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Semi-symmetric metric connections and homology of warped product pointwise semi-slant submanifolds in a Kaehler manifold admitting a concurrent vector field

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Abstract. In this paper, we delve into the study of pointwise semi-slant submanifolds in a Kaehler manifold using a semi-symmetric metric connection within the framework of warped product geometry. Our investigation yields fundamental and significant results that shed light on the properties of these submanifolds. Furthermore, we explore the implications of our findings for the homology of these submanifolds, providing insights into their topological characteristics. Specifically, we establish a compelling proof that, subject to a specific condition, stable currents do not exist for these warped product pointwise semi-slant submanifolds. The outcomes of our research contribute substantial knowledge regarding the stability and behavior of warped product pointwise semi-slant submanifolds equipped with a semi-symmetric metric connection. Moreover, this work establishes a solid foundation for future investigations and advancements in this field of study.

1. Introduction

The geometry of warped product manifolds has long been recognized as a remarkable framework for modeling spacetime near black holes and objects with significant gravitational fields. The concept of warped product manifolds was initially introduced by Bishop and O'Neill [25] as a means to investigate manifolds with negative curvature. These manifolds extend the notion of Riemannian product manifolds by incorporating warping functions. Specifically, a warped product $B \times_b F$ is formed by combining two pseudo-Riemannian manifolds, namely the base manifold (B, g_B) and the fiber (F, g_F), using a smooth function b defined on the base manifold B. This construction results in the metric tensor $g = g_B \oplus b^2 g_F$, where the direct sum symbol $\oplus$ denotes the direct sum of metric tensors. In this context, the base manifold (B, g_B) represents the underlying space on which the warped product is defined, while the fiber (F, g_F) represents an additional space that is warped or scaled by the warping function b. The warping function b is a smooth function that assigns a positive value to each point in the base manifold B. Warped product manifolds with

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a conformal Killing vector have been extensively studied in the framework of Einstein-Weyl geometry. In this setting, the warping function b assumes the role of a conformal factor, influencing the geometry of the manifold. The geometry of such manifolds is determined by a conformal class of metrics, which captures the essential geometric properties shared by the metrics related through conformal transformations. For further insights and details on this subject, refer to the works of Leistner and Nurowski [11, 23].

B.-Y. Chen [4] made significant contributions to the field of submanifold theory by studying warped products. Within the framework of almost Hermitian manifolds, Chen introduced the concept of CR-warped product submanifolds. He provided valuable insights into the warping function and derived an approximation for the norm of the second fundamental form within the expressions of the warping function.

Expanding on Chen's work, Hesigawa and Mihai [19] further explored these submanifolds in the context of contact geometry. They investigated the contact form associated with CR-warped product submanifolds and obtained a comparable approximation for the second fundamental form of a contact CR-warped product submanifold immersed in a Kaehler space form.

In a separate study [6], it was concluded that the homology groups of contact CR-warped product submanifolds immersed in odd-dimensional spheres were trivial. This conclusion was based on the non-existence of stable integral currents and the vanishing of homology, indicating the absence of stable currents in such submanifolds.

Advancing the research, F. Sahin [9, 10] demonstrated that CR-warped product submanifolds in both R^n and S^6 produced identical results. This observation highlighted the similarities in the topological and differentiable structures of CR-warped product submanifolds in these two spaces. It is important to note that different scholars have obtained varying findings regarding the topological and differentiable properties of submanifolds by imposing specific constraints on the second fundamental form [1, 6, 14, 20, 33].

Homology groups provide an algebraic description of manifolds and are fundamental in understanding their topological properties. These groups contain rich topological data related to the components, voids, tunnels, and overall structure of manifolds, making homology theory a powerful tool with numerous applications. It has found relevance in diverse fields such as root construction, molecular docking, image segmentation, and genetic expression analysis. The study of submanifolds and homological theory is closely intertwined. Federer and Fleming [17] established a significant connection by demonstrating that any non-trivial integral homological group $H_p(M, \mathbb{Z})$ is connected to stable currents. This result highlighted the relationship between homology groups and the existence of stable currents in manifolds. Expanding upon this work, Lawson and Simon [18] extended the study to submanifolds of spheres and proved that under a pinching condition on the second fundamental form, integral currents do not exist. This result provided insights into the non-existence of integral currents in specific submanifold scenarios. Leung [24] and Xin [33] furthered this line of research by extending the results from spheres to Euclidean spaces. Their studies explored the relationship between submanifolds in Euclidean spaces and the existence of stable integral currents. In a related investigation, Zhang [32] examined the homology of tori, expanding the understanding of homological properties in this specific context. Additionally, Liu and Zhang [20] made an important contribution by proving that stable integral currents do not exist for certain types of hypersurfaces in Euclidean spaces. This result shed light on the limitations and constraints related to the existence of stable integral currents in specific scenarios.

The concept of a semi-symmetric linear connection on a Riemannian manifold was initially introduced by Friedmann and Schouten (17). Subsequently, Hayden (18) provided a definition for a semi-symmetric connection as a linear connection ∇ existing on an n-dimensional Riemannian manifold (M, g), with a torsion tensor T satisfying $T(Z_1, Z_2) = \pi(Z_2)Z_1 - \pi(Z_1)Z_2$, where π is a 1-form, and $Z_1, Z_2 \in TM$.

Further investigations into semi-symmetric metric connections were conducted by K. Yano (21), who analyzed some of their properties. He showed that a conformally flat Riemannian manifold equipped with a semi-symmetric connection has a curvature tensor that identically vanishes.

Building upon these works, Sular and Ożgür (16) explored warped product manifolds with a semi-symmetric metric connection, focusing specifically on Einstein warped product manifolds with such a connection. They investigated various aspects and properties of these manifolds. Additionally, in (24), they obtained further results related to warped product manifolds with a semi-symmetric metric connection.

Motivated by these previous studies, the present research aims to examine the influence of a semi-symmetric metric connection on warped product pointwise semi-slant submanifolds and their homology within a Kaehler manifold. The objective is to understand how the presence of a semi-symmetric metric connection affects the properties and topological characteristics of these submanifolds in the context of warped product constructions.

2. Preliminaries

Let $(\bar{M}, g)$ denote an even-dimensional Riemannian manifold. An almost Hermitian manifold is defined as a manifold $\bar{M}$ where there exists a tensor field J of type $(1,1)$ on $\bar{M}$ such that the following conditions hold:

$$J^2 Z_1 = -Z_1$$

$$g(JZ_1, JZ_2) = g(Z_1, Z_2),$$

for $Z_1, Z_2 \in T\bar{M}$. The well-known fact states that an almost Hermitian manifold is classified as a Kaehler manifold if and only if the following condition is satisfied:

$$(\bar{\bar{\nabla}}_{Z_1} J)Z_2 = 0, \tag{1}$$

where $Z_1, Z_2 \in T\bar{M}$ and $\bar{\bar{\nabla}}$ is the Riemannian connection with respect to g.
Now, defining a connection $\bar{\nabla}$ as

$$\bar{\nabla}_{Z_1} Z_2 = \bar{\bar{\nabla}}_{Z_1} Z_2 + \pi(Z_2)Z_1 - g(Z_1, Z_2)P \tag{2}$$

such that $\bar{\nabla} g = 0$ for any $Z_1, Z_2 \in T\bar{M}$. The connection $\bar{\nabla}$ is semi-symmetric because $T(Z_1, Z_2) = \pi(Z_2)Z_1 - \pi(Z_1)Z_2$. Using (2) in (1), we have

$$(\bar{\nabla}_{Z_1} J)Z_2 = \pi(JZ_2)Z_1 - g(Z_1, JZ_2)P + \pi(Z_2)JZ_1 - g(Z_1, Z_2)JP. \tag{3}$$

Suppose that the associated vector field P is concurrent [21], that mean

$$\bar{\nabla}_{Z_1} P = Z_1. \tag{4}$$

We define a Kaehler manifold $\bar{M}$ as a complex space form if it possesses a constant J-holomorphic sectional curvature denoted by c, and it is represented as $\bar{M}(c)$.

The curvature tensor $\bar{R}$ associated with the semi-symmetric metric connection $\bar{\nabla}$ is given by:

$$\bar{R}(Z_1, Z_2)Z_3 = \bar{\nabla}_{Z_1} \bar{\nabla}_{Z_2} Z_3 - \bar{\nabla}_{Z_2} \bar{\nabla}_{Z_1} Z_3 - \bar{\nabla}_{[Z_1, Z_2]} Z_3. \tag{5}$$

Similarly, we can define the curvature tensor $\bar{\bar{R}}$ for the Riemannian connection $\bar{\bar{\nabla}}$ as follows:

Suppose

$$\beta(Z_1, Z_2) = (\bar{\nabla}_{Z_1} \pi)Z_2 - \pi(Z_1)\pi(Z_2) + \frac{1}{2} g(Z_1, Z_2)\pi(P). \tag{6}$$

Now, by the application of (2), (5) and (6), we get

$$\begin{aligned} \bar{R}(Z_1, Z_2, Z_3, Z_4) = {} & \bar{\bar{R}}(Z_1, Z_2, Z_3, Z_4) + \beta(Z_1, Z_3)g(Z_2, Z_4) \\ & - \beta(Z_2, Z_3)g(Z_1, Z_4) + \beta(Z_2, Z_4)g(Z_1, Z_3) - \beta(Z_1, Z_4)g(Z_2, Z_3). \end{aligned} \tag{7}$$

When utilizing the value of $\bar{\bar{R}}(Z_1, Z_2, Z_3, Z_4)$, The expression for the curvature tensor $\bar{R}$ of a Kaehler space form $\bar{M}(c)$ equipped with a semi-symmetric metric connection is provided in detail in [21]. This result is further discussed in [34].

$$\begin{aligned}\bar{R}(Z_1, Z_2, Z_3, Z_4) =&\frac{c}{4}\{g(Z_2, Z_3)Z_1 - g(Z_1, Z_3)g(Z_2, Z_4)\\&+ g(Z_1, JZ_3)g(JZ_2, Z_4) - g(Z_2, JZ_3)g(JZ_1, Z_4)\\&+ 2g(Z_1, JZ_2)g(JZ_3, Z_4)\} + \beta(Z_1, Z_3)g(Z_2, Z_4)\\&- \beta(Z_2, Z_3)g(Z_1, Z_4) + \beta(Z_2, Z_4)g(Z_1, Z_3)\\&- \beta(Z_1, Z_4)g(Z_2, Z_3)\end{aligned} \tag{8}$$

for all $Z_1, Z_2, Z_3, Z_4 \in T\bar{M}$.

In the case of a submanifold M isometrically immersed in a differentiable manifold $\bar{M}$, the Gauss and Weingarten formulas for a semi-symmetric metric connection can be derived through a routine calculation. These formulas are given by $\bar{\nabla}_{Z_1}Z_2 = \nabla_{Z_1}Z_2 + h(Z_1, Z_2)$ and $\bar{\nabla}_{Z_1}N = -A_N Z_1 + \nabla^{\perp}_{Z_1}N + \pi(N)Z_1$, where ∇ represents the induced semi-symmetric metric connection on M, N belongs to the normal bundle $T^{\perp}M$, h denotes the second fundamental form of M, $\nabla^{\perp}$ represents the normal connection on $T^{\perp}M$, and A_N is the shape operator. The relationship between the second fundamental form h and the shape operator is given by the following formula:

$$g(h(Z_1, Z_2), N) = g(A_N Z_1, Z_2).$$

For vector fields $Z_1 \in TM$ and $Z_3 \in T^{\perp}M$, we can decompose their relationship as follows:

$$JZ_1 = TZ_1 + FZ_1 \tag{9}$$

and

$$JZ_3 = tZ_3 + fZ_3 \tag{10}$$

where TZ_1 (and tZ_3), FZ_1 (and fZ_3) are the tangential and normal parts of JZ_1 (and JZ_3) respectively.

The equation of Gauss for a semi-symmetric connection can be expressed in terms of the Riemannian curvature tensor R as follows:

$$\bar{R}(Z_1, Z_2, Z_3, Z_4) = R(Z_1, Z_2, Z_3, Z_4) - g(h(Z_1, Z_4), h(Z_2, Z_3)) + g(h(Z_2, Z_4), h(Z_1, Z_3)) \tag{11}$$

for $Z_1, Z_2, Z_3, Z_4 \in TM$.

Sular and Ožgür (16) conducted a study on warped products denoted as $M_1 \times_f M_2$, where M_1 and M_2 are Riemannian manifolds, and f is a positive differentiable function on M_1 referred to as the warping function. Their investigation focused on these warped products in the context of a semi-symmetric metric connection associated with a vector field P on $M_1 \times_f M_2$. The key findings from their work, summarized as a lemma, provide a crucial foundation for subsequent research.

Lemma 2.1. *Let $M_1 \times_f M_2$ be a warped product manifold with semi-symmetric metric connection $\bar{\nabla}$, we have*

1. *if $P \in TM_1$, then*

$$\bar{\nabla}_{Z_1}Z_3 = \frac{Z_1 f}{f}Z_3 \quad \text{and} \quad \bar{\nabla}_{Z_3}Z_1 = \frac{Z_1 f}{f}Z_3 + \pi(Z_1)Z_3,$$

2. *if $P \in TM_2$, then*

$$\bar{\nabla}_{Z_1}Z_3 = \frac{Z_1 f}{f}Z_3 \quad \text{and} \quad \nabla_{Z_3}Z_1 = \frac{Z_1 f}{f}Z_3,$$

where $Z_1 \in TM_1,\ Z_3 \in TM_2$ and π is the 1-form associated with the vector field P.

Consider the warped product submanifold $M = M_1 \times_f M_2$ of a Kaehler manifold $\bar{M}$. In this scenario, we have the curvature tensors R and $\tilde{R}$ associated with the submanifold M and its induced semi-symmetric metric connection ∇ and induced Riemannian connection $\tilde{\nabla}$ respectively. We can express the relationship between these curvature tensors as follows:

$$\begin{aligned} R(Z_1, Z_2)Z_3 =& \tilde{R}(Z_1, Z_2)Z_3 + g(Z_3, \nabla_{Z_1}P)Z_2 - g(Z_3, \nabla_{Z_2}P)Z_1 \\ &+ g(Z_1, Z_3)\nabla_{Z_2}P - g(Z_2, Z_3)\nabla_{Z_1}P \\ &+ \pi(P)[g(Z_1, Z_3)Z_2 - g(Z_2, Z_3)Z_1] \\ &+ [g(Z_2, Z_3)\pi(Z_1) - g(Z_1, Z_3)\pi(Z_2)]P \\ &+ \pi(Z_3)[\pi(Z_2)Z_1 - \pi(Z_1)Z_2], \end{aligned} \tag{12}$$

for any vector field Z_1, Z_2, Z_3 on M [27].

According to part (ii) of Lemma 3.2 in [27], for the warped product submanifold $M = M_1 \times_f M_2$, we obtain the following result

$$\tilde{R}(Z_1, Z_2)Z_3 = \frac{H^f(Z_1, Z_2)}{f} Z_3, \tag{13}$$

where $Z_1, Z_2 \in TM_1$, $Z_3 \in TM_2$ respectively and H^f is the Hessian of the warping function.

By considering equations (12) and (13), we can deduce the following:

$$\begin{aligned} R(Z_1, Z_3)Z_2 = \frac{H^f(Z_1, Z_2)}{f} + \frac{Pf}{f} g(Z_1, Z_2)Z_3 + \pi(P)g(Z_1, Z_2)Z_3 + g(Z_2, \nabla_{Z_1}P)Z_3 \\ - \pi(Z_1)\pi(Z_2)Z_3, \end{aligned} \tag{14}$$

where $Z_1, Z_2 \in TM_1, Z_3 \in TM_2, P \in TM_1$, and H^f is the Hessian of the warping function f.

Utilizing part (1) of Lemma 2.1 and referring to equation (4), we can deduce that $\frac{Pf}{f} = 0$. By substituting this result into the previous equation, we obtain the following expression:

$$R(Z_1, Z_3)Z_2 = \frac{H^f(Z_1, Z_2)}{f} Z_3 + 2g(Z_1, Z_2)Z_3 - \pi(Z_1)\pi(Z_2)Z_3. \tag{15}$$

For the warped product submanifold $M = M_1 \times_f M_2$ of a Riemanian manifold $\bar{M}$, by employing part (i) of Lemma 2.1, we are able to deduce the following relation:

$$\nabla_{Z_1} Z_3 = Z_1(\ln f)Z_3 \tag{16}$$

and

$$\nabla_{Z_3} Z_1 = Z_1(\ln f)Z_3 + \pi(Z_1)Z_3, \tag{17}$$

where $Z_1, P \in TM_1$ and $Z_2 \in TM_2$.

It is straightforward to derive the following expression for the Laplacian Δf of the warping function

$$\frac{\Delta f}{f} = \Delta(\ln f) - \|\nabla(\ln f)\|^2 \tag{18}$$

3. Warped product pointwise Semi-slant submanifolds and their homology

The concept of pointwise slant submanifolds was introduced by F. Etayo in a paper where these submanifolds were referred to as quasi-slant submanifolds [12]. Subsequently, B.-Y. Chen and O. J. Garay [4] further studied the properties of pointwise slant submanifolds within the framework of almost Hermitian manifolds. Pointwise slant submanifolds are defined as submanifolds M of an almost Hermitian manifold $(\bar{M}, g)$ where, for every point $x \in M$, the angle $\mu = \mu(X)$ between JX and T_xM remains constant for all nonzero vectors $X \in T_xM$. This angle, denoted by $\mu(X)$, is known as the slant function on M. The necessary and sufficient condition for M to be pointwise slant is that the endomorphism T satisfies the following relation:

$$T^2Z = -\lambda Z \tag{19}$$

for any $Z \in TM$, where $\lambda = \cos^2\mu$. The following formulae can be deduced by using (19) and (9)

$$g(TZ_1, TZ_2) = \cos^2\mu g(Z_1, Z_2), \tag{20}$$

$$g(FZ_1, FZ_2) = \sin^2\mu g(Z_1, Z_2). \tag{21}$$

In a recent study, Şahin [8] focused on the investigation of warped product pointwise semi-slant submanifolds in the context of Kaehler manifolds. Notably, he established a significant result by proving the nonexistence of warped product pointwise semi-slant submanifolds of the form $M = N_\mu \times_f N_T$, where N_μ represents a proper pointwise slant submanifold and M_T denotes a complex submanifold. Building upon this finding, Şahin shifted his attention to warped products of the form $M = N_T \times_f N_\mu$. Through this analysis, he obtained a range of interesting results, which included characterizations and inequalities. Furthermore, he provided several illustrative examples of pointwise semi-slant submanifolds and their corresponding warped products.

Our analysis begins by considering a specific type of submanifold known as warped product pointwise semi-slant submanifolds. These submanifolds take the form of $N_\mu \times_f N_T$ in a Kaehler manifold equipped with a semi-symmetric metric connection and a concurrent vector field P. Here, N_μ denotes a pointwise slant submanifold, and N_T represents an invariant submanifold satisfying $\xi \in TN_T$. Our investigation results in the following finding

Theorem 3.1. *Let $(\bar{M}, g)$ be a Kaehler manifold with ssm connection. Then there does not exist wppss submanifold of the type $N_\mu \times_f N_T$, such that P is a concurrent vector field tangent to N_T.*

Proof. For any $Z_1, Z_2 \in TN_T$ and $Z_3 \in TN_\mu$, Using part (ii) of Lemma 2.1, the Gauss formula, and equations (2.7) and (2.4), we can derive the following expression:

$$\begin{aligned} Z_3(\ln f)g(Z_1, Z_2) = g(J\bar{\nabla}_{Z_1}Z_3, JZ_2) =& g(\bar{\nabla}_{Z_1}(TZ_3, JZ_2) + g(\bar{\nabla}_{Z_1}(FZ_3, JZ_2) \\ &- g(\pi(JZ_3)Z_1 - g(Z_1, JZ_3)P \\ &+ \pi(Z_3)JZ_1 - g(Z_1, Z_3)JP, JZ_2). \end{aligned} \tag{22}$$

Upon further simplification, we obtain

$$\begin{aligned} Z_3 g(Z_1, Z_2) &= -g(\bar{\nabla}_{Z_1}T^2Z_3, Z_2) - g(\bar{\nabla}_{Z_1}FTZ_3, Z_2) - g(h(Z_1, JZ_2), FZ_3) \\ &= \cos^2\mu g(\nabla_{Z_1}Z_3, Z_2) + g(A_{FTZ_3}Z_1, Z_2) - g(h(Z_1, JZ_2), FZ_3). \end{aligned} \tag{23}$$

Again using equation (17), we find

$$\sin^2\mu Z_3(\ln f)g(Z_1, Z_2) = g(h(Z_1, Z_2), FTZ_3) - g(h(Z_1, JZ_2), FZ_3). \tag{24}$$

By substituting P for Z_1 and Z_2, and utilizing equation (4), we arrive at the expression $\sin^2\mu Z_3 \ln f = 0$. This equation implies that f is a constant, thereby proving the theorem. □

In this study, our focus will be on wppss submanifolds of the form $N_T \times_f N_\mu$ that possess an ssm connection, where $P \in TN_T$ is a concurrent vector field. With this objective in mind, we will now introduce the initial results as follows:

Lemma 3.2. *Let $M = N_T \times_f N_\mu$ be a non-trivial wppss submanifold of a Kaehler manifold with a semi-symmetric metric connection and a concurrent vector fileld P then*

$$g(A_{FZ_3}Z_4, Z_1) = g(A_{FZ_4}Z_3, Z_1), \tag{25}$$

for $\xi, Z_1 \in TN_T$ and $Z_3, Z_4 \in TN_\mu$.

Proof. Making use of Weingarten formula along with (9), we have

$$g(A_{FZ_3}Z_4, Z_1) = -g(\bar{\nabla}_{Z_1}JZ_3, Z_4) + g(\bar{\nabla}_{Z_1}TZ_3, Z_4). \tag{26}$$

Now, using (3) and (16), we get the required result. □

Lemma 3.3. *Let $M = N_T \times_f N_\mu$ be a non-trivial warped product proper pointwise semi-slant submanifold of a Kaehler manifold admitting a semi-symmetric metric connection and a concurrent vector field P then*

$$g(h(Z_1, Z_4), FTZ_3) = -2\pi(JZ_1)g(TZ_3, Z_4) - JZ_1(\ln f)g(TZ_3, Z_4) - Z_1(\ln f)\cos^2\mu g(Z_3, Z_4) \tag{27}$$

for $Z_1 \in TN_T$ and $Z_3, Z_4 \in TN_\mu$.

Proof. By utilizing part (i) of Lemma 3.1 and the Weingarten equation, we obtain the following expression:

$$g(h(Z_1, Z_4)FTZ_3) = g(A_{FTZ_3}Z_4, Z_1) = g(A_{FZ_4}TZ_3, Z_1)$$

using (9), (3) and (17), we obtain

$$\begin{aligned} g(h(Z_1, Z_4), FTZ_3) &= -g(\bar{\nabla}_{TZ_3}(JZ_4 - TZ_4), Z_1) \\ &= -g((\bar{\nabla}_{TZ_3}J)Z_4, Z_1) + g(\bar{\nabla}_{TZ_Z}Z_4, JZ_1) - g(TZ_4, \nabla_{TZ_3}Z_1) \\ &= -g(\pi(Z_4)TZ_3 - g(TZ_3, TZ_4)P + \pi(Z_4)JTZ_3 \\ &- g(TZ_3, Z_4)JP, Z_1) - g(\nabla_{TZ_3}JZ_1, Z_4) - g(\nabla_{TZ_3}Z_1, TZ_4), \end{aligned} \tag{28}$$

using equation (17), we get

$$\begin{aligned} g(h(Z_1, Z_4), FTZ_3) &= \cos^2\mu\pi(Z_1)g(Z_3, Z_4) - g(TZ_3, Z_4)\pi(JZ_1) - g(JZ_1(\ln f)TZ_3 \\ &+ \pi(JZ_1)TZ_3, Z_4) - g(Z_1(\ln f)TZ_3 + \pi(Z_1)TZ_3, TZ_4) \\ &= -2\pi(JZ_1)g(TZ_3, Z_4) - JZ_1(\ln f)g(TZ_3, Z_4) \\ &- Z_1(\ln f)\cos^2\mu g(Z_3, Z_4), \end{aligned} \tag{29}$$

This result is the desired outcome. □

Lemma 3.4. *Let $M = N_T \times_f N_\mu$ be a non-trivial wppss submanifold of a Kaehler manifold admitting a ssm connection and a concurrent vector field P then*

(i) $g(h(Z_1, Z_3), FTZ_3) = -Z_1(\ln f)\cos^2\mu\|Z_3\|^2$,

(ii) $g(h(JZ_1, Z_3), FZ_3) = Z_1(\ln f)\|Z_3\|^2 + 2\pi(Z_1)\|Z_3\|^2$,

for $Z_1 \in TN_T$ and $Z_3 \in TN_\mu$.

Proof. By substituting Z_4 with Z_3 in equation (3.2), we obtain part (i). By employing the Gauss formula in conjunction with equation (2.7), we can derive the following expression:

$$g(h(JZ_1, Z_3), FZ_3) = g(\bar{\nabla}_{Z_3} JZ_1, JZ_3) - g(\bar{\nabla}_{Z_3} JZ_1, TZ_3). \tag{30}$$

On applying Gauss formula and equation (17), we get

$$g(h(JZ_1, Z_3), FZ_3) = g((\bar{\nabla}_{Z_3} J)Z_1, JZ_3) + g(J\bar{\nabla}_{Z_3} Z_1, JZ_3) - g(\nabla_{Z_3} JZ_1, JZ_3).$$

Further using equations (3) and (17), we get the required result. □

In this study, we focus on investigating stable currents on warped product pointwise semi-slant submanifolds. Our main objective is to prove that under certain specific conditions, the existence of stable currents is ruled out. Furthermore, we highlight the notable results established by Simons, Xin, and Lang, which hold significant recognition in the field.

Lemma 3.5. *[18, 20]. Let M^n be a compact submanifold of dimension n in a space form $\bar{M}(c)$ with positive curvature c. If the second fundamental form satisfies the inequality*

$$\sum_{i=1}^{p} \sum_{s=p+1}^{n} (2|h(x_i, x_j)|^2 - g(h(x_i, x_i), h(x_i, x_s))) < pqc, \tag{31}$$

then there are no stable currents in M^n. Here, $p, q \in \mathbb{Z}^+$ with $p + q = n$, $\{x_1, \ldots, x_n\}$ is an orthonormal basis in T_xM, and $x \in M$. Furthermore, we have $\tilde{H}_p(M^n, Z) = 0$ and $\tilde{H}_q(M^n, Z) = 0$, where $H_j(M, Z)$ denotes the j-th homology of M with integer coefficients.

Theorem 3.6. *Let $M^{p+q} = N_T^p \times_f N_\mu^q$ be a compact wppss submanifolds of complex space form $\bar{M}(4)$ with ssm connection and a concurrent vector field P. If the following inequality holds*

$$\Delta f + \sum_{i=1}^{p} \beta(x_i, x_i) + \frac{p}{q} \sum_{j=1}^{q} \beta(x_j, x_j) > (\csc^2 \mu + \cot^2 \mu + 1 - q)\|\nabla(\ln f)\|^2 - \frac{q}{f}\pi(\nabla f) - 3p, \tag{32}$$

Therefore, there are no p-stable currents present in M^{p+q}. Furthermore, the homology groups $H_p(M^n, Z) = 0$ and $H_q(M^n, Z) = 0$ are satisfied, where $H_j(M, Z)$ represents the j-th homology group of M. Here, p and q denote the dimensions of the invariant submanifold N_T^p and the pointwise slant submanifold N_μ^q respectively.

Proof. Suppose $\dim N_T^p = p = 2\alpha$ and $\dim N_\mu^q = q = 2\beta$, where N_T and N_μ are the integral manifolds of invariant distribution D_T and the pointwise slant distribution D_μ. Let $\{x_1, x_2, \ldots, x_\alpha, x_{\alpha+1} = Jx_1, \ldots, x_{2\alpha} = Jx_\alpha\}$ and $\{x_{2\alpha+1} = x_1^*, \ldots, x_{2\alpha+\beta} = x_\beta^*, x_{2\alpha+\beta+1} = x_{\beta+1}^* = \sec\mu T x_1^*, \ldots, x_{p+q} = x_q^* = \sec\mu T x_\beta^*\}$ to be orthonormal basis of TN_T^p and TN_μ^q respectively. Therefore, an orthonormal basis for the normal subbundle FD_μ is $\{x_{n+1} = \bar{x}_1 = \csc\mu F x_1^*, \ldots, x_{n+\beta} = \bar{x}_\beta = \csc\mu F x_1^*, x_{n+\beta+1} = \bar{x}_{\beta+1} = \csc\mu \sec\mu FT x_1^*, \ldots, x_{n+2\beta} = \bar{x}_{2\beta} = \csc\mu \sec\mu FT x_\beta^*\}$.

Thus, we can establish the following relationship

$$\begin{aligned} \sum_{i=1}^{p} \sum_{j=p+1}^{n} \{2\|h(x_i, x_j)\|^2 - g(h(x_i, x_i), h(x_j, x_j))\} &= \sum_{r=n+1}^{p+2q+1} \sum_{i=1}^{p} \sum_{j=p+1}^{n} (h_{ij}^r)^2 \\ &+ \sum_{i=1}^{p} \sum_{j=p+1}^{n} \{\|h(x_i, x_j)\|^2 - g(h(x_i, x_i), h(x_j, x_j))\}. \end{aligned} \tag{33}$$

Applying Gauss equation (11)

$$\sum_{i=1}^{p}\sum_{j=p+1}^{n}\{2\|h(x_i,x_j)\|^2-g(h(x_i,x_i),h(x_j,x_j))\}=\sum_{r=n+1}^{p+2q+1}\sum_{i=1}^{p}\sum_{j=p+1}^{n}(h_{ij}^r)^2 \\ +\sum_{i=1}^{p+1}\sum_{j=1}^{q}g(R(x_i,x_j)x_i,x_j)-\sum_{i=1}^{p+1}\sum_{j=1}^{q}g(\bar{R}(x_i,x_j)x_i,x_j). \tag{34}$$

On making use of formula (8) for a complex space form $\bar{M}(4)$

$$\sum_{i=1}^{p}\sum_{j=p+1}^{n}\{2\|h(x_i,x_j)\|^2-g(h(x_i,x_i),h(x_j,x_j))\}=\sum_{r=n+1}^{p+2q+1}\sum_{i=1}^{p}\sum_{j=p+1}^{n}(h_{ij}^r)^2 \\ -pq-p\sum_{j=1}^{q}\beta(x_j,x_j)-q\sum_{i=1}^{p}\beta(x_i,x_i)+\sum_{i=1}^{p}\sum_{j=1}^{q}g(R(x_i,x_j)x_i,x_j). \tag{35}$$

By considering equation (2.13), we can express the relation for the submanifold $N_T^p\times_f N_\mu^q$ of $\bar{M}^{p+q}(4)$ as follows

$$\sum_{i=1}^{p}\sum_{j=1}^{q}g(R(x_i,x_j)x_i,x_j)=\sum_{i=1}^{p}\sum_{j=1}^{q}\frac{H^f(x_i,x_i)}{f}g(x_j,x_j) \\ +\sum_{i=1}^{p}\sum_{j=1}^{q}\{2g(x_i,x_i)g(x_j,x_j)-\pi(x_i)\pi(x_i)g(x_j,x_j). \tag{36}$$

Ultimately, the subsequent equation is obtained.

$$\sum_{i=1}^{p}\sum_{j=1}^{q}g(R(x_i,x_j)x_i,x_j)=\frac{q}{f}\sum_{i=1}^{p}g(\nabla_{x_i}\nabla f,x_i)+2pq-q. \tag{37}$$

We first compute the term Δf, which represents the Laplacian of f. The derivation is as follows

$$\Delta f=-\sum_{k=1}^{n}g(\nabla_{x_k}\nabla f,x_k)=-\sum_{i=1}^{p}g(\nabla_{x_i}\nabla f,x_i)-\sum_{j=1}^{q}g(\nabla_{x_j^*}\nabla f,x_j^*). \tag{38}$$

Using the adapted orthonormal frame, we can express the components of N_μ^q as follows

$$\Delta f=-\sum_{i=1}^{p}g(\nabla_{x_i}\nabla f,x_i)-\sum_{j=1}^{\beta}g(\nabla_{x_j^*}\nabla f,x_j^*)-\sec^2\mu\sum_{j=1}^{\beta}g(\nabla_{Tx_j^*}\nabla f,Tx_j^*). \tag{39}$$

Since N_T^p is totally geodesic in M^n and $\nabla f\in TN_T$, we can deduce the following:

$$\Delta f=-\frac{1}{f}\sum_{j=1}^{\beta}(g(x_j^*,x_j^*)+\sec^2\mu g(Tx_j^*,Tx_j^*))\|\nabla f\|^2-\sum_{i=1}^{p}g(\nabla_{x_i}\nabla f,x_i), \tag{40}$$

or

$$\frac{\Delta f}{f}=-q\|\nabla(\ln f)\|^2-\frac{1}{f}\sum_{i=1}^{p}g(\nabla_{x_i}\nabla f,x_i)-\frac{q}{f}\pi(\nabla f). \tag{41}$$

By utilizing equation (18), we can determine that

$$\frac{1}{f}\sum_{i=1}^{p} g(\nabla_{x_i}\nabla f, x_i) = -\Delta(\ln f) + (1-q)\|\nabla(\ln f)\|^2 - \frac{q}{f}\pi(\nabla f) \tag{42}$$

or

$$\frac{q}{f}\sum_{i=1}^{p} g(\nabla_{x_i}\nabla f, x_i) = -q\Delta(\ln f) + q(1-q)\|\nabla(\ln f)\|^2 - \frac{q^2}{f}\pi(\nabla f). \tag{43}$$

By plugging in the aforementioned value into equation (37), we get:

$$\sum_{i=1}^{p}\sum_{j=1}^{q} R((x_i, x_j)x_i, x_j) = -q\Delta(\ln f) + q(1-q)\|\nabla(\ln f)\|^2 + 2pq - q - \frac{q^2}{f}\pi(\nabla f). \tag{44}$$

Therefore by equation (35)

$$\begin{aligned}\sum_{i=1}^{p}\sum_{j=p+1}^{n}\{2\|h(x_i, x_j)\|^2 - g(h(x_i, x_i), h(x_j, x_j))\} &= \sum_{r=n+1}^{p+2q+1}\sum_{i=1}^{p}\sum_{j=p+1}^{n}(h_{ij}^r)^2\\ &- (p+1)q - (p+1)\sum_{j=1}^{q}\beta(x_j, x_j) - q\sum_{i=1}^{p}\beta(x_i, x_i)\\ &- q\Delta(\ln f) + q(1-q)\|\nabla(\ln f)\|^2 + 2pq - q - -\frac{q^2}{f}\pi(\nabla f),\end{aligned} \tag{45}$$

or,

$$\begin{aligned}\sum_{i=1}^{p}\sum_{j=p+1}^{n}\{2\|h(x_i, x_j)\|^2 - g(h(x_i, x_i), h(x_j, x_j))\} &= \sum_{r=n+1}^{p+2q+1}\sum_{i=1}^{p}\sum_{j=p+1}^{n}(h_{ij}^r)^2\\ &- (2-p)q - p\sum_{j=1}^{q}\beta(x_j, x_j) - q\sum_{i=1}^{p}\beta(x_i, x_i)\\ &- q\Delta(\ln f) + q(1-q)\|\nabla(\ln f)\|^2 - \frac{q^2}{f}\pi(\nabla f).\end{aligned} \tag{46}$$

Now, let $Z_1 = x_\alpha (1 \leq \alpha \leq p)$ and $Z_3 = x_\beta^* (1 \leq \beta \leq q)$

$$\begin{aligned}\sum_{r=n+1}^{p+2q}\sum_{i=1}^{p}\sum_{j=p+1}^{n}(h_{ij}^r)^2 &= \sum_{r=n+1}^{p+2q+1}\sum_{i=1}^{p}\sum_{j=p+1}^{n} g(h(x_i, x_j^*), \bar{x}_r^*)^2\\ &= \sum_{i=1}^{p}\sum_{j,r=1}^{\beta}\{g(h(x_i, x_j^*), \csc\mu F x_r^*)^2 + g(h(x_i, x_j^*), \csc\mu\sec\mu FT x_r^*)^2\}\\ &= \sum_{i=1}^{\alpha}\sum_{j,r=1}^{\beta}\{g(h(x_i, x_j^*), \csc\mu F x_r^*)^2 + g(h(x_i, x_j^*), \csc\mu\sec\mu FT x_r^*)^2\}\\ &+ \sum_{i=1}^{\alpha}\sum_{j,r=1}^{\beta}\{g(Jx_i, x_j^*), \csc\mu F x_r^*)^2 + g(h(Jx_i, x_j^*), \csc\mu\sec\mu FT x_r^*)^2\}.\end{aligned} \tag{47}$$

Applying Lemma 3.4 to the equation above, we obtain:

$$\sum_{r=n+1}^{p+2q+1}\sum_{i=1}^{p}\sum_{j=p+1}^{n}(h_{ij}^{r})^2 = (\csc^2\mu + \cot^2\mu)\sum_{i=1}^{\alpha+1}\sum_{j=1}^{\beta}(x_i(\ln f))^2 g(x_j^*, x_j^*)^2 + 2\beta + (\csc^2\mu + \cot^2\mu)\sum_{i=1}^{\alpha}\sum_{j=1}^{\beta}(Jx_i(\ln f))^2 g(x_j^*, x_j^*)^2 + 2\beta, \tag{48}$$

or equivalently,

$$\sum_{r=n+1}^{p+2q+1}\sum_{i=1}^{p}\sum_{j=p+1}^{n}(h_{ij}^{r})^2 = q(\csc^2\mu + \cot^2\mu)\|\nabla(\ln f)\|^2 + 2q. \tag{49}$$

By employing equations (46) and (49), we observe that:

$$\sum_{i=1}^{p}\sum_{j=p+1}^{n}\{2\|h(x_i, x_j)\|^2 - g(h(x_i, x_i), h(x_j, x_j))\} - 4pq = q(\csc^2\mu + \cot^2\mu + 1 - q)\|\nabla(\ln f)\|^2 - q\Delta(\ln f) - 3pq - p\sum_{j=1}^{q}\beta(x_j, x_j) - q\sum_{i=1}^{p}\beta(x_i, x_i) - \frac{q^2}{f}\pi(\nabla f). \tag{50}$$

Under the assumption that condition (32) is satisfied, we can deduce the following inequality

$$\sum_{i=1}^{p}\sum_{j=p+1}^{n}\{2\|h(x_i, x_j)\|^2 - g(h(x_i, x_i), h(x_j, x_j))\} < 4pq. \tag{51}$$

Applying Lemma 3.5 to the complex space form with $c = 4$ brings us to the final conclusion of our theorem. □

4. Conclusion

In the realm of Riemannian manifolds, two prominent types of differentiable connections have garnered significant attention: the Levi-Civita connection and the semi-symmetric metric connection. These connections possess distinct characteristics, prompting extensive efforts to compare and contrast the geometric properties of submanifolds associated with each connection type. While considerable research exists on the homology of warped product submanifolds with respect to the Levi-Civita connection, the homology of such submanifolds in the presence of semi-symmetric metric connections remains unexplored. Motivated by this gap in knowledge, our paper embarks on an investigation into the homology and stable currents of poinwise semi-slant warped product submanifolds within Kaehler manifolds, utilizing a semi-symmetric connection. By focusing on this specific setting, we aim to shed light on the topological properties and behavior of generalized warped product submanifolds. We hope that the findings of our study will not only contribute to the understanding of homology and stable currents in the context of semi-invariant warped product submanifolds but also serve as a catalyst for further research in the realm of generalized warped product submanifolds and their associated topological characteristics.

Conflict of interest: The authors declare no conflict of interest in this paper.

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