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The optimal problems for the compatible functional F

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Abstract. Inspired by the definition and properties of geometric measures for convex bodies in Orlicz Brunn-Minkowski theory, such as Orlicz mixed volume, Orlicz mixed p-capacities ($1 < p < n$) and Orlicz mixed torsional rigidity, we will introduce a more general geometric invariant, called the Orlicz L_φ mixed compatible functional $\mathbf{F}_\varphi$. Motivated by the optimal problems for the above three geometric measures, we discuss the optimization problem with respect to Orlicz L_φ mixed compatible functional $\mathbf{F}_\varphi$ and prove the existence of the solution of the problem. Moreover, we consider Orlicz and L_q ($-n \neq q \in \mathbb{R}$) geominimal compatible functional which based on the Orlicz L_φ mixed compatible functional, and we also establish the isoperimetric type inequality about the L_q ($-n \neq q \in \mathbb{R}$) geominimal compatible functional.

1. Introduction

For two convex bodies (compact convex set with nonempty interior) K and L, the L_p ($p \geq 1$) mixed volume $V_p(K, L)$ is defined by (see [12])

$$V_p(K,L) = \frac{1}{n}\int_{S^{n-1}} h_L^p(v) h_K^{1-p}(v) dS(K,v), \tag{1}$$

the special case of $p = 1$, is the (first) mixed volume $V_1(K, L)$ of K and L (see [8]),

$$V_1(K,L) = \frac{1}{n}\int_{S^{n-1}} h_L(v) dS(K,v), \tag{2}$$

where h_L is the support function of L and $S(K,\cdot)$ is the surface area measure of K: for each Borel set $\Sigma \subseteq S^{n-1}$,

$$S(K,\Sigma) = \int_{\nu_K^{-1}(\Sigma)} d\mathcal{H}^{n-1}, \tag{3}$$

where $\nu_K^{-1} : S^{n-1} \to \partial K$ is the inverse Gauss map and $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure on the boundary ∂K of K. Denote by $\mathcal{K}_0$ be the class of convex bodies which contain the origin in their interiors. For $K, L \in \mathcal{K}_0$ and $\lambda > 0$, the Minkowski sum of K and L is $K + L = \{x + y : x \in K, y \in L\}$ and the

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scalar product of λ and K is $\lambda K = \{\lambda x : x \in K\}$. For $K \in \mathcal{K}_0$, denote by $|K|$ be the volume of K. Denote by ω_n and $S^{n-1} = \{x \in \mathbb{R}^n : x \cdot x = 1\}$ be the volume and the boundary of $B_2^n = \{x \in \mathbb{R}^n : x \cdot x \le 1\}$, respectively. For $K \in \mathcal{K}_0$, then $\text{vrad}(K) = (|K|/\omega_n)^{\frac{1}{n}}$ is referred to the volume radius of K.

In [6], Petty introduced the geominimal surface area $G(K)$ of a convex body $K \in \mathcal{K}_0$, is defined by

$$G(K) = \inf\left\{\int_{S^{n-1}} h_L(v)dS(K,v) : L \in \mathcal{K}_0, |L^\circ| = \omega_n\right\}, \tag{4}$$

where L° is the polar body of L (see (14) for the definition). Combining with (2), the optimal problem (4) can be written as

$$G(K) = \inf\{nV_1(K,L) : L \in \mathcal{K}_0, |L^\circ| = \omega_n\}. \tag{5}$$

Petty [6] proved the existence of the solution of the optimal problem (5), so the geominimal surface area $G(K)$ could be defined based on the mixed volume.

In [12], Lutwak extended the geominimal surface area to L_p form associated with (1) for $p > 1$, namely, the p-geominimal surface area $G_p(K)$ of a convex body $K \in \mathcal{K}_0$, is defined by

$$G_p(K) = \inf\left\{nV_p(K,L) : L \in \mathcal{K}_0.|L^\circ| = \omega_n\right\}, \tag{6}$$

and Lutwak proved that the optimal problem (6) has a unique solution in [12]. Later, Ye extended $p > 1$ to $p \in \mathbb{R}$ in [25]. Some other excellent works can be found, see e.g., [7, 11, 19, 20, 22, 23, 27, 31, 33, 34] and the reference therein.

Along the development of the Orlicz Brunn-Minkowski theory, the Orlicz mixed volume was introduced in [9]: Let $\varphi : (0,\infty) \to (0,\infty)$ be a convex function such that $\varphi(0) = 0$ and $\varphi(1) = 1$. For $K, L \in \mathcal{K}_0$, the Orlicz mixed volume $V_\varphi(K,L)$ is defined by

$$V_\varphi(K,L) = \frac{1}{n}\int_{S^{n-1}} \varphi\left(\frac{h_L(v)}{h_K(v)}\right) h_K(v)dS(K,v), \tag{7}$$

and if $\varphi : (0,\infty) \to (0,\infty)$ is a continuous strictly increasing function with $\lim_{t\to 0^+}\varphi(t) = 0$, $\lim_{t\to\infty}\varphi(t) = \infty$ and $\varphi(1) = 1$, the Orlicz mixed volume $\widetilde{V}_\varphi(K,L)$ of $K, L \in \mathcal{K}_0$ is

$$\widetilde{V}_\varphi(K,L) = \inf\left\{\lambda > 0 : \int_{S^{n-1}} \varphi\left(\frac{n|K|h_L(v)}{\lambda h_K(v)}\right) h_K(v)dS(K,v) \le n|K|\right\}.$$

Obviously, when $\varphi(t) = t^p$ ($p \ge 1$), the Orlicz mixed volume (7) is the L_p ($p \ge 1$) mixed volume (1).

In [26], Ye introduced the Orlicz geominimal surface area (see also [24] and [30]) of $K \in \mathcal{K}_0$, which is the extension of the p-geominimal surface area, is defined by

$$G_\varphi^{orlicz}(K) = \inf\left\{nV_\varphi(K,L) : L \in \mathcal{K}_0, |L^\circ| = \omega_n\right\}, \tag{8}$$

$$\widetilde{G}_\varphi^{orlicz}(K) = \inf\left\{\widetilde{V}_\varphi(K,L) : L \in \mathcal{K}_0, |L^\circ| = \omega_n\right\}. \tag{9}$$

In particular, the optimal problems (8) and (9) were proved to have a unique solution in [30]. With the expansion and popularization of the Orlicz-Brunn-Minkowski theory (see e.g., [2, 9, 13, 14, 16, 24, 35]), the Orlicz geominimal surface area was widely considered, see e.g., [28, 29, 36] and the reference therein.

Similarly, there are similar relationships between Orlicz geominimal surface area and the Orlicz mixed volume for other functionals. For example, the Orlicz geominimal p-capacity ($1 < p < n$) was studied by, e.g., [10, 15, 32] and the reference therein. The Orlicz geominimal torsional rigidity was considered by, e.g., [3, 18, 21] and the reference therein.

Inspired by Orlicz geominimal surface area, Orlicz geominimal p-capacity and Orlicz geominimal torsional rigidity, we would like to study a more general functional. As defined in [17], let **F** be a compatible functional defined for every compact convex set $K \subseteq \mathbb{R}^n$ with positively homogeneous of some degree $\alpha \neq 0$. Suppose that for every K there exists a non-negative Borel measure $\mu_{\mathbf{F}}(K, \cdot)$ on S^{n-1} such that:

$$\mathbf{F}(K) = \frac{1}{\alpha} \int_{S^{n-1}} h_K(v) d\mu_{\mathbf{F}}(K, v),$$

$$\frac{d}{d\varepsilon} \mathbf{F}(K + \varepsilon L)\Big|_{\varepsilon=0^+} = \int_{S^{n-1}} h_L(v) d\mu_{\mathbf{F}}(K, v),$$

where L is also a compact convex set. Denote by $\mathbf{F}_1(K, L)$ the mixed compatible functional, i.e.,

$$\mathbf{F}_1(K, L) = \frac{1}{\alpha} \int_{S^{n-1}} h_L(v) d\mu_{\mathbf{F}}(K, v). \tag{10}$$

In Section 3, we will introduce the nonhomogeneous and the homogeneous Orlicz L_φ mixed compatible functionals for $\varphi \in \mathcal{I} \cup \mathcal{D}$ and $K, L \in \mathcal{K}_0$ as follows:

$$\mathbf{F}_\varphi(K, L) = \frac{1}{\alpha} \int_{S^{n-1}} \varphi\left(\frac{h_L(v)}{h_K(v)}\right) h_K(v) d\mu_{\mathbf{F}}(K, v), \tag{11}$$

$$\int_{S^{n-1}} \varphi\left(\frac{\mathbf{F}(K) h_L(v)}{\widetilde{\mathbf{F}}_\varphi(K, L) h_K(v)}\right) d\mu^*_{\mathbf{F}}(K, v) = 1, \tag{12}$$

where $\mu^*_{\mathbf{F}}(K, \cdot)$ is a probability measure defined in (22) and $\mathcal{I}, \mathcal{D}$ are the classes of the nonnegative increasing continuous function and nonnegative decreasing continuous function, respectively (see (18) for the definition). Obviously, when $\varphi(t) = t$, the Orlicz L_φ mixed compatible functional (11) is the mixed compatible functional (10). And we establish the optimal problems associated with the Orlicz L_φ mixed compatible functionals and prove the solution of this problems in Section 3 as follows:

$$\inf / \sup\{\mathbf{F}_\varphi(K, L) : L \in \mathcal{K}_0, |L^\circ| = \omega_n\},$$

$$\inf / \sup\{\widetilde{\mathbf{F}}_\varphi(K, L) : L \in \mathcal{K}_0, |L^\circ| = \omega_n\}.$$

Let $\mathcal{S}_0$ be the class of star bodies. In Section 4, we define the Orlicz and L_q geominimal compatible functionals with respect to $S_0 \subset \mathcal{S}_0$. For $K \in \mathcal{K}_0$, the nonhomogeneous and the homogeneous Orlicz geominimal compatible functionals are given by the following optimal problems:

$$G_\varphi(K, S_0) = \inf / \sup\{\mathbf{F}_\varphi(K, \mathrm{vrad}(L) L^\circ) : L \in S_0\},$$

$$\widetilde{G}_\varphi(K, S_0) = \inf / \sup\{\widetilde{\mathbf{F}}_\varphi(K, \mathrm{vrad}(L) L^\circ) : L \in S_0\}.$$

Based on the Orlicz geominimal compatible functionals, we consider the L_q geominimal compatible functional when $\varphi(t) = t^q$ for $-n \neq q \in \mathbb{R}$.

In this paper, we introduce and establish the optimization problem for Orlicz L_φ mixed compatible functional, and prove the existence of solution of the problem in Section 3. In Section 4, we discuss the Orlicz and L_q geominimal compatible functionals and study the isopermetric type inequalities about them. For example:

Theorem 1.1. *Let $K \in \mathcal{K}_0$ and $\varphi \in \mathcal{I}$, $\mathbf{F}_\varphi(\cdot, \cdot)$ and $\widetilde{\mathbf{F}}_\varphi(\cdot, \cdot)$ be the Orlicz L_φ mixed compatible functionals given in* (11) *and* (12). *Then*
(1) there is $M \in \mathcal{K}_0$ satisfying $|M^\circ| = \omega_n$ and

$$\mathbf{F}_\varphi(K, M) = \inf\{\mathbf{F}_\varphi(K, L) : L \in \mathcal{K}_0, |L^\circ| = \omega_n\}.$$

(2) There is $\widetilde{M} \in \mathcal{K}_0$ satisfying $|\widetilde{M}^\circ| = \omega_n$ and

$$\widetilde{\mathbf{F}}_\varphi(K, \widetilde{M}) = \inf\{\widetilde{\mathbf{F}}_\varphi(K, L) : L \in \mathcal{K}_0, |L^\circ| = \omega_n\}.$$

(3) If $\varphi \in \mathcal{I}$ is a convex function, M and $\widetilde{M}$ existing in (1) and (2) are unique.

2. Background and Preliminaries

A subset $K \subseteq \mathbb{R}^n$ is called convex if for any $x, y \in K$ satisfying $[x, y] \subset K$. A convex set $K \subseteq \mathbb{R}^n$ is a convex body if K is also compact with nonempty interior. Denote by $\mathcal{K}_0$ be the class of convex bodies which contain the origin in their interiors. The usual Euclidean norm is written by $x \cdot y$ for $x, y \in \mathbb{R}^n$ and the origin of $\mathbb{R}^n$ is denoted by o. Let $\{e_1, \ldots, e_n\}$ be the standard orthonormal basis of $\mathbb{R}^n$. Let $C(S^{n-1})$ and $C^+(S^{n-1})$ be the class of all continuous functions on S^{n-1} and all continuous positives functions on S^{n-1}, respectively.

Let K be a convex set of $\mathbb{R}^n$, the support function $h(K, \cdot) : \mathbb{R}^n \to \mathbb{R}$ of K is

$$h_K(x) = \max\{x \cdot y : y \in K\}.$$

For two convex sets K, L and $\lambda > 0$, it is checked that $h_{K+L}(v) = h(K, v) + h(L, v)$ and $h_{\lambda K}(v) = \lambda h_K(v)$ for $v \in S^{n-1}$.

A set $L \subset \mathbb{R}^n$ is called star-shaped set with respect to o if it is not empty and if $[o, x] \subset L$ for all $x \in L$. Let L be a star-shaped set with respect to the origin o, the radial function $\rho(L, \cdot) : S^{n-1} \to [0, \infty)$ is

$$\rho_L(v) = \max\{\lambda \geq 0 : \lambda v \in L\}$$

for $v \in S^{n-1}$. A star-shaped set is called a star body with respect to the origin if the radial function with respect to the origin is continuous and positive. Denote by $\mathcal{S}_0$ be the class of star bodies. Let L be a star body and $\sigma(\cdot)$ be the spherical measure on S^{n-1}, the volume of L is

$$|L| = \frac{1}{n} \int_{S^{n-1}} \rho(L, v)^n d\sigma(v).$$

Let $K \in \mathcal{K}_0$ satisfying the surface area measure $S(K, \cdot)$ is absolutely continuous about $\sigma(\cdot)$, then K has a curvature function $g(\cdot) : S^{n-1} \to \mathbb{R}$, is defined by

$$g(v) = \frac{dS(K, v)}{d\sigma(v)}. \tag{13}$$

The subset $\mathcal{A}_0$ of $\mathcal{K}_0$, is defined by $\mathcal{A}_0 = \{K \in \mathcal{K}_0 : g(v) \in C^+(S^{n-1})\}$.

For $K \in \mathcal{K}_0$, the polar body K° of K is

$$K^\circ = \{x \in \mathbb{R}^n : x \cdot y \leq 1, y \in K\}. \tag{14}$$

Thus it gets that $K^{\circ\circ} = K$, $h_{K^\circ}(v) = \rho_K^{-1}(v)$ and $\rho_{K^\circ}(v) = h_K^{-1}(v)$ for $v \in S^{n-1}$ (see e.g., [8]). Let int$K$ be the interior of $K \in \mathcal{K}_0$ and $x \in \text{int}K$, the polar body K^x of K with respect to x is $K^x = (K - x)^\circ + x$. Moreover, the Santaló point $t(K) \in \text{int}K$ is unique, which satisfies $|K^{t(K)}| = \inf\{|K^x| : x \in \text{int}K\}$ (see e.g., [5]). For $K \in \mathcal{K}_0$, the Blaschke-Santaló inequality is

$$|K| \cdot |K^{t(K)}| \leq \omega_n^2. \tag{15}$$

Equality holds if and only if K is an ellipsoid. The inverse Santaló inequality (see e.g., [1, 4]): there is a constant $\lambda > 0$ satisfying

$$|K| \cdot |K^{s(K)}| \geq \lambda^n \omega_n^2 \tag{16}$$

for $K \in \mathcal{K}_0$.

The following lemmas will be needed.

Lemma 2.1. (see [15, Lemma 2.1]) *If a sequence of measures $\{\mu_i\}_{i=1}^\infty$ on S^{n-1} converges weakly to a finite measure μ on S^{n-1} and a sequence of functions $\{f_i\}_{i=1}^\infty \subseteq C(S^{n-1})$ converges uniformly to a function $f \in C(S^{n-1})$, then*

$$\lim_{i \to \infty} \int_{S^{n-1}} f_i d\mu_i = \int_{S^{n-1}} f d\mu.$$

Lemma 2.2. (see [15, Lemma 2.2]) *Let $\{K_i\}_{i=1}^\infty \subseteq \mathcal{K}_0$ be a uniformly bounded sequence such that the sequence $\{|K_i^\circ|\}_{i=1}^\infty$ is bounded. Then, there exists a subsequence $\{K_{i_j}\}_{j=1}^\infty$ of $\{K_i\}_{i=1}^\infty$ and a convex body $K \in \mathcal{K}_0$ such that $K_{i_j} \to K$. Moreover, if $|K_i^\circ| = \omega_n$ for all $i = 1, 2, \ldots$, then $|K^\circ| = \omega_n$.*

2.1. Orlicz addition of convex bodies

Let $m \in \mathbb{N}$ be an integer number and Φ_m be the class of convex functions $\phi : [0,\infty)^m \to [0,\infty)$ increasing in each variable, and satisfy $\phi(o) = 0$ and $\phi(e_i) = 1$ for $i \in [1,m]$. Let $K_1,\dots,K_m \in \mathcal{K}_0$, the Orlicz L_ϕ sum $+_\phi(K_1,\dots,K_m) \in \mathcal{K}_0$, is defined by (see [9])

$$h_{+_\phi(K_1,\dots,K_m)}(v) = \inf\left\{\lambda > 0 : \phi\left(\frac{h_{K_1}(v)}{\lambda},\dots,\frac{h_{K_m}(v)}{\lambda}\right) \le 1\right\}$$

for any $v \in S^{n-1}$. Thus, the above equation can be described as

$$\phi\left(\frac{h_{K_1}(v)}{h_{+_\phi(K_1,\dots,K_m)}(v)},\dots,\frac{h_{K_m}(v)}{h_{+_\phi(K_1,\dots,K_m)}(v)}\right) = 1$$

for any $v \in S^{n-1}$. Then $K_i \subset +_\phi(K_1,\dots,K_m)$ for $i \in [1,m]$ by $\phi \in \Phi_m$. Let $K, L \in \mathcal{K}_0$ and $\phi_1,\phi_2 \in \Phi_1$, if $t > 0$, consider the convex body $K +_{\phi,t} L \in \mathcal{K}_0$, is defined by,

$$\phi_1\left(\frac{h_K(v)}{h_{K+_{\phi,t}L}(v)}\right) + t\phi_2\left(\frac{h_L(v)}{h_{K+_{\phi,t}L}(v)}\right) = 1$$

for $v \in S^{n-1}$. Let $(\phi_1)'_l(1)$ and $(\phi_1)'_r(1)$ be the left and right derivative of ϕ_1 at $s = 1$, respectively. For $K, L \in \mathcal{K}_0$, the L_{ϕ_2} mixed volume $V_{\phi_2}(K, L)$ is defined by (see [9])

$$V_{\phi_2}(K,L) = \frac{(\phi_1)'_l(1)}{n}\frac{d}{dt}|K +_{\phi,t} L|\bigg|_{t=0^+} = \frac{1}{n}\int_{S^{n-1}} \phi_2\left(\frac{h_L(v)}{h_K(v)}\right) h_K(v)\,dS(K,v) \tag{17}$$

if $(\phi_1)'_l(1)$ exists and is positive. In fact, the assumptions $\phi_1,\phi_2 \in \Phi_1$ in (17) can be extended to more general increasing or decreasing functions in [30]. Thus, we work on the following classes of nonnegative continuous functions:

$$\begin{cases} \mathcal{I} = \{\varphi : \varphi \text{ is strictly increasing with } \lim\limits_{s\to 0}\varphi(s) = 0, \varphi(1) = 1, \lim\limits_{s\to\infty}\varphi(s) = \infty\}, \\ \mathcal{D} = \{\varphi : \varphi \text{ is strictly decreasing with } \lim\limits_{s\to 0}\varphi(s) = \infty, \varphi(1) = 1, \lim\limits_{s\to\infty}\varphi(s) = 0\}. \end{cases} \tag{18}$$

Let $h(v,t)$ be continuous positive function defined on $S^{n-1}\times[0,\delta)$ for some $\delta > 0$ and K_t be the Aleksandrov body associated to $h(v,t)$ for $K \in \mathcal{K}_0$, i.e, $K_t = \{x \in \mathbb{R}^n : x\cdot v \le h(v,t) \text{ for all } v \in S^{n-1}\}$. For $K, L \in \mathcal{K}_0$, the linear Orlicz sum of h_K and h_L is defined by, for $v \in S^{n-1}$,

$$\phi_1\left(\frac{h_K(v)}{h(v,t)}\right) + t\phi_2\left(\frac{h_L(v)}{h(v,t)}\right) = 1 \tag{19}$$

where $\phi_1,\phi_2 \in \mathcal{I}$ or $\phi_1,\phi_2 \in \mathcal{D}$. Obviously, $h_K \le h(\cdot,t)$ when $\phi_1,\phi_2 \in \mathcal{I}$; $h_K \ge h(\cdot,t)$ when $\phi_1,\phi_2 \in \mathcal{D}$; $h_{K+_{\phi,t}L} = h(\cdot,t)$ when $\phi_1,\phi_2 \in \Phi_1$. For $\phi_1,\phi_2 \in \mathcal{I}$ or $\phi_1,\phi_2 \in \mathcal{D}$, one gets the following result in [30], which extends (17) to nonconvex functions,

$$V_{\phi_2}(K,L) = \frac{(\phi_1)'_l(1)}{n}\frac{d}{dt}|K_t|\bigg|_{t=0^+} = \frac{1}{n}\int_{S^{n-1}} \phi_2\left(\frac{h_L(v)}{h_K(v)}\right) h_K(v)\,dS(K,v), \tag{20}$$

if $(\phi_1)'_l(1)$ exists and is positive for $K, L \in \mathcal{K}_0$ and $\phi_1,\phi_2 \in \mathcal{I}$. For $\phi_1,\phi_2 \in \mathcal{D}$, (20) holds with $(\phi_1)'_l(1)$ replaced by $(\phi_1)'_r(1)$ if $(\phi_1)'_r(1)$ exists and is nonzero.

3. The Orlicz mixed L_φ compatible functionals

In this section, we first recall the definition and some properties of the compatible function $\mathbf{F}$ in [17], and introduce the Orlicz L_φ mixed compatible functional $\mathbf{F}_\varphi$ under the assumption $\varphi \in \mathcal{I} \cup \mathcal{D}$.

Denote by $\mathcal{C}$ the class of compact convex sets. Let $\mathbf{F} : \mathcal{C} \to (0, \infty)$ be a real-valued functional with positively homogeneous of some degree $\alpha \neq 0$ and satisfying, for $\alpha > 0$ and $K, L \in \mathcal{C}$,

$$\mathbf{F}(K) = \frac{1}{\alpha} \int_{S^{n-1}} h_K(v) d\mu_{\mathbf{F}}(K, v)$$

and

$$\lim_{\varepsilon \to 0^+} \frac{\mathbf{F}(K + \varepsilon L) - \mathbf{F}(K)}{\varepsilon} = \int_{S^{n-1}} h_L(v) d\mu_{\mathbf{F}}(K, v),$$

where $\mu_{\mathbf{F}}(K, \cdot)$ is called the compatible functional measure on S^{n-1}, given by

$$\mu_{\mathbf{F}}(K, \omega) = \int_{\nu_K^{-1}(\omega)} u(x) d\mathcal{H}^{n-1}(x) \tag{21}$$

for any Borel set $\omega \subseteq S^{n-1}$ and some continuous function $u : K \to (0, \infty)$ which is integrable on the boundary of $K \in \mathcal{C}$.

Combining (3) and (21), it has

$$d\mu_{\mathbf{F}}(K, v) = u(\nu_K^{-1}(v)) dS(K, v) \text{ for } v \in S^{n-1}.$$

Thus the compatible functional measure $\mu_{\mathbf{F}}(K, \cdot)$ is not concentrated on a closed subsphere. For $K \in \mathcal{C}$, define the probability measure $\mu^*_{\mathbf{F}}(K, \cdot)$ of K, by

$$\mu^*_{\mathbf{F}}(K, v) = \frac{1}{\alpha} \cdot \frac{h_K(v)\mu_{\mathbf{F}}(K, v)}{\mathbf{F}(K)} \text{ for } v \in S^{n-1}. \tag{22}$$

Definition 3.1. (see [17, Definition 3.1]) *Let $K, L \in \mathcal{K}$. A functional $\mathbf{F} : \mathcal{K} \to [0, \infty)$ is said to be compatible if $\mathbf{F}$ satisfies the following conditions:*
(i) For a constant $\alpha > 0$ and any $s > 0$,

$$\mathbf{F}(sK) = s^\alpha \mathbf{F}(K).$$

(ii) For any $x \in \mathbb{R}^n$,

$$\mathbf{F}(K + x) = \mathbf{F}(K).$$

(iii) If $K \subseteq L$, then

$$\mathbf{F}(K) \leq \mathbf{F}(L).$$

(iv) For any $t \in [0, 1]$,

$$\mathbf{F}(tK + (1 - t)K)^{\frac{1}{\alpha}} \geq t\mathbf{F}(K)^{\frac{1}{\alpha}} + (1 - t)\mathbf{F}(L)^{\frac{1}{\alpha}} \tag{23}$$

equality holds if and only if K and L are homothetic to each other.
(v) If $V(K) = 0$, then $\mathbf{F}(K) = 0$.
(vi) The compatible functional measure $\mu_{\mathbf{F}}(K, \cdot)$ is weakly convergent.

For $K, L \in C$, denote $\mathbf{F}_1(K, L)$ of the mixed functional of K and L,

$$\mathbf{F}_1(K, L) = \frac{1}{\alpha}\int_{S^{n-1}} h_L(v) d\mu_{\mathbf{F}}(K, v).$$

From (23), it is easy to checked that

$$\mathbf{F}_1(K, L) \geq \mathbf{F}(K)^{\frac{\alpha-1}{\alpha}}\mathbf{F}(L)^{\frac{1}{\alpha}} \tag{24}$$

equality holds if and only if K and L are homothetic to each other. For any $f \in C^+(S^{n-1})$ and $K \in C$, denote $\mathbf{F}_1(K, f)$ of the mixed compatible function of K and f,

$$\mathbf{F}_1(K, f) = \frac{1}{\alpha}\int_{S^{n-1}} f(v) d\mu_{\mathbf{F}}(K, v).$$

It implies that $\mathbf{F}_1(K, h_L) = \mathbf{F}_1(K, L)$ and $\mathbf{F}_1(K, h_K) = \mathbf{F}(K)$ for all $K, L \in C$.

The following three lemmas will be needed:

Lemma 3.2. (see [30, Lemma 5.1]) *Let $K, L \in \mathcal{K}_0$ and $\varphi_1, \varphi_2 \in \mathcal{I}$ be such that $(\varphi_1)'_l(1)$ exists and is positive, and $h(v, t)$ be defined in* (19). *Then*

$$(\varphi_1)'_l(1) \lim_{t\to 0^+} \frac{h(v, t) - h_K(v)}{t} = h(K, v)\varphi_2\left(\frac{h_L(v)}{h_K(v)}\right) \text{ uniformly on } S^{n-1}. \tag{25}$$

For $\varphi_1, \varphi_2 \in \mathcal{D}$, (25) *holds with $(\varphi_1)'_l(1)$ replaced by $(\varphi_1)'_r(1)$.*

Lemma 3.3. (see [17, Lemma 3.1]) *Let $K \in C$ be a compact convex set, the compatible functional measure $\mu_{\mathbf{F}}(K, \cdot)$ is absolutely continuous with respect to the surface area measure $S(K, \cdot)$.*

Lemma 3.4. (see [17, Lemma 3.2]) *If $f \in C^+(S^{n-1})$ and $\mathbf{F}$ is the compatible functional. Let $K \in C$ and K_f be the Aleksandrov body associated with f, then*

$$\mathbf{F}(K_f) = \mathbf{F}_1(K_f, f).$$

Let $h(v, t)$ be a positive continuous function defined on $S^{n-1} \times [0, \delta)$ for some $\delta > 0$. The Aleksandrov body K_t associated with $h(v, t)$ is given by

$$K_t = \{x \in \mathbb{R}^n : x \cdot v \leq h(v, t),\ v \in S^{n-1}\}.$$

By the continuity of $h(v, t)$, K_t converges to K_0 as $t \to 0^+$. Let $K = K_0$.

Theorem 3.5. *Let $K, L \in \mathcal{K}_0$ and $\varphi_1, \varphi_2 \in \mathcal{I}$ satisfying $(\varphi_1)'_l(1)$ exists and is nonzero, $\mathbf{F}$ be the compatible functional given in Definition 3.1. Then*

$$\left.\frac{d}{dt}\mathbf{F}(K_t)\right|_{t=0^+} = \frac{1}{(\varphi_1)'_l(1)}\int_{S^{n-1}} \varphi_2\left(\frac{h_L(v)}{h_K(v)}\right) h_K(v) d\mu_{\mathbf{F}}(K, v).$$

With $(\varphi_1)'_l(1)$ replaced by $(\varphi_1)'_r(1)$ if $(\varphi_1)'_r(1)$ exists and is nonzero, one gets the analogue result for $\varphi_1, \varphi_2 \in \mathcal{D}$.

Proof. Denote $l = \frac{1}{\alpha}\int_{S^{n-1}} \varphi_2\left(\frac{h_K(v)}{h_L(v)}\right) h_K(v) d\mu_{\mathbf{F}}(K, v)$. Since $\mu_{\mathbf{F}}(K_t, \cdot) \to \mu_{\mathbf{F}}(K, \cdot)$ weakly whenever $K_t \to K$ in the Hausdorff distance as $t \to 0^+$, from Lemma 2.1, (24), Lemma 3.3, Lemma 3.4, the fact that $h_K(\cdot) \leq h(\cdot, 0)$ and Lemma 3.2,

$$\begin{aligned}
\liminf_{t\to 0^+} \mathbf{F}(K_t)^{1-\frac{1}{\alpha}} \cdot \frac{\mathbf{F}(K_t)^{\frac{1}{\alpha}} - \mathbf{F}(K)^{\frac{1}{\alpha}}}{t} &= \liminf_{t\to 0^+} \frac{\mathbf{F}(K_t) - \mathbf{F}_1(K_t, K)}{t} \\
&= \frac{1}{\alpha}\liminf_{t\to 0^+} \int_{S^{n-1}} \frac{h(v, t) - h_K(v)}{t} d\mu_{\mathbf{F}}(K_t, v) \\
&\geq \frac{1}{\alpha}\liminf_{t\to 0^+} \int_{S^{n-1}} \frac{h(v, t) - h(v, 0)}{t} d\mu_{\mathbf{F}}(K_t, v) \\
&= \frac{1}{(\varphi_1)'_l(1)}.
\end{aligned}$$

Since $h_{K_t}(\cdot) \le h(\cdot, t)$, then

$$\begin{aligned}\mathbf{F}(K)^{1-\frac{1}{\alpha}} \liminf_{t\to 0^+} \frac{\mathbf{F}(K_t)^{\frac{1}{\alpha}} - \mathbf{F}(K)^{\frac{1}{\alpha}}}{t} &\le \limsup_{t\to 0^+} \frac{\mathbf{F}_1(K, K_t) - \mathbf{F}(K)}{t} \\ &= \frac{1}{\alpha} \limsup_{t\to 0^+} \int_{S^{n-1}} \frac{h_{K_t}(v) - h_K(v)}{t} d\mu_{\mathbf{F}}(K, v) \\ &\le \frac{1}{\alpha} \limsup_{t\to 0^+} \int_{S^{n-1}} \frac{h(v,t) - h(v,0)}{t} d\mu_{\mathbf{F}}(K, v) \\ &= \frac{1}{(\varphi_1)'_l(1)}.\end{aligned}$$

Then

$$\mathbf{F}(K)^{1-\frac{1}{\alpha}} \cdot \lim_{t\to 0^+} \frac{\mathbf{F}(K_t)^{\frac{1}{\alpha}} - \mathbf{F}(K)^{\frac{1}{\alpha}}}{t} = \frac{l}{(\varphi_1)'_l(1)}.$$

Thus

$$l = \frac{1}{\alpha} (\varphi_1)'_l(1) \lim_{t\to 0^+} \frac{\mathbf{F}(K_t) - \mathbf{F}(K)}{t}$$

The result for $\varphi_1, \varphi_2 \in \mathcal{D}$ follows along the same lines. □

3.1. The nonhomogeneous and homogeneous Orlicz L_φ mixed compatible functionals

In this section, let $\varphi \in \mathcal{I} \cup \mathcal{D}$, we will introduce Orlicz L_φ mixed compatible functional $\mathbf{F}_\varphi$ and study some properties of $\mathbf{F}_\varphi$.

Definition 3.6. *Let $K, L \in \mathcal{K}_0$. For $\varphi \in \mathcal{I} \cup \mathcal{D}$,*
i) the nonhomogeneous Orlicz L_φ mixed compatible functional $\mathbf{F}_\varphi(K, L)$ of K and L, is defined by

$$\mathbf{F}_\varphi(K, L) = \frac{1}{\alpha} \int_{S^{n-1}} \varphi\left(\frac{h_L(v)}{h_K(v)}\right) h_K(v) d\mu_{\mathbf{F}}(K, v). \tag{26}$$

And if $L \in \mathcal{S}_0$, (26) is written by

$$\mathbf{F}_\varphi(K, L^\circ) = \frac{1}{\alpha} \int_{S^{n-1}} \varphi\left(\frac{1}{h_K(v)\rho_L(v)}\right) h_K(v) d\mu_{\mathbf{F}}(K, v). \tag{27}$$

ii) the homogeneous Orlicz L_φ mixed compatible functional $\widetilde{\mathbf{F}}_\varphi(K, L)$ of K and L, is defined by

$$\int_{S^{n-1}} \varphi\left(\frac{\mathbf{F}(K) h_L(v)}{\widetilde{\mathbf{F}}_\varphi(K, L) h_K(v)}\right) d\mu^*_{\mathbf{F}}(K, v) = 1. \tag{28}$$

And if $L \in \mathcal{S}_0$, (28) is written by

$$\int_{S^{n-1}} \varphi\left(\frac{\mathbf{F}(K)}{\widetilde{\mathbf{F}}_\varphi(K, L) h_K(v)\rho_L(v)}\right) d\mu^*_{\mathbf{F}}(K, v) = 1. \tag{29}$$

By Definition 3.6 and $\varphi(1) = 1$, it implies that $\mathbf{F}_\varphi(K, K) = \mathbf{F}(K) = \widetilde{\mathbf{F}}_\varphi(K, K)$ for $\varphi \in \mathcal{I} \cup \mathcal{D}$ and $K \in \mathcal{K}_0$. And for $c_1, c_2 > 0$, $K, L_1 \in \mathcal{K}_0$, $L_2 \in \mathcal{S}_0$, it has

$$\mathbf{F}_\varphi(c_1 B_2^n, B_2^n) = c_1^\alpha \varphi(c_1^{-1}) \mathbf{F}(B_2^n), \quad \mathbf{F}_\varphi(B_2^n, c_2 B_2^n) = \varphi(c_2) \mathbf{F}(B_2^n),$$

$$\widetilde{\mathbf{F}}_\varphi(c_1 K, c_2 L_1) = c_1^{\alpha-1} c_2 \widetilde{\mathbf{F}}_\varphi(K, L_1), \quad \widetilde{\mathbf{F}}_\varphi(c_1 K, (c_2 L_2)^\circ) = c_1^{\alpha-1} c_2^{-1} \widetilde{\mathbf{F}}_\varphi(K, L_2).$$

Next we will prove the continuity of $\mathbf{F}_\varphi(\cdot, \cdot)$ and $\widetilde{\mathbf{F}}_\varphi(\cdot, \cdot)$.

Theorem 3.7. *Let $K, L \in \mathcal{K}_0$. Assume that $K_i, L_i \in \mathcal{K}_0$ are two sequences of convex bodies for $i = 1, 2, \ldots$ satisfying $K_i \to K$ and $L_i \to L$ as $i \to \infty$. Then for $\varphi \in \mathcal{I} \cup \mathcal{D}$ and $i \to \infty$,*

$$\mathbf{F}_\varphi(K_i, L_i) \to \mathbf{F}_\varphi(K, L) \text{ and } \widetilde{\mathbf{F}}_\varphi(K_i, L_i) \to \widetilde{\mathbf{F}}_\varphi(K, L).$$

Proof. Since K_i converge to $K \in \mathcal{K}_0$ and L_i converge to $L \in \mathcal{K}_0$, then

$$h_{K_i}(v) \to h_K(v),\ h_{L_i}(v) \to h_L(v) \text{ uniformly},$$

$$\mu_{\mathbf{F}}(K_i, v) \to \mu_{\mathbf{F}}(K, v),\ \mu_{\mathbf{F}}(L_i, v) \to \mu_{\mathbf{F}}(L, v) \text{ weakly},$$

for $v \in S^{n-1}$. Therefore $\lim_{i\to\infty} \mathbf{F}_\varphi(K_i, L_i) = \mathbf{F}_\varphi(K, L)$. Indeed, since $K_i, L_i \in \mathcal{K}_0$, then there are two constants $c_3 > c_4 > 0$, define $c_5 = \frac{c_3}{c_4}$ and $c_6 = \frac{c_4}{c_3}$, satisfying

$$c_4 B_2^n \subseteq K_i, L_i \subseteq c_3 B_2^n \ \Rightarrow\ \frac{h_{L_i}(v)}{h_{K_i}(v)} \in [c_6, c_5] \tag{30}$$

for $v \in S^{n-1}$ and $i \geq 1$. Since φ is a continuous function, combining with Lemma 2.1, it has

$$\lim_{i\to\infty} \frac{1}{\alpha} \int_{S^{n-1}} \varphi\left(\frac{h_{L_i}(v)}{h_{K_i}(v)}\right) h_{K_i}(v) d\mu_{\mathbf{F}}(K_i, v) = \frac{1}{\alpha} \int_{S^{n-1}} \varphi\left(\frac{h_L(v)}{h_K(v)}\right) h_K(v) d\mu_{\mathbf{F}}(K, v).$$

As for $\lim_{i\to\infty} \widetilde{\mathbf{F}}_\varphi(K_i, L_i) = \widetilde{\mathbf{F}}_\varphi(K, L)$, when $\varphi \in \mathcal{I}$ and $\varphi \in \mathcal{D}$, since the proof methods are the same, we only prove the result when $\varphi \in \mathcal{D}$. By the monotonicity of $\mathbf{F}$, it has $\mathbf{F}(c_4 B_2^n) \leq \mathbf{F}(K_i) \leq \mathbf{F}(c_3 B_2^n)$. By (30) and $\varphi \in \mathcal{D}$, it implies that

$$\varphi\left(\frac{\mathbf{F}(c_3 B_2^n) c_3}{\widetilde{\mathbf{F}}_\varphi(K_i, L_i) c_4}\right) \leq \int_{S^{n-1}} \varphi\left(\frac{\mathbf{F}(K_i) h_{L_i}(v)}{\widetilde{\mathbf{F}}_\varphi(K_i, L_i) h_{K_i}(v)}\right) d\mu_{\mathbf{F}}^*(K_i, v) = 1 \leq \varphi\left(\frac{\mathbf{F}(c_4 B_2^n) c_4}{\widetilde{\mathbf{F}}_\varphi(K_i, L_i) c_3}\right).$$

Then $\widetilde{\mathbf{F}}_\varphi(K_i, L_i)$ is bounded, i.e., there exist two constant $a_1, a_2 > 0$ such that $a_1 = \liminf_{i\to\infty} \widetilde{\mathbf{F}}_\varphi(K_i, L_i)$ and $a_2 = \limsup_{i\to\infty} \widetilde{\mathbf{F}}_\varphi(K_i, L_i)$. Indeed, since $\varphi(1) = 1$, for $i \geq 1$, it has $\widetilde{\mathbf{F}}_\varphi(K_i, L_i) \in [\mathbf{F}(c_4 B_2^n) c_4 / c_3, \mathbf{F}(c_3 B_2^n) c_3 / c_4] \subset (0, \infty)$. Then for $m, n \geq 1$, there exist two subsequences of $\widetilde{\mathbf{F}}_\varphi(K_i, L_i)$, called $\widetilde{\mathbf{F}}_\varphi(K_{i_m}, L_{i_m})$ and $\widetilde{\mathbf{F}}_\varphi(K_{i_n}, L_{i_n})$, satisfying $\widetilde{\mathbf{F}}_\varphi(K_{i_m}, L_{i_m}) \to a_1$, $\widetilde{\mathbf{F}}_\varphi(K_{i_n}, L_{i_n}) \to a_2$ as $m, n \to \infty$ and

$$\widetilde{\mathbf{F}}_\varphi(K_{i_n}, L_{i_n}) < \frac{n+1}{n} a_1,\ \ \widetilde{\mathbf{F}}_\varphi(K_{i_m}, L_{i_m}) > \frac{m}{m+1} a_2.$$

By $\varphi \in \mathcal{D}$ and Lemma 2.1, it has

$$\begin{aligned} 1 &= \lim_{m\to\infty} \int_{S^{n-1}} \varphi\left(\frac{\mathbf{F}(K_{i_m}) h_{L_{i_m}}(v)}{\widetilde{\mathbf{F}}_\varphi(K_{i_m}, L_{i_m}) h_{K_{i_m}}(v)}\right) d\mu_{\mathbf{F}}^*(K_{i_m}, v) \\ &\geq \lim_{m\to\infty} \int_{S^{n-1}} \varphi\left(\frac{(m+1)\mathbf{F}(K_{i_m}) h_{L_{i_m}}(v)}{m a_2 h_{K_{i_m}}(v)}\right) d\mu_{\mathbf{F}}^*(K_{i_m}, v) \\ &= \int_{S^{n-1}} \varphi\left(\frac{\mathbf{F}(K) h_L(v)}{a_2 h_K(v)}\right) d\mu_{\mathbf{F}}^*(K, v) \end{aligned} \tag{31}$$

and

$$\begin{aligned} 1 &= \lim_{n\to\infty} \int_{S^{n-1}} \varphi\left(\frac{\mathbf{F}(K_{i_n}) h_{L_{i_n}}(v)}{\widetilde{\mathbf{F}}_\varphi(K_{i_n}, L_{i_n}) h_{K_{i_n}}(v)}\right) d\mu_{\mathbf{F}}^*(K_{i_n}, v) \\ &\leq \lim_{n\to\infty} \int_{S^{n-1}} \varphi\left(\frac{n\mathbf{F}(K_{i_n}) h_{L_{i_n}}(v)}{(n+1) a_1 h_{K_{i_n}}(v)}\right) d\mu_{\mathbf{F}}^*(K_{i_n}, v) \\ &= \int_{S^{n-1}} \varphi\left(\frac{\mathbf{F}(K) h_L(v)}{a_1 h_K(v)}\right) d\mu_{\mathbf{F}}^*(K, v). \end{aligned} \tag{32}$$

Combing (31) with (32), it implies that

$$\limsup_{i\to\infty}\widetilde{\mathbf{F}}_\varphi(K_i,L_i)\leq\widetilde{\mathbf{F}}_\varphi(K,L)\leq\liminf_{i\to\infty}\widetilde{\mathbf{F}}_\varphi(K_i,L_i)\ \Rightarrow\ \lim_{i\to\infty}\widetilde{\mathbf{F}}_\varphi(K_i,L_i)=\widetilde{\mathbf{F}}_\varphi(K,L).$$

□

Theorem 3.8. *Let $K\in\mathcal{K}_0$ and $\varphi\in\mathcal{I}$. Assume that $K_i\in\mathcal{K}_0$ are the sequences of the convex body for $i=1,2,\ldots$ satisfying $K_i\to K$ as $i\to\infty$. If $\{M_i\}_{i\geq1}\subseteq\mathcal{K}_0$ such that $\{\mathbf{F}_\varphi(K_i,M_i)\}_{i\geq1}$ or $\{\widetilde{\mathbf{F}}_\varphi(K_i,M_i)\}_{i\geq1}$ is bounded, then $\{M_i\}_{i\geq1}$ is uniformly bounded.*

Proof. Since $K_i,K\in\mathcal{K}_0$ and K_i converges to K as $i\to\infty$, then for $v\in S^{n-1}$, it has

$$h_{K_i}(v)\to h_K(v)\text{ uniformly},\ \mu_{\mathbf{F}}(K_i,v)\to\mu_{\mathbf{F}}(K,v)\text{ weakly }\Rightarrow\lim_{i\to\infty}\mathbf{F}(K_i)=\mathbf{F}(K).$$

And there exist two positive constant $c_7<c_8$ satisfying

$$c_7B_2^n\subseteq K_i\subseteq c_8B_2^n\ \Rightarrow\ h_{K_i}(v),h_K(v)\in[c_7,c_8],$$

for $v\in S^{n-1}$. Since $\mu_{\mathbf{F}}(K,\cdot)$ is not contained in any closed hemisphere, then there is a constant $c_9>0$ such that

$$\int_{S^{n-1}}(v\cdot w)_+d\mu_{\mathbf{F}}(K,v)\geq c_9,$$

where $(v\cdot w)_+=\max\{0,v\cdot w\}$. Let $v_i\in S^{n-1}$ be a unit vector such that $\rho_{M_i}(v_i)=\max_{v\in S^{n-1}}\rho(M_i,v)$. Then $[0,\rho_{M_i}(v_i)v_i]\subseteq M_i$ and hence $\rho_{M_i}(v_i)(v_i\cdot v)_+\leq h_{M_i}(v)$ for all $v\in S^{n-1}$. Next we will prove that $\{M_i\}_{i\geq1}$ is bounded by the argument of contradiction. Suppose that $\{M_i\}_{i\geq1}$ is not uniformly bounded and v_i converges to $v\in S^{n-1}$ as $i\to\infty$, then $\rho_{M_i}(v_i)=\infty$, furthermore, $\rho_{M_i}(v_i)(v_i\cdot v)_+>c_{10}$ for some constant $c_{10}>0$. Since $\{\mathbf{F}_\varphi(K_i,M_i)\}_{i\geq1}$ or $\{\widetilde{\mathbf{F}}_\varphi(K_i,M_i)\}_{i\geq1}$ is bounded, then there exist constants $c_{11},c_{12}>0$ such that

$$\mathbf{F}_\varphi(K_i,M_i)\leq c_{11},\ \widetilde{\mathbf{F}}_\varphi(K_i,M_i)\leq c_{12}.$$

By (26), (28), Lemma 2.1 and the monotonicity of φ, it has

$$\begin{aligned}
c_{11}&\geq\liminf_{i\to\infty}\frac{1}{\alpha}\int_{S^{n-1}}\varphi\left(\frac{h_{M_i}(v)}{h_{K_i}(v)}\right)h_{K_i}(v)d\mu_{\mathbf{F}}(K_i,v)\\
&\geq\liminf_{i\to\infty}\frac{1}{\alpha}\int_{S^{n-1}}\varphi\left(\frac{c_{10}}{c_8}\right)h_{K_i}(v)d\mu_{\mathbf{F}}(K_i,v)\\
&\geq\frac{1}{\alpha}\int_{S^{n-1}}\varphi\left(\frac{c_{10}}{c_8}\right)h_K(v)d\mu_{\mathbf{F}}(K,v)\\
&\geq\frac{c_7}{\alpha}\varphi\left(\frac{c_{10}}{c_8}\right)\int_{S^{n-1}}(v_i\cdot v)_+d\mu_{\mathbf{F}}(K,v)\\
&\geq\frac{c_7c_9}{\alpha}\varphi\left(\frac{c_{10}}{c_8}\right)\to\infty
\end{aligned}$$

and

$$\begin{aligned} 1 &= \lim_{i\to\infty}\int_{S^{n-1}}\varphi\left(\frac{\mathbf{F}(K_i)h_{M_i}(v)}{\widetilde{\mathbf{F}}_\varphi(K_i,M_i)h_{K_i}(v)}\right)d\mu^*_{\mathbf{F}}(K_i,v)\\ &\geq \lim_{i\to\infty}\int_{S^{n-1}}\varphi\left(\frac{c_{10}\mathbf{F}(K_i)}{c_{12}h_{K_i}(v)}\right)d\mu^*_{\mathbf{F}}(K_i,v)\\ &= \int_{S^{n-1}}\varphi\left(\frac{c_{10}\mathbf{F}(K)}{c_{12}h_K(v)}\right)d\mu^*_{\mathbf{F}}(K,v)\\ &\geq \varphi\left(\frac{c_7^\alpha c_{10}\mathbf{F}(B_2^n)}{c_{12}c_8}\right)\int_{S^{n-1}}(v_i\cdot v)_+d\mu^*_{\mathbf{F}}(K,v)\\ &\geq c_9\cdot\varphi\left(\frac{c_7^\alpha c_{10}\mathbf{F}(B_2^n)}{c_{12}c_8}\right)\to\infty, \end{aligned}$$

as $c_{10}\to\infty$. This proves the theorem. □

3.2. The Orlicz-Petty body for F

In this section, we establish the following optimization problems associated with $\mathbf{F}_\varphi$ and $\widetilde{\mathbf{F}}_\varphi$ and give the solutions to this problems, called Orlicz-Petty bodies for the compatible functional $\mathbf{F}$:

$$I(K)(S(K)) = \inf(\sup)\{\mathbf{F}_\varphi(K,L) : L\in\mathcal{K}_0, |L^\circ| = \omega_n\}, \tag{33}$$

$$\widetilde{I}(K)(\widetilde{S}(K)) = \inf(\sup)\{\widetilde{\mathbf{F}}_\varphi(K,L) : L\in\mathcal{K}_0, |L^\circ| = \omega_n\}. \tag{34}$$

Theorem 3.9. *Let $K\in\mathcal{K}_0$ and $\varphi\in\mathcal{I}$. Then*
(1) there is $M\in\mathcal{K}_0$ satisfying $|M^\circ| = \omega_n$ and

$$\mathbf{F}_\varphi(K,M) = I(K) = \inf\{\mathbf{F}_\varphi(K,L) : L\in\mathcal{K}_0, |L^\circ| = \omega_n\}.$$

(2) there is $\widetilde{M}\in\mathcal{K}_0$ satisfying $|\widetilde{M}^\circ| = \omega_n$ and

$$\widetilde{\mathbf{F}}_\varphi(K,\widetilde{M}) = \widetilde{I}(K) = \inf\{\widetilde{\mathbf{F}}_\varphi(K,L) : L\in\mathcal{K}_0, |L^\circ| = \omega_n\}.$$

(3) if $\varphi\in\mathcal{I}$ is a convex function, M and $\widetilde{M}$ existing in (1) and (2) are unique.

Proof. By the definition of $I(K)$ and $\widetilde{I}(K)$, it has

$$I(K)\leq\mathbf{F}_\varphi(K,B_2^n)<\infty,\ \ \widetilde{I}(K)\leq\widetilde{\mathbf{F}}_\varphi(K,B_2^n)<\infty.$$

Then we can choose two sequences $\{M_i\}_{i\geq1}, \{\widetilde{M}_j\}_{j\geq1}\subseteq\mathcal{K}_0$ such that $\lim_{i\to\infty}\mathbf{F}_\varphi(K,M_i) = I(K)$, $\lim_{j\to\infty}\widetilde{\mathbf{F}}_\varphi(K,\widetilde{M}_j) = \widetilde{I}(K)$ and $|M_i^\circ| = |\widetilde{M}_j^\circ| = \omega_n$. By Theorem 3.8, it implies that $\{M_i\}_{i\geq1}$ and $\{\widetilde{M}_j\}_{j\geq1}$ are uniformly bounded. By Lemma 2.2, there exist two sequences of $\{M_i\}_{i\geq1}$ and $\{\widetilde{M}_j\}_{j\geq1}$, called $\{M_{i_l}\}_{l\geq1}$ and $\{\widetilde{M}_{j_m}\}_{m\geq1}$, respectively, satisfying $M_{i_l}\to M\in\mathcal{K}_0$, $\widetilde{M}_{j_m}\to\widetilde{M}\in\mathcal{K}_0$ and $|M^\circ| = |\widetilde{M}^\circ| = \omega_n$ as $l,m\to\infty$.

By Theorem 3.7, it has

$$I(K) = \lim_{i\to\infty}\mathbf{F}_\varphi(K,M_i) = \lim_{i\to\infty}\mathbf{F}_\varphi(K,M_{i_l}) = \mathbf{F}_\varphi(K,M),$$

$$\widetilde{I}(K) = \lim_{j\to\infty}\widetilde{\mathbf{F}}_\varphi(K,\widetilde{M}_j) = \lim_{m\to\infty}\widetilde{\mathbf{F}}_\varphi(K,\widetilde{M}_{j_m}) = \widetilde{\mathbf{F}}_\varphi(K,\widetilde{M}).$$

Thus the solutions of (33) and (34) are M and $\widetilde{M}$, respectively.

As for uniqueness of the solutions, we prove them by the argument of contradiction. Suppose that there exist two convex bodies $M_1, M_2 \in \mathcal{K}_0$ satisfying $I(K) = \mathbf{F}_\varphi(K, M_1) = \mathbf{F}_\varphi(K, M_2)$ and $|M_1^\circ| = |M_2^\circ| = \omega_n$. Then $M_1 = M_2$. Indeed, let $M_3 = 2^{-1}(M_1 + M_2)$, then $\mathrm{vrad}(M_3^\circ) \le 1$ and inequalities hold if and only if $M_1 = M_2$. It implies that $h_{\mathrm{vrad}(M_3^\circ)M_3}(v) \le h_{M_3}(v)$ for $v \in S^{n-1}$. Since $\varphi \in \mathcal{I}$ is a convex function, it has

$$\begin{aligned} I(K) &\le \mathbf{F}_\varphi(K, \mathrm{vrad}(M_3^\circ)M_3) \\ &= \frac{1}{\alpha}\int_{S^{n-1}} \varphi\left(\frac{h_{\mathrm{vrad}(M_3^\circ)M_3}(v)}{h_K(v)}\right) h_K(v) d\mu_{\mathbf{F}}(K, v) \\ &\le \frac{1}{\alpha}\int_{S^{n-1}} \varphi\left(\frac{h_{M_3}(v)}{h_K(v)}\right) h_K(v) d\mu_{\mathbf{F}}(K, v) \\ &\le \frac{1}{\alpha}\int_{S^{n-1}} \frac{1}{2}\left[\varphi\left(\frac{h_{M_1}(v)}{h_K(v)}\right) + \varphi\left(\frac{h_{M_2}(v)}{h_K(v)}\right)\right] h_K(v) d\mu_{\mathbf{F}}(K, v) \\ &= \frac{1}{2}\left(\mathbf{F}_\varphi(K, M_1) + \mathbf{F}_\varphi(K, M_2)\right) = I(K). \end{aligned}$$

Then $h_{M_1}(v) = h_{M_2}(v)$ for any $v \in S^{n-1}$. Thus $M_1 = M_2$.

Suppose that there exist two convex bodies $\widetilde{M}_1, \widetilde{M}_2 \in \mathcal{K}_0$ satisfying $\widetilde{I}(K) = \widetilde{\mathbf{F}}_\varphi(K, \widetilde{M}_1) = \widetilde{\mathbf{F}}_\varphi(K, \widetilde{M}_2)$ and $|\widetilde{M}_1^\circ| = |\widetilde{M}_2^\circ| = \omega_n$. Then $\widetilde{M}_1 = \widetilde{M}_2$. Indeed, since $\varphi \in \mathcal{I}$ is a convex function and (28), it has

$$\begin{aligned} 1 &= \int_{S^{n-1}} \varphi\left(\frac{\mathbf{F}(K) h_{\widetilde{M}_1}(v)}{\widetilde{\mathbf{F}}_\varphi(K, \widetilde{M}_1) h_K(v)}\right) d\mu_{\mathbf{F}}^*(K, v) = \int_{S^{n-1}} \varphi\left(\frac{\mathbf{F}(K) h_{\widetilde{M}_1}(v)}{\widetilde{I}(K) h_K(v)}\right) d\mu_{\mathbf{F}}^*(K, v), \\ 1 &= \int_{S^{n-1}} \varphi\left(\frac{\mathbf{F}(K) h_{\widetilde{M}_2}(v)}{\widetilde{\mathbf{F}}_\varphi(K, \widetilde{M}_2) h_K(v)}\right) d\mu_{\mathbf{F}}^*(K, v) = \int_{S^{n-1}} \varphi\left(\frac{\mathbf{F}(K) h_{\widetilde{M}_2}(v)}{\widetilde{I}(K) h_K(v)}\right) d\mu_{\mathbf{F}}^*(K, v). \end{aligned}$$

Then $h_{\widetilde{M}_1}(v) = h_{\widetilde{M}_2}(v)$ for any $v \in S^{n-1}$, it means that $\widetilde{M}_1 = \widetilde{M}_2$. □

The solutions M and $\widetilde{M}$ of problems (33) and (34) are called the Orlicz-Petty bodies for $\mathbf{F}$, $I(K) = \mathbf{F}_\varphi(K, M)$ and $\widetilde{I}(K) = \widetilde{\mathbf{F}}_\varphi(K, M)$ are called the geominimal surface area for $\mathbf{F}$. Thus, one can define sets of all Orlicz-Petty bodies for $\mathbf{F}$: let $K \in \mathcal{K}_0$ and $\varphi \in \mathcal{I}$,

$$Q(K) = \{M \in \mathcal{K}_0 : \mathbf{F}_\varphi(K, M) = I(K), |M^\circ| = \omega_n\},$$

$$\widetilde{Q}(K) = \{\widetilde{M} \in \mathcal{K}_0 : \widetilde{\mathbf{F}}_\varphi(K, \widetilde{M}) = \widetilde{I}(K), |\widetilde{M}^\circ| = \omega_n\}.$$

Theorem 3.10. *Suppose that $K \in \mathcal{K}_0$ and $\{K_i\}_{i\ge1} \subseteq \mathcal{K}_0$ are convex bodies sequences satisfying $K_i \to K$ as $i \to \infty$. For $\varphi \in \mathcal{I}$, then*
(1) $I(K_i) \to I(K)$ and $\widetilde{I}(K_i) \to \widetilde{I}(K)$ as $i \to \infty$.
(2) $Q(K_i) \to Q(K)$ and $\widetilde{Q}(K_i) \to \widetilde{Q}(K)$ as $i \to \infty$ if $\varphi \in \mathcal{I}$ is a convex function.

Proof. (1) Let $M \in Q(K)$ and $M_i \in Q(K_i)$, then $\{M_i\}_{i\ge1}$ is uniformly bounded. Indeed, by Theorem 3.7 and (33) , it has

$$I(K) = \mathbf{F}_\varphi(K, M) = \lim_{i\to\infty} \mathbf{F}_\varphi(K_i, M) = \limsup_{i\to\infty} \mathbf{F}_\varphi(K_i, M) \ge \limsup_{i\to\infty} I(K_i), \tag{35}$$

it means that $\{I(K_i)\}_{i\ge1} = \{\mathbf{F}_\varphi(K_i, M_i)\}_{i\ge1}$ is bounded, namely, $\{M_i\}_{i\ge1}$ is uniformly bounded by Theorem 3.8. Let $\{M_{i_j}\}_{j\ge1}$ be a subsequence of $\{M_i\}_{i\ge1}$ satisfying $\lim_{j\to\infty} I(K_{i_j}) = \liminf_{i\to\infty} I(K_i)$. By Lemma 2.2, there exists a sequence of $\{M_{i_j}\}_{j\ge1}$, called $\{M_{i_{j_k}}\}_{k\ge1}$ and a convex body $M_0 \in \mathcal{K}_0$ satisfying $M_{i_{j_k}} \to M_0$ as $k \to \infty$ and $|M_0^\circ| = \omega_n$. By Theorem 3.7, it has

$$\liminf_{i\to\infty} I(K_i) = \lim_{k\to\infty} I(K_{i_{j_k}}) = \lim_{k\to\infty} \mathbf{F}_\varphi(K_{i_{j_k}}, M_{i_{j_k}}) = \mathbf{F}_\varphi(K, M_0) \ge I(K). \tag{36}$$

By (35) and (36), it has $I(K_i) \to I(K)$ as $i \to \infty$. Along the same line, it can prove $\widetilde{I}(K_i) \to \widetilde{I}(K)$ as $i \to \infty$.

(2) By Theorem 3.9, it implies that there exist $M \in Q(K)$ and $M_i \in Q(K_i)$ if $\varphi \in \mathcal{I}$ is convex. Let $\{M_{i_j}\}_{j\geq 1}$ be a sequence of $\{M_i\}_{i\geq 1}$. Then

$$I(K) = \lim_{j\to\infty} I(K_{i_j}) = \lim_{j\to\infty} \mathbf{F}_\varphi(K_{i_j}, M_{i_j}). \tag{37}$$

It means that $\{\mathbf{F}_\varphi(K_{i_k}, M_{i_j})\}_{j\geq 1}$ is bounded. By Theorem 3.8, it implies that $\{M_{i_j}\}_{j\geq 1}$ is uniformly bounded. By Lemma 2.2, there exists a subsequence $\{M_{i_{j_k}}\}_{k\geq 1}$ of $\{M_{i_j}\}_{j\geq 1}$ and a convex body $M_0 \in \mathcal{K}_0$ satisfying $M_{i_{j_k}} \to M_0$ and $|M_0^\circ| = \omega_n$. By Theorem 3.7 and (37), it has

$$I(K) = \lim_{k\to\infty} I(K_{i_{j_k}}) = \lim_{k\to\infty} \mathbf{F}_\varphi(K_{i_{j_k}}, M_{i_{j_k}}) = \mathbf{F}_\varphi(K, M_0).$$

Then $M = M_0$. Thus $M_i \to M$ as $i \to \infty$. Along the same line, it can prove $\widetilde{M}_i \to \widetilde{M}$ as $i \to \infty$. □

Proposition 3.11. *Let $K \in \mathcal{K}_0$ be a polytope and $\varphi \in \mathcal{I}$. Suppose that $M \in Q(K)$ and $\widetilde{M} \in \widetilde{Q}(K)$, then M and $\widetilde{M}$ are polytopes with faces parallel to those of K.*

Proof. Let $m \in \mathbb{N}$ and $\{v_i\}_{i=1}^m \subseteq S^{n-1}$ such that $K = \bigcap_{1\leq i\leq m}\{x \in \mathbb{R}^n : x\cdot v_i \leq h_K(v_i)\}$. Then $\mu_{\mathbf{F}}(K,\cdot)$ is concentrated on $\{v_i\}_{i=1}^m$ by Lemma 3.3. Define a polytope P with faces parallel to those of K by

$$P = \bigcap_{1\leq i\leq m} \{x \in \mathbb{R}^n : x \cdot v_i \leq h_M(v_i)\},$$

where $M \in Q(K)$. It implies that $h_P(v_i) = h_M(v_i)$ for $1 \leq i \leq m$. Thus,

$$\begin{aligned}
\mathbf{F}_\varphi(K, P) &= \frac{1}{\alpha}\int_{S^{n-1}} \varphi\left(\frac{h_P(v)}{h_K(v)}\right) h_K(v) d\mu_{\mathbf{F}}(K, v)\\
&= \frac{1}{\alpha}\sum_{i=1}^m \varphi\left(\frac{h_P(v_i)}{h_K(v_i)}\right) h_K(v_i)\mu_{\mathbf{F}}(K, \{v_i\})\\
&= \frac{1}{\alpha}\sum_{i=1}^m \varphi\left(\frac{h_M(v_i)}{h_K(v_i)}\right) h_K(v_i)\mu_{\mathbf{F}}(K, \{v_i\})\\
&= \frac{1}{\alpha}\int_{S^{n-1}} \varphi\left(\frac{h_M(v)}{h_K(v)}\right) h_K(v) d\mu_{\mathbf{F}}(K, v)\\
&= \mathbf{F}_\varphi(K, M).
\end{aligned}$$

Thus $\mathbf{F}_\varphi(K, P) = \mathbf{F}_\varphi(K, M) = I(K) \leq \mathbf{F}_\varphi(K, \mathrm{vrad}(P^\circ)P)$. It implies that $M = P$, so M is a polytope with faces parallel to those of K. Indeed, since $P^\circ \subseteq M^\circ$, then $\mathrm{vrad}(P^\circ) \leq \mathrm{vrad}(M^\circ) = 1$. And $\varphi \in \mathcal{I}$, then $\mathrm{vrad}(P^\circ) \geq 1$. So $|P^\circ| = |M^\circ|$.

Suppose that $\widetilde{M} \in \widetilde{Q}(K)$, define a polytope $\widetilde{P}$ with faces parallel to those of K by

$$\widetilde{P} = \bigcap_{1\leq i\leq m} \{x \in \mathbb{R}^n : x \cdot v_i \leq h_{\widetilde{M}}(v_i)\}.$$

Then $h_{\widetilde{P}}(v_i) = h_{\widetilde{M}}(v_i)$ for $1 \leq i \leq m$. By (28), it has

$$1 = \int_{S^{n-1}} \varphi\left(\frac{\mathbf{F}(K)h_{\widetilde{P}}(v)}{\widetilde{\mathbf{F}}_\varphi(K, \widetilde{P})h_K(v)}\right) d\mu^*_{\mathbf{F}}(K, v) = \sum_{i=1}^m \varphi\left(\frac{\mathbf{F}(K)h_{\widetilde{P}}(v_i)}{\widetilde{\mathbf{F}}_\varphi(K, \widetilde{P})h_K(v_i)}\right) d\mu^*_{\mathbf{F}}(K, \{v_i\}),$$

$$1 = \int_{S^{n-1}} \varphi\left(\frac{\mathbf{F}(K)h_{\widetilde{M}}(v)}{\widetilde{\mathbf{F}}_\varphi(K, \widetilde{M})h_K(v)}\right) d\mu^*_{\mathbf{F}}(K, v) = \sum_{i=1}^m \varphi\left(\frac{\mathbf{F}(K)h_{\widetilde{M}}(v_i)}{\widetilde{\mathbf{F}}_\varphi(K, \widetilde{M})h_K(v_i)}\right) d\mu^*_{\mathbf{F}}(K, \{v_i\}).$$

Thus $\widetilde{\mathbf{F}}_\varphi(K,\widetilde{P}) = \widetilde{\mathbf{F}}_\varphi(K,\widetilde{M}) = \widetilde{I}(K) \le \widetilde{\mathbf{F}}_\varphi(K, \mathrm{vrad}(\widetilde{P}^\circ)\widetilde{P})$. It implies that $\widetilde{M} = \widetilde{P}$, so $\widetilde{M} \in \widetilde{Q}(K)$ is a polytope with faces parallel to those of K. Indeed, since $\widetilde{P}^\circ \subseteq \widetilde{M}^\circ$, then $\mathrm{vrad}(\widetilde{P}^\circ) \le \mathrm{vrad}(\widetilde{M}^\circ) = 1$. And $\varphi \in \mathcal{I}$, then $\mathrm{vrad}(\widetilde{P}^\circ) \ge 1$. So $|\widetilde{P}^\circ| = |\widetilde{M}^\circ|$. □

Let $\{v_1, v_2, \dots, v_m\}$ be a finite set of S^{n-1} for $m \in \mathbb{N}$, it is proved by some counterexamples that problems (33) and (34) are not always solvable in the following.

Proposition 3.12. *Suppose that $K \in \mathcal{K}_0$ is a polytope with $\{v_1, v_2, \dots, v_m\}$ as the unit normal vectors of its faces.*
(1) If $\varphi \in \mathcal{D}$ and the nth coordinates of $v_1, v_2, \dots, v_m$ are nonzero, then

$$I(K) = 0, \quad \widetilde{S}(K) = \infty.$$

(2) If $\varphi \in \mathcal{I}$, then

$$S(K) = \widetilde{S}(K) = \infty.$$

Proof. (1) For positive numbers $a, b > 0$, let

$$K_a = a^{-1} T_a B_2^n \ \text{ with } \ T_a = \mathrm{diag}(a^n, 1, \dots, 1),$$

$$\widetilde{K}_b = b^{\frac{n-1}{n}} T_b B_2^n \ \text{ with } \ T_b = \mathrm{diag}(b^{-1}, \dots, b^{-1}, 1).$$

It has $K_a^\circ = a(T_a^t)^{-1} B_2^n$ and $|K_a^\circ| = \omega_n$, $K_b^\circ = b^{\frac{1-n}{n}} (T_b^t)^{-1} B_2^n$ and $|K_b^\circ| = \omega_n$. Since the *nth* coordinates of $v_1, v_2, \dots, v_m$ are nonzero, for $1 \le i \le m$, there exist two constants $c_{13}, c_{14} > 0$ satisfying

$$\begin{aligned} h_{K_a}(v_i) &= \max_{w_1 \in K_a} w_1 v_i = \max_{w_2 \in B_2^n} T_a w_2 a^{-1} v_i = a^{-1} \max_{w_2 \in B_2^n} w_2 T_a v_i = a^{-1} |T_a v_i| \\ &= a^{-1} \left(a^{2n} (v_i)_1^2 + (v_i)_2^2 + \cdots + (v_i)_n^2 \right)^{\frac{1}{2}} \ge a^{-1} |(v_i)_n| \ge a^{-1} c_{13} \end{aligned}$$

and

$$\begin{aligned} h_{\widetilde{K}_b}(v_i) &= \max_{w_3 \in \widetilde{K}_b} w_3 v_i = \max_{w_4 \in B_2^n} T_b w_4 b^{\frac{n-1}{n}} v_i = b^{\frac{n-1}{n}} \max_{w_4 \in B_2^n} w_4 T_b v_i = b^{\frac{n-1}{n}} |T_b v_i| \\ &= b^{\frac{n-1}{n}} \left(b^{-2} (v_i)_1^2 + \cdots + b^{-2} (v_i)_{n-1}^2 + (v_i)_n^2 \right)^{\frac{1}{2}} \ge b^{\frac{n-1}{n}} |(v_i)_n| \ge b^{\frac{n-1}{n}} c_{14}. \end{aligned}$$

Since $K \in \mathcal{K}_0$ is a polytope, there is a constant $0 < c_{15} < c_{16}$ such that $c_{15} \le h(K, v_i) \le c_{16}$ for $1 \le i \le m$. By $\varphi \in \mathcal{D}$, it has

$$\begin{aligned} I(K) &\le \frac{1}{\alpha} \int_{S^{n-1}} \varphi\left(\frac{h_{K_a}(v)}{h_K(v)} \right) h_K(v) d\mu_{\mathbf{F}}(K, v) \\ &= \frac{1}{\alpha} \sum_{i=1}^m \varphi\left(\frac{h_{K_a}(v_i)}{h_K(v_i)} \right) h_K(v_i) \mu_{\mathbf{F}}(K, \{v_i\}) \\ &\le \frac{1}{\alpha} \sum_{i=1}^m \varphi\left(\frac{c_{13}}{a c_{16}} \right) c_{16} \mu_{\mathbf{F}}(K, \{v_i\}) \\ &= \frac{c_{16}}{\alpha} \varphi\left(\frac{c_{13}}{a c_{16}} \right) \mu_{\mathbf{F}}(K, S^{n-1}) \to 0 \end{aligned}$$

as $a \to 0$ and

$$\begin{aligned}
1 &= \int_{S^{n-1}} \varphi\left(\frac{\mathbf{F}(K)h_{\widetilde{K}_b}(v)}{\widetilde{\mathbf{F}}_\varphi(K,\widetilde{K}_b)h_K(v)}\right)d\mu^*_{\mathbf{F}}(K,v)\\
&= \sum_{i=1}^m \varphi\left(\frac{\mathbf{F}(K)h_{\widetilde{K}_b}(v_i)}{\widetilde{\mathbf{F}}_\varphi(K,\widetilde{K}_b)h_K(v_i)}\right)d\mu^*_{\mathbf{F}}(K,\{v_i\})\\
&\le \sum_{i=1}^m \varphi\left(\frac{c_{15}^\alpha \mathbf{F}(B_2^n)c_{14}b^{\frac{n-1}{n}}}{\widetilde{S}(K)c_{16}}\right)d\mu^*_{\mathbf{F}}(K,\{u_i\})\\
&\le \varphi\left(\frac{\mathbf{F}(B_2^n)c_{14}c_{15}^\alpha}{c_{16}}\cdot\frac{b^{\frac{n-1}{n}}}{\widetilde{S}(K)}\right)d\mu^*_{\mathbf{F}}(K,\{u_i\}),
\end{aligned}$$

thus $\widetilde{S}(K)\to\infty$ as $b\to 0$.

(2) Assume that $\mu_{\mathbf{F}}(K,\{v_n\})>0$. For positive numbers $\delta,\varepsilon>0$, let

$$K_\delta = \delta T_\delta B_2^n \text{ with } T_\delta = T\mathrm{diag}(1,\dots,1,\delta^{-n})T^t,$$
$$\widetilde{K}_\varepsilon = T_\varepsilon B_2^n \text{ with } T_\varepsilon = T\mathrm{diag}(1,\dots,1,\varepsilon^{-1},\varepsilon)T^t,$$

where T is an orthogonal matrix with v_n as its *nth* column vector. It has $K_\delta^\circ = \delta^{-1}(T_\delta^t)^{-1}B_2^n$, $\widetilde{K}_\varepsilon^\circ = (T_\varepsilon^t)^{-1}B_2^n$ and $|K_\delta^\circ| = |\widetilde{K}_\varepsilon^\circ| = \omega_n$. Then

$$h_{K_\delta}(v_n) = \max_{w_1\in K_\delta} w_1 v_n = \max_{w_2\in B_2^n}\delta T_\delta w_2 v_n = \max_{w_2\in B_2^n} w_2\delta T_\delta v_n = \delta\max_{w_2\in B_2^n} w_2\delta^{-n}v_n = \frac{1}{\delta^{n-1}}.$$

and

$$h_{\widetilde{K}_\varepsilon}(v_n) = \max_{w_1\in \widetilde{K}_\varepsilon} w_1 v_n = \max_{w_2\in B_2^n} T_\varepsilon w_2 v_n = \max_{w_2\in B_2^n} w_2 T_\varepsilon v_n = \max_{w_2\in B_2^n} w_2\varepsilon v_n = \varepsilon.$$

By $\varphi\in\mathcal{I}$, it has

$$\begin{aligned}
S(K) &\ge \frac{1}{\alpha}\int_{S^{n-1}}\varphi\left(\frac{h_{K_\delta}(v)}{h_K(v)}\right)h_K(v)d\mu_{\mathbf{F}}(K,v)\\
&= \frac{1}{\alpha}\sum_{j=1}^m \varphi\left(\frac{h_{K_\delta}(v_j)}{h_K(v_j)}\right)h_K(v_j)\mu_{\mathbf{F}}(K,\{v_j\})\\
&\ge \frac{1}{\alpha}\varphi\left(\frac{h_{K_\delta}(v_n)}{h_K(v_n)}\right)h_K(v_n)\mu_{\mathbf{F}}(K,\{v_n\})\\
&\ge \frac{c_{15}}{\alpha}\varphi\left(\frac{1}{c_{16}\delta^{n-1}}\right)\mu_{\mathbf{F}}(K,\{v_n\})\to\infty
\end{aligned}$$

as $\delta\to\infty$ and

$$\begin{aligned}
1 &= \int_{S^{n-1}} \varphi\left(\frac{\mathbf{F}(K)h_{\widetilde{K}_\varepsilon}(v)}{\widetilde{\mathbf{F}}_\varepsilon(K,\widetilde{K}_\varepsilon)h_K(v)}\right)d\mu^*_{\mathbf{F}}(K,v)\\
&= \sum_{j=1}^m \varphi\left(\frac{\mathbf{F}(K)h_{\widetilde{K}_\varepsilon}(v_j)}{\widetilde{\mathbf{F}}_\varphi(K,\widetilde{K}_\varepsilon)h_K(v_j)}\right)d\mu^*_{\mathbf{F}}(K,\{v_j\})\\
&\ge \varphi\left(\frac{\mathbf{F}(K)h_{\widetilde{K}_\varepsilon}(v_n)}{\widetilde{\mathbf{F}}_\varphi(K,\widetilde{K}_\varepsilon)h_K(v_n)}\right)d\mu^*_{\mathbf{F}}(K,\{v_n\})\\
&\ge \varphi\left(\frac{\mathbf{F}(B_2^n)c_{15}^\alpha}{c_{16}}\cdot\frac{\varepsilon}{\widetilde{S}(K)}\right)d\mu^*_{\mathbf{F}}(K,\{v_n\}),
\end{aligned}$$

thus $\widetilde{S}(K) \to \infty$ as $\varepsilon \to 0$. □

4. The Orlicz and L_q geominimal compatible functionals

In this section, we will introduce the Orlicz and L_q geominimal compatible functionals based on the Orlicz L_φ mixed compatible functionals in Definition 3.6. And some properties of them, such as the isoperimetric type inequalities associated with the L_q geominimal compatible functional will be studied.

4.1. The Orlicz geominimal compatible functional

Let $S_0 \subset \mathcal{S}_0$ be a nonempty subset, $S_1 = \{\varphi : (0,\infty) \to (0,\infty) | \varphi(t^{-1/n})$ is strictly convex$\}$ and $S_2 = \{\varphi : (0,\infty) \to (0,\infty) | \varphi(t^{-1/n})$ is strictly concave$\}$. Define

$$\mathcal{I}_0 = \mathcal{I} \cap S_1, \ \ \mathcal{D}_0 = \mathcal{D} \cap S_2, \ \ \mathcal{D}_1 = \mathcal{D} \cap S_1. \tag{38}$$

Definition 4.1. *Let $K \in \mathcal{K}_0$.*
i) The nonhomogeneous Orlicz geominimal functional $G_\varphi(K, S_0)$ of K with respect to S_0, is defined by

$$G_\varphi(K, S_0) = \inf\{\mathbf{F}_\varphi(K, \mathrm{vrad}(L)L^\circ) : L \in S_0\} \quad \text{if } \varphi \in \mathcal{I} \cup \mathcal{D}_1, \tag{39}$$

$$G_\varphi(K, S_0) = \sup\{\mathbf{F}_\varphi(K, \mathrm{vrad}(L)L^\circ) : L \in S_0\} \quad \text{if } \varphi \in \mathcal{D}_0.$$

ii) The homogeneous Orlicz geominimal functional $\widetilde{G}_\varphi(K, S_0)$ of K with respect to S_0, is defined by

$$\widetilde{G}_\varphi(K, S_0) = \inf\{\widetilde{\mathbf{F}}_\varphi(K, \mathrm{vrad}(L)L^\circ) : L \in S_0\} \quad \text{if } \varphi \in \mathcal{I} \cup \mathcal{D}_0, \tag{40}$$

$$\widetilde{G}_\varphi(K, S_0) = \sup\{\widetilde{\mathbf{F}}_\varphi(K, \mathrm{vrad}(L)L^\circ) : L \in S_0\} \quad \text{if } \varphi \in \mathcal{D}_1.$$

For simplicity, let

$$G_\varphi(K) = G_\varphi(K, \mathcal{K}_0), \ \ \widetilde{G}_\varphi(K) = \widetilde{G}_\varphi(K, \mathcal{K}_0) \quad \text{if } S_0 = \mathcal{K}_0;$$

$$H_\varphi(K) = G_\varphi(K, \mathcal{S}_0), \ \ \widetilde{H}_\varphi(K) = \widetilde{G}_\varphi(K, \mathcal{S}_0) \quad \text{if } S_0 = \mathcal{S}_0.$$

Then $\widetilde{G}_\varphi(c_{17}K) = c_{17}^{\alpha-1}\widetilde{G}_\varphi(K)$ and $\widetilde{H}_\varphi(c_{17}K) = c_{17}^{\alpha-1}\widetilde{H}_\varphi(K)$ for some constant $c_{17} > 0$. Since $\mathcal{K}_0 \subset \mathcal{S}_0$, it implies that

$$G_\varphi(K) \geq H_\varphi(K) \ \text{ if } \ \varphi \in \mathcal{I} \cup \mathcal{D}_1; \ \ G_\varphi(K) \leq H_\varphi(K) \ \text{ if } \ \varphi \in \mathcal{D}_0. \tag{41}$$

$$\widetilde{G}_\varphi(K) \geq \widetilde{H}_\varphi(K) \ \text{ if } \ \varphi \in \mathcal{I} \cup \mathcal{D}_0; \ \ \widetilde{G}_\varphi(K) \leq \widetilde{H}_\varphi(K) \ \text{ if } \ \varphi \in \mathcal{D}_1. \tag{42}$$

4.2. The L_q geominimal compatible functional

In this section, we will introduce the L_q geominimal compatible functional and discuss some properties of them. Based on the Orlicz L_φ mixed compatible functional, let $\varphi(t) = t^q$ in Definition 3.6, we get the following L_q mixed compatible functionals:

$$\mathbf{F}_q(K, L) = \frac{1}{\alpha}\int_{S^{n-1}} \left(\frac{h_L(v)}{h_K(v)}\right)^q h_K(v) d\mu_{\mathbf{F}}(K, v) \text{ for } L \in \mathcal{K}_0,$$

$$\mathbf{F}_q(K, L^\circ) = \frac{1}{\alpha}\int_{S^{n-1}} \left(\frac{1}{h_K(v)\rho_L(v)}\right)^q h_K(v) d\mu_{\mathbf{F}}(K, v) \text{ for } L \in \mathcal{S}_0.$$

Definition 4.2. *Let $K \in \mathcal{K}_0$ and $-n \neq q \in \mathbb{R}$.*
i) The L_q geominimal compatible functional $G_q(K)$ with respect to $\mathcal{K}_0$, is defined by

$$G_q(K) = \inf\left\{\mathbf{F}_q(K,L)^{\frac{n}{(n+q)}}|L^\circ|^{\frac{q}{(n+q)}} : L \in \mathcal{K}_0\right\} \quad \text{if } q \geq 0,$$

$$G_q(K) = \sup\left\{\mathbf{F}_q(K,L)^{\frac{n}{(n+q)}}|L^\circ|^{\frac{q}{(n+q)}} : L \in \mathcal{K}_0\right\} \quad \text{if } -n \neq q < 0.$$

ii) The L_q geominimal compatible functional $H_q(K)$ with respect to $\mathcal{S}_0$, is defined by

$$H_q(K) = \inf\left\{\mathbf{F}_q(K,L^\circ)^{\frac{n}{(n+q)}}|L|^{\frac{q}{(n+q)}} : L \in \mathcal{S}_0\right\} \quad \text{if } q \geq 0,$$

$$H_q(K) = \sup\left\{\mathbf{F}_q(K,L^\circ)^{\frac{n}{(n+q)}}|L|^{\frac{q}{(n+q)}} : L \in \mathcal{S}_0\right\} \quad \text{if } -n \neq q < 0.$$

Remark 4.3. *(1) For $s > 0$, it has $G_q(sK) = s^{\frac{n(\alpha-q)}{n+q}} G_q(K)$ and $H_q(sK) = s^{\frac{n(\alpha-q)}{n+q}} H_q(K)$.*
(2) If $q \neq -n$, then $G_q(B_2^n) = H_q(B_2^n) = \mathbf{F}(B_2^n)^{\frac{n}{(n+q)}}|B_2^n|^{\frac{q}{(n+q)}}$.
(3) If $q \neq 0, -n$, then

$$G_q(K) = \mathbf{F}(K)^{\frac{(q-1)nq}{q(n+q)}} \omega_n^{\frac{q}{n+q}} \widetilde{G}_\varphi(K)^{\frac{nq}{n+q}}, \quad H_q(K) = \mathbf{F}(K)^{\frac{(q-1)nq}{q(n+q)}} \omega_n^{\frac{q}{n+q}} \widetilde{H}_\varphi(K)^{\frac{nq}{n+q}}. \tag{43}$$

For $K \in \mathcal{A}_0$ and $v \in S^{n-1}$, define

$$g_q(K,v) = h_K(v)^{1-q} u(\nu_K^{-1}(v)) g(v)$$

and

$$\xi_q = \left\{K \in \mathcal{A}_0 : \text{exists } L \in \mathcal{S}_0, \text{ s.t. } g_q(K,v) = \rho_L(v)^{n+q}\right\}, \quad q \neq -n,$$

where u is the function defined in (21) and g is the curvature function defined in (13).

Theorem 4.4. *Let $K \in \xi_q$ and $q \neq -n$, then*

$$H_q(K) = \alpha^{-\frac{n}{n+q}} n^{-\frac{q}{n+q}} \int_{S^{n-1}} g_q(K,v)^{\frac{n}{n+q}} d\sigma(v). \tag{44}$$

Proof. For $L \in \mathcal{S}_0$.

(1) If $q = 0$, then $H_0(K) = \frac{1}{\alpha}\int_{S^{n-1}} h_K(v) d\mu_{\mathbf{F}}(K,v) = \mathbf{F}(K)$, the conclusion is true.

(2) Since the proof methods of (44) are the same when $q > 0$ and $q < 0$, we just prove the case $q > 0$. Let $K \in \xi_q$ and $v \in S^{n-1}$, there is $M \in \mathcal{S}_0$ satisfying $\rho_M^{n+q}(v) = g_q(K,v)$. Then by Definition 4.2,

$$\alpha^{-\frac{n}{n+q}} n^{-\frac{q}{n+q}} \int_{S^{n-1}} g_q(K,v)^{\frac{n}{n+q}} d\sigma(v) = \mathbf{F}_q(K,M^\circ)^{\frac{n}{n+q}} \cdot |M|^{\frac{q}{n+q}} \geq H_q(K). \tag{45}$$

On the other hand, by Hölder inequality, it has

$$\begin{aligned}
\alpha^{-\frac{n}{n+q}} n^{-\frac{q}{n+q}} \int_{S^{n-1}} g_q(K,v)^{\frac{n}{n+q}} d\sigma(v) &= \alpha^{-\frac{n}{n+q}} n^{-\frac{q}{n+q}} \int_{S^{n-1}} \left(g_q(K,v)\rho_L^q(v)\rho_L^{-q}(v)\right)^{\frac{n}{n+q}} d\sigma(v) \\
&\leq \left(\frac{1}{\alpha}\int_{S^{n-1}} \frac{g_q(K,v)}{\rho_L^q(v)} d\sigma(v)\right)^{\frac{n}{n+q}} \\
&\quad \cdot \left(\frac{1}{n}\int_{S^{n-1}} \rho_L^n(v) d\sigma(v)\right)^{\frac{q}{n+q}} \\
&= \mathbf{F}_q(K,L^\circ)^{\frac{n}{n+q}} \cdot |L|^{\frac{q}{n+q}},
\end{aligned}$$

with equality if and only if $\rho_L^{n+q}(v) = g_q(K, v)$ for $v \in S^{n-1}$. It implies that

$$\alpha^{-\frac{n}{n+q}} n^{-\frac{q}{n+q}} \int_{S^{n-1}} g_q(K, v)^{\frac{n}{n+q}} d\sigma(v) \le H_q(K). \tag{46}$$

By (45) and (46), it has

$$H_q(K) = \alpha^{-\frac{n}{n+q}} n^{-\frac{q}{n+q}} \int_{S^{n-1}} g_q(K, v)^{\frac{n}{n+q}} d\sigma(v).$$

□

Motivated by Theorem 4.4, we can consider the compatible functional curvature image $C_qK \in \mathcal{S}_0$ of $K \in \xi_q$ such that

$$g_q(K, v) = \frac{\alpha}{n|C_qK|} \rho_{C_qK}^{n+q}(v) \tag{47}$$

and define

$$\eta_q = \left\{K \in \mathcal{A}_0 : \text{exists } L \in \mathcal{K}_0, \text{ s.t. } g_q(K, v) = \rho_L^{n+q}(v)\right\} \subset \xi_q$$

for $v \in S^{n-1}$ and $q \neq -n$. Then

$$H_q(K) = \mathbf{F}_q(K, (C_qK)^\circ)^{\frac{n}{n+q}} |C_qK|^{\frac{q}{n+q}}. \tag{48}$$

Proposition 4.5. *Let $q \neq -n$ and $K \in \eta_q$, then $G_q(K) = H_q(K)$.*

Proof. Since $K \in \eta_q$, there is $L \in \mathcal{K}_0$ satisfying $g_q(K, v) = \rho_L^{n+q}(v)$ for $v \in S^{n-1}$. By (47), it has

$$\frac{\alpha}{n|C_qK|} \rho_{C_qK}^{n+q}(v) = \rho_L^{n+q}(v) \Rightarrow C_qK = \left(\frac{n|C_qK|}{\alpha}\right)^{\frac{1}{n+q}} L \in \mathcal{K}_0.$$

If $q = 0$, the conclusion is true. If $q > 0$, it has $H_q(K) \ge G_q(K)$ by (48) and $C_qK \in \mathcal{K}_0$. And by Definition 4.2, it implies that $G_q(K) \ge H_q(K)$. Thus $G_q(K) = H_q(K)$. If $-n \neq q < 0$, by Definition 4.2 and (48), it implies that $G_q(K) \le H_q(K) \le G_q(K)$. So the conclusion is true. □

Proposition 4.6. *Let $K \in \mathcal{K}_0$.*
(1) If $-n < t < 0 < r < s$, or $-n < s < 0 < r < t$, then

$$G_r(K) \le G_t(K)^{\frac{(r-s)(n+t)}{(t-s)(n+r)}} G_s(K)^{\frac{(r-t)(n+s)}{(s-t)(n+r)}}.$$

(2) If $-n < t < r < s < 0$, or $-n < s < r < t < 0$, then

$$G_r(K) \le G_t(K)^{\frac{(r-s)(n+t)}{(t-s)(n+r)}} G_s(K)^{\frac{(r-t)(n+s)}{(s-t)(n+r)}}.$$

(3) If $t < r < -n < s < 0$, or $s < r < -n < t < 0$, then

$$G_r(K) \ge G_t(K)^{\frac{(r-s)(n+t)}{(t-s)(n+r)}} G_s(K)^{\frac{(r-t)(n+s)}{(s-t)(n+r)}}.$$

Proof. For $K, L \in \mathcal{K}_0$, $s, r, t \in \mathbb{R}$ such that $0 < \dfrac{t-r}{t-s} < 1$, by Hölder inequality, it has

$$\begin{aligned}
\mathbf{F}_r(K,L) &= \frac{1}{\alpha}\int_{S^{n-1}} h_L^r(v) h_K^{1-r}(v)\, d\mu_{\mathbf{F}}(K,v) \\
&\le \frac{1}{\alpha}\left(\int_{S^{n-1}} h_L^s(v) h_K^{1-s}(v)\, d\mu_{\mathbf{F}}(K,v)\right)^{\frac{r-t}{s-t}} \\
&\quad \cdot \left(\int_{S^{n-1}} h_L^t(v) h_K^{1-t}(v)\, d\mu_{\mathbf{F}}(K,v)\right)^{\frac{r-s}{t-s}} \\
&= \mathbf{F}_s(K,L)^{\frac{r-t}{s-t}}\, \mathbf{F}_t(K,L)^{\frac{r-s}{t-s}}.
\end{aligned} \tag{49}$$

(1) If $-n < t < 0 < r < s$, then $\frac{(r-s)(n+t)}{(t-s)(n+t)} > 0$ and $\frac{(r-t)(n+s)}{(s-t)(n+r)} > 0$. By Definition 4.2 and (49), it has

$$\begin{aligned}
G_r(K) &= \inf_{L\in\mathcal{K}_0}\left\{\mathbf{F}_r(K,L)^{\frac{n}{n+r}}|L^\circ|^{\frac{r}{n+r}}\right\} \\
&\le \inf_{L\in\mathcal{K}_0}\{\left(\mathbf{F}_t(K,L)^{\frac{n}{n+t}}|L^\circ|^{\frac{t}{/}n+t}\right)^{\frac{(r-s)(n+t)}{(t-s)(n+r)}} \\
&\quad \cdot \left(\mathbf{F}_s(K,L)^{\frac{n}{n+s}}|L^\circ|^{\frac{s}{n+s}}\right)^{\frac{(r-t)(n+s)}{(s-t)(n+r)}}\} \\
&\le \sup_{L\in\mathcal{K}_0}\{\mathbf{F}_t(K,L)^{\frac{n}{n+t}}|L^\circ|^{\frac{t}{n+t}}\}^{\frac{(r-s)(n+t)}{(t-s)(n+r)}} \\
&\quad \cdot \inf_{L\in\mathcal{K}_0}\{\mathbf{F}_s(K,L)^{\frac{n}{n+s}}|L^\circ|^{\frac{s}{n+s}}\}^{\frac{(r-t)(n+s)}{(s-t)(n+r)}} \\
&= G_t(K,L)^{\frac{(r-s)(n+t)}{(t-s)(n+t)}}\, G_s(K,L)^{\frac{(r-t)(n+s)}{(s-t)(n+r)}}.
\end{aligned}$$

The case $-n < s < 0 < r < t$ can be proved follow along the lines.

(2) If $-n < t < r < s < 0$, then $\frac{(r-s)(n+t)}{(t-s)(n+r)} > 0$ and $\frac{(r-t)(n+s)}{(s-t)(n+r)} > 0$. By Definition 4.2 and (49), it has

$$\begin{aligned}
G_r(K) &= \sup_{L\in\mathcal{K}_0}\left\{\mathbf{F}_r(K,L)^{\frac{n}{n+r}}|L^\circ|^{\frac{r}{n+r}}\right\} \\
&\le \sup_{L\in\mathcal{K}_0}\{\left(\mathbf{F}_t(K,L)^{\frac{r-s}{t-s}}\mathbf{F}_s(K,L)^{\frac{r-t}{s-t}}\right)^{\frac{n}{n+r}}|L^\circ|^{\frac{r}{n+r}}\} \\
&\le \sup_{L\in\mathcal{K}_0}\{\mathbf{F}_t(K,L)^{\frac{n}{n+t}}|L^\circ|^{\frac{t}{n+t}}\}^{\frac{(r-s)(n+t)}{(t-s)(n+r)}} \\
&\quad \cdot \sup_{L\in\mathcal{K}_0}\{\mathbf{F}_s(K,L)^{\frac{n}{n+s}}|L^\circ|^{\frac{s}{n+s}}\}^{\frac{(r-t)(n+s)}{(s-t)(n+r)}} \\
&= G_t(K)^{\frac{(r-s)(n+t)}{(t-s)(n+r)}}\, G_s(K)^{\frac{(r-t)(n+s)}{(s-t)(n+r)}}.
\end{aligned}$$

By transposing s and t, the case $-n < s < r < t < 0$ can be proved.

(3) If $t < r < -n < s < 0$, then $\frac{(r-s)(n+t)}{(t-s)(n+r)} > 0$ and $\frac{(r-t)(n+s)}{(s-t)(n+r)} < 0$. By Definition 4.2 and (49), it has

$$\begin{aligned}
G_r(K) &= \sup_{L\in\mathcal{K}_0}\left\{\mathbf{F}_r(K,L)^{\frac{n}{n+r}}|L^\circ|^{\frac{r}{n+r}}\right\} \\
&\geq \sup_{L\in\mathcal{K}_0}\{\left(\mathbf{F}_t(K,L)^{\frac{r-s}{t-s}}\mathbf{F}_s(K,L)^{\frac{r-t}{s-t}}\right)^{\frac{n}{n+r}}|L^\circ|^{\frac{r}{n+r}}\} \\
&\geq \sup_{L\in\mathcal{K}_0}\{\mathbf{F}_t(K,L)^{\frac{n}{n+t}}|L^\circ|^{\frac{t}{n+t}}\}^{\frac{(r-s)(n+t)}{(t-s)(n+r)}} \\
&\quad \cdot \sup_{L\in\mathcal{K}_0}\{\mathbf{F}_s(K,L)^{\frac{n}{n+s}}|L^\circ|^{\frac{s}{n+s}}\}^{\frac{(r-t)(n+s)}{(s-t)(n+r)}} \\
&= G_t(K)^{\frac{(r-s)(n+t)}{(t-s)(n+r)}}G_s(K)^{\frac{(r-t)(n+s)}{(s-t)(n+r)}}.
\end{aligned}$$

By transposing s and t, the case $s < r < -n < t < 0$ can be proved. □

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