delta_f u - \chi^4\Delta_f|\nabla u|^2 \\ &\quad -2\chi^2|\nabla u|^2(\chi\Delta_f\chi + \chi|\nabla\chi|^2). \end{aligned} \tag{9}$$

Simplifying we get

$$\begin{aligned}(\partial_t - \chi^2\Delta_f)(\chi^2|\nabla u|^2) &= -2\chi^4|\text{Hess } u|^2 - 2\chi^4 Ric_f(\nabla u, \nabla u) + 4\chi^3\langle\nabla u, \nabla\chi\rangle - 2\chi^2|\nabla u|^2(\chi\Delta_f\chi + |\nabla\chi|^2) \\ &\quad -8\chi^2\langle\nabla\chi\nabla u, \text{Hess } u\rangle. \end{aligned} \tag{10}$$

For $m > n$ by Cauchy-Schwarz and Young's inequality we find that

$$4\chi^3\langle\nabla u, \nabla\chi\rangle\Delta_f u \leq 16m\chi^2|\nabla u|^2|\nabla\chi|^2 + \chi^4|\text{Hess } u|^2 + \frac{2\chi^4}{m-n}\langle\nabla f, \nabla u\rangle^2, \tag{11}$$

where we used $\frac{1}{m}(\Delta_f u)^2 \leq |\text{Hess } u|^2 + \frac{1}{m-n}\langle\nabla f, \nabla u\rangle^2$ (For an explicit proof one can see [13]). We also used the fact that the quantity $\frac{\chi^4}{m-n}\langle\nabla f, \nabla u\rangle^2 \geq 0$. In similar way we deduce

$$-8\chi^2\langle\nabla\chi\nabla u, \text{Hess } u\rangle \leq 16\chi^2|\nabla\chi|^2|\nabla u|^2 + \chi^4|\text{Hess } u|^2. \tag{12}$$

Using (11) and (12) in (10) we get

$$(\partial_t - \chi^2\Delta_f)(\chi^2|\nabla u|^2) \leq 2\chi^2|\nabla u|^2(-\chi\Delta_f\chi + (7+8m)|\nabla\chi|^2) - 2\chi^4 Ric_f^{m-n}(\nabla u, \nabla u). \tag{13}$$

Applying the bound of Ric_f^{m-n} in the above equation and assuming that the functions $-\chi\Delta_f\chi$, $|\nabla\chi|^2$ and χ^2 are all bounded above by a constant $C_1 < \infty$ we infer

$$\begin{aligned}(\partial_t - \chi^2\Delta_f)(\chi^2 t|\nabla u|^2) &= t(\partial_t - \chi^2\Delta_f)(\chi^2|\nabla u|^2) + \chi^2|\nabla u|^2 \\ &\leq 2\chi^2|\nabla u|^2(C_2 t + \frac{1}{2}), \end{aligned} \tag{14}$$

where $C_2 := (1 + (7+8m) + K(m-1))C_1$. Using the idea of (6) in (14) we deduce

$$(\partial_t - \chi^2\Delta_f)\left(\chi^2 t|\nabla u|^2 + (C_2 T + \frac{1}{2})u^2\right) \leq 0. \tag{15}$$

Since $\chi^2 t|\nabla u|^2 = 0$ on $(\Omega \times \{0\}) \cup (\partial\Omega \times [0, T])$ and $u = u_0$ on $\partial\Omega \times [0, T]$ so by maximum principle we find that

$$\chi^2 t|\nabla u|^2 \leq \chi^2 t|\nabla u|^2 + (C_2 T + \frac{1}{2})u^2 \leq (C_2 T + \frac{1}{2})\max_{\Omega} u_0. \tag{16}$$

Set $B = C_2 T + \frac{1}{2}$ we get (5). This completes the proof. □

2.2. Bernstein type estimate on evolving manifold

In this subsection we derive Bernstein type estimate on weighted Riemannian manifold $(M^n, g(t), e^{-f}d\mu)$, where $g(t)$ is an one parameter family of Riemannian metrics evolving along an abstract geometric flow

$$\frac{\partial}{\partial t}g_{ij}(t) = 2h_{ij}(t), \tag{17}$$

where $h_{ij} = \mathcal{H}(e_i, e_j)(t)$ for some orthonormal frame $\{e_i : i = 1, 2, \cdots, n\}$ on M and $h_{ij}(t) \geq 0, \forall t \in (0, T)$ where T is the maximum time of existence of the flow (17). We assume that such T exist for (17). Observe that we have taken the flow in backward sense, one can consider $\frac{\partial}{\partial t}g_{ij} = -2h_{ij}$ and derive similar results.

Lemma 2.2. *If u is a smooth solution of* (3) *along the flow* (17) *then the quantity $\chi^2|\nabla u|^2$ satisfies*

$$\begin{aligned}(\partial_t - \chi^2\Delta_f)(\chi^2|\nabla u|^2) \quad = \quad & -2\chi^2 h(\nabla u, \nabla u) - 2\chi^4|Hess\ u|^2 - 2\chi^4 Ric_f(\nabla u, \nabla u) + 4\chi^3\langle\nabla u, \nabla\chi\rangle\Delta_f u \\ & -2\chi^2|\nabla u|^2(\chi\Delta_f\chi + |\nabla\chi|^2) - 8\chi^3\langle\nabla u\nabla\chi, Hess\ u\rangle.\end{aligned} \tag{18}$$

Proof. Using weighted Bochner formula (Lemma 1.3) we find that

$$\begin{aligned}\partial_t|\nabla u|^2 &= \partial_t(g^{ij}\nabla_i u\nabla_j u) \\ &= -2h(\nabla u, \nabla u) + \langle\nabla u, \nabla u_t\rangle \\ &= -2h(\nabla u, \nabla u) + \langle\nabla u, \nabla(\chi^2\Delta_f u)\rangle \\ &= -2h(\nabla u, \nabla u) - 2\chi^2|\text{Hess}\ u|^2 - 2\chi^2 Ric_f(\nabla u, \nabla u) + 4\chi\langle\nabla u, \nabla\chi\rangle\Delta_f u + \chi^2\Delta_f|\nabla u|^2.\end{aligned}$$

Consequently we get

$$\partial_t(\chi^2|\nabla u|^2) \quad = \quad -2\chi^2 h(\nabla u, \nabla u) - 2\chi^4|\text{Hess}\ u|^2 - 2\chi^4 Ric_f(\nabla u, \nabla u) + 4\chi^3\langle\nabla u, \nabla\chi\rangle\Delta_f u + \chi^4\Delta_f|\nabla u|^2. \tag{19}$$

From (8) we already have

$$\Delta_f(\chi^2|\nabla u|^2) \quad = \quad \chi^2\Delta_f|\nabla u|^2 + 2|\nabla u|^2(\chi\Delta_f\chi + |\nabla\chi|^2) + 8\chi\langle\nabla\chi\ \nabla u, \text{Hess}\ u\rangle. \tag{20}$$

Combining (19) and (20) gives (18). □

For the next theorem we state that by means of 'independent of Ricci curvature restriction' means there is no assumption on bounds for Ric_f, Ric_f^{m-n} or Ric.

Theorem 2.3 (Bernstein type estimate along local Ricci flow). *If u is a solution to the local heat equation* (3) *along the local Ricci flow*

$$\frac{\partial}{\partial t}g_{ij} = -2\chi^2 R_{ij}, \tag{21}$$

with $|\nabla f| \leq K_1$ and Hess $f \geq -K_2 g$, for some $K_1, K_2 \geq 0$, then we can find a constant, independent of Ricci curvature restriction, $\tilde{B} < \infty$ such that

$$|\nabla u|^2 \leq \frac{\tilde{B}}{\chi^2 t}, \tag{22}$$

where $\tilde{B} < \infty$ is a constant depends only on m, T, n, $\max\limits_{\Omega} u_0$, $\max\limits_{\Omega} \chi^2$, $\max\limits_{\Omega}(-\chi\Delta_f\chi)$ and $\max\limits_{\Omega}|\nabla\chi|^2$.

Proof. For local Ricci flow set $h_{ij} = -\chi^2 R_{ij}$ and apply Lemma 2.2, we get

$$\begin{aligned}(\partial_t - \chi^2\Delta_f)(\chi^2|\nabla u|^2) \quad = \quad & -2\chi^4\text{Hess}\ f(\nabla u, \nabla u) - 2\chi^4|\text{Hess}\ u|^2 + 4\chi^3\langle\nabla u, \nabla\chi\rangle\Delta_f u \\ & -2\chi^2|\nabla u|^2(\chi\Delta_f\chi + |\nabla\chi|^2) - 8\chi^3\langle\nabla u\nabla\chi, \text{Hess}\ u\rangle.\end{aligned} \tag{23}$$

By Cauchy-Schwarz and Young's inequality we find that

$$
\begin{aligned}
4\chi^3\langle\nabla u,\nabla\chi\rangle\Delta_f u &\leq 4m\chi^2|\nabla u|^2|\nabla\chi|^2+\frac{\chi^4}{m}(\Delta_f u)^2\\
&\leq 4m\chi^2|\nabla u|^2|\nabla\chi|^2+\chi^4|\text{Hess } u|^2+\frac{\chi^4}{m-n}\langle\nabla f,\nabla u\rangle^2,
\end{aligned}
\tag{24}
$$

where we have used $|\text{Hess } u|^2 \geq \frac{1}{m}(\Delta_f u)^2 - \frac{1}{m-n}\langle\nabla f,\nabla u\rangle^2$. Finally using (12) in (24), we infer

$$
\begin{aligned}
(\partial_t-\chi^2\Delta_f u)(\chi^2|\nabla u|^2) \leq\ & -2\chi^4\text{Hess } f(\nabla u,\nabla u)+4m\chi^2|\nabla u|^2|\nabla\chi|^2\\
&+\frac{\chi^4}{m-n}\langle\nabla f,\nabla u\rangle^2-2\chi^2|\nabla u|^2(\chi\Delta_f\chi+|\nabla\chi|^2)+16|\nabla u|^2|\nabla\chi|^2\chi^2.
\end{aligned}
\tag{25}
$$

Assuming $|\nabla f| \leq K_1$ and Hess $f \geq -K_2 g$, the above inequality reduces to

$$
(\partial_t-\chi^2\Delta_f u)(\chi^2|\nabla u|^2) \leq 2\chi^2|\nabla u|^2\left(-\chi\Delta_f\chi-|\nabla\chi|^2+2m|\nabla\chi|^2+\chi^2K_2+\frac{\chi^2}{2(m-n)}K_1+8|\nabla\chi|^2\right). \tag{26}
$$

Further suppose that the functions $-\chi\Delta_f\chi$, $|\nabla\chi|^2$, χ^2 are all bounded above by a constant $C_1<\infty$ then the above equation further reduces to

$$
(\partial_t-\chi^2\Delta_f)(\chi^2|\nabla u|^2) \leq 2\chi^2|\nabla u|^2B_1, \tag{27}
$$

where $B_1 := (2m-6+K_2+\frac{K_1}{2(m-n)})C_1$ is a constant depending only on C_1, m, n, K_1, K_2. Thus we obtain

$$
(\partial_t-\chi^2\Delta_f)(\chi^2 t|\nabla u|^2) \leq 2\chi^2|\nabla u|^2(B_1 t+\frac{1}{2}). \tag{28}
$$

Using the idea of (6) we deduce

$$
(\partial_t-\chi^2\Delta_f)(\chi^2 t|\nabla u|^2+(B_1 t+\frac{1}{2})u^2) \leq 0. \tag{29}
$$

Since $\chi^2 t|\nabla u|^2 = 0$ on $(\Omega\times\{0\})\cap(\partial\Omega\times[0,T])$ and $u=u_0$ on $\partial\Omega\times[0,T]$ so by maximum principle we infer

$$
\chi^2 t|\nabla u|^2 \leq (B_1 T+\frac{1}{2})\max_{\Omega} u_0. \tag{30}
$$

Set $\tilde{B} = (B_1+\frac{1}{2})\max_\Omega u_0$. This completes the proof. □

Theorem 2.4 (Bernstein type estimate along extended local Ricci flow). *For any $\alpha>0$, a real number, if u is a solution to the local heat equation* (3) *along the extended local Ricci flow*

$$
\frac{\partial}{\partial t}g = -2\chi^2 Ric+2\alpha\nabla\phi\otimes\nabla\phi, \tag{31}
$$

with Hess $f \geq -K_1 g$, $|\nabla f|^2 \leq K_2$ and $|\nabla\phi|^2 \leq K_3$, for some $K_1, K_2, K_3 \geq 0$, then we can find a constant depends on α, K_1, K_2, K_3, m, T, n, $\max\limits_{\Omega} u_0$, $\max\limits_{\Omega}\chi^2$, $\max\limits_{\Omega}(-\chi\Delta_f\chi)$, $\max\limits_{\Omega}\chi^2|\nabla\chi|^2$ and $\max\limits_{\Omega}|\nabla\chi|^2$, and independent of Ricci curvature restriction, $B'<\infty$ such that

$$
|\nabla u|^2 \leq \frac{B'}{\chi^2 t}. \tag{32}
$$

Proof. For $h = -Ric + \alpha\nabla\phi \otimes \nabla\phi$, we have from Lemma 2.2

$$\begin{aligned}(\partial_t - \chi^2\Delta_f)(\chi^2|\nabla u|^2) &= 2\chi^4 Ric(\nabla u, \nabla u) - 2\chi^4\alpha\langle\nabla\phi, \nabla u\rangle^2 - 2\chi^4|\text{Hess } u|^2 - 2\chi^4 Ric_f(\nabla u, \nabla u) \\ &\quad +4\chi^3\langle\nabla u, \nabla\chi\rangle\Delta_f u - 2\chi^2|\nabla u|^2(\chi^2\Delta_f\chi + |\nabla\chi|^2) - 8\chi^3\langle\nabla u\nabla\chi, \text{Hess } u\rangle. \end{aligned} \tag{33}$$

In view of $Ric_f(\nabla u, \nabla u) = Ric(\nabla u, \nabla u) + \text{Hess } f(\nabla u, \nabla u)$ the above relation further reduces to

$$\begin{aligned}(\partial_t - \chi^2\Delta_f)(\chi^2|\nabla u|^2) &= -2\chi^4\alpha\langle\nabla\phi, \nabla u\rangle^2 - 2\chi^4|\text{Hess } u|^2 + 4\chi^3\langle\nabla u, \nabla\chi\rangle\Delta_f u - 2\chi^4\text{Hess } f(\nabla u, \nabla u) \\ &\quad -2\chi^2|\nabla u|^2(\chi^2\Delta_f\chi + |\nabla\chi|^2) - 8\chi^3\langle\nabla u\nabla\chi, \text{Hess } u\rangle. \end{aligned} \tag{34}$$

By Cauchy-Schwarz and Young's inequality we find that

$$-8\chi^3\langle\nabla u\nabla\chi, \text{Hess } u\rangle \leq \chi^2|\text{Hess } u|^2 + 16\chi^2|\nabla u|^2|\nabla\chi|^2, \tag{35}$$

$$4\chi^3\langle\nabla u, \nabla\chi\rangle\Delta_f u \leq 4m\chi^2|\nabla u|^2|\nabla\chi|^2 + \chi^4|\text{Hess } u|^2 + \frac{\chi^4}{m-n}\langle\nabla f, \nabla u\rangle^2. \tag{36}$$

Using the above two equation and applying the bounds for Hess f, $|\nabla\phi|^2$ and $|\nabla f|^2$ in (33), we infer

$$(\partial_t - \chi^2\Delta_f)(\chi^2|\nabla u|^2) \leq 2\chi^2|\nabla u|^2\left(-\chi\Delta_f\chi - |\nabla\chi|^2 + \alpha\chi^2 K_3 + \chi^2 K_1 + 2m|\nabla\chi|^2 + \frac{\chi^2 K_2}{2(m-n)} + 8\chi^2|\nabla\chi|^2\right). \tag{37}$$

Finally assuming that the functions $-\chi\Delta_f\chi$, $|\nabla\chi|^2$, χ^2 and $\chi^2|\nabla\chi|^2$ are all bounded above by a constant $C < \infty$ then the above equation further reduces to

$$(\partial_t - \chi^2\Delta_f)(\chi^2|\nabla u|^2) \leq 2\chi^2|\nabla u|^2 B_3,$$

where $B_3 = C(10 + \alpha K_3 + K_1 + 2m + \frac{K_3}{2(m-n)})$, is a constant independent of curvature restriction. Thus we obtain

$$(\partial_t - \chi^2\Delta_f)(\chi^2 t|\nabla u|^2) \leq 2\chi^2|\nabla u|^2(B_3 t + \frac{1}{2}).$$

Using the idea of (6) we see that

$$(\partial_t - \chi^2\Delta_f)(\chi^2 t|\nabla u|^2 + (B_3 t + \frac{1}{2})u^2) \leq 0.$$

Since $\chi^2 t|\nabla u|^2 = 0$ on $(\Omega \times \{0\}) \cap (\partial\Omega \times [0, T])$ and $u = u_0$ on $\partial\Omega \times [0, T]$ so by maximum principle we infer

$$\chi^2 t|\nabla u|^2 \leq (B_3 T + \frac{1}{2})\max_{\Omega} u_0. \tag{38}$$

Take $B' = (B_3 + \frac{1}{2})\max_\Omega u_0$, we see B' satisfies the criteria. This completes the proof. □

2.3. Bernstein type estimate for local weighted heat equation with exponential potential on static manifold

In this subsection we consider a local weighted heat equation with exponential potential on static weighted Riemannian manifold, given by

$$(\partial_t - \chi^2\Delta_f)u = ae^u, \ a > 0 \text{ real number}, \tag{39}$$

and we wish to find the Bernstein type gradient estimate for a bounded solution u of (39).

Theorem 2.5 (Bernstein type estimate with exponential potential). *If u is a bounded solution, with $u \leq \ln b_1$ for some $b_1 > 1$, to the heat equation* (39) *on $(M^n, g, e^{-f}d\mu)$ with $Ric_f^{m-n} \geq -K(m-1)g$ then*

$$|\nabla u|^2 \leq \frac{C_3(b_1)}{\chi^2 t}, \text{ on } \Omega \times (0, T], \tag{40}$$

where $C_3(b_1) < \infty$ is a constant depends only on K, m, T, $\max_\Omega u_0$, $\max_\Omega \chi^2$, $\max_\Omega(-\chi\Delta_f\chi)$, $\max_\Omega |\nabla\chi|^2$ and b_1.

Proof. Similar to the proof of Theorem 2.1 we have from weighted Bochner formula (Lemma 1.3)

$$\partial_t|\nabla u|^2 = \chi^2\Delta_f|\nabla u|^2 - 2\chi^2|\text{Hess } u|^2 - 2\chi^2 Ric_f(\nabla u, \nabla u) + 4\chi\langle\nabla u, \nabla\chi\rangle\Delta_f u + 2ae^u|\nabla u|^2. \tag{41}$$

Consequently

$$\partial_t(\chi^2|\nabla u|^2) = \chi^4\left(\Delta_f|\nabla u|^2 - 2|\text{Hess } u|^2 - 2Ric_f(\nabla u, \nabla u)\right) + 4\chi^3\langle\nabla u, \nabla\chi\rangle\Delta_f u + 2\chi^2 ae^u|\nabla u|^2. \tag{42}$$

Again in similar way we get

$$\Delta_f(\chi^2|\nabla u|^2) = \chi^2\Delta_f|\nabla u|^2 + 2|\nabla u|^2(\chi\Delta_f\chi + |\nabla\chi|^2) + 8\chi\langle\nabla\chi\ \nabla u, \text{Hess } u\rangle. \tag{43}$$

Subtracting (43) from (42) and simplifying we get

$$\begin{aligned}(\partial_t - \chi^2\Delta_f)(\chi^2|\nabla u|^2) = {} & -2\chi^4|\text{Hess } u|^2 - 2\chi^4 Ric_f(\nabla u, \nabla u) + 4\chi^3\langle\nabla u, \nabla\chi\rangle - 2\chi^2|\nabla u|^2(\chi\Delta_f\chi + |\nabla\chi|^2)\\ & -8\chi^2\langle\nabla\chi\nabla u, \text{Hess } u\rangle + 2a\chi^2e^u|\nabla u|^2.\end{aligned} \tag{44}$$

From (11) and (12) we already have the following relations

$$4\chi^3\langle\nabla u, \nabla\chi\rangle\Delta_f u \le 16m\chi^2|\nabla u|^2|\nabla\chi|^2 + \chi^4|\text{Hess } u|^2 + \frac{2\chi^4}{m-n}\langle\nabla f, \nabla u\rangle^2, \tag{45}$$

and

$$-8\chi^2\langle\nabla\chi\nabla u, \text{Hess } u\rangle \le 16\chi^2|\nabla\chi|^2|\nabla u|^2 + \chi^4|\text{Hess } u|^2. \tag{46}$$

Using the above relations together with $u \le \ln b_1$ in (44), we find that

$$(\partial_t - \chi^2\Delta_f)(\chi^2|\nabla u|^2) \le 2\chi^2|\nabla u|^2(-\chi\Delta_f\chi + (7+8m)|\nabla\chi|^2 + 2ab_1) - 2\chi^4 Ric_f^{m-n}(\nabla u, \nabla u). \tag{47}$$

Applying $Ric_f^{m-n} \ge -K(m-1)g$ in the above equation and assuming that the functions $-\chi\Delta_f\chi$, $|\nabla\chi|^2$ and χ^2 are all bounded above by a constant $C_1 < \infty$, we deduce

$$\begin{aligned}(\partial_t - \chi^2\Delta_f)(\chi^2 t|\nabla u|^2) &= t(\partial_t - \chi^2\Delta_f)(\chi^2|\nabla u|^2) + \chi^2|\nabla u|^2\\ &\le 2\chi^2|\nabla u|^2(C_3(b_1)t + \frac{1}{2}),\end{aligned} \tag{48}$$

where $C_3(b_1) := (1 + (7+8m) + K(m-1))C_1 + 2ab_1$. With the same idea as in (6) we see that the above equation reduces to

$$(\partial_t - \chi^2\Delta_f)\left(\chi^2 t|\nabla u|^2 + (C_3(b_1)T + \frac{1}{2})u^2\right) \le 0. \tag{49}$$

Since $\chi^2 t|\nabla u|^2 = 0$ on $(\Omega \times \{0\}) \cup (\partial\Omega \times [0, T])$ and $u = u_0$ on $\partial\Omega \times [0, T]$ so by maximum principle we find that

$$\chi^2 t|\nabla u|^2 \le \chi^2 t|\nabla u|^2 + (C_3(b_1)T + \frac{1}{2})u^2 \le (C_3(b_1)T + \frac{1}{2})\max_{\Omega} u_0. \tag{50}$$

Redefining the constant $C_3(b_1) := C_3(b_1)T + \frac{1}{2}$ we get (40). This completes the proof. □

To encourage the reader we mention that Theorem 2.5 can also be deduced on evolving manifold but the results will be similar to Theorem 2.3 and Theorem 2.4 for local Ricci flow and extended local Ricci flow. Thus we skip those results and leave it to the reader to explore further.

3. Concluding remark

In this section we summarize the paper briefly and provide some future aspects of this estimation. The gradient estimation of partial differential equations over Riemannian manifolds is an active field of research in modern times, and local heat equations offer interesting discoveries as well. In this article we derived Bernstein type gradient estimation for local heat equation on different cases. For example, on static weighted Riemannian manifold, evolving weighted Riemannian manifold along local Ricci flow and extended local Ricci flow and local heat equation with non-linear exponential potential. We showed that the gradient estimation derived in Theorem 2.3 and Theorem 2.4 are independent of Bakry-Émery Ricci curvature restriction.

3.1. *Future aspect*

One can use this method or an improved one to find gradient estimation for system of parabolic equations of the form

$$\begin{cases} (\partial_t - \chi^2\Delta_f)u = ae^v \\ (\partial_t - \chi^2\Delta_f)v = be^u, \end{cases} \tag{51}$$

where a, b are positive real constants. Further, gradient estimation on local heat equations involving p-Laplacian, weighted p-Laplacian will be a good contribution to this field.

Data availibility: The authors confirm that the data supporting the findings of this study are available within the article.

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