Filomat 38:17 (2024), 6147–6167
https://doi.org/10.2298/FIL2417147H

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Real hypersurfaces in the complex hyperbolic quadric with pseudo-Ricci-Bourguignon solitons

Doo Hyun Hwang[a], Changhwa Woo[b,*]

[a]*Department of Mathmatics & RIRCM, Kyungpook National University, Daegu 41566, Republic of Korea*
[b]*Department of Applied Mathematics, Pukyong National University, Busan 48547, Republic Of Korea*

Abstract. By using the property of generalized pseudo-anti-commuting Ricci tensor, that is, $\mathrm{Ric}\phi + \phi\mathrm{Ric} = f\phi$, for real hypersurfaces in the complex hyperbolic quadric Q^{m*}, we give a non-existence theorem for Hopf pseudo-Ricci-Bouguignon soliton real hypersurfaces in the complex hyperbolic quadric Q^{m*}. Next as an application we obtain a classification of gradient pseudo-Ricci-Bouguignon solitons on Hopf real hypersurfaces in Q^{m*}.

1. Introduction

The complex hyperbolic quadric $Q^{m*} = SO^0_{2,m}/SO_2SO_m$ is the typical example of Hermitian symmetric space of noncompact type with rank 2. Here $SO^0_{2,m}$ denotes the identity component of the indefinite special orthogonal group with respect to the metric g (see Kobayashi and Nomizu [18], Smyth [38] and [39], Suh [47]). Then for $m \geq 2$ the triple (Q^{m*}, J, g) is a Hermitian symmetric space of noncompact type and its minimal sectional curvature is equal to -4 which derives the negative of the curvature tensor (see Klein [15] and Reckziegel [35]).

Q^{m*} has two remarkable geometric structures, first one is Kähler structure J. When we consider a Kähler structure tensor J for any vector field X on M, then JX is given by

$$JX = \phi X + \eta(X)N,$$

where $\phi X = (JX)^T$ is the tangential component of the vector field JX, $\eta(X) = g(\xi, X)$, $\xi = -JN$, and N denotes a unit normal vector field on M.

The other one is a real structure A, which act as complex conjugations A on the tangent spaces of Q^{m*}. This geometric structure determines a maximal $\mathfrak{A}$-invariant subbundle $\mathcal{Q}$ of the tangent bundle TM of a real hypersurface M in Q^{m*}. It is denoted by

$$\mathfrak{A}_{[z]} = \{A_{\lambda\bar{z}} | \lambda \in S^1 \subset \mathbb{C}\},\ [z] \in Q^{m*},$$

2020 *Mathematics Subject Classification*. Primary 53C40. Secondary 53C55.
Keywords. Ricci soliton, Ricci-Bourguignon soliton, generalized pseudo-anti-commuting property, $\mathfrak{A}$-isotropic, $\mathfrak{A}$-principal, complex conjugation, complex hyperbolic quadric.
Received: 13 September 2023; Accepted: 05 November 2023
Communicated by Mića S. Stanković
The first author was supported by KNU Development Project Research Fund, 2022, and the second by grant Proj. No. NRF-2020-R1A2C1A-01101518 from National Research Foundation of Korea.
* Corresponding author: Changhwa Woo
Email addresses: engus0322@knu.ac.kr (Doo Hyun Hwang), legalgwch@pknu.ac.kr (Changhwa Woo)

moreover, it is the set of all complex conjugations defined on Q^{m*}.

Moreover, the derivative of the complex conjugation A on Q^{m*} is given by

$$(\bar{\nabla}_X A)Y = q(X)JAY$$

for any vector fields X and Y on M, where $\bar{\nabla}$ and q denote a connection and a certain 1-form defined on $T_{[z]}Q^{m*}$, $[z] \in Q^{m*}$ respectively.

Let M be a Hopf hypersurface in the complex hyperbolic quadric Q^{m*}. Then we have

$$S\xi = \alpha\xi$$

for the shape operator S with the Reeb function $\alpha = g(S\xi, \xi)$ on M in Q^{m*}.

For real hypersurfaces in the complex hyperbolic quadric Q^{m*} Suh, [43], has classified the problem of isomeric Reeb flow as follows:

Theorem A 1. *Let M be a real hypersurface of the complex hyperbolic quadric Q^{m*}, $m \geq 3$. The Reeb flow on M is isometric if and only if m is even, say $m = 2k$, and M is an open part of a tube around a totally geodesic $\mathbb{C}H^k$ in the complex hyperbolic quadric Q^{2k^*}, $k \geq 2$, or a horosphere in Q^{2k^*} whose center at infinity is in the equivalent class of an $\mathfrak{A}$-isotropic singular geodesic in Q^{2k^*}.*

When examining a hypersurface denoted as M within the complex hyperbolic quadric Q^{m*}, the unit normal vector field N associated with M in Q^{m*} can fall into two categories, namely, being $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal (see [43], [45], [49] and [50]). In the first case, where N is $\mathfrak{A}$-isotropic, our Theorem A establishes that M can be locally transformed into either a tubular structure over a completely geodesic complex hyperbolic space denoted as $\mathbb{C}H^k$ in Q^{2k^*} or a horosphere.

In the second case, when the unit normal vector field N is $\mathfrak{A}$-principal, we have given a complete classification of contact hypersurfaces M in Q^{m*} due to Klein and Suh [16] as follows:

Theorem B 1. *Let M be a connected orientable real hypersurface in the complex hyperbolic quadric $Q^{m*} = SO^0_{2,m}/SO_2SO_m$, $m \geq 3$. Then M is a contact hypersurface if and only if M is congruent to an open part of one of the following real hypersurfaces in Q^{m*}:*

(i) a tube of radius r around the Hermitian symmetric space Q^{m-1^*} which is embedded in Q^{m*} as a totally geodesic complex hypersurface,

(ii) a horosphere in Q^{m*} whose center at infinity is the equivalence class of an $\mathfrak{A}$-principal geodesic in Q^{m*},

(iii) a tube of radius r around the m-dimensional real hyperbolic space $\mathbb{R}H^m$ which is embedded in Q^{m*} as a real space form of Q^{m*}.

Inspired by these findings, we present various descriptions of real hypersurfaces within the complex hyperbolic quadric Q^{m*} concerning a set of geometric flows. Indeed, we know that a solution of the Ricci flow equation $\frac{\partial}{\partial t}g(t) = -2\mathrm{Ric}(g(t))$ is given by

$$\frac{1}{2}(\mathcal{L}_V g)(X,Y) + \mathrm{Ric}(X,Y) = \Omega g(X,Y),$$

where Ω is a constant and $\mathcal{L}_V$ denotes the Lie derivative along the direction of the vector field V (see Chaubey-Suh [9], Chaubey-Lee-Suh [10], Chaubey-De-Suh [11], Jeong-Suh [13], Morgan-Tian [25], Perelman [28], Wang [51],[52],[53],[54] and [55]). Then this solution (M, V, Ω, g) is said to be a *Ricci soliton* with potential vector field V and Ricci soliton constant Ω.

As a generalization of the notion of Ricci flow, the Ricci-Bourguignon flow (see Bourguignon [3] and [4], Catino-Cremaschi-Djadli-Mantegazza-Mazzieri [5]) is given by

$$\frac{\partial}{\partial t}g(t) = -2(\mathrm{Ric}(g(t)) - \theta\gamma g(t)), \quad g(0) = g_0.$$

This family of geometric flows with $\theta = 0$ reduces to the Ricci flow $\frac{\partial}{\partial t}g(t) = -2\mathrm{Ric}(g(t))$, $g(0) = g_0$. If the constant $\theta = \frac{1}{2}$, it is said to be *Einstein flow*. The critical point of the following Einstein flow

$$\frac{\partial}{\partial t}g(t) = -2(\mathrm{Ric}(g(t)) - \frac{1}{2}\gamma g(t)), \quad g(0) = g_0,$$

implies that the Einstein gravitational tensor $\mathrm{Ric}(g(t)) - \frac{1}{2}\gamma g(t)$ vanishes. For a four-dimensional space-time M^4, this is equivalent to the vanishing Ricci tensor by virtue of $d\gamma = 2\mathrm{div}(\mathrm{Ric})$. In this case M^4 becomes a vacuum. That is, $g(t) = g(0)$, the metric is constant along the time (see O'Neill [27]). For $\theta = \frac{1}{n}$, the tensor $\mathrm{Ric} - \frac{\gamma}{n}g$ is said to be traceless Ricci tensor, and for $\theta = \frac{1}{2(n-1)}$, it is said to be the Schouten tensor.

Now we introduce the Ricci-Bourguignon soliton $(M, \xi, \Omega, \theta, \gamma, g)$, which is a solution of the Ricci-Bourguignon flow as follows:

$$\frac{1}{2}(\mathcal{L}_V g)(X, Y) + \mathrm{Ric}(X, Y) = (\Omega + \theta\gamma)g(X, Y), \tag{1.1}$$

for any tangent vector fields X and Y on M, where Ω is a soliton constant, θ any constant and γ the scalar curvature on M, and $\mathcal{L}_V$ denotes the Lie derivative along the direction of the vector field V (see Bourguignon [3], [4], Morgan-Tian [25], and Suh [47]). Then (M, g) is said to be a *Ricci-Bourguignon soliton* with potential vector field V and Ricci-Bourguignon soliton constant Ω.

If the Ricci operator Ric of a real hypersurface M in Q^{m^*} satisfies

$$\mathrm{Ric}(X) = aX + b\eta(X)\xi \tag{1.2}$$

for smooth functions a, b on M, then M is said to be *pseudo-Einstein*. Then we introduce a complete classification of pseudo-Einstein Hopf real hypersurfaces in the complex hyperbolic quadric Q^{m^*} as follows:

Theorem C 1. *There does not exist any Hopf pseudo-Einstein real hypersurface in the complex hyperbolic quadric Q^{m^*}, $m \geq 3$.*

Let M be a Hopf hypersurface in the complex hyperbolic quadric Q^{m^*}. Then we have

$$S\xi = \alpha\xi$$

for the shape operator A with the Reeb function $\alpha = g(S\xi, \xi)$ on M in Q^{m^*}. When we consider a tensor field J for any vector field X on M , which is a Kähler structure on the tangent space T_zM, $z \in M$, then JX is given by

$$JX = \phi X + \eta(X)N,$$

where $\phi X = (JX)^T$ is the tangential component of the vector field JX, $\eta(X) = g(\xi, X)$, $\xi = -JN$, and N denotes a unit normal vector field on M.

In this paper we introduce a new notion named *generalized pseudo-anti-commuting property* for the Ricci tensor of a real hypersurface M in the complex hyperbolic quadric Q^{m^*} as follows:

$$\mathrm{Ric}\,\phi + \phi\,\mathrm{Ric} = f\phi \tag{1.3}$$

for a smooth function f on M in Q^{m^*} (see Suh [44], [45]).

It is a well-established that Einstein and pseudo-Einstein real hypersurfaces denoted as M residing within the complex hyperbolic quadric Q^{m^*} adhere to a condition known as the generalized pseudo-anti-commuting Ricci tensor condition. This condition is expressed as $\mathrm{Ric}\phi + \phi\mathrm{Ric} = f\phi$, where f represents a smooth function defined on M within Q^{m^*} (see Suh [44] and [45]).

Moreover, Hopf hypersurfaces featuring an $\mathfrak{A}$-principal unit normal vector field, which includes cases like a horosphere with an $\mathfrak{A}$-principal unit normal vector field within the complex hyperbolic quadric Q^{m^*}, are characterized by the equation $S\phi + \phi S = k\phi$, where k is a non-zero constant. Importantly, these hypersurfaces also satisfy the aforementioned formula for the generalized pseudo-anti-commuting Ricci tensor as indicated in equation (1.3).

Under the assumption of Hopf it is classified in the complex hyperbolic quadric Q^{m^*} as follows:

Theorem D 1. *Let M be a pseudo-anti-commuting Hopf real hypersurfaces in the complex hyperbolic quadric Q^{m^*}, $m \geq 3$. Then M is locally congruent to one of the following:*

(i) *a tube around a totally geodesic $\mathbb{C}H^k \subset Q^{2k^*}$, where $m = 2k$,*
(ii) *a horosphere whose center at infinity is $\mathfrak{A}$-isotropic singular,*
(iii) *a tube around a totally geodesic Hermitian symmetric space Q^{m-1^*} embedded in Q^{m^*},*
(iv) *a horosphere in Q^{m^*} whose center at infinity is the equivalence class of an $\mathfrak{A}$-principal geodesic in Q^{m^*},*
(v) *a tube around the m-dimensional real hyperbolic space $\mathbb{R}H^m$ which is embedded in Q^{m^*} as a real space form.*

Then in this paper we consider a pseudo-Ricci-Bourguignon soliton $(M, V, \eta, \Omega, \theta, \gamma, g)$ as follows:

$$\frac{1}{2}(\mathcal{L}_V g)(X, Y) + \mathrm{Ric}(X, Y) + \psi\eta(X)\eta(Y) = (\Omega + \theta\gamma)g(X, Y), \tag{1.4}$$

for any tangent vector fields X and Y on M, where Ω is said to be a pseudo-Ricci-Bourguignon soliton constant, the functions θ and ψ are any constants and γ the scalar curvature on M, and $\mathcal{L}_V$ denotes the Lie derivative along the direction of the vector field V (see Blaga-Tastan [2], Chaubey-Siddiqi-Prakasha [12], and Suh-Woo [48]).

When the function ψ identically vanishes, the pseudo-Ricci-Bourguignon soliton $(M, V, \eta, \Omega, \theta, \gamma, g)$ is said to be a Ricci-Bourguignon soliton $(M, V, \Omega, \theta, \gamma, g)$. We also say that the pseudo-Ricci-Bourguignon soliton is shrinking, steady, and expanding according to the pseudo-Ricci-Bourguignon soliton constant function $\Omega > 0$, $\Omega = 0$, and $\Omega < 0$ respectively.

By virtue of Theorem C, we also know that there does not exist any Hopf Einstein real hypersurface in the complex hyperbolic quadric Q^{m^*}. This fact will be used in the proof of our Main Theorems. It can be easily checked that a Ricci-Bourguignon soliton $(M, \xi, \eta, \Omega, \theta, \gamma, g)$ satisfies the pseudo-anti-commuting Ricci tensor. Then the unit normal vector field N becomes $\mathfrak{A}$-principal or $\mathfrak{A}$-isotropic (see Proposition 5.3 in [45]). Moreover, the contact hypersurfaces in Q^{m^*} also satisfy the pseudo-anti commuting property and N is $\mathfrak{A}$-principal. By virtue of such a situation we obtain the following

Main Theorem 1 1. *Let M be a Hopf real hypersurface in the complex hyperbolic quadric Q^{m^*}, $m \geq 3$. Then there does not exist a pseudo-Ricci-Bourguignon soliton $(M, \xi, \eta, \Omega, \theta, \gamma, g)$ in the complex hyperbolic quadric Q^{m^*}, $m \geq 3$.*

If the pseudo-Ricci-Bourguignon constant ψ identically vanishes, then it becomes a Ricci-Bourguignon soliton. So the pseudo-Ricci-Bourguignon soliton $(M, \xi, \eta, \Omega, \theta, \gamma, g)$ is a general notion weaker than the Ricci-Bourguignon soliton $(M, \xi, \Omega, \theta, \gamma, g)$. From such a view point, we give the following

Corollary 1.1. *There does not exist a Hopf Ricci-Bourguignon soliton $(M, \xi, \Omega, \theta, \gamma, g)$ in the complex hyperbolic quadric Q^{m^*}, $m \geq 3$.*

We denote Df by the gradient vector field of the function f on M defined by $g(Df, X) = g(\mathrm{grad} f, X) = X(f)$ for any tangent vector field X on M. We consider the *gradient Ricci-Bourguignon soliton* $(M, Df, \nu, \rho, \gamma, g)$ (see Catino-Mazzieri [6], Cernea-Guan [8]) defined by

$$\mathrm{Hess}(f) + \mathrm{Ric} = (\Omega + \theta\gamma)g,$$

where Hess(f) is defined by Hess(f) = ∇Df and for any tangent vector fields X and Y on M

$$\text{Hess}(f)(X,Y) = XY(f) - (\nabla_X Y)f.$$

Then a gradient pseudo-Ricci-Bourguignon soliton is given by

$$\nabla_X Df + \text{Ric}X + \psi\eta(X)\xi = (\Omega + \theta\gamma)X$$

for any vector field X tangent to M in Q^{m^*}. Then first by virtue of Theorem A we can give a non-existence theorem for gradient pseudo-Ricci-Bourguignon solitons $(M, \xi, \eta, \Omega, \theta, \gamma, g)$ as follows:

Main Theorem 2 1. *There does not exist a real hypersurface with isomeric Reeb flow in the complex hyperbolic quadric Q^{m^*}, $m \geq 3$, being a gradient pseudo-Ricci-Bourguignon soliton.*

Next by Theorem B for a contact real hypersurface in the complex hyperbolic quadric Q^{m^*}, we can assert the following

Main Theorem 3 1. *There does not exist a contact real hypersurface in the complex hyperbolic quadric Q^{m^*}, $m \geq 3$, being a gradient pseudo-Ricci-Bourguignon soliton.*

2. The complex hyperbolic quadric

In this section, let us introduce the complex hyperbolic quadric Q^{m^*}. This section is due to Klein and Suh [16]. In more detail, we also refer to Kim and Suh [14], Pérez [31], and Suh [45] and [46].

Now let us realize the complex hyperbolic quadric Q^{m^*} as the quotient symmetric manifold $SO^0_{2,m}/SO_2SO_m$. As Q^{1^*} is isomorphic to the real hyperbolic space $\mathbb{R}H^2 = SO^0_{1,2}/SO_2$, and Q^{2^*} is isomorphic to the Hermitian product of complex hyperbolic spaces $\mathbb{C}H^1 \times \mathbb{C}H^1$, we suppose $m \geq 3$ in the sequel and throughout this paper. Let $G := SO^0_{2,m}$ be the transvection group of Q^{m^*} and $K := SO_2SO_m$ be the isotropy group of Q^{m^*} at the "origin" $p_0 := eK \in Q^{m^*}$. Then

$$\sigma : G \to G,\ g \mapsto sgs^{-1} \quad \text{with} \quad s := \begin{pmatrix} -1 & & & & & \\ & -1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$$

is an involutive Lie group automorphism of G with $\text{Fix}(\sigma)_0 = K$, and therefore $Q^{m^*} = G/K$ is a Riemannian symmetric space. The center of the isotropy group K is isomorphic to SO_2, and therefore Q^{m^*} is in fact a Hermitian symmetric space.

The Lie algebra $\mathfrak{g} := \mathfrak{so}_{2,m}$ of G is given by

$$\mathfrak{g} = \left\{ X \in \mathfrak{gl}(m+2, \mathbb{R}) \middle| X^t \cdot s = -s \cdot X \right\}$$

(see [17, p. 59]). In the sequel we will write members of $\mathfrak{g}$ as block matrices with respect to the decomposition $\mathbb{R}^{m+2} = \mathbb{R}^2 \oplus \mathbb{R}^m$ in the form

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix},$$

where X_{11}, X_{12}, X_{21}, X_{22} are real matrices of the dimensions 2×2, $2 \times m$, $m \times 2$ and $m \times m$, respectively. Then

$$\mathfrak{g} = \left\{ \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \middle| X^t_{11} = -X_{11},\ X^t_{12} = X_{21},\ X^t_{22} = -X_{22} \right\}.$$

The linearisation $\sigma_L = \mathrm{Ad}(s) : \mathfrak{g} \to \mathfrak{g}$ of the involutive Lie group automorphism σ induces the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, where the Lie subalgebra

$$\begin{aligned}\mathfrak{k} =&\mathrm{Eig}(\sigma_*, 1) = \{X \in \mathfrak{g} | sXs^{-1} = X\}\\ =&\left\{ \left(\begin{smallmatrix} X_{11} & 0 \\ 0 & X_{22} \end{smallmatrix} \right) \middle| \; X_{11}^t = -X_{11}, \; X_{22}^t = -X_{22} \right\}\\ \cong&\mathfrak{so}_2 \oplus \mathfrak{so}_m\end{aligned}$$

is the Lie algebra of the isotropy group K, and the $2m$-dimensional linear subspace

$$\mathfrak{m} = \mathrm{Eig}(\sigma_*, -1) = \{X \in \mathfrak{g} | sXs^{-1} = -X\} = \left\{ \left(\begin{smallmatrix} 0 & X_{12} \\ X_{21} & 0 \end{smallmatrix} \right) \middle| \; X_{12}^t = X_{21} \right\}$$

is canonically isomorphic to the tangent space $T_{p_0}Q^{m*}$. Under the identification $T_{p_0}Q^{m*} \cong \mathfrak{m}$, the Riemannian metric g of Q^{m*} (where the constant factor of the metric is chosen so that the formulae become as simple as possible) is given by

$$g(X, Y) = \frac{1}{2}\,\mathrm{tr}(Y^t \cdot X) = \mathrm{tr}(Y_{12} \cdot X_{21}) \quad \text{for} \quad X, Y \in \mathfrak{m}.$$

g is clearly $\mathrm{Ad}(K)$-invariant, and therefore corresponds to an $\mathrm{Ad}(G)$-invariant Riemannian metric on Q^{m*}. The complex structure J of the Hermitian symmetric space is given by

$$JX = \mathrm{Ad}(j)X \quad \text{for} \quad X \in \mathfrak{m}, \quad \text{where} \quad j := \begin{pmatrix} 0 & 1 & & & & \\ -1 & 0 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} \in K\,.$$

Because j is in the center of K, the orthogonal linear map J is $\mathrm{Ad}(K)$-invariant, and thus defines an $\mathrm{Ad}(G)$-invariant Hermitian structure on Q^{m*}. By identifying the multiplication with the unit complex number i with the application of the linear map J, the tangent spaces of Q^{m*} thus become m-dimensional complex linear spaces, and we will adopt this point of view in the sequel.

As in the complex quadric, the Riemannian curvature tensor $\bar{R}$ of Q^{m*} can be fully described in terms of the "fundamental geometric structures" g, J and $\mathfrak{A}$, where the set $\mathfrak{A}_{[z]}$, $[z] \in Q^{m*}$, is mentioned in the introduction.

In fact, under the correspondence $T_{p_0}Q^{m*} \cong \mathfrak{m}$, the curvature $\bar{R}(X, Y)Z$ corresponds to $-[[X, Y], Z]$ for $X, Y, Z \in \mathfrak{m}$, see [18, Chapter XI, Theorem 3.2(1)]. By evaluating the latter expression explicitly, one can show that one has

$$\begin{aligned}\bar{R}(X, Y)Z = &-g(Y, Z)X + g(X, Z)Y\\ &- g(JY, Z)JX + g(JX, Z)JY + 2g(JX, Y)JZ\\ &- g(AY, Z)AX + g(AX, Z)AY\\ &- g(JAY, Z)JAX + g(JAX, Z)JAY\end{aligned} \tag{2.1}$$

for arbitrary $A \in \mathfrak{A}_{[z]}$, $[z]{\in}Q^{m*}$. Therefore the curvature of Q^{m*} is the negative of that of the complex quadric Q^m, compare [35, Theorem 1]. This confirms that the symmetric space Q^{m*} which we have constructed here is indeed the non-compact dual of the complex quadric.

3. Some general equations

In this section we want to refer to [20], [30], [31], [32], [29], [45], [46] and [49]. Let M be a real hypersurface in the complex hyperbolic quadric Q^{m*} and denote by (ϕ, ξ, η, g) the induced almost contact metric structure. Note that $\xi = -JN$, where N is a (local) unit normal vector field of M. The tangent bundle TM of M splits orthogonally into $TM = \mathcal{C} \oplus \mathbb{R}\xi$, where $\mathcal{C} = \ker(\eta)$ is the maximal complex subbundle of TM. The structure tensor field ϕ restricted to $\mathcal{C}$ coincides with the complex structure J restricted to $\mathcal{C}$, and $\phi\xi = 0$.

Now at each point $[z] \in M$ let us consider a maximal $\mathfrak{A}$-invariant subspace $\mathcal{Q}_{[z]}$ of $T_{[z]}M$, $[z]{\in}M$, defined by

$$\mathcal{Q}_{[z]} = \{X \in \mathcal{C}_{[z]} \mid AX \in T_zM \text{ for all } A \in \mathfrak{A}_{[z]}\}.$$

Thus if the unit normal vector field N is $\mathfrak{A}$-isotropic it can be easily checked that the orthogonal complement $\mathcal{Q}^{\perp}_{[z]} = \mathcal{C}_{[z]} \ominus \mathcal{Q}_{[z]}$, $[z]{\in}M$, of the distribution $\mathcal{Q}$ in the complex subbundle $\mathcal{C}$, becomes $\mathcal{Q}^{\perp}_{[z]} = \text{Span}[A\xi, AN]$. Here the vector fields $A\xi$ and AN belong to the tangent space T_zM, $z{\in}M$ due to Suh [43] and [45]. Then we introduce the following lemma for real hypersurfaces in the complex hyperbolic quadric Q^{m*}.

Lemma 3.1. *Let M be a real hypersurface in the complex hyperbolic quadric Q^{m*}. Then the following statements are equivalent:*

(i) *The normal vector $N_{[z]}$ of M is $\mathfrak{A}$-principal,*
(ii) $\mathcal{Q}_{[z]} = \mathcal{C}_{[z]}$,
(iii) *There exists a real structure $A \in \mathfrak{A}_{[z]}$ such that $AN_{[z]} \in \mathbb{C}\nu_{[z]}M$, where $\mathfrak{A}_{[z]} = \{A_{\lambda\bar{z}} | \lambda{\in}S^1{\subset}\mathbb{C}\}$, $[z]{\in}Q^{m*}$.*

Then from the curvature tensor $\bar{R}(X,Y)Z$ in (2.1) in section 2 we get the equation of Codazzi as follows:

$$\begin{aligned} g((\nabla_X S)Y - (\nabla_Y S)X, Z) = & -\eta(X)g(\phi Y, Z) + \eta(Y)g(\phi X, Z) + 2\eta(Z)g(\phi X, Y) \\ & - g(X, AN)g(AY, Z) + g(Y, AN)g(AX, Z) \\ & - g(X, A\xi)g(JAY, Z) + g(Y, A\xi)g(JAX, Z), \end{aligned}$$

where S denotes the shape operator of M in Q^{m*}.

At each point $z \in M$ we can choose $A \in \mathfrak{A}_z$ such that

$$N = \cos(t)Z_1 + \sin(t)JZ_2$$

for some orthonormal vectors $Z_1, Z_2 \in V(A)$ and $0 \leq t \leq \frac{\pi}{4}$ (see Proposition 3 in [35]). Note that t is a function on M. First of all, since $\xi = -JN$, we have

$$\begin{aligned} AN &= \cos(t)Z_1 - \sin(t)JZ_2, \\ \xi &= \sin(t)Z_2 - \cos(t)JZ_1, \\ A\xi &= \sin(t)Z_2 + \cos(t)JZ_1. \end{aligned} \tag{3.1}$$

This implies $g(\xi, AN) = 0$ and hence

$$\begin{aligned} 0 = & 2g(S\phi SX, Y) - \alpha g((\phi S + S\phi)X, Y) + 2g(\phi X, Y) \\ & - g(X, AN)g(Y, A\xi) + g(Y, AN)g(X, A\xi) \\ & + g(X, A\xi)g(JY, A\xi) - g(Y, A\xi)g(JX, A\xi) \\ & + 2g(X, AN)g(\xi, A\xi)\eta(Y) - 2g(Y, AN)g(\xi, A\xi)\eta(X). \end{aligned} \tag{3.2}$$

On the other hand, we have $JA\xi = -AJ\xi = -AN$, and inserting this formula into the previous equation implies

Lemma 3.2. *(Suh [45]) Let M be a Hopf hypersurface in Q^{m*} with (local) unit normal vector field N. For each point $z \in M$ we choose $A \in \mathfrak{A}_z$ such that $N_z = \cos(t)Z_1 + \sin(t)JZ_2$ holds for some orthonormal vectors $Z_1, Z_2 \in V(A)$ and $0 \leq t \leq \frac{\pi}{4}$. Then*

$$\begin{aligned} 0 = & 2g(S\phi SX, Y) - \alpha g((\phi S + S\phi)X, Y) + 2g(\phi X, Y) - 2g(X, AN)g(Y, A\xi) \\ & + 2g(Y, AN)g(X, A\xi) - 2g(\xi, A\xi)\{g(Y, AN)\eta(X) - g(X, AN)\eta(Y)\} \end{aligned}$$

holds for all vector fields X and Y on M.

4. Some important key lemmas

By the equation of Gauss, the curvature tensor $R(X,Y)Z$ for a real hypersurface M in Q^{m*} in (2.1) induced from the curvature tensor $\bar{R}$ of Q^{m*} can be described in terms of the complex structure J and the complex conjugations $A \in \mathfrak{A}$ as follows: for any tangent vector fields X, Y and Z on M in Q^{m*}

$$\begin{aligned} R(X,Y)Z = &-g(Y,Z)X + g(X,Z)Y - g(JY,Z)(JX)^T + g(JX,Z)(JY)^T \\ &+ 2g(JX,Y)(JZ)^T - g(AY,Z)(AX)^T + g(AX,Z)(AY)^T \\ &- g(JAY,Z)(JAX)^T + g(JAX,Z)(JAY)^T \\ &+ g(SY,Z)SX - g(SX,Z)SY, \end{aligned} \tag{4.1}$$

where $(\cdots)^T$ denotes the tangential component of the vector $(\cdots)$ in Q^{m*}.

Let $\{e_1, e_2, \cdots, e_{2m-1}, e_{2m} := N\}$ be a basis of the tangent vector space T_zQ^{m*} of Q^{m*} at $z \in Q^{m*}$. By the definition of the Ricci operator of M in Q^{m*}, it is given by $\mathrm{Ric}(X) = \Sigma_{i=1}^{2m-1} R(X,e_i)e_i$. So from contracting the curvature tensor in (4.1) it follows that

$$\begin{aligned} \mathrm{Ric}(X) = &-(2m-1)X + 3\eta(X)\xi + g(AN,N)(AX)^T - g(AX,N)(AN)^T \\ &+ g(JAN,N)(JAX)^T - g(JAX,N)(JAN)^T \\ &+ (\mathrm{Tr}\, S)SX - S^2X. \end{aligned} \tag{4.2}$$

In the proof of our Main Theorems 1 and 2, we want to give more information on Hopf hypersurfaces in the complex quadric with either $\mathfrak{A}$-principal or $\mathfrak{A}$-isotropic normal vector field. By using the formulas given in section 3 we want to introduce three important lemmas for real hypersurfaces in the complex hyperbolic quadric Q^{m*}. First let us introduce one of them due to Suh, Pérez and Woo [50] as follows:

Lemma 4.1. *Let M be a Hopf hypersurface in the complex hyperbolic quadric Q^{m*}, $m \geq 3$, such that the normal vector field N is $\mathfrak{A}$-principal everywhere. Let A be the section of the S^1-bundle $\mathfrak{A}$ so that $AN = N$ holds. Then we have the following:*

(i) The Reeb flow function α is constant.

(ii) If $X \in \mathcal{C}$ is a principal vector field on M with principal curvature λ, then $\alpha = \pm 2$, $\lambda = \pm 1$ for $\alpha = 2\lambda$ or ϕX is a principal curvature vector with principal curvature $\mu = \frac{\alpha\lambda - 2}{2\lambda - \alpha}$ for $\alpha \neq 2\lambda$.

(iii) $\bar{\nabla}_X A = 0$ for any $X \in \mathcal{C}$, where $\bar{\nabla}$ denotes a connection on Q^{m*}.

(iv) $ASX = SX$ for any $X \in \mathcal{C}$.

(v) The shape operator commutes with the complex conjugation, that is, $AS = SA$.

(vi) $q(\xi) = 2\alpha$.

We also introduce a well known lemma for a real hypersurface in Q^{m*} due to Suh [43] and [45] as follows:

Lemma 4.2. *Let M be a Hopf hypersurface in the complex hyperbolic quadric Q^{m*}, $m \geq 3$. Then the tensor field*

$$2S\phi S - \alpha(\phi S + S\phi)$$

leaves $\mathcal{Q}$ and $\mathcal{C} \ominus \mathcal{Q}$ invariant and we have

$$2S\phi S - \alpha(\phi S + S\phi) = -2\phi \text{ on } \mathcal{Q}$$

and

$$2S\phi S - \alpha(\phi S + S\phi) = -2\delta^2\phi \text{ on } \mathcal{C} \ominus \mathcal{Q},$$

where the function δ is given by

$$\delta = g(N, AN) = -g(\xi, A\xi) = -(\sin^2 t - \cos^2 t) = \cos 2t.$$

If the normal vector field N is $\mathfrak{A}$-isotropic, the tangent vector bundle TM is decomposed by

$$TM = \text{Span}\{\xi\} \oplus \text{Span}\{A\xi, AN\} \oplus \mathcal{Q},$$

where $\mathcal{C} \ominus \mathcal{Q} = \mathcal{Q}^{\perp} = \text{Span}\{A\xi, AN\}$. Then by Lemma 4.2, for any $X \in T_{\lambda} \subset \mathcal{Q}$ we get the following

$$(2\lambda - \alpha)S\phi X = (\alpha\lambda - 2)\phi X, \tag{4.3}$$

where T_{λ} denotes the eigenspace corresponding to the principal curvature λ. If $2\lambda - \alpha = 0$, then (4.3) gives $\alpha\lambda - 2 = 0$. So it implies that $\alpha = \pm 2$ and $\lambda = \pm 1$. Otherwise we have $\phi X \in T_{\mu} \subset \mathcal{Q}$, where $\mu = \frac{\alpha\lambda - 2}{2\lambda - \alpha}$. Then finally, summing up the above facts, we can assert the following due to Suh, Lee and Woo [49]:

Lemma 4.3. *Let M be a Hopf hypersurface with $\mathfrak{A}$-isotropic unit normal vector field N in the complex hyperbolic quadric Q^{m*}. Then the tangent vector fields $A\xi$ and $AN = -\phi A\xi$ of M are principal under the shape operator S such that $SA\xi = 0$ and $SAN = S\phi A\xi = 0$. Moreover, if $X \in \mathcal{Q}$ is a principal vector field of M with principal curvature λ, then $\alpha = \pm 1$ and $\lambda = \pm 1$ for $2\lambda = \alpha$, or its corresponding vector field ϕX is also principal such that $S\phi X = \frac{\alpha\lambda - 2}{2\lambda - \alpha}\phi X$ for $2\lambda \neq \alpha$.*

5. Pseudo-Ricci-Bourguignon soliton real hypersurfaces with $\mathfrak{A}$-principal normal vector field

Now in this section we want to check whether the pseudo-Ricci-Bourguignon soliton real hypersurface M in the complex hyperbolic quadric Q^{m*} satisfy that the unit normal vector field N is singular, that is, N is either $\mathfrak{A}$-principal or $\mathfrak{A}$-isotropic.

If $(M, \xi, \eta, \Omega, \theta, \gamma, g)$ is a pseudo-Ricci-Bourguignon soliton,

$$\frac{1}{2}(\mathcal{L}_{\xi} g)(X, Y) + \text{Ric}(X, Y) + \psi\eta(X)\eta(Y) = (\Omega + \theta\gamma)g(X, Y), \tag{5.1}$$

where Ω is the Ricci soliton constant, θ any constant and γ the scalar curvature on M, then the first term in (5.1) is given by

$$\frac{1}{2}(\mathcal{L}_{\xi} g)(X, Y) = \frac{1}{2}g((\phi S - S\phi)X, Y), \tag{5.2}$$

because for any vector fields X and Y on M in Q^{m*} we get

$$\begin{aligned}(\mathcal{L}_{\xi} g)(X, Y) &= \xi(g(X, Y)) - g(\mathcal{L}_{\xi}X, Y) - g(X, \mathcal{L}_{\xi}Y) \\ &= g(\nabla_{\xi}X, Y) + g(X, \nabla_{\xi}Y) - g([\xi, X], Y) - g(X, [\xi, Y]) \\ &= g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) \\ &= g((\phi S - S\phi)X, Y).\end{aligned}$$

Then the formula (5.1) can be given by

$$\text{Ric}(X) = \frac{1}{2}(S\phi - \phi S)X - \psi\eta(X)\xi + (\Omega + \theta\gamma)X. \tag{5.3}$$

From this, by applying the structure tensor ϕ to both sides, we get the following two formulas

$$\text{Ric}(\phi X) = \frac{1}{2}(S\phi^2 - \phi S\phi)X - \psi\eta(\phi X)\xi + (\Omega + \theta\gamma)\phi X,$$

and

$$\phi\text{Ric}(X) = \frac{1}{2}(\phi S\phi - \phi^2 S)X - \psi\eta(X)\phi\xi + (\Omega + \theta\gamma)\phi X.$$

By using the almost contact structure (ϕ, ξ, η, g) in the right side above, we know that the *generalized pseudo-anti-commuting property* holds as follows:

$$\mathrm{Ric}(\phi X) + \phi \mathrm{Ric}(X) = 2(\Omega + \theta\gamma)\phi X. \tag{5.4}$$

Now in this section we want to introduce an important proposition which will be used in the proof of our Main Theorem 1 as follows:

Proposition 5.1. *(see [45]) Let M be a Hopf real hypersurface in the complex hyperbolic quadric Q^{m*} such that $(M, \xi, \eta, \Omega, \theta, \gamma, g)$ is a pseudo-Ricci-Bourguignon soliton. Then the unit normal vector field N becomes singular, that is, N is either $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal.*

The proof of this proposition is similar to the proof of Proposition 5.3 in [45]. So we want to omit this proof. Then by virtue of Proposition 5.1, naturally we can consider two cases that N is $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal for a Hopf pseudo-Ricci-Bourguignon soliton $(M, \xi, \nu, \rho, \gamma, g)$ in Q^{m*}. So in section 5 we give a complete classification of pseudo-Ricci-Bourguignon soliton $(M, \xi, \eta, \Omega, \theta, \gamma, g)$ real hypersurfaces in Q^{m*} when the unit normal vector field is $\mathfrak{A}$-principal and in section 6 we will complete the proof of our Main Theorem 1 for the case where N is $\mathfrak{A}$-isotropic.

In order to do this, we will consider a remarkable proposition which will be useful in the proof of Main Theorem 1. By virtue of Lemma 4.1, some geometric properties of Hopf hypersurfaces in Q^{m*} are being investigated when the unit normal vector field N is $\mathfrak{A}$-principal. Among them, as a new characterization of contact hypersurfaces in the complex hyperbolic quadric Q^{m*}, we want to give one remarkable result as follows:

Proposition 5.2. *(see [49]) Let M be a Hopf real hypersurface in the complex hyperbolic quadric Q^{m*}, $m \geq 3$. Then M has an $\mathfrak{A}$-principal singular normal vector field N if and only if M is locally congruent to one of the following:*

(i) a tube of radius r around the Hermitian symmetric space $Q^{*(m-1)}$ which is imbedded in Q^{m*} as a totally geodesic complex hypersurface,

(ii) a horosphere in Q^{m*} whose center at infinity is the equivalence class of an $\mathfrak{A}$-principal geodesic in Q^{m*},

(iii) a tube of radius r around the m-dimensional real hyperbolic space $\mathbb{R}H^m$ which is embedded in Q^{m*} as a real space form of Q^{m*}.

Then by virtue of Proposition 5.2 a Hopf pseudo-Ricci-Bourguignon soliton real hypersurface M with $\mathfrak{A}$-principal unit normal vector field N in Q^{m*} can be regarded as a contact hypersurface in Theorem B. By Lemma 4.1 in section 4, the expression of the shape operator S of M in Q^{m*} is given either by

$$S = \begin{bmatrix} 2 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & 0 \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 1 \end{bmatrix},$$

or

$$S = \begin{bmatrix} \alpha & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \frac{2}{\alpha} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & 0 \\ 0 & 0 & \cdots & \frac{2}{\alpha} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix},$$

where α, $\frac{2}{\alpha}$ and 0 are the principal curvatures with multiplicities $1, m-1$ and $m-1$, respectively. This means that the shape operator satisfies $S\phi + \phi S = k\phi$, where $k = \frac{2}{\alpha}$. Then by Theorem B due to Suh [43], M is locally congruent to a tube of radius r around the Hermitian symmetric space Q^{m-1^*}, a horosphere in Q^{m^*} whose center at infinity is the equivalence class of an $\mathfrak{A}$-principal geodesic in Q^{m^*}, or a tube of radius r around the m-dimensional real hyperbolic space $\mathbb{R}H^m$ which is embedded in Q^{m^*} as a real space form of Q^{m^*}.

On the other hand, for a contact hypersurface in complex hyperbolic quadric Q^{m^*}, the contact constant function k is given by $\alpha k = 2$. That is, $k = \frac{2}{\alpha}$.

Since the unit normal vector field N of M in Q^{m^*} is $\mathfrak{A}$-principal, that is, $AN = N$, and $A\xi = -\xi$, the Ricci tensor becomes

$$\mathrm{Ric}(X) = -(2m-1)X + 2\eta(X)\xi + AX + hSX - S^2X, \tag{5.5}$$

where $h = \mathrm{Tr}S$ denotes the trace of the shape operator and the mean curvature of M in Q^{m^*}. From this it follows that

$$\begin{aligned}(\mathrm{Ric}\phi + \phi\mathrm{Ric})X = &-(4m-2)\phi X + (A\phi + \phi A)X \\ &+ h(S\phi + \phi S)X - (S^2\phi + \phi S^2)X.\end{aligned} \tag{5.6}$$

We know that M is contact if and only if $S\phi + \phi S = k\phi$, $k \neq 0$ constant. So it implies $S\phi S + \phi S^2 = k\phi S$ and $S^2\phi + S\phi S = kS\phi$ respectively, we get the following

$$S^2\phi + 2S\phi S + \phi S^2 = k(\phi S + S\phi).$$

Moreover, by virtue of the results given in Lemmas 3.1, 3.2 and 4.2 due to Suh [43] and [45], the following holds for a contact hypersurface in Q^{m^*} with $\mathfrak{A}$-principal normal vector field

$$\begin{aligned}2S\phi SX &= \alpha(S\phi + \phi S)X - 2\phi X \\ &= (\alpha k - 2)\phi X \\ &= 0,\end{aligned}$$

where in the above third equality we have used $\alpha k = 2$.

On the other hand, let us show that $(A\phi + \phi A)X = 0$. From the anti-commuting of $AJ = -JA$ between the Kähler structure J and the real structure A it follows that

$$\begin{aligned}0 &= AJX + JAX \\ &= A(\phi X + \eta(X)N) + \phi AX + \eta(AX)N \\ &= A\phi X + \phi AX + \eta(X)N + \eta(AX)N.\end{aligned}$$

Then, as N is $\mathfrak{A}$-principal, we have $(A\phi + \phi A)X = 0$, because $A\phi X$ and AX are tangent vector fields from $g(A\phi X, N) = g(\phi X, N) = 0$ and $g(AX, N) = g(X, AN) = g(X, N) = 0$ for any tangent vector field X on M. Then from this property, together with $S^2\phi + \phi S^2 = k^2\phi$, (5.5) becomes

$$(\mathrm{Ric}\phi + \phi\mathrm{Ric})X = \{-(4m-2) + hk - k^2\}\phi X. \tag{5.7}$$

Since $\mathrm{Ric}(\xi,\xi)+\frac{1}{2}(\mathcal{L}_\xi g)(\xi,\xi)+\psi=\Omega+\theta\gamma$, the Ricci-Bourguignon soliton constant $\Omega+\theta\gamma$, if the unit normal $\mathfrak{A}$-principal, is given by

$$\begin{aligned}\Omega+\theta\gamma &= \mathrm{Ric}(\xi,\xi)+\psi=g(\mathrm{Ric}(\xi),\xi)+\psi=-2(m-1)+h\alpha-\alpha^2+\psi\\ &=-2(m-1)+2(m-1)+\psi=\psi,\end{aligned} \tag{5.8}$$

where we have used $h=\alpha+\frac{2}{\alpha}(m-1)$, and (5.2) and (5.4), and $A\xi=-\xi$. By virtue of the pseudo-Ricci-Bourguignon soliton $(M,\xi,\eta,\Omega,\theta,\gamma,g)$ in the complex hyperbolic quadric Q^{m*}, and using (5.4) and (5.8), we have

$$(\mathrm{Ric}\phi+\phi\mathrm{Ric})X=2\psi\phi X. \tag{5.9}$$

On the other hand, by (5.6), for a contact real hypersurface in the complex hyperbolic quadric Q^{m*}, we give the following

$$\begin{aligned}(\mathrm{Ric}\phi+\phi\mathrm{Ric})X &= \{-(4m-2)+hk-k^2\}\phi X\\ &=\Big\{-(4m-2)+\{\alpha+\frac{2}{\alpha}(m-1)\}k-k^2\Big\}\phi X\\ &=\{-4(m-1)+(m-2)k^2\}\phi X,\end{aligned} \tag{5.10}$$

where in the third equality we have used $\alpha k=2$.

Then by comparing two equations (5.8) and (5.9), it gives us

$$0\leq k^2=\frac{4(m-1)}{m-2}+\frac{2}{m-2}\psi. \tag{5.11}$$

Then by Theorem B if N is the $\mathfrak{A}$-principal, a contact hypersurface M is locally congruent to a tube over a totally geodesic and totally complex submanifold Q^{m-1^*} in Q^{m*}, a horosphere , or a tube over a totally geodesic totally real submanifold $\mathbb{R}H^m$ in complex hyperbolic quadric Q^{m*}. From (5.10) and $\tanh^2(\sqrt{2}r)\leq 1$ we can determine the soliton constant ψ. Moreover, in such tubes the Reeb function α, $\lambda=0$, and the non-vanishing principal curvature μ are respectively given as follows:

Lemma 5.3. *Let M be a contact real hypersurface in the complex hyperbolic quadric Q^{m*}. Then its principal curvatures are respectively given by*

(i) $\alpha=\sqrt{2}\coth(\sqrt{2}r)$, *and* $\lambda=0$, $\mu=\sqrt{2}\tanh(\sqrt{2}r)$, *and* $-2(m-1)\leq\psi\leq-m$,

(ii) $\alpha=\sqrt{2}$, *and* $\lambda=0$, $\mu=\sqrt{2}$, *and* $\psi=-m$,

(iii) $\alpha=\sqrt{2}\tanh(\sqrt{2}r)$, *and* $\lambda=0$, $\mu=\sqrt{2}\coth(\sqrt{2}r)$ *and* $\psi\geq m$.

Proof. For the first (i): $k=\frac{2}{\alpha}=\sqrt{2}\tanh(\sqrt{2}r)$, from this, together with (5.11), it follows that

$$1\leq\tanh^2(\sqrt{2}r)=\frac{2(m-1)}{m-2}+\frac{1}{m-2}\psi\geq 0.$$

Then it follows that $-2(m-1)\leq\psi\leq-m$.

For the second (ii): $k^2=2$. From this, together with (5.11), it follows that $\psi=-m$.

For the third (iii): $k^2=2\coth^2(\sqrt{2}r)$. From this and (5.11) it follows that $\psi\geq m$. □

Then from (5.1), (5.4) and (5.7) it follows that

$$-(2m-1)X+2\eta(X)\xi+AX+hSX-S^2X-\frac{1}{2}(S\phi-\phi S)X+\psi\eta(X)\xi=\psi X. \tag{5.12}$$

Consider $X \in T_\lambda$, $\lambda = 0$. Then by Lemma 4.1, $S\phi X = \frac{2}{\alpha}\phi X$.

Now we consider the following subcases.

Subcase 3.1. $X \in V(A) \cap T_\lambda$, $\lambda = 0$.

Then $SX = 0$ and $AX = X$. So (5.12) becomes $-(2m-1)X + X - \frac{1}{2}\mu\phi X = \psi X$. From this, by taking the inner product with ϕX, we get $\mu = \sqrt{2}\coth\sqrt{2}r = 0$. But this gives a contradiction.

Subcase 3.2. $X \in JV(A) \cap T_\lambda$, $\lambda = 0$.

In this subcase, $AX = -X$, and $SX = 0$. Then (5.12) gives the following

$$-(2m-1)X - X - \frac{1}{2}\mu\phi X = \psi X.$$

So it implies also a contradiction.

Subcase 3.3. $X \in (V(A) \oplus JV(A)) \cap T_\lambda$, $\lambda = 0$.

Now we may put $X = \frac{1}{\sqrt{2}}(Y+Z)$, $Y, Z \in T_\lambda$, $\lambda = 0$, where $Y \in V(A)$ and $Z \in JV(A)$. Of course, $SY = SZ = 0$ and $S\phi Y = \mu\phi Y$, and $S\phi Z = \mu\phi Z$, $\mu = \sqrt{2}\coth(\sqrt{2}r)$ in Lemma 5.3. Now we can use $g(Y, \phi Z) = 0$, because $\phi Z \in T_\mu$. Then $AX = \frac{1}{\sqrt{2}}(Y - Z)$. Moreover, we get the following

$$\begin{aligned}\frac{1}{2}(S\phi - \phi S)X &= \frac{1}{2\sqrt{2}}(S\phi - \phi S)(Y+Z) = \frac{1}{2\sqrt{2}}(S\phi Y + S\phi Z)\\ &= \frac{1}{2}\coth(\sqrt{2}r)(\phi Y + \phi Z) = \frac{1}{\sqrt{2}\alpha}(\phi Y + \phi Z).\end{aligned}$$

Then by virtue of this formula, (5.12) gives that

$$-(2m-1)(Y+Z) + (Y-Z) - \frac{1}{\alpha}(\phi Y + \phi Z) = \psi X = \frac{1}{\sqrt{2}}\psi(Y+Z).$$

From this, let us take the inner product with the vector field ϕY and use $g(\phi Y, Z) = 0$, because $\phi Y \in T_\mu$ and $Z \in T_\lambda$. Then we get

$$\tanh(\sqrt{2}r) = 0 \text{ or } \coth(\sqrt{2}r) = 0.$$

But $0 < \tanh(\sqrt{2}r) \leq 1$ or $\coth(\sqrt{2}r) \geq 1$ for $r > 0$. This gives a contradiction. So this Subcase 3.3 also can not appear.

Accordingly, by virtue of these three subcases, we can assert that there does not exist any Hopf pseudo-Ricci-Bouguignon soliton in the complex hyperbolic quadric Q^{m*}.

Consequently, summing up Propositions 5.1 and 5.2, and Lemma 5.3, together with above facts, we obtain a complete proof of our Main Theorem 1 when the unit normal vector field N is $\mathfrak{A}$-principal.

6. Pseudo-Ricci-Bourguignon soliton real hypersurfaces with $\mathfrak{A}$-isotropic normal vector field

By Proposition 5.1, in this section we consider a Hopf pseudo-Ricci-Bourguignon soliton real hypersurface $(M, \xi, \eta, \Omega, \theta, \gamma, g)$ in the complex hyperbolic quadric Q^{m*} with $\mathfrak{A}$-isotropic unit normal vector field. Here, ξ, Ω, ψ, θ and γ denote respectively soliton vector field, soliton constant, any constant, and the scalar curvature of M in Q^{m*}.

Since N is $\mathfrak{A}$-isotropic, we know that $g(A\xi, \xi) = 0$, $g(AN, N) = 0$ and $g(A\xi, N) = 0$. In this case the Ricci tensor becomes

$$\begin{aligned}\text{Ric}(X) = &-(2m-1)X + 3\eta(X)\xi - g(AX, N)AN - g(AX, \xi)A\xi\\ &+ hSX - S^2X.\end{aligned} \tag{6.1}$$

Then our assumption of Hopf pseudo-Ricci-Bourguignon soliton $(M, \xi, \eta, \Omega, \theta, \gamma, g)$ gives

$$\frac{1}{2}(\mathcal{L}_\xi g)(X,Y) + \mathrm{Ric}(X,Y) + \psi\eta(X)\eta(Y) = (\Omega + \theta\gamma)g(X,Y), \tag{6.2}$$

where Ω is the Ricci soliton constant, θ any constant and γ the scalar curvature on M. Then it follows that

$$\mathrm{Ric}(X) + \psi\eta(X)\eta(Y) = \frac{1}{2}(S\phi - \phi S)X + (\Omega + \theta\gamma)X. \tag{6.3}$$

Now let us consider the distribution $\mathcal{Q}^\perp$, which is an orthogonal complement of the maximal $\mathfrak{A}$-invariant subspace $\mathcal{Q}$ in the complex subbundle $\mathcal{C}$ of T_zM, $z \in M$ in Q^m.

Then by Lemmas 4.2 and 4.3 in section 4, we know that $\alpha = \pm 2$ and $\lambda = \pm 1$ for any vector field $X \in T_\lambda \subset \mathcal{Q}$. In this case the expression of the shape operator can be given by

$$S = \begin{bmatrix} 2 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

From this it follows that the shape operator S commutes with the structure tensor ϕ, that is, $S\phi = \phi S$. Or otherwise, by Lemmas 4.2 and 4.3, for $2\lambda \neq \alpha$, the vector field ϕX belongs to the distribution $T_\mu \subset \mathcal{Q}$ such that $SX = \lambda X$ and $S\phi X = \mu\phi X$, where $\mu = \frac{\alpha\lambda - 2}{2\lambda - \alpha}$.

Then putting $X \in T_\lambda$ in (6.3) and using (6.1) gives

$$\frac{1}{2}(\mu - \lambda)\phi X = -(2m-1)X + (h\lambda - \lambda^2)X + \psi\eta(X)\xi - (\Omega + \theta\gamma)X. \tag{6.4}$$

By taking the inner product (6.4) with ϕX, we get $\lambda = \mu = \frac{\alpha\lambda - 2}{2\lambda - \alpha}$ from Lemma 4.1, which gives that

$$\lambda^2 - \alpha\lambda + 1 = 0.$$

Then we have two distinct roots λ_1 and λ_2 such that $\lambda_1 = \frac{\alpha + \sqrt{\alpha^2 - 4}}{2}$ and $\lambda_2 = \frac{\alpha - \sqrt{\alpha^2 - 4}}{2}$. By using this formula, we may put $\lambda_1 = \coth(t)$ and $\lambda_2 = \tanh(t)$ respcetively. Then the expression of the shape operator becomes the following:

$$S = \begin{bmatrix} 2\coth(2t) & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \coth(t) & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \coth(t) & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \tanh(t) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & \tanh(t) \end{bmatrix}.$$

From this expression, together with the horosphere, we can assert the following

Lemma 6.1. *Let M be a Hopf real hypersurface with $\mathfrak{A}$-isotropic normal vector field N in complex hyperbolic quadric Q^{m*}, $m \geq 3$. If it admits the pseudo-Ricci-Bourguignon soliton of type $(M, \xi, \eta, \Omega, \theta, \gamma, g)$, then the shape operator S commutes with the structure tensor ϕ, that is, $S\phi = \phi S$.*

Then summing up two cases for $\alpha = 2\lambda$ and $\alpha \neq 2\lambda$, Lemma 6.1 means that M has isometric Reeb flow. Then by Theorem A we get a complete classification of isometric Reeb flow.

Consequently, by virtue of (6.3), Lemma 6.1 implies that

$$\mathrm{Ric}(X) = (\Omega + \theta\gamma)X - \psi\eta(X)\xi.$$

This means that the Hopf pseudo-Ricci-Bourguignon soliton real hypersurface M in Q^{m*} becomes pseudo-Einstein. But by Theorem C in the introduction, there does not exist a Hopf pseudo-Einstein real hypersurface in Q^{m*}. Then, together with the result in section 5 for $\mathfrak{A}$-principal unit normal vector field, we give a complete proof of our Main Theorem 1 in the introduction.

7. Gradient pseudo-Ricci-Bourguignon soliton real hypersurfaces with $\mathfrak{A}$-isotropic unit normal

Now in this section let us consider a real hypersurface M in the complex hyperbolic quadric Q^{m*} with isometric Reeb flow, and let us assume that it is a gradient pseudo-Ricci-Bourguignon soliton $(M, Df, \eta, \Omega, \theta, \gamma, g)$, where Df denotes the gradient of the smooth function f on M. Then from the $\mathfrak{A}$-isotropy of the unit normal vector field N, it follows that $g(A\xi, \xi) = 0$, $g(AN, N) = 0$ and $g(A\xi, N) = 0$. Then the Ricci operator becomes

$$\mathrm{Ric}(X) = -(2m-1)X + 3\eta(X)\xi - g(AX, N)AN - g(AX, \xi)A\xi + hSX - S^2X.$$

From this, let us put $X = \xi$. Then by virtue of M being Hopf and the properties of $\mathfrak{A}$-isotropy, we get the following

$$\mathrm{Ric}(\xi) = \ell\xi,$$

where the constant ℓ is given by

$$\ell = -2(m-2) + h\alpha - \alpha^2,$$

because we have assumed that M has isometric Reeb flow. Then by taking the covariant derivative we get the following two formulas

$$(\nabla_X \mathrm{Ric})\xi = \ell\phi SX - \mathrm{Ric}(\phi SX),$$

and

$$\begin{aligned}(\nabla_\xi \mathrm{Ric})X = &-g(X, \nabla_\xi(AN))AN - g(X, AN)\nabla_\xi(AN) - g(X, \nabla_\xi(A\xi))A\xi \\ &- g(X, A\xi)\nabla_\xi(A\xi) + h(\nabla_\xi S)X - (\nabla_\xi S^2)X.\end{aligned}$$

From here, let us use the gradient pseudo-Ricci-Bourguignon soliton on M in the complex hyperbolic quadric Q^{m*}. As it is mentioned in the introduction, the gradient pseudo-Ricci-Bourguignon soliton is given by

$$\nabla_X Df + \mathrm{Ric}(X) + \psi\eta(X)\xi = (\Omega + \theta\gamma)X$$

for any vector field X tangent to M in Q^{m*}. From this, together with the above two formulas, it follows that

$$\begin{aligned}R(\xi, Y)Df &= \nabla_\xi\nabla_Y Df - \nabla_Y\nabla_\xi Df - \nabla_{[\xi,Y]}Df \\ &= (\nabla_Y \mathrm{Ric})\xi - (\nabla_\xi \mathrm{Ric})Y + \psi\phi SY \\ &= (\ell + \psi)\phi SY - \mathrm{Ric}(\phi SY) + g(Y, \nabla_\xi(AN))AN + g(Y, AN)\nabla_\xi(AN) \\ &\quad + g(Y, \nabla_\xi(A\xi))A\xi + g(Y, A\xi)\nabla_\xi(A\xi) \\ &\quad - h(\nabla_\xi S)Y + (\nabla_\xi S^2)Y,\end{aligned} \tag{7.1}$$

where we have used that the scalar curvature γ is constant, because the Reeb flow of M is isometric in Theorem A.

On the other hand, since N is $\mathfrak{A}$-isotropic, the vector fields $A\xi$ and AN are tangent vector fields on M. So by the equation of Gauss it follows that

$$\begin{aligned}\nabla_X(A\xi) &= \bar{\nabla}_X(A\xi) - \sigma(X, A\xi)\\ &= \{(\bar{\nabla}_X A)\xi + A\bar{\nabla}_X\xi\} - \sigma(X, A\xi)\\ &= q(X)JA\xi + A\phi SX + g(SX,\xi)AN - g(SX, A\xi)N\end{aligned}$$

and

$$\begin{aligned}\nabla_X(AN) &= \bar{\nabla}_X(AN) - \sigma(X, AN)\\ &= \{(\bar{\nabla}_X A)N + A\bar{\nabla}_X N\} - \sigma(X, AN)\\ &= q(X)JAN - ASX - g(SX, A\xi)N,\end{aligned}$$

where q denotes a certain 1-form defined on M in Q^{m*}. From this, if we put $X = \xi$ into the above two formulas, we get the following

$$\nabla_\xi(A\xi) = -\{q(\xi) - \alpha\}AN, \quad \text{and} \quad \nabla_\xi(AN) = \{q(\xi) - \alpha\}A\xi.$$

Then (7.1) can be written as follows:

$$\begin{aligned}R(\xi, Y)Df =&(\ell + \psi)\phi SY - \mathrm{Ric}(\phi SY) - h(\nabla_\xi S)Y + (\nabla_\xi S^2)Y\\ &+ (q(\xi) - \alpha)\{g(Y, A\xi)AN + g(Y, AN)A\xi\\ &- g(Y, AN)A\xi - g(Y, A\xi)AN\}.\end{aligned} \tag{7.2}$$

On the other hand, from the curvature tensor of M in Q^{m*} it follows that

$$\begin{aligned}R(\xi, Y)Df = &-g(Y, Df)\xi + g(\xi, Df)Y - g(AY, Df)A\xi\\ &+ g(A\xi, Df)AY - g(JAY, Df)JA\xi + g(JA\xi, Df)JAY\\ &+ g(SY, Df)S\xi - g(S\xi, Df)SY.\end{aligned} \tag{7.3}$$

From this formula, we can take $Y \in \mathcal{Q}$ which is orthogonal to ξ, $A\xi$, and AN such that $SY = \coth(r)Y$. Then $Y \in T_\lambda \subset V(A)$, $\lambda = \coth(r)$ and $\phi Y \in T_\lambda \subset V(A)$, because of the commuting property $S\phi = \phi S$ in Theorem A. That is, $SY = \coth(r)Y$, $AY = Y$, $A\phi Y = \phi Y$, $JAY = \phi AY = \phi Y$ and $JA\xi = -AN$. Using these properties into (7.2) and (7.3), it follows that

$$\begin{aligned}&(\ell + \psi)\phi SY - \lambda\mathrm{Ric}(\phi Y) - h(\nabla_\xi S)Y + (\nabla_\xi S^2)Y\\ &= -g(Y, Df)\xi + g(\xi, Df)Y - g(Y, Df)A\xi\\ &\quad + g(A\xi, Df)Y + g(\phi Y, Df)AN - g(AN, Df)\phi Y\\ &\quad + \alpha\lambda g(Y, Df)\xi - \alpha g(\xi, Df)SY,\end{aligned} \tag{7.4}$$

where the Reeb function α is given by $\alpha = 2\coth(2r) = \coth(r) + \tanh(r)$.

From this, by taking the inner product of (7.4) with the Reeb vector field ξ, it follows that

$$0 = (-1 + \alpha\lambda)g(Y, Df) = \coth^2 r g(Y, Df)$$

for any $Y \in T_\lambda$, $\lambda = \coth r$. This means that $g(Y, Df) = 0$ for any $Y \in T_\lambda$.

Next, let us take $Y \in \mathcal{Q}$ which is orthogonal to ξ, $A\xi$, and AN such that $SY = \tanh(r)Y$. Then $Y \in T_\mu \subset JV(A)$, $\mu = \tanh(r)$ and $\phi Y \in T_\mu \subset JV(A)$, because of the commuting property $S\phi = \phi S$ in Theorem A. That is, $SY = \tanh(r)Y$, $AY = -Y$, $A\phi Y = -\phi Y$, $JAY = \phi AY = -\phi Y$ and $JA\xi = -AN$. In this case from (7.2) and (7.3) it follows that

$$\begin{aligned}&(\ell + \psi)\phi SY - \mu \mathrm{Ric}(\phi Y) - h(\nabla_\xi S)Y + (\nabla_\xi S^2)Y \\ &= -g(Y, Df)\xi + g(\xi, Df)Y + g(Y, Df)A\xi \\ &\quad - g(A\xi, Df)Y - g(\phi Y, Df)AN + g(AN, Df)\phi Y \\ &\quad + \alpha\mu g(Y, Df)\xi - \alpha g(\xi, Df)SY.\end{aligned} \tag{7.5}$$

Then, by taking also the inner product (7.5) with the Reeb vector field ξ, we have

$$0 = (-1 + \alpha\mu)g(Y, Df) = \tanh^2 r g(Y, Df)$$

for any $Y \in T_\mu$. It gives $g(Y, Df) = 0$ for $Y \in T_\mu$. From this together with $g(Y, Df) = 0$ for any $Y \in T_\lambda$, the gradient vector field Df can be written as

$$Df = g(Df, \xi)\xi + g(Df, AN)AN + g(Df, A\xi)A\xi. \tag{7.6}$$

Moreover, by taking the inner product of (7.4) with the vector fields $Y \in T_\lambda$ and $\phi Y \in T_\lambda$ respectively , we get the following

$$g(Df, A\xi) = -(1 - \alpha\lambda)g(\xi, Df) = \coth^2(r)g(\xi, Df)$$

and

$$g(AN, Df) = -(\ell + \psi)\coth(r) + \coth(r)\{-(2m - 1) + h\coth(r) - \coth^2(r)\} = f(r),$$

where we have denoted the right side by $f(r)$ and $\ell = -2(m - 2) + h\alpha - \alpha^2$, and used $g(\mathrm{Ric}(\phi Y), Y) = 0$, $g((\nabla_\xi S)Y, Y) = 0$, and $g((\nabla_\xi S^2)Y, Y) = 0$ for any $Y \in T_\lambda$. Then (7.6) and these two formulas imply the following

$$Df = g(\xi, Df)\{\xi + \coth^2(r)A\xi\} + f(r)AN. \tag{7.7}$$

On the other hand, by taking the inner product of (7.5) with $Y \in T_\mu$, $\mu = \tanh r$ and $\phi Y \in T_\lambda$ respectively, we have

$$g(Df, A\xi) = (1 - \alpha\mu)g(\xi, Df) = -\tanh^2(r)g(\xi, Df)$$

and

$$g(AN, Df) = (\ell + \psi)\tanh(r) - \tanh(r)\{-(2m - 1) + h\tanh(r) - \tanh^2(r)\} = g(r),$$

where the right side is denoted by $g(r)$, and used $g(\mathrm{Ric}(\phi Y), Y) = 0$, $g((\nabla_\xi S)Y, Y) = 0$, and $g((\nabla_\xi S^2)Y, Y) = 0$ for any $Y \in T_\mu$. Then these two formulas and (7.6) give another expression of the gradient vector field Df as follows:

$$Df = g(\xi, Df)\{\xi - \tanh^2(r)A\xi\} + g(r)AN. \tag{7.8}$$

Then substracting the equations (7.8) from (7.7) implies

$$(\coth^2(r) + \tanh^2(r))A\xi + (f(r) - g(r))AN = 0.$$

By virtue of the independency of the vector fields $A\xi$ and AN, we know that $\coth(r) = \tanh(r) = 0$. But this gives a contradiction. From this, we give a complete proof of our Main Theorem 2 in the introduction

8. Gradient pseudo-Ricci-Bourguignon soliton real hypersurfaces with $\mathfrak{A}$-principal unit normal vector field

In this section, we want to give a property for a gradient pseudo-Ricci-Bourguignon soliton on a contact real hypersurface M in the complex hyperbolic quadric Q^{m*}. Then the gradient Ricci-Bourguignon soliton $(M, Df, \Omega, \theta, \gamma, g)$ satisfies the following for any tangent vector field X on M

$$\nabla_X Df + \mathrm{Ric}(X) + \psi\eta(X)\xi = (\Omega + \theta\gamma)X. \tag{8.1}$$

Then by differentiating (8.1), the curvature tensor of grad$f$ is given by the following

$$\begin{aligned}R(X,Y)Df =&\nabla_X\nabla_Y Df - \nabla_Y\nabla_X Df - \nabla_{[X,Y]}Df\\ =&-(\nabla_X\mathrm{Ric})Y - \mathrm{Ric}(\nabla_X Y) - \psi(\nabla_X\eta)(Y)\xi - \psi\eta(\nabla_X Y)\xi\\ &- \psi\eta(Y)\nabla_X\xi + (\Omega+\theta\gamma)\nabla_X Y\\ &+ (\nabla_Y\mathrm{Ric})X + \mathrm{Ric}(\nabla_Y X) + \psi(\nabla_Y\eta)(X)\xi + \psi\eta(\nabla_Y X)\xi\\ &+ \psi\eta(X)\nabla_Y\xi - (\Omega+\theta\gamma)\nabla_X Y\\ &+ \mathrm{Ric}([X,Y]) - (\Omega+\theta\gamma)[X,Y] + \psi\eta([X,Y])\xi\\ =&(\nabla_Y\mathrm{Ric})X - (\nabla_X\mathrm{Ric})Y - \psi(\nabla_X\eta)(Y)\xi + \psi(\nabla_Y\eta)(X)\xi\\ &- \psi\eta(Y)\nabla_X\xi + \psi\eta(X)\nabla_Y\xi\end{aligned} \tag{8.2}$$

where we have used that the functions Ω, θ and the scalar curvature γ are constant on M in Q^{m*}.

Now let us assume that M is a contact real hypersurface in Q^{m*}. Then it is Hopf and $\mathfrak{A}$-principal. So the Ricci operator is given by

$$\mathrm{Ric}(X) = -(2m-1)X + 2\eta(X)\xi + AX + hSX - S^2X$$

for any tangent vector field X on M. From this, let us put $X=\xi$. Then M being Hopf and $A\xi=-\xi$ imply

$$\mathrm{Ric}(\xi) = d\xi,$$

where $d = -2(m-1) + h\alpha - \alpha^2$ is constant, and the mean curvature $h = \mathrm{Tr}S$ is constant for a contact hypersurface M in Q^{m*}. Then by taking the covariant derivative of the Ricci operator, we have

$$(\nabla_X\mathrm{Ric})\xi = \nabla_X(\mathrm{Ric}(\xi)) - \mathrm{Ric}(\nabla_X\xi) = d\phi SX - \mathrm{Ric}(\phi SX),$$

and

$$\begin{aligned}(\nabla_\xi\mathrm{Ric})X &= \nabla_\xi(\mathrm{Ric}X) - \mathrm{Ric}(\nabla_\xi X)\\ &= -(\nabla_\xi A)X + h(\nabla_\xi S)X - (\nabla_\xi S^2)X\\ &= h(\nabla_\xi S)X - (\nabla_\xi S^2)X,\end{aligned}$$

where we have used $\nabla_\xi A = 0$, because $(\nabla_\xi A)A + A(\nabla_\xi A) = 2(\nabla_\xi A)A = 0$ from $A^2 = I$ and $A \in \mathrm{End}(TQ^m)$ for an $\mathfrak{A}$-principal unit normal N. From (8.2), together with above formula, by putting $X=\xi$ we have the following for a contact hypersurface M in Q^{m*}

$$\begin{aligned}R(\xi,Y)Df =&(\nabla_Y\mathrm{Ric})\xi - (\nabla_\xi\mathrm{Ric})Y\\ &- \psi(\nabla_\xi\eta)(Y)\xi + \psi(\nabla_Y\eta)(\xi)\xi - \psi\eta(Y)\nabla_\xi\xi + \psi\eta(\xi)\nabla_Y\xi\\ =&(d+\psi)\phi AY - \mathrm{Ric}(\phi AY) - h(\nabla_\xi A)Y + (\nabla_\xi A^2)Y.\end{aligned} \tag{8.3}$$

where we have used that the scalar curvature γ of contact real hypersurfaces in Theorem B is constant. Then the diagonalization of the shape operator S of the contact real hypersurface in complex hyperbolic quadric Q^{m*} is given by

$$S = \begin{bmatrix} \alpha & 0 & \cdots & 0 & 0 & \cdots & 0\\ 0 & \frac{2}{\alpha} & \cdots & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & 0\\ 0 & 0 & \cdots & \frac{2}{\alpha} & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Here by Lemma 5.3 the principal curvatures are given by $\alpha = \sqrt{2}\coth(\sqrt{2}r)$, $\lambda = \frac{2}{\alpha} = \sqrt{2}\tanh(\sqrt{2}r)$ and $\mu = 0$ for the case (i), $\alpha = \sqrt{2}$, $\lambda = \sqrt{2}$ and $\mu = 0$, for the case (ii) and $\alpha = \sqrt{2}\coth(\sqrt{2}r)$, $\lambda = \frac{2}{\alpha} = \sqrt{2}\tanh(\sqrt{2}r)$ and $\mu = 0$ for the case (iii) in Theorem B in the Introduction with multiplicities 1, $2m-1$ and $2m-1$ respectively. All of these principal curvatures satisfy $\alpha\lambda = 2$.

On the other hand, the curvature tensor $R(X, Y)Z$ of M induced from the curvature tensor $\bar{R}(X, Y)Z$ of the complex hyperbolic quadric Q^{m*} gives

$$\begin{aligned} R(\xi, Y)Df &= -g(Y, Df)\xi + g(\xi, Df)Y - g(AY, Df)A\xi + g(A\xi, Df)AY \\ &\quad - g(JAY, Df)\phi A\xi + g(\phi A\xi, Df)JAY \\ &\quad + g(SY, Df)S\xi - g(S\xi, Df)SY \\ &= \alpha g(SY, Df)\xi - \alpha\eta(Df)SY \end{aligned} \tag{8.4}$$

for any $Y \in T_\lambda \subset V(A)$, $\lambda = \sqrt{2}\tanh(\sqrt{2}r)$, $\lambda = \sqrt{2}$, or $\lambda = \sqrt{2}\coth(\sqrt{2}r)$ such that $SY = \lambda Y$, $AY = Y$ and $A\xi = -\xi$ for a contact real hypersurface M in the complex hyperbolic quadric Q^{m*}. Consequently, (8.3) and (8.4) give

$$(\ell + \psi)\phi SY - \mathrm{Ric}(\phi SY) - h(\nabla_\xi S)Y + (\nabla_\xi S^2)Y = \alpha g(SY, Df)\xi - \alpha\eta(Df)SY.$$

From this, by taking the inner product with the Reeb vector field ξ, we have

$$\alpha g(SY, Df) - \alpha^2\eta(Df)\eta(Y) = 0. \tag{8.5}$$

So for any $Y \in T_\lambda \subset V(A)$ in (8.5) it follows that

$$0 = \alpha g(SY, Df) = \alpha\lambda g(Y, Df) = 2g(Y, Df). \tag{8.6}$$

Accordingly, the gradient vector field Df is orthogonal to the eigenspace T_λ for principal curvatures, $\lambda = \sqrt{2}\tanh\sqrt{2}r$, $\lambda = \sqrt{2}$, and $\lambda = \sqrt{2}\coth\sqrt{2}r$, respectively.

Next, we consider $Y \in T_\mu \subset JV(A)$, $\mu = 0$. Then it follows that $SY = \mu Y = 0$, $A\xi = -\xi$ and $AY = -Y$. Then these properties, (8.3) and (8.4) imply the following

$$(d + \psi)\phi SY - \mathrm{Ric}(\phi SY) - h(\nabla_\xi S)Y + (\nabla_\xi S^2)Y = 2g(Y, Df)\xi - 2g(\xi, Df)Y, \tag{8.7}$$

where we have used $\alpha\lambda = 2$. From this, by taking the inner product with the Reeb vector field ξ, we get

$$g(Y, Df) = 0 \quad \text{for any} \quad Y \in T_\mu. \tag{8.8}$$

Moreover, if we take the inner product the above formula with $Y \in T_\mu$, and use $SY = 0$, we have

$$\begin{aligned} -2g(\xi, Df) &= (d + \psi)g(\phi SY, Y) - g(\mathrm{Ric}(\phi SY), Y) - hg((\nabla_\xi S)Y, Y) + g((\nabla_\xi S^2)Y, Y) \\ &= -hg(\nabla_\xi(SY) - S\nabla_\xi Y, Y) + g(\nabla_\xi(S^2Y) - S^2\nabla_\xi Y, Y) \\ &= 0. \end{aligned} \tag{8.9}$$

Consequently, from (8.6), (8.8) and (8.9) it follows that the gradient vector field Df identically vanishes. Then $Df = 0$ in (8.1) means that

$$\mathrm{Ric}(X) = (\Omega + \theta\gamma)X - \psi\eta(X)\xi.$$

That is, M is pseudo-Einstein. But Theorem C in the introduction gives that there does not exist a Hopf pseudo-Einstein real hypersurface in the complex hyperbolic quadric Q^{m*}. From this, we give a complete proof of our Main Theorem 3 in the introduction.

Remark 1. *The metric g of a Riemannian manifold M of dimension $m \geq 3$ is said to be a gradient Einstein soliton [6] if there exists a smooth function f on M such that*

$$Ric - \frac{1}{2}\gamma g + \nabla^2 f = \rho g,$$

where γ denotes the scalar curvature of M and ρ a constant on M. Here $\nabla^2 f$ denotes the Hessian operator of g and f the Einstein potential function of the gradient Einstein soliton. So this soliton is an example of gradient pseudo-Ricci-Bourguignon soliton.

Data Availability

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declaration of Competing Interest

The present authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

[1] A. L. Besse, Einstein Manifolds, Classics in Mathematics, Springer-Verlag, Berlin, 2008.

[2] A. M. Blaga and H. M. Tastan, *Some results on almost η-Ricci-Bourguignon soliton*, J. Geom. Phys. **168** (2021), 104316.

[3] J.P. Bourguignon, *Une stratification de l'espace des structures riemanniennes*, Compositio Math. **30** (1975), 1-41.

[4] J. P. Bourguignon, *Ricci curvature and Einstein metrics*, Global differential geometry and global analysis (Berlin,1979), pp. 42-63, Lecture Nots in Math., 838, Springer, Berlin-New York, 1981.

[5] G. Catino, L. Cremaschi, L. Cremaschi, Z. Djadli, C. Mantegazza and L. Mazzieri, *The Ricci-Bourguignon flow*, Pacific J. Math. **287** (2017), no. 2, 337-370.

[6] G. Catino and L. Mazzieri, *Gradient Einstein solitons*, Nonlinear Anal. **132** (2016), 66-94.

[7] T.E. Cecil and P.J. Ryan, *Focal sets and real hypersurfaces in complex projective space*, Trans. Amer. Math. Soc. **269** (1982), no. 2, 481-499.

[8] P. Cernea and D. Guan, *Killing fields generated by multiple solutions to the Fischer-Marsden equation*, Internat. J. Math. **26** (2015), no. 4, 1540006, 18 pp.

[9] S.K. Chaubey and Y.J. Suh, *Ricci-Bourguignon solitons and Fischer-Marsden conjecture on generalized Sasakian-Space-Forms with β-Kenmotsu structure*, J. of Korean Math. Soc. 60 (2023), no. 2, 341–358.

[10] S.K. Chaubey, H.Lee and Y.J.Suh, *Yamabe and gradient Yamabe soliton on real hypersurfaces in the complex hyperbolic quadric*, Int. J. of Geom. Methods in Modern Phys.**19** (2022), 2250026.

[11] S. K. Chaubey, U. C. De and Y. J. Suh, *Kenmotsu manifolds satisfying the Fischer-Marsden equation*, J. Korean Math. Soc. **58** (2021), no. 3, 597-607.

[12] S.K. Chaubey, M.D. Siddiqi and D.G. Prakasha, *Invariant submanifolds of hyperbolic Sasakian manifolds and η-Ricci-Bourguignon solitons*, Filomat **36** (2022), no 2, 409-421.

[13] I. Jeong and Y.J. Suh, *Pseudo anti-commuting and Ricci soliton real hypersurfaces in complex two-plane Grassmannians*, J. Geom. Phys. **86** (2014), 258-272.

[14] G.J. Kim and Y.J. Suh, *Real hypersurfaces in the complex hyperbolic quadrics with parallel Ricci tensor*, Results in Math. **74** (2019), 33(30 pages).

[15] S. Klein, *Totally geodesic submanifolds in the complex quadric*, Diff. Geom. Appl. **26** (2008), 79–96.

[16] S. Klein and Y.J. Suh, *Contact real hypersurfaces in the complex hyperbolic quadric*, Ann. Mat. Pura Applicata **198** (2019), 1481-1494.

[17] A.W. Knapp, Lie Groups Beyond an Introduction, Progress in Math., Birkhäuser, 2002.

[18] S. Kobayashi and K. Nomizu, Foundations of Differential Geometry, Vol. II, Wiley Classics Library, A Wiley-Interscience Publication, John Wiley & Sons, Inc., New York, 1996.

[19] M. Kon, *Pseudo-Einstein real hypersurfaces in complex space forms*, J. Differential Geometry **14** (1979), no. 3, 339-354.

[20] J.D. P´erez, H. Lee and Y.J. Suh, *Derivatives of normal Jacobi operator on real hypersurfaces in the complex quadric* Bull. London Math. Soc. **52** (2020) no. 6, 1122-1133.

[21] H. Lee and Y.J. Suh, *Real hypersufaces with recurrent normal Jacobi operator in the complex quadric*, J. Geom. Phys. **123** (2018), 463-474.

[22] H. Lee and Y.J. Suh, *Commuting Jacobi operators on real hypersurfaces of type B in the complex quadric*, Math. Phys. Anal. Geom. **23** (2020), no. 4, Paper No. 44, 21 pp.

[23] S. Montiel and A. Romero, *Complex Einstein hypersurfaces of indefinite complex space forms*, Proc. Camb. Phil. Soc. **94**(1983), 495-508.

[24] S. Montiel and A. Romero, *On some real hypersurfaces of a complex hyperbolic space*, Geom. Dedicata **20** (1986), no. 2, 245-261.

[25] J. Morgan and G. Tian, Ricci Flow and The Poincaré Conjecture, Mathematics Monographs, 3, American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2007.

[26] M. Okumura, *On some real hypersurfaces of a complex projective space*, Trans. Amer. Math. Soc. **212** (1975), 355-364.

[27] B. O'Neill, Semi-Riemannian geometry with applications to relativity, Academic Press, New York, Boston, London, Tokyo, 1983.

[28] G. Perelman, *Ricci flow with surgery on three-manifolds*, math.DG/0303109, 2003.

[29] J.D. P´erez, *On the structure vector field of a real hypersurface in complex quadric*, Open Mathematics, **16** (2018) no. 1, 185-189.

[30] J.D. P´erez, *Commutativity of torsion and normal Jacobi operators on real hypersurfaces in the complex quadric*, Publ. Math. Debrecen **95** (2019), no. 1-2, 157–168.

[31] J.D. P´erez, *Some real hypersurfaces in complex and complex hyperbolic quadrics*, Bull. Malays. Math. Sci. Soc. **43** (2020), no. 2, 1709–1718.
[32] J.D. P´erez and Y.J. Suh, *Derivatives of the shape operator of real hypersurfaces in the complex quadric*, Results Math. **73** (2018), no. 3, Paper No. 126, 10 pp.
[33] J.D. P´erez and Y.J. Suh, *New Conditions on Normal Jacobi Operator of Real Hypersurfaces in the Complex Quadric*, Bull. Malays. Math. Sci. Soc. **44** (2021), 891–903.
[34] J.D. Pérez, Y.J. Suh and Y. Watanabe, *Generalized Einstein Real hypersurfaces in complex two-plane Grassmannians*, J. Geom. Phys. **60** (2010), no. 11, 1806-1818.
[35] H. Reckziegel, *On the geometry of the complex quadric*, in: Geometry and Topology of Submanifolds VIII (Brussels/Nordfjordeid 1995), 302-315, World Sci. Publ., River Edge, NJ, 1996.
[36] A. Romero, *Some examples of indefinite complete complex Einstein hypersurfaces not locally symmetric*, Proc. Amer. Math. Soc. **98** (1986), 283—286.
[37] A. Romero, *On a certain class of complex Einstein hypersurfaces in indefinite complex space forms*, Math. Z. **192** (1986), 627—635.
[38] B. Smyth, *Differential geometry of complex hypersurfaces*, Ann. of Math. **85** (1967), 246-266.
[39] B. Smyth, *Homogeneous complex hypersurfaces*, J. Math. Soc. Japan **20** (1968), 643-647.
[40] Y.J. Suh, *Real hypersurfaces of type B in complex two-plane Grassmannians*, Monatsh. Math. **147** (2006), no. 4, 337-355.
[41] Y.J. Suh, *Hypersurfaces with isometric Reeb flow in complex hyperbolic two-plane Grassmannians*, Adv. in Appl. Math. **50** (2013), no. 4, 645–659.
[42] Y.J. Suh, *Pseudo-anti commuting Ricci tensor and Ricci soliton real hypersurfaces in the complex quadric*, J. Math. Pures Appl. **107** (2017), 429–450.
[43] Y.J. Suh, *Real hypersurfaces in the complex hyperbolic quadrics with isometric Reeb flow*, Comm. in Contemp. Math. **20** (2018), 1750031(20 pages).
[44] Y.J. Suh, *Pseudo-anti commuting Ricci tensor for real hypersurfaces in the complex hyperbolic quadric*, Sci. China, **62** (2019), 679-698.
[45] Y.J. Suh, *Ricci soliton and pseudo-Einstein real hypersurfaces in the complex hyperbolic quadric*, J. of Geom. and Phys. **162** (2021), 103888(16 pages).
[46] Y.J. Suh, *Real hypersurfaces in the complex hyperbolic quadrics with harmonic curvature*, Math. Nachr. **294** (2021), 145-176.
[47] Y.J. Suh, *Fischer-Marsden conjecture on real hypersurfaces in the complex quadric*, J. Math. Pures Appl. (2023), https://doi.org/10.1016/j.matpur.2023.06.010.
[48] Y.J. Suh and C. Woo, *Real hypersurfaces with pseudo-Ricci-Bourguignon soliton in the complex two-plane Grassmannians*, J. of Geom. and Phys. **192** (2023), 104913(19 pages).
[49] Y.J. Suh, H. Lee and C. Woo, *Generalized Killing Ricci tensor for real hypersurfaces in the complex hyperbolic quadric*, Rev. Real Acad. Cienc. Exactas, Fis. Nat.Ser. A-Mat. **115** (2021), 117(31 pages).
[50] Y.J. Suh, J.D. Pérez and C. Woo, *Real hypersurfaces in the complex hyperbolic quadric with parallel structure Jacobi operator*, Publ. Math. Debrecen. **94**(2019), 75-107.
[51] Y. Wang, *Ricci solitons on almost Kenmotsu 3-manifolds*, Open Math. **15** (2017), no. 1, 1236-1243.
[52] Y. Wang, *Ricci solitons on almost co-Kähler manifolds*, Canad. Math. Bull. **62** (2019), no. 4, 912–922.
[53] Y. Wang and X. Dai *Cyclic-parallel Ricci tensors on a class of almost Kenmotsu 3-manifolds*, Int. J. Geom. Methods Mod. Phys. **16** (2019), no. 6, 1950092.
[54] Y. Wang, *Real hypersurfaces in* $\mathbb{C}P^2$ *with constant Reeb sectional curvature*, Differ. Geom. Appl. **73** (2020) 101683.
[55] Y. Wang and Y. Zhang, *Generalized D-Einstein real hypersurfaces with constant coefficient*, J. Math. Anal. Appl. **519** (2023) no. 1, 126760.
[56] K. Yano and M. Kon, CR Submanifolds of Kaehlerian and Sasakian Manifolds, Progress in Mathematics, 30, Birkhäuser, Boston, Mass., 1983.