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Well-posedness and analyticity for quasi-geostrophic equation in the Besov-Morrey spaces characterized by semi-group

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Abstract. In this work, we show the existence and uniqueness, the analyticity of the solutions and the decay estimates of the solutions of the quasi-geostrophic equation (QG) in the Besov-Morrey spaces charterized by the semigroup $\mathrm{T}_\alpha := e^{-t(-\Delta)^\alpha}$, noted by $\mathrm{N}^s_{p,\lambda}$. If we assume that the initial data θ_0 are small and belong to the Besov-Morrey critical spaces, we obtain the global well-posedness results of the QG equation.

1. Introduction

In this paper, we are concerned with the two-dimensional dissipative quasi-geostrophic (QG) equation:

$$\begin{cases} \partial_t\theta + V_\theta\cdot\nabla\theta + \mu\Lambda^{2\alpha}\theta = 0, \quad x\in\mathbb{R}^2, t>0, \\ V_\theta = \left(-\mathcal{R}_2\theta, \mathcal{R}_1\theta\right), \\ \theta(0,x) = \theta_0(x). \end{cases} \tag{1}$$

Where $\mu > 0$ is the dissipative coefficient, $\alpha \geq \frac{1}{2}$ is a real number, $\theta(t,x)$ is a real-valued function of two space variables t and x. The function θ represents the potential temperature. The velocity V_θ is incompressible and determined from θ by a stream function g,

$$V_\theta = \left(-\frac{\partial g}{\partial x_2}, \frac{\partial g}{\partial x_1}\right), \tag{2}$$

where the function g is satisfies

$$\Lambda g = -\theta. \tag{3}$$

We define the operator Λ by the fractional power of $-\Delta$:

$$\Lambda v = (-\Delta)^{1/2}v, \quad \mathcal{F}(\Lambda v) = \mathcal{F}((-\Delta)^{1/2}v) = |\xi|\mathcal{F}(v),$$

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and more generally

$$\mathcal{F}(\Lambda^{2\alpha}v) = \mathcal{F}((-\Delta)^{\alpha}v) = |\xi|^{2\alpha}\mathcal{F}(v),$$

where $\mathcal{F}$ is the Fourier transform. The relation between (2) and (3) can be determined by using the Riesz transform as follows

$$\begin{aligned} V_\theta &= \left(\partial_{x_2}\Lambda^{-1}\theta, -\partial_{x_1}\Lambda^{-1}\theta\right) \\ &= \left(-\mathcal{R}_2\theta, \mathcal{R}_1\theta\right), \end{aligned}$$

where $\mathcal{R}_j$, $j = 1, 2$, is the Riesz transform defined by

$$\mathcal{R}_j = \partial_{x_j}(-\Delta)^{-1/2}.$$

The QG equation is a partial differential equation that describes the evolution of large-scale atmospheric and oceanic flows. With important assumptions that allow for a more tractable mathematical formulation, the QG equation is a simplified version of the Navier-Stokes equations, which govern fluid motion in general.

Our study is driven by two primary motivations. The first motivation stems from the examination of actual geophysical flows in the atmosphere, specifically focusing on the phenomenon of frontogenesis. Frontogenesis refers to the formation of strong boundaries between hot and cold air masses, and we aim to study this process within the framework of quasigeostrophic approximations, without explicitly considering ageostrophic effects [19]. Frontogenesis is a term commonly used by atmospheric scientists to describe the development of a discontinuous temperature front within a finite time period. The second motivation for our study arises from the remarkable physical and mathematical similarities observed between the behavior of strongly nonlinear solutions to these equations in two dimensions and the behavior of potentially singular solutions to the Euler equations governing three-dimensional, incompressible flow [18]. This connection presents an intriguing unresolved problem within the theoretical turbulence community, and we seek to explore and understand this analogy further (for more detail see [3, 14, 19]).

The intriguing aspect lies in comprehending the case of small values, or more precisely, when the dissipation approaches zero, the quasi-geostrophic equation converges to the three dimensional Navier-Stokes equations. Although artificial, this model is physically interesting in the case $\alpha = \frac{1}{2}$. Mathematically, the power $\alpha = \frac{1}{2}$ corresponds to the index for which the nonlinear term and the dissipation are of the same order (for more detail see [4]) . The case $\alpha > \frac{1}{2}$ is called sub-critical, and the case $\alpha = \frac{1}{2}$ is critical, while the case $0 \leq \alpha < \frac{1}{2}$ is super-critical, respectively.

There exists an extensive body of literature on the global-in-time well-posedness of fluid dynamics partial differential equations in various spaces. The conditions for smallness are typically imposed in critical spaces, which are invariant under the scaling associated with the model. Notably, for models such as Navier-Stokes equations, well-posedness outcomes have been demonstrated in critical spaces, including Lebesgue space L^p [8, 11], Marcinkiewicz space $L^{p,\infty}$ [15], Morrey spaces $\mathcal{M}_{p,\mu}$ [12], Besov spaces $\mathcal{B}^{2-2\alpha}_{2,1}$ [13, 17], Fourier-Besov spaces $\mathcal{F}\mathcal{B}^{s}_{p,q}$ [5, 10, 14], Fourier- Besov-Morrey spaces $\mathcal{F}\mathcal{N}^{s}_{p,\lambda q}$ [1, 6, 9].

In order to solve the equation (1), we consider the following equivalent integral equation coming from Duhamel's principle

$$\theta(t) = \mathrm{T}_\alpha(t)\theta_0 + \mathbb{B}(\theta, \theta)(t), \tag{4}$$

The operator $\mathrm{T}_\alpha := e^{-t(-\Delta)^\alpha}$ is defined as the fractional heat semigroup operator. It can be interpreted as a convolution operator using the kernel $k_t(x) = \mathcal{F}^{-1}(e^{-t|\xi|^{2\alpha}})$, and

$$\mathbb{B}(\theta, \xi)(t) = \int_0^t \mathrm{T}_\alpha(t-\varepsilon)\left(V_\theta \cdot \nabla\xi)\right)(\varepsilon)d\varepsilon. \tag{5}$$

By utilizing the equivalent integral equation (4) and applying the contraction principle, we can demonstrate the existence of global solutions for (1) with dissipation, provided that the initial data is sufficiently small and belongs to the space $\mathrm{N}^s_{p,\lambda}$. Then we show the analyticity of this solution by Gevrey estimates.

Allow us to present the function space $\mathrm{N}^s_{p,\lambda}$ of the Besov-Morrey type, which is defined by the linear semigroup T_α.

Definition 1.1. *Let $\alpha > 0$, $1 \le p \le \infty$, $s > 0$ and $0 \le \lambda < 2$, the function space $\mathrm{N}^s_{p,\lambda}(\mathbb{R}^2)$ is defined as follows:*

$$\mathrm{N}^s_{p,\lambda}(\mathbb{R}^2) = \{\varphi \in \mathcal{S}', \|\varphi\|_{\mathrm{N}^s_{p,\lambda}} < \infty\},$$

$$\|\varphi\|_{\mathrm{N}^s_{p,\lambda}} = \sup_{t>0} t^s \|T_\alpha(t)\varphi\|_{\mathcal{M}^\lambda_p}.$$

In Theorem 1.2, we show the well-posedness in the scaling critical spaces $\mathrm{N}^s_{p,\lambda}(\mathbb{R}^2)$ with $\frac{1}{2} \le \alpha < \min(1, \frac{1}{2} + \frac{2-\lambda}{p})$. The following is our result in subcritical cases.

Theorem 1.2. *Let α, λ, p satisfy $p > 0, 0 \le \lambda < 2$ and $\frac{1}{2} \le \alpha < min(1, \frac{1}{2} + \frac{2-\lambda}{p})$.*
Then, there exists a positive constant $\nu = \nu(\alpha, \lambda, p)$, such that for the initial velocity $\theta_0 \in \mathrm{N}^s_{p,\lambda}(\mathbb{R}^2)$ satisfying div $\theta_0 = 0$ and

$$\|\theta_0\|_{\mathrm{N}^s_{p,\lambda}} \le \nu,$$

the Eq (1) has a unique global solution θ satisfying

$$\sup_{t>0} t^{-s} \|\theta(t)\|_{\mathcal{M}^\lambda_p} \le 2\|\theta_0\|_{\mathrm{N}^s_{p,\lambda}}.$$

Where $s = \frac{1}{2\alpha}(1 - 2\alpha + \frac{2-\lambda}{p})$.

The following Theorem provides the result regarding analyticity.

Theorem 1.3. *Under the conditions of Theorem 1.2, there exists a positive constant $\beta = \beta(\alpha, \lambda, p)$, such that for the initial velocity $\theta_0 \in \mathrm{N}^s_{p,\lambda}(\mathbb{R}^2)$ satisfying*

$$\|\theta_0\|_{\mathrm{N}^s_{p,\lambda}} \le \beta,$$

the Eq (1) has a unique analytic solution $e^{\sqrt{t}|\Lambda|^\alpha}\theta \in \mathrm{N}^s_{p,\lambda}$ such that

$$\|e^{\sqrt{t}|\Lambda|^\alpha}\theta\|_{\mathrm{N}^s_{p,\lambda}} \le C\|\theta_0\|_{\mathrm{N}^s_{p,\lambda}},$$

where C is a positive constant.

Furthermore, In Theorem 1.3, we proved the analyticity of the solutions, which allows us to obtain an estimate of the decay in time of the solutions.

Theorem 1.4. *Under the conditions of Theorem 1.2, for any $\gamma > 0$, the global solution $\theta \in \mathrm{N}^s_{p,\lambda}$ and $e^{\sqrt{t}|\Lambda|^\alpha}\theta \in \mathrm{N}^s_{p,\lambda}$ satysfing the decay in time estimate*

$$\|(-\Delta)^\gamma\theta\|_{\mathrm{N}^s_{p,\lambda}} \le Ct^{\frac{\gamma}{\alpha}}\|\theta_0\|_{\mathrm{N}^s_{p,\lambda}},$$

where C is a constant depend α and γ.

The structure of this paper is as follows: Section 2 revisits the Morrey space definition and outlines some of its properties that will be applied in the sequel. Our result on global solutions are presented in Section 3, and Section 4 contains the proof of the solutions analyticity and the decay in time estimate.

2. PRELIMINARIES

In this section, we recall the definition of the Morrey space and some of their properties which will be used throughout the paper, and then we give some lemmas concerning the Gevrey estimate.

We start by recalling the functional spaces $\mathcal{M}_q^\lambda$.

Definition 2.1. *Let $1 \le q \le \infty$ and $0 \le \lambda < d$. The homogeneous Morrey space $\mathcal{M}_q^\lambda$ are defined by*

$$\mathcal{M}_q^\lambda(\mathbb{R}^d) := \{\varphi \in L^1_{loc}(\mathbb{R}^d), \|\varphi\|_{\mathcal{M}_q^\lambda} < \infty\}$$

with

$$\|\varphi\|_{\mathcal{M}_q^\lambda} := \sup_{x_0 \in \mathbb{R}^d} \sup_{R>0} R^{-\frac{\lambda}{q}} \|\varphi\|_{L^p(B(x_0,R))}, \tag{6}$$

where $B(y,R)$ is the open ball in $\mathbb{R}^d$ centered at y and with radius $R > 0$.

The space $\mathcal{M}_q^\lambda$ endowed with the norm $\|\varphi\|_{\mathcal{M}_q^\lambda}$ is a Banach space and has the following scaling property

$$\|\varphi(\beta x)\|_{\mathcal{M}_q^\lambda} = \beta^{-\frac{n-\lambda}{q}} \|\varphi(x)\|_{\mathcal{M}_q^\lambda} \qquad for \qquad \mu > 0.$$

In the case of $p = 1$, the norm $|\cdot|_{L^1}$ in equation (6) corresponds to the total variation of the measure φ on the ball $B(y,R)$, and the space $\mathcal{M}_p^\lambda$ is regarded as a subset of Radon measures. When $\lambda = 0$, $\mathcal{M}_q^\lambda$ is equal to L^q.

Lemma 2.2. *[16] (**Hölder's inequality**)*
Let $0 \le \lambda_1,\ \lambda_2,\ \lambda_3 < n$ and $1 \le r_1, r_2, r_3 < \infty$, such that $\frac{1}{r_3} = \frac{1}{r_1} + \frac{1}{r_2}$ and $\frac{\lambda_3}{r_3} = \frac{\lambda_1}{r_1} + \frac{\lambda_2}{r_2}$, then we have

$$\|fg\|_{\mathcal{M}_{r_3}^{\lambda_3}} \le \|f\|_{\mathcal{M}_{r_1}^{\lambda_1}} \|g\|_{\mathcal{M}_{r_2}^{\lambda_2}}. \tag{7}$$

Lemma 2.3. *[16] Let $1 \le r_1 \le r_2 \le \infty$ and $\frac{d-\lambda_1}{r_1} \le \frac{d-\lambda_2}{r_2}$, then*

$$\mathcal{M}_{r_2}^{\lambda_2}(\mathbb{R}^d) \hookrightarrow \mathcal{M}_{r_1}^{\lambda_1}(\mathbb{R}^d).$$

By combining [22, Lemma 3] with [21, Lemma 2.3], we can derive the subsequent lemma

Lemma 2.4. *Let $1 \le p_1 \le p_2 \le \infty$, $0 \le \lambda < d$ and $\beta = (\beta_1, \beta_2) \in (\mathbb{N} \cup \{0\})^2$.*
If $f \in \mathcal{S}'(\mathbb{R}^d)$, then there exists a constant C depending only on n such that

$$\|e^{-t(-\Delta)^\alpha} f\|_{\mathcal{M}_{p_2}^\lambda} \le C t^{-\frac{1}{2\alpha}(\frac{d-\lambda}{p_1} - \frac{d-\lambda}{p_2})} \|f\|_{\mathcal{M}_{p_1}^\lambda}.$$

$$\|\partial^\beta e^{-t(-\Delta)^\alpha} f\|_{\mathcal{M}_{p_2}^\lambda} \le C t^{-\frac{|\beta|}{2\alpha} - \frac{1}{2\alpha}(\frac{d-\lambda}{p_1} - \frac{d-\lambda}{p_2})} \|f\|_{\mathcal{M}_{p_1}^\lambda}.$$

Finally, we will use the following three lemmas to obtain the Gevrey estimates.

Lemma 2.5. *[23] If the operator $O = e^{-[\sqrt{\varepsilon_1 - \varepsilon_2} + \sqrt{\varepsilon_2} - \sqrt{\varepsilon_1}]|\Lambda|^\alpha}$ for $0 \le \varepsilon_2 \le \varepsilon_1$, then O is either the identity operator or an $L^1(\mathbb{R}^d)$ kernel whose $L^1(\mathbb{R}^d)$ norm is bounded independent of $\varepsilon_2, \varepsilon_1$.*

Lemma 2.6. *[2] The operator $O = e^{\frac{1}{2}b(-\Delta)^\alpha + \sqrt{b}|\Lambda|^\alpha}$ is a Fourier multiplier which maps boundedly $\mathcal{M}_p^\lambda\left(\mathbb{R}^d\right) \to \mathcal{M}_p^\lambda\left(\mathbb{R}^d\right)$, $1 < p < \infty$, and its operator norm is uniformly bounded with respect to $b \ge 0$.*

At the end of this section, we will introduce a bounded estimate that involves the bilinear operator $L_t(\Phi_1, \Phi_2)$ in the following form:

$$\begin{aligned} L_t(\Phi_1, \Phi_2) : &= e^{\sqrt{t}|\Lambda|^\alpha}\left(e^{-\sqrt{t}|\Lambda|^\alpha}\Phi_1 e^{-\sqrt{t}|\Lambda|^\alpha}\Phi_2\right) \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{ix\zeta} e^{\sqrt{t}(|\zeta|_1 - |\zeta-\nu|_1 - |\nu|_1)} \widehat{\Phi}_1(\zeta - \nu)\widehat{\Phi}_2(\eta) d\nu d\zeta. \end{aligned}$$

Lemma 2.7. *[21] Let* $\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p}$ *and* $1<p<\infty$*. Then we have*

$$\|L_t(\Phi_1,\Phi_2)\|_{\mathcal{M}^{\lambda}_{p}(\mathbb{R}^d)} \leq C\|\Phi_1\|_{\mathcal{M}^{\lambda}_{p_1}(\mathbb{R}^d)}\|\Phi_2\|_{\mathcal{M}^{\lambda}_{p_2}(\mathbb{R}^d)}.$$

Where C is a positive constant independent of Φ_1 *and* Φ_2.

3. WELL-POSEDNESS.

In this section we give a proof of Theorem 1.2. To do that we will start by giving the bilinear estimate defined in (4). We denote

$$Z := \{f \in \mathcal{S}', \|f\|_Z < \infty\},$$

with

$$\|f\|_Z = \quad sup \quad t^{-\frac{s}{2\alpha}}\|f\|_{\mathcal{M}^{\lambda}_{p}},$$

where $s=1-2\alpha+\frac{2-\lambda}{p}$. Then, we obtain the bilinear estimate for $\mathbb{B}(\theta,\xi)$ in Z.

Lemma 3.1. *Let* $\alpha,\lambda,\ p$ *satisfy* $0\leq\lambda<2$ *and* $\frac{1}{2}\leq\alpha<min(1,\frac{1}{2}+\frac{2-\lambda}{p})$.

Then, there exists a positive constant C_1 *such that*

$$\|\mathbb{B}(\theta,\xi)\|_{\mathrm{N}^{s}_{p,\lambda}} \leq C_1\|\theta\|_Z\|\xi\|_Z, \tag{8}$$

for all $\theta,\xi\in Z$.

Proof : Applying Lemma 2.4 and Hölder's inequality we get

$$\begin{aligned}\Big\|\int_0^t \mathrm{T}_\alpha(t-\varepsilon)\left(V_\theta\cdot\nabla\xi\right)\right)(\varepsilon)d\varepsilon\Big\|_{\mathcal{M}^{\lambda}_{p}} &\leq \int_0^t \Big\|\mathrm{T}_\alpha(t-\varepsilon)\left(V_\theta\cdot\nabla\xi\right)\right)(\varepsilon)\Big\|_{\mathcal{M}^{\lambda}_{p}}d\varepsilon\\ &\leq C\int_0^t (t-\varepsilon)^{-\frac{1}{2\alpha}-\frac{1}{2\alpha}(\frac{4-2\lambda}{p}-\frac{2-\lambda}{p})}\Big\|\theta\cdot\xi\Big\|_{\mathcal{M}^{\lambda}_{\frac{p}{2}}}d\varepsilon\\ &\leq C\int_0^t (t-\varepsilon)^{-\frac{1}{2\alpha}(1+\frac{2-\lambda}{p})}\Big\|\theta\Big\|_{\mathcal{M}^{\lambda}_{p}}\Big\|\xi\Big\|_{\mathcal{M}^{\lambda}_{p}}d\varepsilon\\ &\leq C\int_0^t (t-\varepsilon)^{-\frac{1}{2\alpha}(1+\frac{2-\lambda}{p})}t^{\frac{s}{\alpha}}\|\theta\|_Z\|\xi\|_Z d\varepsilon.\end{aligned}$$

Multiplying $t^{-\frac{s}{2\alpha}}$ on the both sides of the above two inequalities, we get

$$t^{-\frac{s}{2\alpha}}\Big\|\int_0^t \mathrm{T}_\alpha(t-\varepsilon)\left(V_\theta\cdot\nabla\xi\right)\right)(\varepsilon)d\varepsilon\Big\|_{\mathcal{M}^{\lambda}_{p}} \leq Ct^{\frac{s}{2\alpha}}\|\theta\|_Z\|\xi\|_Z\int_0^t (t-\varepsilon)^{-\frac{1}{2\alpha}(1+\frac{3-\lambda}{p})}d\varepsilon,$$

where $s=1-2\alpha+\frac{2-\lambda}{p}$, then

$$\|\mathbb{B}(\theta,\xi)\|_{\mathrm{N}^{s}_{p,\lambda}} \leq C_1\|\theta\|_Z\|\xi\|_Z.$$

Since C_1 is independent of t, we have now completed the proof of Lemma 3.1. □

Proof of Theorem 1.2: It is simple to verify that the indices α,λ and p provided in the Theorem 1.2 satisfy the assumptions of Lemma 3.1 and Lemma 2.4. Suppose $\theta_0\in\mathrm{N}^{s}_{p,\lambda}(\mathbb{R}^2)$ with divergence free.
By the definitions of $\|\cdot\|_Z$, we see that $\|T_\alpha(\cdot)\theta_0\|_Z=\|\theta_0\|_{\mathrm{N}^{s}_{p,\lambda}}$. Then, let us introduce the map Φ and the complete metric space $(\mathbf{Y},d)$, defined as follows:

$$\mathbf{Y} := \left\{\theta\in\mathrm{N}^{s}_{p,\lambda}(\mathbb{R}^2)^2, \|\theta\|_Z\leq 2\|\theta_0\|_{\mathrm{N}^{s}_{p,\lambda}}\right\},$$

$$d(\theta,\xi) := \|\theta-\xi\|_Z,$$
$$\Theta(\theta)(t) := T_\alpha(t)\theta_0 - \mathbb{B}(\theta,\theta)(t), \quad t\in\mathbb{R}^+,$$

where $\mathbb{B}(\cdot,\cdot)(t)$ is defined in (5). Applying the inequality (8), then for all $\theta\in\mathbf{Y}$, There is a positive constant C_2 such that

$$\begin{aligned}\|\Theta(\theta)\|_Z &\le \|\theta_0\|_{N^s_{p,\lambda}} + C_2\|\theta\|_Z\|\theta\|_Z\\ &\le \|\theta_0\|_{N^s_{p,\lambda}}\{1+4C_2\|\theta_0\|_{N^s_{p,\lambda}}\}.\end{aligned} \tag{9}$$

Furthermore, utilizing Lemma 3.1, we can find a positive constant C_3 such that for any $\theta,\xi\in\mathbf{Y}$, we have:

$$\begin{aligned}\|\Theta(\theta)-\Theta(\xi)\|_Z &= \|\mathbb{B}(\theta,\theta-\xi)+\mathbb{B}(\theta-\xi,\xi)\|_Z\\ &\le C_3(\|\theta\|_Z+\|\xi\|_Z)\|\theta-\xi\|_Z\\ &\le 4C_3\|\theta_0\|_{N^s_{p,\lambda}}\|\theta-\xi\|_Z.\end{aligned} \tag{10}$$

Now, let us assume that initial velocity $\theta_0\in N^s_{p,\lambda}(\mathbb{R}^2)$ satisfies

$$\|\theta_0\|_{N^s_{p,\lambda}} \le \min\{\frac{1}{8C_3},\frac{1}{4C_2}\},$$

we obtain from (9) and (10) that

$$\|\Theta(\theta)\|_Z \le 2\|\theta_0\|_{N^s_{p,\lambda}},$$
$$\|\Theta(\theta)-\Theta(\xi)\|_Z \le \frac{1}{2}\|\theta-\xi\|_Z.$$

for all $\theta,\xi\in\mathbf{Y}$.

Thus, applying the contraction mapping principle, we can conclude that there exists a unique global solution $\theta\in\mathbf{Y}$ that satisfies (4) for all $t>0$. This completes the proof of Theorem 1.2. □

4. ANALYTICITY AND DECAY IN TIME ESTIMATE

This section is dedicated to proving the Gevrey class regularity and the decay in time estimate for the 2D quasi-geostrophic equations in the Besov-Morrey spaces charterized by semigroup.

- **THE ANALYTICITY**

  Inspired by [20, 21], we have the following specific results.

Lemma 4.1. *Let α,λ,p satisfy $0\le\lambda<2$ and $\frac{1}{2}\le\alpha<min(1,\frac{1}{2}+\frac{2-\lambda}{p})$. Then,*
(i) There is a positive constant C_4 such that

$$\|e^{\sqrt{t}|\Lambda|^\alpha}T_\alpha(t)\theta_0\|_{N^s_{p,\lambda}} \le C_4\|\theta_0\|_{N^s_{p,\lambda}}.$$

(ii) There is a positive constant C_5 such that

$$\|\overline{\mathbb{B}}(\overline{\theta},\overline{g})(t)\|_{N^s_{p,\lambda}} \le C_5\|\overline{\theta}\|_{N^s_{p,\lambda}}\|\overline{g}\|_{N^s_{p,\lambda}},$$

where

$$\overline{\mathbb{B}}(\overline{\theta},\overline{g})(t) = e^{\sqrt{t}|\Lambda|^\alpha}\int_0^t T_\alpha(t-\varepsilon)\left(e^{-\sqrt{\varepsilon}|\Lambda|^\alpha}V_\theta\cdot\nabla e^{-\sqrt{\varepsilon}|\Lambda|^\alpha}g)\right)(\varepsilon)d\varepsilon.$$

Proof: Applying Lemma 2.4 and Lemma 2.6, we get

$$\begin{aligned}\left\|e^{\sqrt{t}|\Lambda|^{\alpha}}T_{\alpha}(t)\theta_0\right\|_{\mathcal{M}_q^{\lambda}} &= \left\|e^{\sqrt{t}|\Lambda|^{\alpha}}e^{-t(-\Delta)^{\alpha}}\theta_0\right\|_{\mathcal{M}_q^{\lambda}}\\ &= \left\|e^{\sqrt{t}|\Lambda|^{\alpha}}e^{\frac{1}{2}t(-\Delta)^{\alpha}}e^{-\frac{3}{2}t(-\Delta)^{\alpha}}\theta_0\right\|_{\mathcal{M}_q^{\lambda}}\\ &\leq \left\|e^{-t(-\Delta)^{\alpha}}\theta_0\right\|_{\mathcal{M}_q^{\lambda}}\\ &\leq C\left\|\theta_0\right\|_{\mathcal{M}_q^{\lambda}}.\end{aligned}$$

Then

$$\|e^{\sqrt{t}|\Lambda|^{\alpha}}T_{\alpha}(t)\theta_0\|_{\mathrm{N}^s_{p,\lambda}} \leq C\|\theta_0\|_{\mathrm{N}^s_{p,\lambda}}.$$

And on the other hand applying Lemma 2.5, Lemma 2.6 and Lemma 2.7, we get

$$\begin{aligned}&\left\|\overline{\mathbb{B}}(\overline{\theta},\overline{g})(t)\right\|_{\mathcal{M}_q^{\lambda}}\\ &= \left\|e^{\sqrt{t}|\Lambda|^{\alpha}}\int_0^t \mathrm{T}_{\alpha}(t-\varepsilon)\left(e^{-\sqrt{\varepsilon}|\Lambda|^{\alpha}}V_{\theta}\cdot\nabla e^{-\sqrt{\varepsilon}|\Lambda|^{\alpha}}g)\right)(\varepsilon)d\varepsilon\right\|_{\mathcal{M}_q^{\lambda}}\\ &= \left\|e^{\sqrt{t}|\Lambda|^{\alpha}}\int_0^t \mathrm{T}_{\alpha}(t-\varepsilon)e^{-\sqrt{\varepsilon}|\Lambda|^{\alpha}}e^{\sqrt{\varepsilon}|\Lambda|^{\alpha}}\left(e^{-\sqrt{\varepsilon}|\Lambda|^{\alpha}}V_{\theta}\cdot\nabla e^{-\sqrt{\varepsilon}|\Lambda|^{\alpha}}g)\right)(\varepsilon)d\varepsilon\right\|_{\mathcal{M}_q^{\lambda}}\\ &= \left\|\int_0^t e^{\sqrt{t}|\Lambda|^{\alpha}}e^{-(t-\varepsilon)(-\Delta)^{\alpha}}e^{-\sqrt{\varepsilon}|\Lambda|^{\alpha}}\nabla L_t(V_{\theta},g)(\varepsilon)d\varepsilon\right\|_{\mathcal{M}_q^{\lambda}}\\ &= \left\|\int_0^t e^{-(\sqrt{t-\varepsilon}+\sqrt{\varepsilon}-\sqrt{t})|\Lambda|^{\alpha}}e^{\sqrt{t-\varepsilon}|\Lambda|^{\alpha}}e^{-(t-\varepsilon)(-\Delta)^{\alpha}}\nabla L_t(V_{\theta},g)(\varepsilon)d\varepsilon\right\|_{\mathcal{M}_q^{\lambda}}\\ &= \left\|\int_0^t e^{-(\sqrt{t-\varepsilon}+\sqrt{\varepsilon}-\sqrt{t})|\Lambda|^{\alpha}}e^{\sqrt{t-\varepsilon}|\Lambda|^{\alpha}+\frac{1}{2}(t-\varepsilon)(-\Delta)^{\alpha}}e^{-\frac{3}{2}(t-\varepsilon)(-\Delta)^{\alpha}}\nabla L_t(V_{\theta},g)(\varepsilon)d\varepsilon\right\|_{\mathcal{M}_q^{\lambda}}\\ &\leq C\left\|\int_0^t e^{-\frac{3}{2}(t-\varepsilon)(-\Delta)^{\alpha}}\nabla L_t(V_{\theta},g)(\varepsilon)d\varepsilon\right\|_{\mathcal{M}_q^{\lambda}}.\end{aligned}$$

Applying the Lemma 2.7 and following the same argument presented in the proof of Theorem 1.2, we can readily establish the desired result.

- **DECAY IN TIME ESTIMATE**

To show Theorem 1.4, we need the following lemma.

Lemma 4.2. *[7] The Fourier multipliers associated with the symbols $A(\xi) = |\xi|^{\beta}e^{-\sqrt{t}|\xi|^{\alpha}}$ are obtained by convolving with their respective kernels $\mathbb{K}$. These kernels are functions in L^1 with $\|\widehat{\mathbb{K}}\|_{L^1} \leq Ct^{-\frac{\beta}{2\alpha}}$.*

Proof of Theorem 1.4: Using the Lemma 4.2, Lemma 2.6 and Theorem 1.3, we obtain

$$\begin{aligned}\left\|(-\Delta)^{\gamma}e^{-t(-\Delta)^{\alpha}}\theta(t)\right\|_{\mathcal{M}_q^{\lambda}} &= \left\|(-\Delta)^{\gamma}e^{-\sqrt{t}|\Lambda|^{\alpha}}e^{\sqrt{t}|\Lambda|^{\alpha}}e^{-t(-\Delta)^{\alpha}}\theta(t)\right\|_{\mathcal{M}_q^{\lambda}}\\ &\leq Ct^{-\frac{\gamma}{\alpha}}\left\|e^{\sqrt{t}|\Lambda|^{\alpha}-t(-\Delta)^{\alpha}}\theta(t)\right\|_{\mathcal{M}_q^{\lambda}}\\ &\leq Ct^{-\frac{\gamma}{\alpha}}\left\|e^{\frac{1}{2}t(-\Delta)^{\alpha}+\sqrt{t}|\Lambda|^{\alpha}e^{-\frac{3}{2}t(-\Delta)^{\alpha}}}\theta(t)\right\|_{\mathcal{M}_q^{\lambda}}\\ &\leq Ct^{-\frac{\gamma}{\alpha}}\left\|e^{-\frac{3}{2}t(-\Delta)^{\alpha}}\theta(t)\right\|_{\mathcal{M}_q^{\lambda}}.\end{aligned}$$

Multiplying $t^{-\frac{s}{2\alpha}}$ on the both sides of the above two inequalities, we get

$$\left\|(-\Delta)^{\gamma}\theta(t)\right\|_{\mathrm{N}^s_{p,\lambda}} \leq Ct^{-\frac{\gamma}{\alpha}}\left\|\theta(t)\right\|_{\mathrm{N}^s_{p,\lambda}}.$$

Then, by Theorem 1.2, we have

$$\|(-\Delta)^{\gamma}\theta(t)\|_{\mathcal{N}^{s}_{p,\lambda}} \le Ct^{\frac{\gamma}{\alpha}}\|\theta_0\|_{\mathcal{N}^{s}_{p,\lambda}}.$$

This finishes the proof of Theorem 1.4 .

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