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Topological method for coupled systems of impulsive neutral functional differential inclusions driven by a fractional Brownian motion and Wiener process

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Abstract. In this paper we prove the existence of mild solutions for a first-order impulsive semilinear stochastic differential inclusion with an infinite-dimensional fractional Brownian motion and a standard cylindrical Wiener process. We focus on convex-valued case. The results are obtained by using two different fixed point theorems for multivalued mappings, more precisely, the technique is based on multivalued version of a nonlinear alternative of Leray-Schauder's fixed point theorem in generalized Banach spaces.

1. Introduction

The theory of stochastic differential and partial differential inclusions has become an active area of investigation due to their applications in several fields in the applied sciences such as mechanics, electrical engineering, medical biology, ecology amongst others.

Recently, stochastic differential and partial differential inclusions have been extensively studied. For instance, in [22, 24] the authors investigated the existence of solutions of nonlinear stochastic differential inclusions by means of a Banach fixed point theorem and a semigroup approach. Balasubramaniam [23] obtained existence of solutions of functional stochastic differential inclusions by Kakutani's fixed point theorem,[24] initiated the study of existence of solutions of semilinear stochastic evolution inclusions in a Hilbert space by using the nonlinear alternative of Leray-Schauder type [18], some existence results for impulsive neutral stochastic evolution inclusions in Hilbert Space, where a class of second-order evolution inclusions with a convex case are considered. In [12] the authors study the existence results for impulsive neutral stochastic evolution inclusions in Hilbert spaces where they considered a class of first-order evolution inclusions with convex by using Leray-Schauder's Alternative fixed point theorem.

That is why in recent years they have been the objectives of many investigations. We refer to the monographs by Benchohra et al. [21], amongst others, to see several studies on the properties of their

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solutions. The reader can also find a detailed and extensive bibliography in the previously mentioned books (see also Da Prato and Zabczyk [10], Gard [28], Gikhman and Skorokhod [14], Sobzyk [13]). As a motivating example, let us refer to a stochastic model for drug distribution in a biological system which was described by Tsokos and Padgett [6] as a closed system with a simplified heart, one organ or capillary bed, and re-circulation of a blood with a constant rate of flow, where the heart is considered as a mixing chamber of constant volume. For the basic theory concerning stochastic differential inclusions see the monographs of Bharucha-Reid [3], Mao[31], Φksendal, [5], Tsokos and Padgett [6], Sobczyk [13] and Da Prato and Zabczyk [10]. In many realistic cases, it is advantageous to treat the first order differential (see Bao [15]). Motivated by [32, 33] we will generalize the existence of the solution to impulsive stochastic partial functional differential equations.
Recently, inspired by the works of Boudaoui et al. [2], Blouhi et al.
Motivated by the previous works, in the present paper, it is interesting to show more general existence result to that in [32, 33]. To the best of our acknowledge, there is no result concerning coupled System of Impulsive Neutral Functional Differential Inclusions Driven by a Fractional Brownian Motion and Wiener Process,
in this paper we are interested in proving the existence of solutions for a system of stochastic impulsive differential inclusions of the following

$$\begin{cases} dx(t) & \in & (Ax(t) + F^1(t, x_t, y_t)dt + \sum_{l=1}^{\infty} \sigma_l^1(t)dB_l^H(t) \\ & + & g^1(t)dW(t),\ t \in J := [0, b], t \neq t_k, \\ dy(t) & \in & (Ay(t) + F^2(t, x_t, y_t)dt + \sum_{l=1}^{\infty} \sigma_l^2(t)dB_l^H(t) \\ & + & g^2(t)dW(t),\ t \in J := [0, b], t \neq t_k, \\ \Delta x(t) & = & I_k(x(t_k)), \quad t = t_k \quad k = 1, 2, \ldots, m \\ \Delta y(t) & = & \bar{I}_k(y(t_k)), \\ x(t) & = & \phi(t), \quad t \in J_0 = (-\infty, 0], \\ y(t) & = & \bar{\phi}(t), \quad t \in J_0 = (-\infty, 0], \end{cases} \tag{1}$$

where X is a real separable Hilbert space with inner product $\langle\cdot,\cdot\rangle$ induced by norm $\|\cdot\|$, $A : D(A) \subset X \longrightarrow X$ is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $(S(t))_{t\geq 0}$ in X .Here, ,B_l^H is an infinite sequence of independent fractional Brownian motions, $l = 1, 2, \ldots$, with Hurst parameter X, $I_k, \bar{I}_k \in C(X, X)$ $(k = 1, 2, \ldots, m)$, $\sigma_l^i : J \times \mathcal{D}_{\mathcal{F}_0} \times \mathcal{D}_{\mathcal{F}_0} \to L^0_{Q_i}(Y, X)$.Here, $\mathcal{D}_{\mathcal{F}_0}$ is a linear space of family of $\mathcal{F}_0$-measurable functions from $(-\infty, 0]$ into X,which will be also defined in the next section and $L^0_{Q_i}(Y, X)$ denotes the space of all Q_i-Hilbert-Schmidt operators from Y into X for each $i = 1, 2$, $g^i : J \times \mathcal{D}_{\mathcal{F}_0} \times \mathcal{D}_{\mathcal{F}_0} \to L^0(Y, X)$ Here, $L^0(Y, X) = L_2(Q^{1/2}Y, X)$ be a separable Hilbert space with respect to the Hilbert- Schmidt norm $\|.\|_{L^0}$ and Q-Wiener process on $(\Omega, \mathcal{F}, \mathbb{P})$ with a linear bounded covariance operator Q such that $trQ < \infty$. Let $\{W(t), t \in \mathbb{R}\}$ be a standard cylindrical Wiener process with values in Y and defined on $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, which will be also defined in the next section. Moreover, the fixed times t_k satisfies $0 < t_1 < t_2 < \ldots < t_m < b$, $x(t_k^-)$ and $x(t_k^+)$ denotes the left and right limits of $x(t)$ at $t = t_k$.As for x_t we mean the segment solution which is defined in the usual way, that is, if $y(\cdot,\cdot) : (-\infty, b] \times \Omega \to X$, then for any $t \geq 0$, $y_t(\cdot,\cdot) : (-\infty, 0] \times \Omega \to X$ is given by:

$$x_t(\theta, \omega) = x(t + \theta, \omega),\ \text{for}\ \theta \in (-\infty, 0],\ \omega \in \Omega,$$

$$\begin{cases} \sigma(t) = (\sigma_1(t), \sigma_2(t), \ldots), \\ \|\sigma(t)\|^2 = \sum_{l=1}^{\infty} \|\sigma_l(t)\|^2_{L_Q^0} < \infty, \end{cases} \tag{2}$$

where $\sigma(\cdot) \in l^2$ and l^2 is given as

$$l^2 = \{\phi = (\phi_l)_{l\geq 1} : \ [0,T] \to L_Q^0(Y,X) \quad : \|\phi(t)\|^2 = \sum_{l=1}^{\infty} \|\phi_l(t)\|_{L_Q^0}^2 < \infty\}.$$

It is obvious that system (1) can be seen as a fixed point problem:

$$\begin{cases} dz(t) &= (A_* z(t) + F(t,z_t))dt \\ &\quad + \sum_{l=1}^{\infty} \sigma_l(t) dB_l^H(t) + g(t)dW(t), \ t \in [0,b], t \neq t_k, \\ \Delta z(t) &= I_k^*(z(t_k)), \quad t = t_k \quad k = 1,2,\dots,m \\ z(t) &= z_0, \quad t \in J_0 = (-\infty, 0], \end{cases} \tag{3}$$

where

$$z_t = \begin{bmatrix} x_t \\ y_t \end{bmatrix}, A_* = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}, \ F(t,z_t) = \begin{bmatrix} F^1(t,x_t,y_t) \\ F^2(t,x_t,y_t) \end{bmatrix}, \ \sigma_l(t) = \begin{bmatrix} \sigma_l^1(t) \\ \sigma_l^2(t) \end{bmatrix}$$

and

$$z_0 = \begin{bmatrix} \phi(t) \\ \bar{\phi}(t) \end{bmatrix}, \ g(t) = \begin{bmatrix} g^1(t) \\ g^2(t) \end{bmatrix}$$

Some results on the existence of solutions for differential equations with infinite Brownian motion were obtained in [7, 26]. Some existence and uniqueness of mild solutions to neutral stochastic delay functional integro-differential equations perturbed by a fractional Brownian motion can be found in [2].
Very recently in the case without delay and B_l^H is a fractional Brownian motion, the problem (1) was studied by Blouhi *et. al.* [8] and Boudaoui et al. [2] proved the existence of mild solutions to stochastic impulsive evolution equations with time delays, driven by fractional Brownian motion and Krasnoselski-Schaefer type fixed point theorem. Recently Precup [25] proved the role of matrix convergence and vector metric in the study of semilinear operator systems.
Before describing the properties fulfilled by the operators f^i, h^i, σ^i and $I_k, \bar{I}_k$, we need to introduce some notation and describe some spaces.

In this work, we will use an axiomatic definition of the phase space $\mathcal{D}_{\mathcal{F}_0}$ introduced by Hale and Kato [16].

Definition 1.1. *$\mathcal{D}_{\mathcal{F}_0}$ is a linear space of family of $\mathcal{F}_0$-measurable functions from $(-\infty,0]$ into X endowed with a norm $\|\cdot\|_{\mathcal{D}_{\mathcal{F}_0}}$, which satisfies the following axioms:*

(i) *If $x : (-\infty, b] \longrightarrow X$, $b > 0$ is such that $z_0 = (x_0, y_0) \in \mathcal{D}_{\mathcal{F}_0} \times \mathcal{D}_{\mathcal{F}_0}$, then for every $t \in [0,b)$ the following conditions hold*

- **(a)** $x_t \in \mathcal{D}_{\mathcal{F}_0}$,
- **(b)** $\|x(t)\| \leq L\|x_t\|_{\mathcal{D}_{\mathcal{F}_0}}$,
- **(c)** $\|x_t\|_{\mathcal{D}} \leq K(t)\sup\{\|x(s)\| : 0 \leq s \leq t\} + N(t)\|x_0\|_{\mathcal{D}_{\mathcal{F}_0}}$,
  where $L > 0$ is a constant; $K, N : [0,\infty) \longrightarrow [0,\infty)$, K is continuous, N is locally bounded and K, N are independent of $x(\cdot)$.

(ii) *For the function $x(\cdot)$ in (i), x_t is a $\mathcal{D}_{\mathcal{F}_0}$-valued function $[0,b)$.*

(iii) *The space $\mathcal{D}_{\mathcal{F}_0}$ is complete.*

Denote

$$\widehat{K} = \sup\{K(t) : t \in J\} \text{ and } \widehat{N} = \sup\{N(t) : t \in J\}.$$

Now, for a given $b > 0$, we define

$$\mathcal{D}_{\mathcal{F}_b} = \Big\{x\colon (-\infty, b] \times \Omega \to X,\ x_k \in C(J_k, X) \text{ for } k = 1, \dots m,\ x_0 \in \mathcal{D}_{\mathcal{F}_0}, \text{ and there exist } x(t_k^-) \text{ and } x(t_k^+) \text{ with } x(t_k) = x(t_k^-),\ k = 1, \cdots, m, \text{ and } \sup_{t\in[0,b]} \mathbb{E}(|x(t)|^2) < \infty\Big\},$$

endowed with the norm

$$\|x\|_{\mathcal{D}_{\mathcal{F}_b}} = \|\phi\|_{\mathcal{D}_{\mathcal{F}_0}} + \sup_{0\le s\le b}(\sqrt{\mathbb{E}\|x(s)\|^2}),$$

where x_k denotes the restriction of x to $J_k = (t_{k-1}, t_k]$, $k = 1, 2, \cdots, m$, and $J_0 = (-\infty, 0]$, $i = 1, 2$.

The paper is organized as follows. In Section 2 we recall some definitions and facts which will be needed in our analysis. Section 3 we prove some existence results based on a nonlinear alternative of Leray-Schauder type theorem in generalized Banach spaces in the convex case.

2. Preliminaries

In this section, we introduce some notations, recall some definitions, and preliminary facts which are used throughout this paper. Actually we will borrow it from [29]. Although we could simply refer to this paper whenever we need it, we prefer to include this summary in order to make our paper as much self-contained as possible.

2.1. Some results on stochastic integrals with respect to fractional Brownian motions

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $(\mathcal{F} = \mathcal{F}_t)_{t\ge 0}$ satisfying the usual conditions (i.e. right continuous and $\mathcal{F}_0$ containing all $\mathbb{P}$-null sets).

For a stochastic process $x(\cdot, \cdot) : [0, T] \times \Omega \to X$ we will write $x(t)$ (or simply x when no confusion is possible) instead of $x(t, \omega)$.

Definition 2.1. *Given $H \in (0, 1)$, a continuous centered Gaussian process B^H is said to be a two-sided one-dimensional fractional Brownian motion (fBm) with Hurst parameter H, if its covariance function $R_H(t, s) = E[B^H(t))B^H(s)]$ satisfies*

$$R_H(t, s) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t - s|^{2H}) \quad t, s \in [0, T].$$

It is known that $B^H(t)$ with $H > \frac{1}{2}$ admits the following Volterra representation

$$B^H(t) = \int_0^t K_H(t, s)dB(s), \tag{4}$$

where B is a standard Brownian motion given by

$$B(t) = B^H((K_H^*)^{-1}\xi_{[0,t]}),$$

and the Volterra kernel the kernel $K(t, s)$ is given by

$$K_H(t, s) = c_H s^{1/2-H} \int_s^t (u - s)^{H-\frac{3}{2}} \left(\frac{u}{s}\right)^{H-\frac{1}{2}} du, \quad t \ge s,$$

where $c_H = \sqrt{\frac{H(2H-1)}{\beta(2H-2,H-\frac{1}{2})}}$ and $\beta(\cdot,\cdot)$ denotes the Beta function, $K(t,s) = 0$ if $t \le s$, and it holds

$$\frac{\partial K_H}{\partial t}(t,s) = c_H \left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{H-\frac{3}{2}},$$

and the kernel K_H^* is defined as follows. Denote by $\mathcal{E}$ the set of step functions on $[0,T]$. Let $\mathcal{H}$ be the Hilbert space defined as the closure of $\mathcal{E}$ with respect to the scalar product

$$\langle \chi_{[0,t]}, \chi_{[0,s]} \rangle_{\mathcal{H}} = R_H(t,s),$$

and consider the linear operator K_H^* from $\mathcal{E}$ to $L^2([0,T])$ defined by,

$$(K_H^*\phi)(t) = \int_s^T \phi(t)\frac{\partial K_H}{\partial t}(t,s)dt.$$

Notice that,

$$(K_H^*\chi_{[0,t]})(s) = K_H(t,s)\chi_{[0,t]}(s).$$

The operator K_H^* is an isometry between $\mathcal{E}$ and $L^2([0,T])$ which can be extended to the Hilbert space $\mathcal{H}$. In fact, for any $s,t \in [0,T]$ we have

$$\langle K_H^*\chi_{[0,t]}, K_H^*\chi_{[0,t]}\rangle_{L^2([0,T])} = \langle \chi_{[0,t]}, \chi_{[0,s]}\rangle_{\mathcal{H}} = R_H(t,s).$$

In addition, for any $\phi \in \mathcal{H}$,

$$\int_0^T \phi(s)dB^H(s) = \int_0^T (K_H^*\phi)(s)dB(s),$$

if and only if $K_H^*\phi \in L^2([0,T])$.
Next we are interested in considering an fBm with values in a Hilbert space and giving the definition of the corresponding stochastic integral.

Definition 2.2. *An $\mathcal{F}_t$-adapted process ϕ on $[0,T]\times\Omega \to X$ is an elementary or simple process if for a partition $\psi = \{\bar{t}_0 = 0 < \bar{t}_1 < \ldots < \bar{t}_n = T\}$ and $(\mathcal{F}_{\bar{t}_i})$-measurable X-valued random variables $(\phi_{\bar{t}_i})_{1\le i\le n}$, ϕ_t satisfies*

$$\phi_t(\omega) = \sum_{i=1}^{n} \phi_i(\omega)\chi_{(\bar{t}_{i-1},\bar{t}_i]}(t), \text{ for } 0 \le t \le T, \ \omega \in \Omega.$$

The Itô integral of the simple process ϕ is defined as

$$I_H(\phi) = \int_0^T \phi_l(s)dB_l^H(s) = \sum_{i=1}^{n} \phi_l(\bar{t}_i)(B_l^H(\bar{t}_i) - B_l^H(\bar{t}_{i-1})), \tag{5}$$

whenever $\phi_{\bar{t}_i} \in L^2(\Omega, \mathcal{F}_{\bar{t}_i}\mathbb{P}, X)$ for all $i \le n$.

Let $(X, \langle\cdot,\cdot\rangle, |\cdot|_X)$, $(Y, \langle\cdot,\cdot\rangle, |\cdot|_Y)$ be separable Hilbert spaces. Let $\mathcal{L}(Y,X)$ denote the space of all linear bounded operators from Y into X. Let $e_n, n = 1,2,\ldots$ be a complete orthonormal basis in Y and $Q \in \mathcal{L}(Y,X)$ be an operator defined by $Qe_n = \lambda_n e_n$ with finite trace $trQ = \sum_{n=1}^{\infty}\lambda_n < \infty$ where λ_n, $n = 1,2,\ldots$, are non-negative real numbers. Let $(\beta_n^H)_{n\in N}$ be a sequence of two-sided one-dimensional standard fractional Brownian motions mutually independent on $(\Omega, \mathcal{F}, \mathbb{P})$. If we define the infinite dimensional fBm on Y with covariance Q as

$$B^H(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\beta_n^H(t)e_n, \tag{6}$$

then it is well defined as an Y-valued Q-cylindrical fractional Brownian motion (see [10]) and we have

$$E\langle\beta_l^H(t),x\rangle\langle\beta_k^H(s),y\rangle = R_{H_{lk}}(t,s)\langle Q(x),y\rangle, \quad x,y\in Y \quad and\, s,t\in[0,T],$$

such that

$$R_{H_{lk}} = \frac{1}{2}\{\mid t\mid^{2H} + \mid s\mid^{2H} + \mid t-s\mid^{2H}\}\delta_{lk} \quad t,s\in[0,T],$$

where

$$\delta_{lj} = \begin{cases} 1 & j=l,\\ 0, & j\neq l.\end{cases}$$

In order to define Wiener integrals with respect to a $Q-fBm$, we introduce the space $L_Q^0 := L_Q^0(Y,X)$ of all Q−Hilbert-Schmidt operators $\varphi : Y \longrightarrow X$. We recall that $\varphi \in L(Y,X)$ is called a Q−Hilbert-Schmidt operator, if

$$\|\varphi\|^2_{L_Q^0} = \|\varphi Q^{1/2}\|^2_{HS} = tr(\varphi Q\varphi^*) < \infty.$$

Definition 2.3. *Let $\phi(s), s\in[0,T]$, be a function with values in $L_Q^0(Y,X)$. The Wiener integral of ϕ with respect to fBm given by* (6) *is defined by*

$$\int_0^T \phi(s)dB^H(s) = \sum_{n=1}^{\infty}\int_0^t \sqrt{\lambda_n}\phi(s)e_n d\beta_n^H$$

$$= \sum_{n=1}^{\infty}\int_0^T \sqrt{\lambda_n}K_H^*(\phi e_n)(s)d\beta_n. \tag{7}$$

Notice that if

$$\sum_{n=1}^{\infty}\|\phi Q^{1/2}e_n\|_{L^{1/H}([0,T];X)} < \infty, \tag{8}$$

the next result ensures the convergence of the series in the previous definition.

Lemma 2.4. *[29] For any $\phi : [0,T]\to L_Q^0(Y,X)$ such that (8) holds, and for any $\alpha,\beta\in[0,T]$ with $\alpha>\beta$,*

$$E\left|\int_\alpha^\beta \phi(s)dB^H(s)\right|_X^2 \leq c_H H(2H-1)(\alpha-\beta)^{2H-1}\sum_{n=1}^{\infty}\int_\alpha^\beta \left|\phi(s)Q^{1/2}e_n\right|_X^2 ds. \tag{9}$$

If in addition

$$\sum_{n=1}^{\infty}|\phi Q^{1/2}e_n|_X \text{ is uniformly convergent for } t\in[0,T],$$

then,

$$E\left|\int_\alpha^\beta \phi(s)dB^H(s)\right|_X^2 \leq c_H H(2H-1)(\alpha-\beta)^{2H-1}\int_\alpha^\beta \left\|\phi(s)\right\|^2_{L_Q^0} ds. \tag{10}$$

2.2. Some results on fixed point theorems and set-valued analysis

For $x, y \in \mathbb{R}^n$, $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, by $x \leq y$ we mean $x_i \leq y_i$ for all $i = 1, \ldots, n$. Also $|x| = (|x_1|, \ldots, |x_n|)$ and $\max(x, y) = \max(\max(x_1, y_1), \ldots, \max(x_n, y_n))$. If $c \in \mathbb{R}$, then $x \leq c$ means $x_i \leq c$ for each $i = 1, \ldots, n$.

Definition 2.5. *A square matrix of real numbers M is said to be convergent to zero if its spectral radius $\rho(M)$ is strictly less than 1. In other words, this means that all the eigenvalues of M are in the open unit disc. (i.e. $|\lambda| < 1$, for every $\lambda \in \mathbb{C}$ with $det(M - \lambda I) = 0$, where I denote the unit matrix of $\mathcal{M}_{n\times n}(\mathbb{R})$).*

Some examples of matrices convergent to zero can be seen in [29].

$\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : y \text{ closed }\}$,

$\mathcal{P}_{b}(X) = \{Y \in \mathcal{P}(X) : y \text{ bounded }\}$,

$\mathcal{P}_{c}(X) = \{Y \in \mathcal{P}(X) : y \text{ convex }\}$,

$\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : y \text{ compact }\}$.

Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \longrightarrow \mathbb{R}^n_+ \cup \{\infty\}$ defined by

$$H_d(A, B) \quad := \quad \begin{pmatrix} H_{d_1}(A, B) \\ \ldots \\ H_{d_n}(A, B) \end{pmatrix}.$$

Let (X, d) be a generalized metric space with

$$d(x, y) \quad := \quad \begin{pmatrix} d_1(x, y) \\ \ldots \\ d_n(x, y) \end{pmatrix}.$$

Notice that d is a generalized metric space on X if and only if d_i, $i = 1, \ldots, n$ are metrics on X, $H_d(A, B) = max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\right\}$,
where $d(A, b) = \inf_{a\in A} d(a, b), d(a, B) = \inf_{b\in B} d(a, b)$. Then, $(\mathcal{P}_{b,cl}(X), H_d)$ is a metric space and $(\mathcal{P}_{cl}(X), H_d)$ is a generalized metric space.

A multivalued map $F : X \longrightarrow \mathcal{P}(X)$ is convex (closed) valued if $F(y)$ is convex (closed) for all $y \in X$, F is bounded on bounded sets if $F(B) = \bigcup_{y\in B} F(y)$ is bounded in X for all $B \in \mathcal{P}_b(X)$. F is called upper semi-continuous (u.s.c. for short) on X if for each $y_0 \in X$ the set $F(y_0)$ is a nonempty, subset of X, and for each open set $\mathcal{U}$ of X containing $F(y_0)$, there exists an open neighborhood $\mathcal{V}$ of y_0 such that $F(\mathcal{V}) \in \mathcal{U}$. F is said to be completely continuous if $F(B)$ is relatively compact for every $B \in \mathcal{P}_b(X)$. F is quasicompact if, for each subset $A \subset X$,$F(A)$ is relatively compact.

If the multivalued map F is completely continuous with nonempty compact valued, then F is *u.s.c.* if and only if F has a closed graph, i.e., $x_n \longrightarrow x_*$, $y_n \longrightarrow y_*$, $y_n \in F(x_n)$ imply $y_* \in F(x_*)$.

A multi-valued map $F : J \longrightarrow \mathcal{P}_{cp,c}(X)$ is said to be measurable if for each $y \in X$, the mean-square distance between y and $F(t)$ is measurable.

Definition 2.6. *The set-valued map $F : J \times X \times X \to \mathcal{P}(X \times X)$ is said to be L^2-Carathéodory if*

(i) *$t \mapsto F(t, v)$ is measurable for each $v \in X \times X$;*

(ii) *$v \mapsto F(t, v)$ is u.s.c. for almost all $t \in J$;*

(iii) *for each $q > 0$, there exists $h_q \in L^1(J, \mathbb{R}^+)$ such that*

$$\|F(t, v)\|^2 := \sup_{f\in F(t,v)} \|f\|^2 \leq h_q(t), \quad \text{for all } \|v\|^2 \leq q \text{ and for a.e. } t \in J.$$

Remark 2.7. **(a)** *For each $x \in C(J, X)$, the set $S_{F,x}$ is closed whenever F has closed values. It is convex if and only if $F(t, x(t))$ is convex for a.e. $t \in J$.*

Lemma 2.8. *[4] Let I be a compact interval and X be a Hilbert space. Let F be an L^2-Carathéodory multi-valued map with $S_{F,y} \neq \emptyset$. and let Γ be a linear continuous mapping from $L^2(I, X)$ to $C(I, X)$. Then, the operator*

$$\Gamma \circ S_F : C(I, X) \longrightarrow \mathcal{P}_{cp,c}(L^2([0, T], X)), \quad y \longmapsto (\Gamma \circ S_F)(y) = \Gamma(S_F, y),$$

is a closed graph operator in $C(I, X) \times C(I, X)$, where $S_{F,y}$ is known as the selectors set from F and given by $f \in S_{F,y} = \{f \in L^2([0, T], X) : f(t) \in F(t, y) \text{ for a.e.} t \in [0, T]\}$.

We denote the graph of G to be the set $gr(G) = \{(x, y) \in X \times Y, \quad y \in G(x)\}$.

Lemma 2.9. *[19] If $G : X \to \mathcal{P}_{cl}(Y)$ is u.s.c., then $gr(G)$ is a closed subset of $X \times Y$. Conversely, if G is locally compact and has nonempty compact values and a closed graph, then it is u.s.c.*

Lemma 2.10. *[27] If $G : X \to \mathcal{P}_{cp}(Y)$ is quasicompact and has a closed graph, then G is u.s.c.*

The following two results are easily deduced from the limit properties

Lemma 2.11. *(See e.g. [17], Theorem 1.4.13) If $G : X \to \mathcal{P}_{cp}(X)$ is u.s.c., then for any $x_0 \in X$,*

$$\limsup_{x \to x_0} G(x) = G(x_0).$$

Lemma 2.12. *(See e.g. [17], Lemma 1.1.9) If Let $(K_n)_{n \in N} \subset K \subset X$ be a sequence of subsets where K is compact in the separable Banach space X. Then*

$$\overline{co}(\limsup_{n \to \infty} K_n) = \cap_{N>0} \overline{co}(\cup_{n \geq N} K_n)$$

where $\overline{co}A$ refers to the closure of the convex hull of A.

The second one is due to Mazur, 1933:

Lemma 2.13. *(Mazur's Lemma, ([20] [Theorem 21.4])) Let X be a normed space and $\{x_k\}_{k \in N} \subset X$ be a sequence weakly converging to a limit $x \in X$. Then there exists a sequence of convex combinations $y_m = \sum_{k=1}^{m} \alpha_{mk} x_k$ with $\alpha_{mk} > 0$ for $k = 1, 2, \ldots, m$ and $\sum_{k=1}^{m} \alpha_{mk} = 1$, which converges strongly to x.*

Recall that a set-valued operator G possesses a fixed point if there exists $y \in X$ such that $y \in G(y)$.

By above lemma we can easily prove the following so-called nonlinear alternatives of Leray and Schauder will be needed in the proof of our result (see [18]).

Lemma 2.14. *Let $(X, \|\cdot\|)$ be a generalized Banach and $G : X \longrightarrow \mathcal{P}_{cl,cv}(X)$ be an upper semicontinuous and compact map. Then either,*

(a) *G has at least one fixed point, or*

(b) *the set $\mathcal{M} = \{x \in X \text{ and } \lambda \in (0, 1), \text{with} \quad x \in \lambda G(u)\}$ is unbounded.*

Our next result describes a basic theorem of reflexive spaces with $A : E \to E$ be a linear operator:

Theorem 2.15. *[11] E is reflexive if and only if $B_E = \{x \in E; \|x\| \leq 1\}$ is compact in the weak topology.*

Now, let us state the following well-known lemma [10], which will be used in the sequel in the proofs of the main results.

Lemma 2.16. *For any $r \geq 1$ and for arbitrary L_0^2-valued predictable process $g(.)$,*

$$\sup_{s \in [0,t]} \Big\| \int_0^s g(u) dW(u) \Big\|_X^{2r} \leq (r(2r-1))^r \Big(\int_0^t \|g(s)\|_{L_0^2}^{2r} ds \Big)^r. \tag{11}$$

3. Mild Solutions

In this section we prove the existence of mild solution of the problem (1). Our approach is based on multivalued versions of Schaefer's fixed point theorem .

3.1. The convex case

First, we define what we mean by a mild solution.

Definition 3.1. *A stochastic process $x, y : (-\infty, b] \times \Omega \longrightarrow X$ is called a mild solution of the system (1) if*

- *$u(t) = (x(t), y(t))$ is measurable and $\mathcal{F}_t$-adapted, for each $t \geq 0$;*
- *$(x(t), y(t)) \in X \times X$ has càdlàg paths on $t \in [0, b]$ a.s., for every $0 \leq s < t \leq b$, there exist $f^i \in S_{F^i,x,y}$ for each $i = 1, 2$ such that the following integral equation holds*

$$\begin{cases} x(t) &= S(t)\phi(0) + \displaystyle\int_0^t S(t-s)f^1(s)ds + \int_0^t S(t-s)\sigma_l^1(s)dB_l^H(s) \\ &+ \displaystyle\int_0^t S(t-s)g^1(s)dW(s) + \sum_{0<t_k<t} S(t-t_k)I_k(x(t_k^-), y(t_k^-)), \\ y(t) &= S(t)\bar{\phi}(0) + \displaystyle\int_0^t S(t-s)f(s)ds + \int_0^t S(t-s)\sigma_l^1(s)dB_l^H(s) \\ &+ \displaystyle\int_0^t S(t-s)g^2(s)dW(s) + \sum_{0<t_k<t} S(t-t_k)\bar{I}_k(x(t_k^-), y(t_k^-)), \quad t \in J. \end{cases}$$

- *$(x_0(\cdot), y_0(\cdot)) = (\phi, \bar{\phi}) \in \mathcal{D}_{\mathcal{F}_0} \times \mathcal{D}_{\mathcal{F}_0}$ on $J_0 := (-\infty, 0]$ satisfies $\|\phi\|_{\mathcal{D}_{\mathcal{F}_0}} < \infty$ and $\|\bar{\phi}\|_{\mathcal{D}_{\mathcal{F}_0}} < \infty$.*

We are now in a position to state and prove our existence result for the problem (1). First we will list the following hypotheses which will be imposed in our main theorem.
Consider the following assumptions: In all this part, we assume that $S(t)$ is compact for $t > 0$ and that there exists $M > 0$ such that

$$\|S(t)\| \leq M, \quad \text{for every} \quad t \in [0, b].$$

(H_1) The function $g^i : J \longrightarrow L^0(Y, X)$ and $\sigma^i : J \longrightarrow L_Q^0(Y, X)$ satisfies

$$\int_0^b \|g^i(s)\|_{L^0}^2 ds = C_1 < \infty, \quad t \in J,$$

and

$$\int_0^b \|\sigma^i(s)\|^2 ds = C_2 < \infty, \quad t \in J.$$

(H_2) $F^i : [0, b] \times \mathcal{D}_{\mathcal{F}_0} \times \longrightarrow \mathcal{P}_{cv,cp}(X)$ is an integrably bounded multi-valued map,i.e.,there exists $p_i \in L^2(J, X)$ and $\psi_i : \mathbb{R}^+ \longrightarrow (0, \infty)$ is continuous and increasing such that

$$\mathbb{E}|F^i(t, x, y)|_X^2 = \sup_{f^i \in F^i(t,x,y)} \mathbb{E}|f^i(t)|_X^2 \leq p_i(t)\psi_i(\|x\|_{\mathcal{D}_{\mathcal{F}_0}}^2 + \|y\|_{\mathcal{D}_{\mathcal{F}_0}}^2), \quad t \in J, \quad x, y \in \mathcal{D}_{\mathcal{F}_0},$$

(H_3) There exist constants $d_k, \overline{d}_k \geq 0$ for each $k = 1, \ldots, m$ such that

$$|I_k(x)|_X^2 \leq d_k, \quad |\bar{I}_k(y)|_X^2 \leq \overline{d}_k \text{ for all } x, y \in X.$$

Lemma 3.2. *Assume that $F^i : J \times \mathcal{D}_{\mathcal{F}_0} \times \mathcal{D}_{\mathcal{F}_0} \longrightarrow \mathcal{P}_{cv,cp}(X)$ is a Carathèodory map satisfying (H_1)-(H_3) hold. Then the operator is completely continuous and u.s.c.*

Proof. The proof will be given in several steps.
Step 1. Consider the problem (1) on $(-\infty, t_1]$.

$$\begin{cases} dx(t) & \in & (Ax(t) + F^1(t, x_t, y_t)dt + \sum_{l=1}^{\infty} \sigma_l^1(t)dB_l^H(t) \\ & + & g^1(t)dW(t), \ t \in J := [0, t_1] \\ dy(t) & \in & (Ay(t) + F^2(t, x_t, y_t)dt + \sum_{l=1}^{\infty} \sigma_l^2(t)dB_l^H(t) \\ & + & g^2(t)dW(t), \ t \in J := [0, t_1], \\ x(t) & = & \phi(t) \in \mathcal{D}_{\mathcal{F}_0}, \quad J_0 = (-\infty, 0], \\ y(t) & = & \bar{\phi}(t) \in \mathcal{D}_{\mathcal{F}_0}, \quad J_0 = (-\infty, 0], \end{cases} \tag{12}$$

Let

$$C_0 = \{x \in C([0, t_1], X) : \sup_{t \in [0, t_1]} E(|x(t)|^2) < \infty\}.$$

Put

$$C_0^* = \mathcal{D}_{\mathcal{F}_0} \cap C_0.$$

Consider the multivalued operator $N^0 : C_0^* \times C_0^* \to \mathcal{P}(C_0^* \times C_0^*)$.We will prove that N^0 the operator is completely continuous and *u.s.c.* with $(N_1^0(x, y), N_2^0(x, y))$, $(x, y) \in C_0^* \times C_0^*$ defined by

$$N^0(x, y) = \left\{(h^0, \overline{h}^0) \in C_0^* \times C_0^*\right\},$$

given by

$$N_1^0(x, y) = \left\{ h^0 \in C_0^* : \ h^0(t) = \begin{cases} \phi(t), \quad \text{if } t \in (-\infty, 0], \\ S(t)\phi(0) + \int_0^t S(t - s)f^1(s)ds \\ + \sum_{l=1}^{\infty} \int_0^t S(t - s)\sigma_l^1(s)dB_l^H(s) \\ + \int_0^t S(t - s)g^1(s)dW(s), \text{if } \ t \in [0, t_1] \end{cases} \right\}$$

and

$$N_2^0(x, y) = \left\{ \bar{h}^0 \in C_0^* : \ \bar{h}^0(t) = \begin{cases} \bar{\phi}(t), \quad \text{if } t \in (-\infty, 0], \\ S(t)\bar{\phi}(0) + \int_0^t S(t - s)f^2(s)ds \\ + \sum_{l=1}^{\infty} \int_0^t S(t - s)\sigma_l^2(s)dB_l^H(s) \\ + \int_0^t S(t - s)g^2(s)dW(s), \qquad \text{if } \ t \in [0, t_1] \end{cases} \right\}$$

where

$$f^i \in S_{F^i,u} = \{f^i \in L^2(J, X) : f^i(t) \in F^i(t, x, y) \quad \text{for a.e} \quad t \in [0, t_1]\}.$$

From the assumption it is easy to see that N^0 is well defined.Let $\theta, \bar{\theta} : (-\infty, t_1] \longrightarrow X$ be the function defined by

$$\theta(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ S(t)\phi(0), & t \in [0, t_1]. \end{cases}$$

and

$$\bar{\theta}(t) = \begin{cases} \bar{\phi}(t), & t \in (-\infty, 0], \\ S(t)\bar{\phi}(0), & t \in [0, t_1]. \end{cases}$$

It is clear that $(\theta,\bar{\theta})$ is an element of $C_0^*\times C_0^*$. Set $(x(t),y(t))=(z(t)+\theta(t),\bar{z}(t)+\bar{\theta}(t)),-\infty<t\le t_1$.Obviously, if x,y satisfies (12) if and only if $(z,\bar{z})$ satisfies $(z_0,\bar{z}_0)=(0,0)$ if $t\in(-\infty,0]$ and

$$\begin{cases} z(t)=\int_0^t S(t-s)f^1(s)ds+\sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^2(s)dB_l^H(s)\\ \qquad +\int_0^t S(t-s)g^1(s)dW(s), \quad \text{if} \quad t\in[0,t_1],\\ \bar{z}(t)=\int_0^t S(t-s)f^2(s)ds+\sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^2(s)dB_l^H(s)\\ \qquad +\int_0^t S(t-s)g^2(s)dW(s), \quad \text{if} \quad t\in[0,t_1], \end{cases}$$

where $f^i(t)\in F^i(t,z_t+\theta_t,\bar{z}_t+\bar{\theta}_t)$ for a.e. $t\in[0,t_1]$.
Put

$$\widehat{C_0^*}=\left\{z,\bar{z}\in C_0^*, \quad \text{such that} \quad z_0=0\in\mathcal{D}_{\mathcal{F}_0} \quad \text{and} \quad \bar{z}_0=0\in\mathcal{D}_{\mathcal{F}_0}\right\}$$

and for any $z,\bar{z}\in\widehat{C_0^*}$ we have

$$\|x\|_{\widehat{C_0^*}}=\|z_0\|_{\mathcal{D}_{\mathcal{F}_0}}+\sup_{t\in[0,t_1]}\sqrt{E\|z(t)\|^2}.$$

It is not difficult to check that $(\widehat{C_0^*},\|.\|_{\widehat{C_0^*}})$ is a Banach space. Consider the multivalued operator $\underline{N}^0:\widehat{C_0^*}\times\widehat{C_0^*}\to\mathcal{P}(\widehat{C_0^*}\times\widehat{C_0^*})$ defined by

$$\underline{N}^0(z,\bar{z})=(\underline{N}_1^0(z,\bar{z}),\underline{N}_2^0(z,\bar{z})),\ (z,\bar{z})\in\widehat{C_0^*}\times\widehat{C_0^*}$$

where

$$\underline{N}^0(z,\bar{z})=\left\{(\underline{h}^0,\underline{\bar{h}}^0)\in\widehat{C_0^*}\times\widehat{C_0^*}\right\}$$

given by

$$\underline{N}_1^0(z,\bar{z})=\left\{\underline{h}^0\in\widehat{C_0^*}:\ \underline{h}^0(t)=\begin{cases}0, \text{if} \quad t\in(-\infty,0],\\ \int_0^t S(t-s)f^1(s)ds\\ +\sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^1(s)dB_l^H(s)\\ +\int_0^t S(t-s)g^1(s)dW(s), \quad \text{if } t\in[0,t_1]\end{cases}\right\}$$

and

$$\underline{N}_2^0(z,\bar{z})=\left\{\underline{\bar{h}}^0\in\widehat{C_0^*}:\ \underline{\bar{h}}^0(t)=\begin{cases}0, \text{if} \quad t\in(-\infty,0],\\ \int_0^t S(t-s)f^2(s)ds\\ +\sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^2(s)dB_l^H(s)\\ +\int_0^t S(t-s)g^2(s)dW(s), \quad \text{if } t\in[0,t_1]\end{cases}\right\}$$

Clearly, that the operator N^0 is equivalent to $\underline{N}^0$, and so we turn our attention to proving that $\underline{N}^0$ does in fact have a fixed point. We shall show that $\underline{N}^0$ Then the operator is completely continuous and *u.s.c.* on $t\in(-\infty,t_1]$.We divide the proof into several claims.

Claim 1. $\underline{N}^0$ is convex for each $z,\bar{z}\in\widehat{C_0^*}$.
In fact, if $\underline{h}_1^0,\underline{h}_2^0\in\underline{N}_1^0$ and $\underline{\bar{h}}_1^0,\underline{\bar{h}}_2^0\in\underline{N}_2^0$, then there exist $f_1^1,f_2^1\in S_{F^1,z+\theta,\bar{z}+\bar{\theta}}$ and $f_1^2,f_2^2\in S_{F^2,z+\theta,\bar{z}+\bar{\theta}}$ such that

$$\underline{h}_i^0(t)=\int_0^t S(t-s)f_i^1(s)ds+\sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^1(s)dB_l^H(s)+\int_0^t S(t-s)g^1(s)dW(s).$$

and

$$\underline{\bar{h}}_i^0(t)=\int_0^t S(t-s)f_i^2(s)ds+\sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^2(s)dB_l^H(s)+\int_0^t S(t-s)g^2(s)dW(s).$$

Let $0 \le \alpha \le 1$. Then, for each $t \in [0, t_1]$, we have

$$(\alpha \underline{h}_1^0 + (1-\delta)\underline{h}_2^0)(t) = \int_0^t S(t-s)[\alpha f_1^1(s) + (1-\alpha) f_2^1(s)]ds + \int_0^t S(t-s)g(s)dB^H(s)$$
$$+ \int_0^t S(t-s)g^1(s)dW(s) \in \underline{N}_1^0(z,\bar{z}).$$

Similarly, we have

$$(\alpha \underline{\bar{h}}_1^0 + (1-\delta)\underline{\bar{h}}_2^0)(t) = \int_0^t S(t-s)[\alpha f_1^2(s) + (1-\alpha) f_2^2(s)]ds + \int_0^t S(t-s)g(s)dB^H(s) \in \underline{N}_2^0(z,\bar{z})$$
$$+ \int_0^t S(t-s)g^2(s)dW(s) \in \underline{N}_2^0(z,\bar{z}).$$

Since $S_{F,z+\theta,\bar{z}+\bar{\theta}} = (S_{F^1,z+\theta,\bar{z}+\bar{\theta}}, S_{F^2,z+\theta,\bar{z}+\bar{\theta}})$ is convex (because $F(t,z,\bar{z})$ has convex values).

Claim 2. $\underline{N}^0$ maps bounded sets into bounded sets in $\widehat{C_0^*} \times \widehat{C_0^*}$.
Indeed, it is enough to show that for any $q > 0$ there exists a positive constant $l = (l_1, l_2)$ such that for each $(z,\bar{z}) \in B_q = \{(z,\bar{z}) \in \widehat{C_0^*} \times \widehat{C_0^*} : \|z\|^2_{\widehat{C_0^*}} \le q,\ \|\bar{z}\|^2_{\widehat{C_0^*}} \le q\}$ one has

$$\|\underline{h}^0\|^2_{\widehat{C_0^*}} \le l_1, \quad \|\underline{\bar{h}}^0\|^2_{\widehat{C_0^*}} \le l_2.$$

If $(\underline{h}^0, \underline{\bar{h}}^0) \in (\underline{N}_1^0, \underline{N}_2^0)$ there exists $f^i(t) \in F^i(t, z+\theta, \bar{z}+\bar{\theta})$ for each $t \in (-\infty, t_1]$, we get

$$E|\underline{h}^0(t)|^2 = E\Big|\int_0^t S(t-s)f^1(s)ds + \sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^1(s)dB_l^H(s) + \int_0^t S(t-s)g^1(s)dW(s)\Big|^2.$$

and

$$E|\underline{\bar{h}}^0(t)|^2 = E\Big|\int_0^t S(t-s)f^2(s)ds + \sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^2(s)dB_l^H(s) + \int_0^t S(t-s)g^2(s)dW(s)\Big|^2.$$

This, together with (H_1)-(H_3) , Lemma 2.16 and 2.4 yields that,

$$\begin{aligned}
&E|\underline{h}^0(t)|^2 \\
&= E\Big|\int_0^t S(t-s)f^1(s)ds + \textstyle\sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^1(s)dB_l^H(s) + \int_0^t S(t-s)g^1(s)dW(s)\Big|^2 \\
&\quad \le 3E\Big|\int_0^t S(t-s)f^1(s)ds\Big|^2 + 3E\Big|\textstyle\sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^1(s)dB_l^H(s)\Big|^2 \\
&+3E\Big|\int_0^t S(t-s)g^1(s)dW(s)\Big|^2 \\
&\quad \le 3Mt_1\int_0^t E|f^1(s)|^2ds + 3Mc_H H(2H-1)t_1^{2H-1}\int_0^{t_1}\|\sigma^1(s)\|^2ds \\
&+3MC_{2*}\int_0^{t_1}\|g^1(s)\|^2_{L_0^2}ds \\
&\quad \le 3Mt_1\psi_1(2C_{std})\int_0^t p_1(s)ds + 3Mc_H H(2H-1)t_1^{2H-1}C_2 + 3MC_1 \\
&:= l_1,
\end{aligned}$$

where

$$\begin{aligned}
&\|z_t+\theta_t\|^2_{\mathcal{D}_{\mathcal{F}_0}}+\|\bar{z}_t+\bar{\theta}_t\|^2_{\mathcal{D}_{\mathcal{F}_0}}\\
&\leq 2(\|z_t\|^2_{\mathcal{D}_{\mathcal{F}_0}}+\|\theta_t\|^2_{\mathcal{D}_{\mathcal{F}_0}})+2(\|\bar{z}_t\|^2_{\mathcal{D}_{\mathcal{F}_0}}+\|\bar{\theta}_t\|^2_{\mathcal{D}_{\mathcal{F}_0}})\\
&\quad\leq 4(\widetilde{N}^2(\|\phi\|^2_{\mathcal{D}_{\mathcal{F}_0}}+\|\bar{\phi}\|^2_{\mathcal{D}_{\mathcal{F}_0}})+\widetilde{K}^2(2q+M(E|\phi(0)|^2+E|\bar{\phi}(0)|^2))\\
&=C_{std}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&E|\underline{\bar{h}}^0(t)|^2\\
&\leq 3Mt_1\int_0^t E|f^2(s)|^2ds+3Mc_HH(2H-1)t_1^{2H-1}\int_0^{t_1}\|\sigma^2(s)\|^2ds+3MC_{2*}\int_0^{t_1}\|g^2(s)\|^2ds\\
&\quad\leq 3Mt_1\psi_2(2C_{std})\int_0^t p_2(s)ds+3Mc_HH(2H-1)t_1^{2H-1}C_2+3MC_1\\
&:=l_2,
\end{aligned}$$

$$\begin{pmatrix} E|\underline{h}^0(t)|^2\\ E|\underline{\bar{h}}^0(t)|^2\end{pmatrix}\leq\begin{pmatrix} l_1\\ l_2\end{pmatrix}$$

Claim 3. $\underline{N}^0$ maps bounded sets into equicontinuous sets of $\widehat{C_0^*}\times\widehat{C_0^*}$.

Let B_q be a bounded set in $\widehat{C_0^*}\times\widehat{C_0^*}$ as in **Claim 2**. Let $\tau_1,\tau_2\in J,\tau_1<\tau_2$ and $(x,y)\in B_q$,there exists $f^i(t)\in F^i(t,z+\theta,\bar{z}+\bar{\theta}),\ i=1,2,$ such that

$$\begin{aligned}
&E|\underline{h}^0(\tau_2)-\underline{h}^0(\tau_1)|^2\\
&\leq 6E|\int_0^{\tau_2}S(\tau_2-s)-S(\tau_1-s)f^1(s)ds|^2+6E|\int_{\tau_1}^{\tau_2}|S(\tau_1-s)f^1(s)ds|^2\\
&\quad+6E|\sum_{l=1}^{\infty}\int_0^{\tau_2}S(\tau_2-s)-S(\tau_1-s)\sigma_l^1(s)dB_l^H(s)|^2+6E|\sum_{l=1}^{\infty}\int_{\tau_1}^{\tau_2}|S(\tau_1-s)\sigma_l^1(s)dB_l^H(s)ds|^2\\
&+\quad 6E|\int_0^{\tau_2}S(\tau_2-s)-S(\tau_1-s)g^1(s)dW(s)|^2+6E|\int_{\tau_1}^{\tau_2}|S(\tau_1-s)g^1(s)dW(s)|^2
\end{aligned}$$

From (H_1)-(H_3) , Lemma 2.16 and 2.4, we obtain

$$\begin{aligned}
&E|\underline{h}^0(\tau_2)-\underline{h}^0(\tau_1)|^2\\
&\leq 6t_1\psi_1(2C_{std})\int_0^{\tau_2}|S(\tau_2-s)-S(\tau_1-s)|^2p_1(s)ds+6t_1\psi_1(2C_{std})E|\int_{\tau_1}^{\tau_2}|S(\tau_1-s)|^2p_1(s)ds\\
&\quad+6C_2c_HH(2H-1)t_1^{2H-1}\int_0^{\tau_2}|S(\tau_2-s)-S(\tau_1-s)|^2\\
&+6C_2c_HH(2H-1)t_1^{2H-1}\int_{\tau_1}^{\tau_2}|S(\tau_1-s)|^2ds+6C_1\int_0^{\tau_2}|S(\tau_2-s)-S(\tau_1-s)|^2ds\\
&\quad+6C_1E|\int_{\tau_1}^{\tau_2}|S(\tau_1-s)|^2ds
\end{aligned}$$

Similarly, we obtain that

$$\begin{aligned}
&E|\underline{\bar{h}}^0(\tau_2)-\underline{\bar{h}}^0(\tau_1)|^2\\
&\leq 6t_1\psi_2(2C_{std})\int_0^{\tau_2}|S(\tau_2-s)-S(\tau_1-s)|^2p_2(s)ds+6t_1\psi_1(2C_{std})E|\int_{\tau_1}^{\tau_2}|S(\tau_1-s)|^2p_1(s)ds\\
&\quad+6C_2c_HH(2H-1)t_1^{2H-1}\int_0^{\tau_2}|S(\tau_2-s)-S(\tau_1-s)|^2
\end{aligned}$$

$$+6C_2c_HH(2H-1)t_1^{2H-1}\int_{\tau_1}^{\tau_2}|S(\tau_1-s)|^2ds+6C_1\int_0^{\tau_2}|S(\tau_2-s)-S(\tau_1-s)|^2ds$$
$$+6C_1E|\int_{\tau_1}^{\tau_2}|S(\tau_1-s)|^2ds$$

The right-hand term tends to zero as $|\tau_2-\tau_1|\to 0$ since $S(t)$ is strongly continuous operator and the compactness of $S(t)$ for $t>0$ implies the continuity in the uniform operator topology [1]. This proves the equicontinuity.

Claim 4. $(\underline{N}^0(B_q)(t)$ is precompact in $\widehat{C_0^*}\times\widehat{C_0^*}$. As a consequence of Steps **Claim 2** and **Claim 3**, together with the Arzelá-Ascoli theorem, it suffices to show that $\underline{N}^0$ maps B_q into a precompact set in $\widehat{C_0^*}\times\widehat{C_0^*}$. Let $0<t<t_1$ be fixed and let ϵ be a real number satisfying $0<\epsilon<t_1$. For $(z,\bar{z})\in B_q$ we define

$$\underline{h}_\epsilon^0(z,\bar{z})(t)=S(\epsilon)\int_0^{t-\epsilon}S(t-s-\epsilon)f^1(s)ds+\sum_{l=1}^{\infty}S(\epsilon)\int_0^{t-\epsilon}S(t-s-\epsilon)\sigma_l^1(s)dB_l^H(s)$$
$$+S(\epsilon)\int_0^{t-\epsilon}S(t-s-\epsilon)g^1(s)dW(s)$$

and

$$\underline{\bar{h}}_\epsilon^0(z,\bar{z})(t)=S(\epsilon)\int_0^{t-\epsilon}S(t-s-\epsilon)f^2(s)ds+\sum_{l=1}^{\infty}S(\epsilon)\int_0^{t-\epsilon}S(t-s-\epsilon)\sigma_l^2(s)dB_l^H(s)$$
$$+S(\epsilon)\int_0^{t-\epsilon}S(t-s-\epsilon)g^2(s)dW(s)$$

Since $S(t)$ is a compact operator, the set

$$H_\epsilon=\{\widetilde{h}_{1\epsilon}(t)=(\underline{h}_\epsilon^0,\underline{\bar{h}}_\epsilon^0):\widetilde{h}_{1\epsilon}\in\underline{N}^0(z,\bar{z})\ \ (z,\bar{z})\in B_q\}$$

Using (H_1)-(H_3),Lemma 2.16 and 2.4, we have

$$\begin{aligned}
&E\Big|\underline{h}^0(t)-\underline{h}_\epsilon^0(t)\Big|^2\\
&\le 3E\Big|\int_{t-\epsilon}^{t}S(t-s)f^1(s)ds)\Big|^2+3E\Big|\sum_{l=1}^{\infty}\int_{t-\epsilon}^{t}S(t-s)\sigma_l^1(s))dB_l^H(s)\Big|^2\\
&\quad+3E\Big|\int_{t-\epsilon}^{t}S(t-s)g^1(s)ds)\Big|^2\\
&\le 3M^2\psi_1(2C_{std})\int_{t-\epsilon}^{t}p_1(s)ds+3M^2(c_HH(2H-1)\epsilon^{2H-1}\int_{t-\epsilon}^{t}\|\sigma^1(s))\|ds\\
&\quad+3M^2\int_{t-\epsilon}^{t}\|g^1(s))\|_{L^0}^2ds
\end{aligned}$$

Similarly,

$$\begin{aligned}
E\Big|\underline{\bar{h}}^0(t)-\underline{\bar{h}}_\epsilon^0(t)\Big|^2 &\le 3M^2\psi_2(2C_{std})\int_{t-\epsilon}^{t}p_2(s)ds\\
&+\ 3M^2(c_HH(2H-1)\epsilon^{2H-1}\int_{t-\epsilon}^{t}\|\sigma^2(s))\|^2ds\\
&+\ 3M^2C_{1*}\int_{t-\epsilon}^{t}\|g^2(s))\|_{L_0}^2ds
\end{aligned}$$

The right-hand side tends to 0, as $\epsilon \to 0$. Therefore, there are precompact sets arbitrarily close to the set $H = \{\widetilde{h_1}(t) = (\underline{h}^0, \bar{\underline{h}}^0) : \widetilde{h_1} \in \underline{N}^0(z, \bar{z})\ \ (z, \bar{z}) \in B_q\}$. This set is then precompact in $X \times X$.
Claim 5. $\underline{N}^0 = (\underline{N}_1^0, \underline{N}_2^0)$ has a closed graph. Let $u_n = (z_n, z_n) \longrightarrow u_* = (z_*, z_*)$,$(\underline{h}_n^0, \bar{\underline{h}}_n^0) \in \underline{N}^0(u_n)$ and $(\underline{h}_n^0, \bar{\underline{h}}_n^0) \longrightarrow (\underline{h}_*^0, \bar{\underline{h}}_*^0)$ as $n \longrightarrow \infty$, we shall prove that $\underline{h}_*^0 \in N_1^0(u_*)$ and $\bar{\underline{h}}_*^0 \in N_2^0(u_*)$. The fact that $\bar{\underline{h}}_n^0 \in \underline{N}_1^0(u_n)$ and $\bar{\underline{h}}_n^0 \in \underline{N}_2^0(u_n)$ means that there exists $f_n^i \in S_{F^i, u_n}$ for each $i = 1, 2$ such that

$$\underline{h}_n^0(z, \bar{z})(t) = \int_0^t S(t-s) f_n^1(s) ds + \sum_{l=1}^{\infty} \int_0^t S(t-s) \sigma_l^1(s) dB_l^H(s) + \int_0^t S(t-s) g^1(s) dW(s)$$

and

$$\bar{\underline{h}}_n^0(z, \bar{z})(t) = \int_0^t S(t-s) f_n^2(s) ds + \sum_{l=1}^{\infty} \int_0^t S(t-s) \sigma_l^2(s) dB_l^H(s) + \int_0^t S(t-s) g^2(s) dW(s)$$

We must prove that there exists $f_*^i \in S_{F^i, z_* + \theta, \bar{z}_* + \bar{\theta}}$ such that

$$\underline{h}_*^0(t) = \int_0^t S(t-s) f_*^1(s) ds + \sum_{l=1}^{\infty} \int_0^t S(t-s) \sigma_l^1(s) dB_l^H(s) + \int_0^t S(t-s) g^1(s) dW(s), \quad t \in [0, t_1].$$

and

$$\bar{\underline{h}}_*^0(t) = \int_0^t S(t-s) f_*^2(s) ds + \sum_{l=1}^{\infty} \int_0^t S(t-s) \sigma_l^2(s) dB_l^H(s) + \int_0^t S(t-s) g^2(s) dW(s), \quad t \in [0, t_1].$$

Now, consider the continuous linear operator $\Gamma : L^2([0, t_1], X) \longrightarrow \widehat{C_0^*}$ defined for each $i = 1, 2$, by

$$\Gamma(f^i)(t) = \int_0^t S(t-s) f^i(s) ds.$$

From the definition of Γ we know that

$$\underline{h}_n^0(t) - \sum_{l=1}^{\infty} \int_0^t S(t-s) \sigma_l^1(s) dB_l^H(s) - \int_0^t S(t-s) g^1(s) dW(s) \in \Gamma(S_{F^1, z_n + \theta, \bar{z}_n + \bar{\theta}}).$$

and

$$\bar{\underline{h}}_n^0(t) - \sum_{l=1}^{\infty} \int_0^t S(t-s) \sigma_l^2(s) dB_l^H(s) - \int_0^t S(t-s) g^2(s) dW(s) \in \Gamma(S_{F^2, z_n + \theta, \bar{z}_n + \bar{\theta}}).$$

Since $(z_n, \bar{z}_n) \longrightarrow (z_*, \bar{z}_*)$ and $(\underline{h}_n^0, \bar{\underline{h}}_n^0) \longrightarrow (\underline{h}_*^0, \bar{\underline{h}}_*^0)$, there is $f_*^i \in S_{F^i, z_* + \theta, \bar{z}_* + \bar{\theta}}$ such that

$$\underline{h}_*^0(t) = \int_0^t S(t-s) f_*^1(s) ds + \sum_{l=1}^{\infty} \int_0^t S(t-s) \sigma_l^1(s) dB_l^H(s) + \int_0^t S(t-s) g^1(s) dW(s), \quad t \in [0, t_1].$$

and

$$\underline{h}_*^0(t) = \int_0^t S(t-s) f_*^2(s) ds + \sum_{l=1}^{\infty} \int_0^t S(t-s) \sigma_l^2(s) dB_l^H(s) + \int_0^t S(t-s) g^2(s) dW(s), \quad t \in [0, t_1].$$

Hence $(\underline{h}_*^0, \bar{\underline{h}}_*^0) \in (\underline{N}_1^0(u_*), \underline{N}_2^0(u_*))$, proving our claim. Lemma 2.10 yields that $\underline{N}^0$ is upper semicontinuous on $t \in [0, t_1]$,denote this solution by $(x_0, y_0) \in C_0^* \times C_0^*$

Step 2. Now consider the problem on $(-\infty, t_2]$

$$\begin{cases} dx(t) &= (Ax(t) + F^1(t, x_t, y_t)dt \\ & + \sum_{l=1}^{\infty} \sigma_l^1(t)dB_l^H(t) + g^1(t)dW(t),\ t \in (t_1, t_2], \\ dy(t) &= (Ay(t) + F^2(t, x_t, y_t)dt \\ & + \sum_{l=1}^{\infty} \sigma_l^2(t)dB_l^H(t) + g^2(t)dW(t),\ t \in (t_1, t_2], \\ x(t_1^+) &= x_0(t_1^-) + I_1(x_1(t_1)) \\ y(t_1^+) &= y_0(t_1^-) + \bar{I}_1(y_1(t_1)) \\ x(t) &= x_0(t) \text{ if } t \in (-\infty, t_1] \\ y(t) &= y_0(t) \text{ if } t \in (-\infty, t_1] \end{cases} \tag{13}$$

Let

$$C_1 = \{y \in C([t_1, t_2], X) : \quad : x(t_1^+) \text{ exists },\ \sup_{t\in[t_1,t_2]} E(|x(t)|^2) < \infty\}.$$

Put

$$C_1^* = \mathcal{D}_{\mathcal{F}_0} \cap C_0 \cap C_1.$$

Consider the multivalued operator $N^1 : C_1^* \times C_1^* \to \mathcal{P}(C_1^* \times C_1^*)$ with $N^1(x, y) = (N_1^1(x, y), N_2^1(x, y)),\ (x, y) \in C_1^* \times C_1^*$ defined by

$$N^1(x, y) = \left\{(h^1, \bar{h}^1) \in C_1^* \times C_1^*\right\},$$

given by

$$N_1^1(x, y) = \left\{ h^1 \in C_1^* : \ h^1(t) = \begin{cases} x_0(t), & \text{if} t \in (-\infty, t_1], \\ x_0(t_1^-) + S(t - t_1)I_1(y_0(t_1^-)) & \\ + \int_{t_1}^t S(t - s)f^2(s)ds & \\ + \sum_{l=1}^{\infty} \int_{t_1}^t S(t - s)\sigma_l^2(s)dB_l^H(s) & \\ + \int_{t_1}^t S(t - s)g^2(s)dW(s), & \text{if } t \in (t_1, t_2] \end{cases} \right\}$$

and

$$N_2^1(x, y) = \left\{ \bar{h}^1 \in C_1^* : \ \bar{h}^1(t) = \begin{cases} y_0(t), & \text{if} t \in (-\infty, t_1], \\ y_0(t_1^-) + S(t - t_1)\bar{I}_1(y_0(t_1^-)) & \\ + \int_{t_1}^t S(t - s)f^2(s)ds & \\ + \sum_{l=1}^{\infty} \int_{t_1}^t S(t - s)\sigma_l^2(s)dB_l^H(s) & \\ + \int_{t_1}^t S(t - s)g^2(s)dW(s), & \text{if } t \in (t_1, t_2] \end{cases} \right\}$$

where

$$f^i \in S_{F^i,u} = \{f^i \in L^2((t_1, t_2], X) : f^i(t) \in F^i(t, x, y) \quad \text{for a.e} \quad t \in (t_1, t_2]\}.$$

Let $\theta : (-\infty, t_2] \longrightarrow X$ be the function defined by

$$\theta(t) = \begin{cases} x_0(t), & t \in (-\infty, t_1], \\ x_0(t_1^-) + S(t - t_1)I_1(x_0(t_1^-)), & \text{if } t \in (t_1, t_2] \end{cases}$$

and

$$\bar{\theta}(t) = \begin{cases} y_0(t), & t \in (-\infty, t_1], \\ y_0(t_1^-) + S(t - t_1)\bar{I}_1(y_0(t_1^-)), & \text{if } t \in (t_1, t_2] \end{cases}$$

Thus we have $(\theta,\bar{\theta})$ is an element of $C_1^*\times C_1^*$. Let$(x(t),y(t))=(z(t)+\theta(t),\bar{z}(t)+\bar{\theta}(t)),-\infty<t\le t_2$.Obviously, if (x,y) satisfies the integral equation

$$\begin{cases} x(t)=x_0(t_1^-)+S(t-t_1)I_1(x_0(t_1^-))+\int_{t_1}^t S(t-s)f^1(s)ds+\sum_{l=1}^\infty\int_{t_1}^t S(t-s)\sigma_l^2(s)dB_l^H(s)\\ \quad +\int_{t_1}^t S(t-s)g^1(s)dW(s) \quad \text{if} \quad t\in(t_1,t_2]\\ y(t)=y_0(t_1^-)+S(t-t_1)\bar{I}_1(y_0(t_1^-))+\int_{t_1}^t S(t-s)f^2(s)ds+\sum_{l=1}^\infty\int_{t_1}^t S(t-s)\sigma_l^2(s)dB_l^H(s)\\ \quad +\int_{t_1}^t S(t-s)g^2(s)dW(s) \quad \text{if} \quad t\in(t_1,t_2]\end{cases}$$

By replacing $(z,\bar{z})$ in previous equation and satisfies $(z_0,\bar{z}_0)=(0,0)$ if $t\in(-\infty,t_1]$ we have

$$\begin{cases} z(t)=S(t-t_1)I_1(z_0(t_1^-)+\theta(t_1^-),\bar{z}_0(t_1^-)+\bar{\theta}(t_1^-))\\ +\int_0^t S(t-s)f^1(s)ds+\sum_{l=1}^\infty\int_{t_1}^t S(t-s)\sigma_l^2(s)dB_l^H(s)\\ \quad +\int_{t_1}^t S(t-s)g^1(s)dW(s) \quad \text{if} \quad t\in(t_1,t_2]\\ \bar{z}(t)=S(t-t_1)I_1(z_0(t_1^-)+\theta(t_1^-),\bar{z}_0(t_1^-)+\bar{\theta}(t_1^-))\\ +\int_0^t S(t-s)f^2(s)ds+\sum_{l=1}^\infty\int_{t_1}^t S(t-s)\sigma_l^2(s)dB_l^H(s)\\ \quad +\int_{t_1}^t S(t-s)g^2(s)dW(s) \quad \text{if} \quad t\in(t_1,t_2]\end{cases}$$

where $f^i(t)\in F^i(t,z_t+\theta_t,\bar{z}_t+\bar{\theta}_t)$ for a.e. $t\in(t_1,t_2]$. Put

$$\widehat{C_1^*}=\left\{z,\bar{z}\in C_1^*, \quad \text{such that} \quad z_{t_1}=0 \quad \text{and} \quad \bar{z}_{t_1}=0\right\}$$

and for any $z,\bar{z}\in\widehat{C_1^*}$.
Consider the multivalued operator $\underline{N}^1:\widehat{C_1^*}\times\widehat{C_1^*}\to\mathcal{P}(\widehat{C_1^*}\times\widehat{C_1^*})$ with $\underline{N}^1(z,\bar{z})=(\underline{N}_1^1(z,\bar{z}),\underline{N}_2^1(z,\bar{z}))$, $(z,\bar{z})\in\widehat{C_1^*}\times\widehat{C_1^*}$ defined by

$$\underline{N}^1(z,\bar{z})=\left\{(\underline{h}^1,\underline{\bar{h}}^1)\in\widehat{C_1^*}\times\widehat{C_1^*}\right\},$$

given by

$$\underline{N}_1^1(z,\bar{z})=\left\{\underline{h}^1\in C_1^*:\ \underline{h}^1(t)=\begin{cases}0, \quad \text{if} \quad t\in(-\infty,t_2]\\ S(t-t_1)I_1(z_0(t_1^-)+\theta(t_1^-),\bar{z}_0(t_1^-)+\bar{\theta}(t_1^-))\\ \int_{t_1}^t S(t-s)f^1(s)ds\\ +\sum_{l=1}^\infty\int_{t_1}^t S(t-s)\sigma_l^1(s)dB_l^H(s)\\ +\int_{t_1}^t S(t-s)g^1(s)dW(s), \quad \text{if} \quad t\in(t_1,t_2]\end{cases}\right\}$$

and

$$\underline{N}_2^1(z,\bar{z})=\left\{\underline{\bar{h}}^1\in C_1^*:\ \underline{\bar{h}}^1(t)=\begin{cases}0, \quad \text{if} \quad t\in(-\infty,t_2]\\ S(t-t_1)\bar{I}_1(z_0(t_1^-)+\theta(t_1^-),\bar{z}_0(t_1^-)+\bar{\theta}(t_1^-))\\ +\int_{t_1}^t S(t-s)f^2(s)ds\\ +\sum_{l=1}^\infty\int_{t_1}^t S(t-s)\sigma_l^2(s)dB_l^H(s)\\ +\int_{t_1}^t S(t-s)g^2(s)dW(s), \quad \text{if} \quad t\in(t_1,t_2]\end{cases}\right\}$$

As in Step 1, we can show that $\underline{N}^1$ is upper semicontinuous on $t\in(t_1,t_2]$,denote this solution by $(x_1,y_1)\in C_1^*\times C_1^*$.

Step3. We continue this process taking into account that $(z_m, \bar{z}_m) := (z|_{[t_m,b]}, \bar{z}|_{[t_m,b]})$ is a solution of the problem

$$\begin{cases} dx(t) &= (Ax(t) + F^1(t, x_t, y_t)dt \\ &+ \sum_{l=1}^{\infty} \sigma_l^1(t)dB_l^H(t) + g^1(t)dW(t),\ t \in (t_m, b], \\ dy(t) &= (Ay(t) + F^2(t, x_t, y_t)dt \\ &+ \sum_{l=1}^{\infty} \sigma_l^2(t)dB_l^H(t) + g^2(t)dW(t),\ t \in (t_m, b], \\ x(t_m^+) &= x_m(t_m^-) + I_m(x_m(t_m), \\ y(t_m^+) &= y_m(t_m^-) + \bar{I}_m(y_m(t_1)), \\ x(t) &= x_{m-1}(t) \text{ if } \quad t \in (-\infty, t_m] \\ y(t) &= y_{m-1}(t) \text{ if } \quad t \in (-\infty, t_m] \end{cases} \tag{14}$$

Let

$$C_m = \{x \in C([t_m, b], X) : \quad : x(t_m^+) \text{ exists }, \sup_{t\in[t_m,b]} E(|x(t)|^2) < \infty\}.$$

Set

$$C_m^* = \mathcal{D}_{\mathcal{F}_0} \cap \cap_{j=0}^{j=m} C_j.$$

Consider the multivalued operator $N^m : C_m^* \times C_m^* \to \mathcal{P}(C_m^* \times C_m^*)$ with $N^m(x, y) = (N_1^m(x, y), N_2^m(x, y)),\ (x, y) \in C_m^* \times C_m^*$ defined by

$$N^m(x, y) = \left\{(h^m, \overline{h}^m) \in C_m^* \times C_m^*\right\}$$

given by

$$N_1^m(x, y) = \left\{ h' \in C_1^* : \ h'(t) = \begin{cases} x_m(t), \quad \text{if} \quad t \in (-\infty, t_m], \\ x_m(t_1^-) + S(t - t_1)I_m(x_m(t_1^-)) \\ + \int_{t_m}^t S(t - s)f^2(s)ds \\ + \sum_{l=1}^{\infty} \int_{t_1}^t S(t - s)\sigma_l^2(s)dB_l^H(s) \\ + \int_{t_m}^t S(t - s)g^2(s)dW(s), \quad \text{if} \quad t \in (t_m, b] \end{cases} \right\}$$

and

$$N_2^m(x, y) = \left\{ \bar{h}^m \in C_m^* : \ \bar{h}^m(t) = \begin{cases} y_m(t), \quad \text{if} \quad t \in (-\infty, t_m], \\ y_m(t_1^-) + S(t - t_1)\bar{I}_1(y_m(t_1^-)) \\ + \int_{t_m}^t S(t - s)f^2(s)ds \\ + \sum_{l=1}^{\infty} \int_{t_1}^t S(t - s)\sigma_l^2(s)dB_l^H(s) \\ + \int_{t_m}^t S(t - s)g^2(s)dW(s), \quad \text{if} \quad t \in (t_m, b] \end{cases} \right\}$$

where

$$f^i \in S_{F^i,u} = \{f^i \in L^2((t_1, t_2], X) : f^i(t) \in F^i(t, x, y) \quad \text{for a.e} \quad t \in (t_m, b]\}.$$

Let $\theta : (-\infty, b] \longrightarrow X$ be the function defined by

$$\theta(t) = \begin{cases} x_m(t), & t \in (-\infty, t_m], \\ x_m(t_1^-) + S(t - t_1)I_1(x_m(t_1^-)), & \text{if } t \in (t_m, b] \end{cases}$$

and

$$\bar{\theta}(t) = \begin{cases} y_m(t), & t \in (-\infty, t_m], \\ y_m(t_1^-) + S(t - t_1)\bar{I}_1(y_m(t_1^-)), & \text{if } t \in (t_m, b] \end{cases}$$

Observe that $(\theta, \bar{\theta})$ is an element of $C_m^* \times C_m^*$. Let $(x(t), y(t)) = (z(t) + \theta(t), \bar{z}(t) + \bar{\theta}(t)), -\infty < t \leq b$. Obviously, if (x, y) satisfies the integral equation

$$\begin{cases} x(t) = x_m(t_m^-) + S(t - t_m)I_m(x_m(t_1^-)) + \int_{t_1}^t S(t - s)f^1(s)ds + \sum_{l=1}^{\infty} \int_{t_m}^t S(t - s)\sigma_l^2(s)dB_l^H(s) \\ \qquad + \int_{t_m}^t S(t - s)g^1(s)dW(s) \quad t \in (t_m, b] \\ y(t) = y_m(t_m^-) + S(t - t_m)\bar{I}_m(y_m(t_1^-)) + \int_{t_1}^t S(t - s)f^2(s)ds + \sum_{l=1}^{\infty} \int_{t_m}^t S(t - s)\sigma_l^2(s)dB_l^H(s) \\ \qquad + \int_{t_1}^t S(t - s)g^2(s)dW(s) \quad t \in (t_m, b] \end{cases}$$

Put $(z_m, \bar{z}_m) := (z_{t_m}, \bar{z}_{t_m}) = (0,0)$ for each $t \in (-\infty, t_m]$ we have

$$\begin{cases} z(t) = S(t-t_m)I_1(z_0(t_1^-) + \theta(t_1^-), \bar{z}_0(t_1^-) + \bar{\theta}(t_1^-)) \\ + \int_{t_m}^t S(t-s)f^1(s)ds + \sum_{l=1}^{\infty} \int_{t_m}^t S(t-s)\sigma_l^2(s)dB_l^H(s) \\ \quad + \int_{t_m}^t S(t-s)g^1(s)dW(s) \quad t \in (t_m, b] \\ \bar{z}(t) = S(t-t_m)I_m(z_m(t_1^-) + \theta(t_m^-), \bar{z}_m(t_1^-) + \bar{\theta}(t_m^-)) \\ + \int_{t_m}^t S(t-s)f^2(s)ds + \sum_{l=1}^{\infty} \int_{t_1}^t S(t-s)\sigma_l^2(s)dB_l^H(s) \\ \quad + \int_{t_m}^t S(t-s)g^2(s)dW(s) \quad t \in (t_m, b] \end{cases}$$

where $f^i(t) \in F^i(t, z_t + \theta_t, \bar{z}_t + \bar{\theta}_t)$ for a.e. $t \in (t_m, b]$.
Put

$$\widehat{C_m^*} = \{z, \bar{z} \in C_m^*, \quad \text{such that} \quad z_{t_m} = 0 \quad \text{and} \quad \bar{z}_{t_m} = 0\}$$

and for any $z, \bar{z} \in \widehat{C_1^*}$.
Consider the multivalued operator $\underline{N}^m \widehat{C_m^*} \times \widehat{C_m^*} \to \mathcal{P}(\widehat{C_m^*} \times \widehat{C_m^*})$ with $\underline{N}^m(z,\bar{z}) = (\underline{N}_1^m(z,\bar{z}), \underline{N}_2^m(z,\bar{z}))$, $(z,\bar{z}) \in \widehat{C_m^*} \times \widehat{C_m^*}$ defined by

$$\underline{N}^m(z,\bar{z}) = \left\{(\underline{h}^m, \underline{\bar{h}}^m) \in \widehat{C_m^*} \times \widehat{C_m^*}\right\}$$

given by

$$\underline{N}_1^m(z,\bar{z}) = \left\{ \underline{h}^m \in \widehat{C_m^*} : \underline{h}^m(t) = \begin{cases} 0, & \text{if} \quad t \in (-\infty, t_m] \\ S(t-t_1)I_m(z_m(t_1^-) + \theta(t_m^-), \bar{z}_m(t_m^-) + \bar{\theta}(t_m^-)) \\ \int_{t_m}^t S(t-s)f^1(s)ds \\ + \sum_{l=1}^{\infty} \int_{t_m}^t S(t-s)\sigma_l^1(s)dB_l^H(s) \\ + \int_{t_m}^t S(t-s)g^1(s)dW(s), & \text{if} \quad t \in (t_m, b] \end{cases} \right\}$$

and

$$\underline{N}_2^m(z,\bar{z}) = \left\{ \underline{\bar{h}}^1 \in \widehat{C_m^*} : \underline{\bar{h}}^1(t) = \begin{cases} 0, & \text{if} \quad t \in (-\infty, t_m] \\ S(t-t_m)\bar{I}_m(z_m(t_m^-) + \theta(t_m^-), \bar{z}_m(t_m^-) + \bar{\theta}(t_m^-)) \\ + \int_{t_m}^t S(t-s)f^2(s)ds \\ + \sum_{l=1}^{\infty} \int_{t_m}^t S(t-s)\sigma_l^2(s)dB_l^H(s) \\ + \int_{t_m}^t S(t-s)g^2(s)dW(s), & \text{if} \quad t \in (t_m, b] \end{cases} \right\}$$

As in Step 1, we can show that $\underline{N}^1$ is upper semicontinuous on $t \in (t_m, b]$,denote this solution by $(x_m, y_m) \in C_m^* \times C_m^*$.
The desired result is then complete. □

Now, we present the fist our existence and compactness of solution set of the Problem (1).

Theorem 3.3. *Assume that $F^i : [0,b] \times \mathcal{D}_{\mathcal{F}_0} \times \mathcal{D}_{\mathcal{F}_0} \longrightarrow \mathcal{P}_{cv,cp}(X)$ is a Caratheodory map satisfying (H_1)-(H_3) hold. Then the (1) has at least one mild solution on J. If further X is a reflexive space, then the solution set is compact in $\mathcal{D}_{\mathcal{F}_b} \times \mathcal{D}_{\mathcal{F}_b}$.*

Proof. **Part 1**. Existence of solutions.
We transform the problem (1) into a fixed point problem. Consider the multi- valued operator $N : \mathcal{D}_{\mathcal{F}_b} \times \mathcal{D}_{\mathcal{F}_b} \to \mathcal{P}(\mathcal{D}_{\mathcal{F}_b} \times \mathcal{D}_{\mathcal{F}_b})$ defined in lemma 3.2. It is clear that all solutions of Problem (1) are fixed points of the multi-valued operator defined by We shall show that N satisfies assumptions of Lemma 2.14. Since for each $(x,y) \in \mathcal{D}_{\mathcal{F}_b} \times \mathcal{D}_{\mathcal{F}_b}$, the nonlinearity F^i takes convex values, the selection set $S_{F^i,u}$ is convex, and therefore N has convex values. From lemma 3.2, N is completely continuous and u.s.c.
Claim 5. There exist a priori bounds on solutions.

Consider the multivalued operator $N : \mathcal{D}_{\mathcal{F}_b} \times \mathcal{D}_{\mathcal{F}_b} \to \mathcal{P}(\mathcal{D}_{\mathcal{F}_b} \times \mathcal{D}_{\mathcal{F}_b})$.We will prove that N the operator is completely continuous and *u.s.c.* with $(N_1(x,y), N_2(x,y)),\ (x,y) \in \mathcal{D}_{\mathcal{F}_b} \times \mathcal{D}_{\mathcal{F}_b}$ defined by

$$N(x,y) = \left\{(h,\overline{h}) \in \mathcal{D}_{\mathcal{F}_b} \times \mathcal{D}_{\mathcal{F}_b}\right\}$$

given by

$$N_1(x,y) = \left\{ h \in \mathcal{D}_{\mathcal{F}_b} : \ h(t) = \begin{cases} \phi(t), & \text{if } t \in (-\infty, 0], \\ S(t)\phi(0) + \int_0^t S(t-s) f^1(s)ds & \\ + \sum_{l=1}^{\infty} \int_0^t S(t-s)\sigma_l^1(s)dB_l^H(s) & \\ + \int_0^t S(t-s) g^1(s)dW(s) & \\ + \sum_{0<t_k<t} S(t-t_k) I_k(x(t_k^-)), & \text{if } \ t \in [0,b] \end{cases} \right\}$$

and

$$N_2(x,y) = \left\{ \bar{h} \in \mathcal{D}_{\mathcal{F}_b} : \ \bar{h}(t) = \begin{cases} \bar{\phi}(t), & \text{if } t \in (-\infty, 0], \\ S(t)\bar{\phi}(0) + \int_0^t S(t-s) f^2(s)ds & \\ + \sum_{l=1}^{\infty} \int_0^t S(t-s)\sigma_l^2(s)dB_l^H(s) & \\ + \int_0^t S(t-s) g^2(s)dW(s) & \\ + \sum_{0<t_k<t} S(t-t_k) \bar{I}_k(y(t_k^-)), & \text{if } \ t \in [0,b] \end{cases} \right\}$$

where

$$f^i \in S_{F^i,u} = \{f^i \in L^2(J,X) : f^i(t) \in F^i(t,x,y) \quad \text{for a.e} \quad t \in J\}.$$

Let $\theta, \bar{\theta} : (-\infty, b] \longrightarrow X$ be the function defined by

$$\theta(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ S(t)\phi(0), & t \in [0, t_1]. \end{cases}$$

and

$$\bar{\theta}(t) = \begin{cases} \bar{\phi}(t), & t \in (-\infty, 0], \\ S(t)\bar{\phi}(0), & t \in [0, t_1]. \end{cases}$$

It is clear that $(\theta, \bar{\theta})$ is an element of $\mathcal{D}_{\mathcal{F}_b} \times \mathcal{D}_{\mathcal{F}_b}$. Set $(x(t), y(t)) = (z(t) + \theta(t), \bar{z}(t) + \bar{\theta}(t)), -\infty < t \leq b$.Obviously, if x, y satisfies (1) if and only if $(z, \bar{z})$ satisfies $(z_0, \bar{z}_0) = (0,0)$ if $t \in (-\infty, 0]$ and

$$\begin{cases} z(t) = \int_0^t S(t-s) f^1(s)ds + \sum_{l=1}^{\infty} \int_0^t S(t-s)\sigma_l^2(s)dB_l^H(s) \\ \qquad + \int_0^t S(t-s) g^1(s)dW(s) + \sum_{0<t_k<t} S(t-t_k) I_k(z(t_k^-) + \theta(t_k^-)), \text{if } t \in [0,b] \\ \bar{z}(t) = \int_0^t S(t-s) f^2(s)ds + \sum_{l=1}^{\infty} \int_0^t S(t-s)\sigma_l^2(s)dB_l^H(s) \\ \qquad + \int_0^t S(t-s) g^2(s)dW(s) + \sum_{0<t_k<t} S(t-t_k) I_k(z(t_k^-) + \theta(t_k^-)), \text{if } t \in [0,b] \end{cases}$$

where $f^i(t) \in F^i(t, z_t + \theta_t, \bar{z}_t + \bar{\theta}_t)$ for a.e. $t \in [0,b]$.
Put

$$\widehat{\mathcal{D}}_{\mathcal{F}_b} = \left\{ z, \bar{z} \in \mathcal{D}_{\mathcal{F}_b}, \quad \text{such that} \quad z_0 = 0 \in \mathcal{D}_{\mathcal{F}_0} \quad \text{and} \quad \bar{z}_0 = 0 \in \mathcal{D}_{\mathcal{F}_0} \right\}$$

and for any $z, \bar{z} \in \widehat{\mathcal{D}}_{\mathcal{F}_b}$ we have

$$\|x\|_{\widehat{\mathcal{D}}_{\mathcal{F}_b}} = \|z_0\|_{\mathcal{D}_{\mathcal{F}_0}} + \sup_{t \in [0,b]} \sqrt{E\|z(t)\|^2}.$$

It is not difficult to check that $(\widehat{\mathcal{D}}_{\mathcal{F}_b}, \|.\|_{\widehat{\mathcal{D}}_{\mathcal{F}_b}})$ is a Banach space. Consider the multivalued operator $\underline{N}$: $\widehat{\mathcal{D}}_{\mathcal{F}_b} \times \widehat{\mathcal{D}}_{\mathcal{F}_b} \to \mathcal{P}(\widehat{\mathcal{D}}_{\mathcal{F}_b} \times \widehat{\mathcal{D}}_{\mathcal{F}_b})$ defined by

$$\underline{N}(z, \bar{z}) = (\underline{N}_1(z, \bar{z}), \underline{N}_2(z, \bar{z})),\ (z, \bar{z}) \in \widehat{\mathcal{D}}_{\mathcal{F}_b} \times \widehat{\mathcal{D}}_{\mathcal{F}_b}$$

where

$$\underline{N}(z,\bar{z}) = \Big\{(\underline{h},\underline{\bar{h}}) \in \widehat{\mathcal{D}}_{\mathcal{F}_b} \times \widehat{\mathcal{D}}_{\mathcal{F}_b}\Big\}$$

given by

$$\underline{N}_1(z,\bar{z}) = \left\{ \underline{h} \in \widehat{\mathcal{D}}_{\mathcal{F}_b} : \ \underline{h}(t) = \begin{cases} 0, \quad \text{if} \quad t \in (-\infty, 0], \\ \int_0^t S(t-s)f^1(s)ds \\ +\sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^1(s)dB_l^H(s) \\ +\int_0^t S(t-s)g^1(s)dW(s) \\ +\sum_{0<t_k<t} S(t-t_k)\bar{I}_k(\bar{z}(t_k^-)+\bar{\theta}(t_k^-)), \text{if } t \in [0,b] \end{cases} \right\}$$

and

$$\underline{N}_2(z,\bar{z}) = \left\{ \underline{\bar{h}}^0 \in \widehat{\mathcal{D}}_{\mathcal{F}_b} : \ \underline{\bar{h}}(t) = \begin{cases} 0, \quad \text{if} \quad t \in (-\infty, 0], \\ \int_0^t S(t-s)f^2(s)ds \\ +\sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^2(s)dB_l^H(s) \\ +\int_0^t S(t-s)g^2(s)dW(s) \\ +\sum_{0<t_k<t} S(t-t_k)\bar{I}_k(\bar{z}(t_k^-)+\bar{\theta}(t_k^-)), \text{if } t \in [0,b] \end{cases} \right\}$$

Clearly, that the operator N is equivalent to $\underline{N}$.Let z be a possible solution of the equation $(z,\bar{z}) \in \lambda\underline{N}(z,\bar{z})$ and $(z_0,\bar{z}_0) = (\phi,\bar{\phi})$, for some $\lambda \in (0,1)$. Then,

$$\begin{aligned} z(t) &= \int_0^t S(t-s)f^1(s)ds + \sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^1(s)dB^H(s) + \int_0^t S(t-s)g^1(s)dW(s) \\ &+ \sum_{0<t_k<t} S(t-t_k)I_k(z(t_k^-)+\theta(t_k^-)) \end{aligned}$$

and

$$\begin{aligned} \bar{z}(t) &= \int_0^t S(t-s)f^2(s)ds + \sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^2(s)dB^H(s) + \int_0^t S(t-s)g^2(s)dW(s) \\ &+ \sum_{0<t_k<t} S(t-t_k)\bar{I}_k(z(t_k^-)+\theta(t_k^-)) \end{aligned}$$

Thus, for $t \in [0,b]$, namely:

$$\begin{aligned} &E|z(t)|^2 \\ &\leq 4E\Big|\int_0^t S(t-s)f^1(s)ds\Big|^2 + 4E\Big|\sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^1(s)dB^H(s)\Big|^2 + 4E\Big|\int_0^t S(t-s)g^1(s)dW(s)\Big|^2 \\ &\quad +4E\Big|\sum_{0<t_k<t} S(t-t_k)I_k(z(t_k^-)+\theta(t_k^-),\bar{z}(t_k^-)+\bar{\theta}(t_k^-))\Big|^2 \end{aligned}$$

which immediately yields:

$$\begin{aligned} &E|z(t)|^2 \\ &\leq 4Mb\int_0^t p_1(s)\psi_1(\|z_s+\theta_s\|^2_{\mathcal{D}_{\mathcal{F}_0}} + \|\bar{z}_s+\bar{\theta}_s\|^2_{\mathcal{D}_{\mathcal{F}_0}})ds + 4Mc_H H(2H-1)t^{2H-1}\int_0^t \|\sigma^1(s)\|^2 ds \\ &\quad +4M\int_0^t \|g^1(s)\|^2_{L^0}ds + 6M\Big(\sum_{k=1}^m d_k\Big)^2 \end{aligned}$$

which immediately yields

$$E|z(t)|^2 \quad \leq \quad A_1 + 4Mb\int_0^t p_1(s)\psi_1(\|z_s+\theta_s\|^2_{\mathcal{D}_{\mathcal{F}_0}} + \|\bar{z}_s+\bar{\theta}_s\|^2_{\mathcal{D}_{\mathcal{F}_0}})ds$$

and

$$E|\bar{z}(t)|^2 \quad \leq \quad A_2 + 4Mb\int_0^t p_2(s)\psi_2(\|z_s+\theta_s\|^2_{\mathcal{D}_{\mathcal{F}_0}} + \|\bar{z}_s+\bar{\theta}_s\|^2_{\mathcal{D}_{\mathcal{F}_0}})ds$$

where

$$A_1 = 4Mc_H H(2H-1)t^{2H-1}\int_0^t \|\sigma^1(s)\|^2 ds + 4M\int_0^t \|g^1(s)\|^2_{L^0} ds + 6M\Big(\sum_{k=1}^m d_k\Big)^2,$$

$$A_2 = 4Mc_H H(2H-1)t^{2H-1}\int_0^t \|\sigma^2(s)\|^2 ds + 4M\int_0^t \|g^2(s)\|^2_{L^0} ds + 6M\Big(\sum_{k=1}^m \bar{d}_k\Big)^2.$$

But

$$\begin{aligned}\|z_t+\theta_t\|^2_{\mathcal{D}_{\mathcal{F}_0}} + \|\bar{z}_t+\bar{\theta}_t\|^2_{\mathcal{D}_{\mathcal{F}_0}} \quad &\leq \quad 4\widetilde{K}^2\sup_{s\in[0,b]}\Big(E|z(s)|^2 + E|\bar{z}(s)|^2\Big)\\ &+ \quad 4\widetilde{K}^2 M\Big(E|\phi(0)|^2 + E|\bar{\phi}(0)|^2\Big)\\ &+ \quad 4\widetilde{N}^2\Big(\|\phi\|^2_{\mathcal{D}_{\mathcal{F}_0}} + \|\bar{\phi}\|^2_{\mathcal{D}_{\mathcal{F}_0}}\Big).\end{aligned}$$

Adding these we obtain

$$E|z(t)|^2 + E|\bar{z}(t)|^2 \leq B_* + 4Mb\int_0^t p_*(s)\phi_*(\|z_s+\theta_s\|^2_{\mathcal{D}_{\mathcal{F}_0}} + \|\bar{z}_s+\bar{\theta}_s\|^2_{\mathcal{D}_{\mathcal{F}_0}})ds \tag{15}$$

where

$$B_* = A_1 + A_2 \quad \text{and} \quad p_*(t) = \sup\{p_1(t), p_2(t)\} \qquad \phi_* = \psi_1 + \psi_2.$$

If we set $v(t)$ the right hand side of the above inequality we have that

$$\|z_t+\theta_t\|^2_{\mathcal{D}_{\mathcal{F}_0}} + \|\bar{z}_t+\bar{\theta}_t\|^2_{\mathcal{D}_{\mathcal{F}_0}} \leq v(t),$$

and therefore (15) becomes

$$E|z(t)|^2 + E|\bar{z}(t)|^2 \leq B_* + 6Mb\int_0^t p_*(s)\phi(v(s))ds. \tag{16}$$

Using (16) in the definition of v, we have that

$$\begin{aligned}v(t) \leq \quad & 4\widetilde{K}^2\Big(B_* + Mb\textstyle\int_0^t p_*(s)\phi(v(s)ds\Big) + 4\widetilde{K}^2 M\Big(E|\phi(0)|^2 + E|\bar{\phi}(0)|^2\Big)\\ & + 4\widetilde{N}^2\Big(\|\phi\|^2_{\mathcal{D}_{\mathcal{F}_0}} + \|\bar{\phi}\|^2_{\mathcal{D}_{\mathcal{F}_0}}\Big)\end{aligned} \tag{17}$$

and therefore

$$v(t) \leq L_1 + L_2\int_0^t p_*(s)\phi(v(s))ds, \tag{18}$$

where

$$L_1 = 4\widetilde{K}^2 M\Big(E|\phi(0)|^2 + E|\bar{\phi}(0)|^2\Big) + 4\widetilde{N}^2\Big(\|\phi\|^2_{\mathcal{D}_{\mathcal{F}_0}} + \|\bar{\phi}\|^2_{\mathcal{D}_{\mathcal{F}_0}}\Big) + 4\widetilde{K}^2 B_*$$

and

$$L_2 = 4\widetilde{K}^2 Mb.$$

Let us denote the right-hand side of the inequality (18) by $v(t)$. Then we have

$$w(0) = L_1,\ v(t) \leq w(t),\ \ t \in J,$$

and

$$w'(t) = L_2 p_*(t)\phi(v(t)), \ \ t \in J.$$

Using the increasing character of ψ we obtain

$$w'(t) \leq L_2 p_*(t)\phi(w(t)), \ \ \text{for } \ a.e. \ \ t \in J.$$

This implies, for each $t \in J$, we have

$$\Gamma(w(t)) = \int_{w(0)}^{w(t)} \frac{ds}{\psi(s)} \leq L_2 \int_0^b m_*(s)ds < \int_{L_1}^{\infty} \frac{ds}{\psi(s)}.$$

Consequently, there exists a constant K such that

$$v(t) \leq w(t) \leq C_{st}, \ t \in J.$$

Thus

$$\|z\|^2_{\widehat{\mathcal{D}}_{\mathcal{F}_b}} \leq C_{st} \qquad \text{and} \qquad \|\bar{z}\|^2_{\widehat{\mathcal{D}}_{\mathcal{F}_b}} \leq C_{st}.$$

As a consequence of Lemma 2.14 we deduce that N has a fixed point, since $(x(t), y(t)) = (z(t) + \theta(t), \bar{z}(t) + \bar{\theta}(t))$ if $t \in (-\infty, b]$. Then (x, y) is a fixed point of the operator N which is a mild solution of the problem (1).

Part 2 Compactness of the solution set. Let

$$S_F = \{(z, \bar{z}) \in \widehat{\mathcal{D}}_{\mathcal{F}_b} \times \widehat{\mathcal{D}}_{\mathcal{F}_b} : \quad (z, \bar{z}) \quad \text{is a solution of Problem(1)}\}$$

From Part 1, $S_F \neq \emptyset$ and there exists M such that for every $(z, \bar{z}) \in S_F, \|z\|^2_{\widehat{\mathcal{D}}_{\mathcal{F}_b}} \leq M$ and $\|\bar{z}\|^2_{\widehat{\mathcal{D}}_{\mathcal{F}_b}} \leq M$. Since $\underline{N}$ is completely continuous, then $\underline{N}(S_F) = (\underline{N}_1(S_{F^1}), \underline{N}_2(S_{F^2}))$ is relatively compact in $\widehat{\mathcal{D}}_{\mathcal{F}_b} \times \widehat{\mathcal{D}}_{\mathcal{F}_b}$. Let $(z, \bar{z}) \in S_F$ then $(z, \bar{z}) \in \underline{N}(z, \bar{z})$ and $S_F \subset \overline{N(S_F)}$. It remains to prove that S_F is a closed set in $\widehat{\mathcal{D}}_{\mathcal{F}_b} \times \widehat{\mathcal{D}}_{\mathcal{F}_b}$. Let $(z_n, \bar{z}_n) \in S_F$ such that $(z_n, \bar{z}_n)$ converge to $(z, \bar{z})$. For every $n \in N$, there exists $v_n^i(t) \in F^i(t, z_n + \theta, \bar{z}_n + \bar{\theta})$ a.e. $t \in J$ for each $i \in \{1, 2\}$ such that

$$\begin{aligned} z_n(t) &= \int_0^t S(t-s)f_n^1(s)ds + \sum_{l=1}^{\infty} \int_0^t S(t-s)\sigma_l^1(s)dB^H(s) + \int_0^t S(t-s)g^1(s)dW(s) \\ &+ \sum_{0<t_k<t} S(t-t_k)I_k(z_n(t_k^-) + \theta(t_k^-)), \end{aligned}$$

and

$$\begin{aligned} \bar{z}_n(t) &= \int_0^t S(t-s)f_n^2(s)ds + \sum_{l=1}^{\infty} \int_0^t S(t-s)\sigma_l^2(s)dB^H(s) + \int_0^t S(t-s)g^2(s)dW(s) \\ &+ \sum_{0<t_k<t} S(t-t_k)\bar{I}_k(z_n(t_k^-) + \theta(t_k^-)) \end{aligned}$$

(H_2) implies that for a.e. $t \in J$,$f_n^i \in p_i(t)\psi_i(2C_{st}), \quad i = 1, 2$ hence $(f_n^i)_{n\in N}$ is integrably bounded. Note that this still remains true holds for S_F is a bounded set. Since X is reflexive, by Theorem 2.15, there exists a subsequence, still denoted by $(f_n^i)_{n\in N}$, which converges weakly to some limit $f^i \in L^2(J, X)$. Moreover, the mapping $\Gamma : L^2(J, X) \longrightarrow X$ defined by

$$\Gamma(f^i)(t) = \int_0^t S(t-s)f^i(s)ds$$

is a continuous linear operator. Then it remains continuous if these spaces are endowed with their weak topologies [11]. Therefore for a.e. $t \in J$, the sequence $(z_n(t), \bar{z}_n(t))$ converges to $(z(t), \bar{z}(t))$ and by the continuity of $(I_k, \bar{I}_k)$ it follows that

$$\begin{aligned} z(t) &= \int_0^t S(t-s)f^1(s)ds + \sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^1(s)dB^H(s) + \int_0^t S(t-s)g^1(s)dW(s) \\ &+ \sum_{0<t_k<t} S(t-t_k)I_k(z(t_k^-)+\theta(t_k^-)) \end{aligned}$$

and

$$\begin{aligned} \bar{z}(t) &= \int_0^t S(t-s)f^2(s)ds + \sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^2(s)dB^H(s) + \int_0^t S(t-s)g^2(s)dW(s) \\ &+ \sum_{0<t_k<t} S(t-t_k)\bar{I}_k(z(t_k^-)+\theta(t_k^-)). \end{aligned}$$

Now we need to prove that $f^i(t) \in F^i(t, z(t)+\theta(t), \bar{z}(t)+\bar{\theta}(t))$, for a.e. $t \in J$. Lemma 2.13 yields the existence of constants $\alpha_i^n \geq 0$, $j = 1, 2 \ldots, k(n)$ and $i = 1, 2$ such that $\sum_{j=1}^{k(n)} \alpha_i^n = 1$ and the sequence of convex combinations $h_n^i(.) = \sum_{j=1}^{k(n)} \alpha_j^n f_j^i(.)$ converges strongly to some limit $f^i \in L^2(J, X)$. Since F^i takes convex values, using Lemma 2.12, we obtain that

$$\begin{aligned} f^i(t) &\in \bigcap_{n\geq 1} \overline{\{h_k^i(t) : k \geq n\}}, \quad a.e \quad t \in J \\ &\subseteq \bigcap_{n\geq 1} \overline{co}\{f_k^i(t), \quad k \geq n\} \\ &\subseteq \bigcap_{n\geq 1} \overline{co}\{\bigcup_{k\geq n} F^i(t, z_k(t)+\theta(t), \bar{z}_k(t)+\bar{\theta})\}. \end{aligned}$$

Thus

$$f^i(t) \subseteq \overline{co}\{\limsup_{k\to\infty} F^i(t, z_k(t)+\theta(t), \bar{z}_k(t)+\bar{\theta}(t))\}. \tag{19}$$

Since F^i is u.s.c. and has compact values, then by Lemma 2.11, we have

$$\limsup_{n\to\infty} F^i(t, z_n(t)+\theta(t), \bar{z}_n(t)+\bar{\theta}(t)) \subseteq F^i(t, z(t)+\theta(t), \bar{z}(t)+\bar{\theta}(t)) \quad \text{for a.e} \quad t \in J.$$

This and (19) imply that $f^i(t) \in \overline{co}(F^i(t, z(t)+\theta(t), \bar{z}(t)+\bar{\theta}(t))$. Since, for each $i = 1, 2$, $F^i(.,.)$ has closed, convex values, we deduce that $f^i(t) \in F^i(t, z(t)+\theta(t), \bar{z}(t)+\bar{\theta}(t))$ for a.e. $t \in J$, for each $i = 1, 2$ as claimed. Hence $(z, \bar{z}) \in S_{F^i}$ which proves that S_{F^i}, for each $i = 1, 2$, is closed, hence compact in $\widehat{\mathcal{D}}_{\mathcal{F}_b} \times \widehat{\mathcal{D}}_{\mathcal{F}_b}$. □

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