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Weak solution for obstacle problem with variable growth and weak monotonicity in Sobolev spaces with variable exponent

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Abstract. In this article, we investigate the presence of weak solutions for obstacle problems $\int_\Omega \mathcal{A}(z,u,Du) : D(v-u) + \phi(u) : D(v-u)\,\mathrm{d}z \geq 0$, for v belonging to the following convex set $\mathcal{K}_{\psi,\theta}$, applying the Young measure theory and a theorem by Kinderlehrer and Stampacchia, the desired outcome is achieved.

1. Introduction

We are interested in the study of existence and uniqueness of weak solutions for obstacle problems:

$$\begin{cases} \displaystyle\int_\Omega \mathcal{A}(z,u,Du) : D(v-u) + \phi(u) : D(v-u)\,\mathrm{d}z \quad \geq 0 \\ v \in \mathcal{K}_{\psi,\theta} \end{cases} \tag{1}$$

where

$$\mathcal{K}_{\psi,\theta} = \Big\{ v \in W^{1,p}(\Omega;\mathbb{R}^m) : \; v-\theta \in W_0^{1,p}(\Omega;\mathbb{R}^m),\; v \geq \psi \text{ a.e. in } \Omega \Big\}. \tag{2}$$

Here, Ω is a bounded open domain in $\mathbb{R}^n (n \geq 2)$ and $u : \Omega \to \mathbb{R}^m$ is a vector-valued function.
Research into obstacle problems dates back to the 1960s, when G. Stampacchia [38] and G. Fichera [22] made pioneering discoveries. It was determined that solutions to the obstacle problem cannot be of class C^2, regardless of the regularity of the obstacle, prompting the development of the concept of weak solutions and the theory of variational inequalities through the work of J.L. Lions and G. Stampacchia [31]. Functional analysis methods are currently used to solve these issues, and the goal is to find conditions in which weak solutions can become classical ones (see [16]). For further information, please refer to the monographs [1, 2, 12, 17, 21, 26, 29, 36, 39–41]. Junxia and Yuming [28] studied the boundary regularity of weak solutions to a nonlinear obstacle problem with a $C^{1,\beta}$-obstacle function and found a $C^{1,\alpha}_{\mathrm{loc}}$ boundary

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regularity. Jacques-Louis Lions [32] studied the presence of solutions to parabolic obstacle problems via variational inequalities. In [37], the author examined obstacle problems with measure data related to p-Laplace type elliptic equations and checked the relationships between the solutions' low order regularity characteristics and the nonlinear potential of the data. H. El Hammar et al. in [24, 25, 27] verified the existence of a weak solution to the quasilinear elliptic system under regularity, growth and coercivity conditions for $\mathcal{A}$ by utilizing Galerkin's approximation and the theory of Young measures. Many papers have been devoted to the study of the existence and uniqueness of weak solutions for the obstacle problem (1) using classical monotone methods developed by [3, 4, 43]. In [23], the author studied the scalar version of problem (1) and showed the existence of a weak solution with variable growth. For further works on related topics, see [15, 20]. The use of Young measures in elliptic systems is discussed in [6, 24, 25].
E. Azroul and F. Balaadich in [10], the following quasilinear elliptic system was considered:

$$\begin{cases} -\operatorname{div}(\sigma(x, Du) + \phi(u)) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where f belongs to the dual space $W^{-1,p'}(\Omega;\mathbb{R}^m)$ of $W_0^{1,p}(\Omega;\mathbb{R}^m)$, the authors proved the existence of weak solutions under weak monotonicity assumptions on the stress tensor $\sigma : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ and by the theory of Young measures.
By taking into consideration the works of [10], this paper proves the existence and uniqueness of weak solutions for obstacle problems (1). The result is extended by incorporating a general source term with constant growth and weak monotonicity, through the concept of Young measure and the Kinderlehrer and Stampacchia theorem.
We denote by $\mathbb{M}^{m\times n}$ the set of real m by n matrices equipped with the usual inner product $S : G = \sum_{i,j} S_{ij}G_{ij}$. The obstacle function $\psi : \Omega \to \mathbb{R}^m$ defined in (2) and $\theta \in W^{1,p}(\Omega;\mathbb{R}^m)$ is a function which gives the boundary values. We will study the solution $u \in \mathcal{K}_{\psi,\theta}$ for (1) under the following hypotheses:

(f_0) $\mathcal{A} : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ is a Carathéodory function (i.e., measurable with respect to z and continuous with respect to the last variables).

(f_1) There exist $N_1 \in L^{p'}(\Omega)$, $N_2 \in L^1(\Omega)$ and $c_1, c_2 > 0$ such that

$$|\mathcal{A}(z, S, G)| \le N_1(z) + c_1\Big(|S|^{p-1} + |G|^{p-1}\Big), \tag{3}$$

$$\mathcal{A}(z, S, G) : G \ge -N_2(z) + c_2|G|^p, \tag{4}$$

for a.e. $z \in \Omega$ and all $(S, G) \in \mathbb{R}^m \times \mathbb{M}^{m\times n}$.

(f_2) $\mathcal{A}$ satisfies one of the following conditions:

(a) The map $G \mapsto \mathcal{A}(z, u, G)$ is strictly quasimonotone, i.e., there exists constants $c_3 > 0$ such that

$$\int_\Omega \Big(\mathcal{A}(z, u, G) - \mathcal{A}(z, u, K)\Big) : (G - K)\,\mathrm{d}z \ge c_3 \int_\Omega |G - K|^p\,\mathrm{d}z$$

for all $z \in \Omega$ and $G, K \in \mathbb{M}^{m\times n}$.

(b) There exists a function $Z : \Omega\times\mathbb{R}^m\times\mathbb{M}^{m\times n} \to \mathbb{R}$ such that $\mathcal{A}(x, u, G) = \frac{\partial Z}{\partial G}(z, u, G)$, and $G \to \mathcal{A}(z, u, G)$ is convex and C^1.

(c) For all $x \in \Omega$, the map $G \mapsto \mathcal{A}(z, u, G)$ is a C^1-function and is monotone, i.e.

$$\Big(\mathcal{A}(z, u, G) - \mathcal{A}(z, u, K)\Big) : (G - K) \ge 0$$

for all $x \in \Omega$ and $G, K \in \mathbb{M}^{m\times n}$.

(f_3) $\phi : \mathbb{R}^m \to \mathbb{M}^{m\times n}$ is linear and continuous and there exists a constant $c_1 > \alpha_0 > 0$ such that

$$|\phi(u)| \leq \alpha_0.$$

We will demonstrate the existence of a solution for the obstacle problem (1)-(2).

Theorem 1.1. *Suppose $\mathcal{K}_{\psi,\theta} \neq \emptyset$ and $\mathcal{A}$ satisfies the conditions (f_0)-(f_2). Then, there exists a weak solution $u \in \mathcal{K}_{\psi,\theta}$ to the obstacle problem* (1)-(2)*. In other words, there exists a function $u \in \mathcal{K}_{\psi,\theta}$ satisfying*

$$\int_\Omega \mathcal{A}(z,u,Du) : D(v-u) + \phi(u) : D(v-u)\,\mathrm{d}z \geq 0$$

for each $v \in \mathcal{K}_{\psi,\theta}$.

We rapidly outline the contents of this work in the following way: Section 2 sets out the basis of Sobolev spaces, including the Kinderlehrer and Stampacchia theorem and a concise explanation of Young measures. Section 3 gives the proof of the existence of solutions to obstacle problems, while Section 4 provides the proof of the uniqueness of solution to obstacle problems.

2. Mathematical Preliminaries

In this section, we review the properties of Lebesgue and Sobolev spaces which will be employed in what follows. Consider a bounded open domain Ω in $\mathbb{R}^N$ (with $N \geq 2$) having a smooth boundary $\partial\Omega$. We will start by discussing a theorem by Kinderlehrer and Stampacchia and then present a review of Young measures along with some of its properties that will be necessary later.

2.1. Spaces of Lebesgue and Sobolev

We define the Lebesgue space $L^p(\Omega)$ by

$$L^p(\Omega) = \left\{ w : \Omega \to \mathbb{R} : w \text{ is measurable and } \int_\Omega |w|^p dx < \infty \right\},$$

endowed with the norm

$$\|w\|_p = \left(\int_\Omega |w|^p dz \right)^{\frac{1}{p}}.$$

We denote by $L^{p'}(\Omega)$ the dual space of $L^p(\Omega)$, where

$$\frac{1}{p} + \frac{1}{p'} = 1$$

The classical Sobolev space is defined by

$$W^{1,p}(\Omega) = \{ w \in L^p(\Omega) \text{ and } |\nabla w| \in L^p(\Omega) \},$$

with the norm

$$\|w\|_{1,p} = \|w\|_p + \|\nabla w\|_p \quad \forall w \in W^{1,p}(\Omega).$$

For $1 < p < \infty$, $W^{1,p}(\Omega)$ is a reflexive Banach space. The space $W_0^{1,p}(\Omega)$ is well defined as the closure of $\mathcal{D}(\Omega)$ in $W^{1,p}(\Omega)$ with respect to the norm $\|w\|_{1,p}$. We can identify the dual of $W_0^{1,p}(\Omega)$ to a subspace of the space of distributions $\mathcal{D}'(\Omega)$ by:

$$W^{-1,p'}(\Omega) = \left(W_0^{1,p}(\Omega) \right)', \quad \left(p' = \frac{p}{p-1} \right).$$

The manipulation of Sobolev spaces often involves the use of specific Sobolev injections, such as the Rellich-Kondrachov theorem.

Proposition 2.1. *Assume Ω of class C^∞ and $p < N$. Then*

$$W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega), \forall q \in \left[1, p^*\right[\text{ with } p^* = \frac{Np}{N-p}.$$

In particular, $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega)$ for all $p \in [1, +\infty)$. In the sequel, the Hölder inequality and the following Poincare inequality (see [33, Lemma 2.2]), there exists a positive constant β such that

$$\|w\|_p \leq \frac{\beta}{2}\|Dw\|_p, \quad \forall w \in W_0^{1,p}(\Omega; \mathbb{R}^m)$$

are central to establish the required estimates to prove the desired results.

2.2. Essential information on Young measures -Theorem of Kinderlehrer-Stampacchia

Let Y be a reflexive Banach space and Y' its dual. The duality pairing between Y' and Y is denoted by

$$\langle \mathcal{G}, \mathcal{H} \rangle = \int_\Omega \mathcal{G}\mathcal{H} \, \mathrm{d}z, \quad \mathcal{H} \in Y, \mathcal{G} \in Y'.$$

Recalling the following theorem of Kinderlehrer and Stampacchia:

Theorem 2.2. *(Kinderlehrer and Stampacchia[29]) Let $\mathcal{K}$ be a nonempty closed convex subset of Y and let $\mathcal{A} : \mathcal{K} \to Y'$ be monotone, coercive and strong-weakly continuous on $\mathcal{K}$. Then there exists an element $u \in \mathcal{K}$ such that*

$$\langle \mathcal{L}(u), v - u \rangle \geq 0 \quad \text{for all } v \in \mathcal{K}.$$

One can use a Young measure to gain insight into and manage the issues that come up when weak convergence does not act in line with expectations concerning nonlinear functions and operators.

Definition 2.3. *Assume that the sequence $\left\{\Lambda_j\right\}_{j\geq 1}$ is bounded in $L^\infty(\Omega; \mathbb{R}^m)$. Then there exist a subsequence $\{\Lambda_k\}_{k\geq 1} \subset \left\{\Lambda_j\right\}_{j\geq 1}$ and a Borel probability measure v_z on $\mathbb{R}^m$ for a.e. $z \in \Omega$, such that for each $\psi \in C(\mathbb{R}^m)$ we have*

$$\psi(\Lambda_k) \to^* \bar{\psi} \quad \text{weakly}^* \text{ in } L^\infty(\Omega),$$

where $\bar{\psi}(z) := \int_{\mathbb{R}^m} \psi(\eta) dv_z(\eta)$ for a.e. $z \in \Omega$. We call $\{v_z\}_{z\in\Omega}$ the family of Young measure associated with $\{\Lambda_k\}_{k\geq 1}$.

Lemma 2.4. *Let $\Omega \subset \mathbb{R}^n$ be Lebesgue measurable (not necessarily bounded) and $\varrho_j : \Omega \to \mathbb{R}^m$, $j = 1, 2, \ldots$ be a sequence of Lebesgue measurable functions. Then there exist a subsequence w_k and a family $\{\mathcal{V}_z\}$ of nonnegative Radon measures on $\mathbb{R}^n$, such that*

(I_1) $\|\mathcal{V}_z\|_{\mathcal{M}} := \int_{\mathbb{R}^m} d\mathcal{V}_z(\eta) \leq 1$ *for almost every $z \in \Omega$.*

(I_2) $\psi(\varrho_k) \to^* \bar{\psi}$ *weakly* * *in $L^\infty(\Omega)$ for any $\psi \in C_0(\mathbb{R}^m)$,*
where $\bar{\psi} = \langle \mathcal{V}_z, \psi \rangle$ *and* $C_0(\mathbb{R}^m) = \left\{\psi \in C(\mathbb{R}^m) : \lim_{|\varrho|\to\infty} |\psi(\varrho)| = 0\right\}$.

(I_3) If for any $R > 0$

$$\lim_{L\to\infty} \sup_{k\in\mathbb{N}} \left|\left\{z \in \Omega \cap B_R(0) : \left|\varrho_k(z)\right| \geq L\right\}\right| = 0,$$

then $\|\mathcal{V}_z\|_{\mathcal{M}} = 1$ for almost every $z \in \Omega$, and for any measurable $\Omega' \subset \Omega$ we have $\psi(\varrho_k) \to \bar{\psi} = \langle \mathcal{V}_z, \psi \rangle$ weakly in $L^1(\Omega')$ for continuous ψ provided the sequence $\psi(\varrho_k)$ is weakly precompact in $L^1(\Omega')$.

The fundamental theorem of Young measure, Lemma 2.4, serves as the basis for the following Fatou-type lemma, which is useful for our purposes.

Lemma 2.5. *([19]). Let $O : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{R}$ be a Carathéodory function and $u_k : \Omega \to \mathbb{R}^m$ a sequence of measurable functions such that Du_k generates the Young measure ν_z. Then*

$$\liminf_{k\to\infty} \int_\Omega O\left(z, u_k(z), Du_k(z)\right) \, \mathrm{d}z \geq \int_\Omega \int_{\mathbb{M}^{m\times n}} O(z, u, \lambda) d\mathcal{V}_z(\lambda) \, \mathrm{d}z,$$

provided that the negative part $O^-\left(z, u_k(z), Du_k(z)\right)$ is equiintegrable.

3. Weak Solution of Obstacle Problem

We will utilize the concept of Young measure to demonstrate the existence of weak solutions for the obstacle problem stated in (1)-(2), by defining a mapping $\mathcal{L} : \mathcal{K}_{\psi,\theta} \to W^{-1,p'}(\Omega; \mathbb{R}^m)$ by

$$\langle \mathcal{L}(u), v\rangle = \int_\Omega \mathcal{A}(z, u, Du) : Dv + \phi(u) : Dv \, \mathrm{d}z$$

satisfy the hypothesis of Theorem 2.2.

3.1. Proof of Existence The Weak Solution

For this, we can solve the problem (1). We first show the following Assertion:

Assertion 3.1.

i) $\mathcal{K}_{\psi,\theta}$ is a closed convex set.

ii) For each $v \in \mathcal{K}_{\psi,\theta}$, $\mathcal{L}u \in W^{-1,p'}(\Omega; \mathbb{R}^m)$.

Proof.

i) Is immediate that $\mathcal{K}_{\psi,\theta}$ is a closed convex set.

ii) Since, Hölder, growth condition in (f_1), we have

$$\begin{aligned}
|\langle \mathcal{L}u, v\rangle| &= \left| \int_\Omega \mathcal{A}(z, u, Du) : Dv + \phi(u) : Dv \, \mathrm{d}x \right| \\
&\leq \left| \int_\Omega \mathcal{A}(z, u, Du) : Dv \, \mathrm{d}z \right| + \left| \int_\Omega \phi(u) : Dv \, \mathrm{d}z \right| \\
&\leq \left(\|N_1\|_{p'} + C_1 \|u\|_p^{p-1} + C_1 \|Du\|_p^{p-1} \right) \|Dv\|_p + \alpha_0 \|v\|_p \\
&\leq \left(\|N_1\|_{p'} + C_1 \|u\|_p^{p-1} + C_1 \|Du\|_p^{p-1} \right) \|v\|_{1,p} + \alpha_0 \|v\|_{1,p} \\
&\leq \left(\|N_1\|_{p'} + C_1 \|u\|_p^{p-1} + C_1 \|Du\|_p^{p-1} + \alpha_0 \right) \|v\|_{1,p} \\
&\leq C \|v\|_{1,p} .
\end{aligned}$$

So, we get $\mathcal{L}u \in W^{-1,p'}(\Omega; \mathbb{R}^m)$.

□

Assertion 3.2. *$\mathcal{L}$ is monotone and coercive on $\mathcal{K}_{\psi,\theta}$.*

Proof. For fixed $v \in \mathcal{K}_{\psi,\theta}$, by the monotonicity of $\mathcal{A}$, we have

$$\begin{aligned}\langle \mathcal{L}u - \mathcal{L}v, u - v\rangle &= \int_\Omega \Big(\mathcal{A}(x,u,Du) - \mathcal{A}(x,v,Dv\Big) : (Du - Dv)dx + \int_\Omega \Big(\phi(u) - \phi(v)\Big) : (Du - Dv)\, \mathrm{d}x \\ &\geq \int_\Omega \Big(\mathcal{A}(x,u,Du) - \mathcal{A}(x,v,Dv\Big) : (Du - Dv)dx \quad \text{(in view of } f_2\text{(c))} \\ &\geq 0.\end{aligned}$$

Then, $\mathcal{L}$ is monotone on $\mathcal{K}_{\psi,\theta}$.
Next, we show that $\mathcal{L}$ is coercive. Indeed, for fixed element $v \in \mathcal{K}_{\psi,\theta}$, in view of the condition (f_2)(a), we have

$$\begin{aligned}\langle \mathcal{L}u - \mathcal{L}v, u - v\rangle &= \int_\Omega \Big(\mathcal{A}(x,u,Du) - \mathcal{A}(x,v,Dv)\Big) : (Du - Dv)\, \mathrm{d}x + \int_\Omega \Big(\phi(u) - \phi(v)\Big) : (Du - Dv)\, \mathrm{d}x \\ &\geq \int_\Omega c_3 |Du - Dv|\, \mathrm{d}x\end{aligned}$$

which implies that

$$\frac{\langle \mathcal{L}(u) - \mathcal{L}(v), u - v\rangle}{\|u - v\|_{1,p}} \geq c\|u - v\|_{1,p}^{p-1} \to \infty$$

as $\|u - v\|_{1,p} \to \infty$ and therefore $\mathcal{L}$ is coercive. □

Assertion 3.3. *$\mathcal{L}$ is strongly-weakly continuous.*

Proof. We choose a sequence $u_k \in \mathcal{K}_{\psi,\theta}$ such that $u_k \to u \in \mathcal{K}_{\psi,\theta}$ in $W^{1,p}(\Omega; \mathbb{R}^m)$. Then $\|u_k\|_{1,p} \leq C$ for some constant C. In virtue of Lemma 2.4, there exists a Young measure $\mathcal{V}_z$ generated by $\{Du_k\}$ such that $\|\mathcal{V}_z\|_{\mathcal{M}(\mathbb{M}^{m\times n})} = 1$ and

$$Du_k \to \langle \mathcal{V}_z, id\rangle = \int_{\mathbb{M}^{m\times n}} \lambda d\mathcal{V}_z(\lambda) \quad \text{in } L^1(\Omega). \tag{5}$$

Since $L^p(\Omega; \mathbb{M}^{m\times n})$ is reflexive, then $Du_k \rightharpoonup Du$ in $L^p(\Omega; \mathbb{M}^{m\times n}) \subset L^1(\Omega; \mathbb{M}^{m\times n})$ and thus $Du(z) = \langle \mathcal{V}_z, \mathrm{id}\rangle$ for a.e. $z \in \Omega$ (by uniqueness of limit, see also [7, Lemma 4.1]). □

The following lemmas allow us to prove the Assertion 3.3.

Lemma 3.4. *(div-curl inequality). Suppose $\mathcal{A}$ satisfies (f_0)-(f_2) and $\{Du_k\}$ generates the Young measure $\mathcal{V}_z$, then*

$$\int_\Omega \int_{\mathbb{M}^{m\times n}} \Big(\mathcal{A}(z,u,\lambda) - \mathcal{A}(z,u,Du)\Big) : (\lambda - Du)\, d\mathcal{V}_z(\lambda)\, \mathrm{d}z \leq 0.$$

Proof. Let consider the sequence

$$\begin{aligned}I_k &:= \Big(\mathcal{A}(z,u_k,Du_k) - \mathcal{A}(z,u,Du)\Big) : (Du_k - Du) \\ &= \mathcal{A}(z,u_k,Du_k) : (Du_k - Du) - \mathcal{A}(z,u,Du) : (Du_k - Du) \\ &=: I_{k,1} + I_{k,2}.\end{aligned}$$

Since $Du \in L^p(\Omega; \mathbb{M}^{m\times n})$, it follows by the growth condition in (f_1) that $\mathcal{A} \in L^{p'}(\Omega; \mathbb{M}^{m\times n})$. Using the weak convergence of $\{Du_k\}$ defined in Lemma 3.1, we obtain

$$I_{k,2} \to 0 \text{ as } k \to \infty.$$

Hence,

$$I = \liminf_{k\to\infty} \int_\Omega I_k dz = \liminf_{k\to\infty} \int_\Omega I_{k,1}\, dz.$$

To get the equiintegrability of $I_{k,1}$, we take a measurable subset $\Omega' \subset \Omega$ and by the Hölder inequality, one gets

$$\begin{aligned}\int_{\Omega'} |\mathcal{A}(z,u_k,Du_k) : Du|\; dz &\leq \int_{\Omega'} |\mathcal{A}(z,u_k,Du_k)| \cdot |Du|\, dz \\ &\leq \|\mathcal{A}(z,u_k,Du_k)\|_{p',\Omega'} \|Du\|_{p',\Omega'}.\end{aligned}$$

Since $\{u_k\}$ is bounded in $W_0^{1,p}(\Omega;\mathbb{R}^m)$, the growth condition stated in (f_1) leads to the inequality:

$$\int_\Omega |\mathcal{A}(z,u_k,Du_k)|^{p'}\; dz \leq c \int_\Omega |d_1(z)|^{p'} + |u_k|^p + |Du_k|^p\; dz \leq c.$$

It's worth noting that the term $\int_{\Omega'} |Du|^p\, dz$ can be made arbitrarily small by choosing a sufficiently small measure for Ω'. Furthermore, it's important to observe that:

$$\mathcal{A}(z,u_k,Du_k) : Du_k \geq -N_2(z) + \alpha\, |Du_k|^p \geq -N_2(z)$$

and

$$\int_{\Omega'} (\mathcal{A}(z,u_k,Du_k) : Du_k)^-\; dz \leq \int_{\Omega'} |N_2(z)|\; dz$$

Consequently, $I_{k,1}^-$is equiintegrable. We infer from Lemma (2.5) that

$$\begin{aligned} I &= \liminf_{k\to\infty} \int_\Omega \mathcal{A}(z,u_k,Du_k) : (Du_k - Du)\; dz \\ &\geq \int_\Omega \int_{\mathbb{M}^{m+n}} \mathcal{A}(z,u,\lambda) : (\lambda - Du) d\mathcal{V}_z(\lambda)\, dz.\end{aligned}$$

New, we prove that $I \leq 0$. Indeed, to Mazur's theorem (see, e.g., [43, Theorem 2, page 120]) there exists $(\vartheta_k) \in W^{1,p}(\Omega;\mathbb{R}^m)$ where each ϑ_k is a convex linear combination of $\{\hbar_1,\dots,\hbar_k\}$ such that $v_k \to \hbar$ in $W^{1,p}(\Omega;\mathbb{R}^m)$. This implies that ϑ_k belongs to the same space as $\hbar_k$. Hence,

$$\begin{aligned} I &= \liminf_{k\to\infty} \int_\Omega \mathcal{A}(z,u_k,Du_k) : (Du_k - Du)\; dz \\ &= \liminf_{k\to\infty} \left[\int_\Omega \mathcal{A}(z,u_k,Du_k) : D(u_k - u - v_k)\; dz + \int_\Omega \mathcal{A}(z,u_k,Du_k) : Dv_k dz\right] \\ &= \liminf_{k\to\infty} \left[\int_\Omega \mathcal{A}(z,u_k,Du_k) : D(u_k - u - v_k)\; dz - \int_\Omega \phi(u_k) : (Du_k - Du)\, dz\right] \\ &\leq \liminf_{k\to\infty} \left[\||\mathcal{A}(z,u_k,Du_k)\||_{p'} \|D(u_k - u - v_k)\|_p + c_1 \|D(u_k - v_k)\|_p\right].\end{aligned}$$

On one hand, we have that $\|\mathcal{A}(z,u_k,Du_k)\|_{p'}$ is bounded by the growth condition (f_1). On the other hand, by choosing $v_k \in V_k$ such that $\|u_k - u - v_k\|_{1,p} < \epsilon$ for any $k > k_0$, the term $\|D(u_k - u - v_k)\|_p$ is bounded by ϵ. Notice that since ϕ is linear and continuous and (u_k) is bounded then $\phi(u_k)$ is bounded. By Hölder's inequality, we have

$$\left|\int_\Omega \phi(u_k) : (Du_k - Dv_k)\; dz\right| \leq c_1 \|Du_k - Dv_k\|_p \longrightarrow 0$$

by definition of v_k, $1 < p$ and

$$\|Du_k - Dv_k\|_p \leq \|Du_k - Du\|_p + \|Dv_k - Du\|_p \longrightarrow 0 \quad \text{as } k\to\infty.$$

and $\|v_k\|_p \le \|v_k - (u_k - u)\|_p + \|u_k - u\|_p \le \epsilon + o(k)$. Hence

$$I = \liminf_{k\to\infty} \int_\Omega I_k \, dz \le 0,$$

as desired. □

Remark 3.5. *An intermidiary result is the following inequality:*

$$\liminf_{k\to\infty} \int_\Omega \left(\mathcal{A}\left(z, u_k, Du_k\right) - \mathcal{A}(z, u, Du)\right) : \left(Du_k - Du\right) \, \mathrm{d}z \le 0.$$

To see this, it is sufficient to repeat the proof of Lemma 3.4.

Lemma 3.6. *For almost every $z \in \Omega$, we have*

$$(\mathcal{A}(z, u, \lambda) - \mathcal{A}(z, u, Du)) : (\lambda - Du) = 0 \quad \text{on } \operatorname{supp} \mathcal{V}_z.$$

Proof. By Lemma 3.4, we have

$$\int_\Omega \int_{\mathbb{M}^{m\times n}} (\mathcal{A}(z, u, \lambda) - \mathcal{A}(z, u, Du)) : (\lambda - Du) \, d\mathcal{V}_z(\lambda) \, \mathrm{d}z \le 0.$$

By the monotonicity of $\mathcal{A}$, the above integrand is nonnegative, thus must vanish with respect to the product measure $\mathrm{d}\mathcal{V}_z(\lambda) \otimes \mathrm{d}z$. Therefore,

$$(\mathcal{A}(z, u, \lambda) - \mathcal{A}(z, u, Du)) : (\lambda - Du) = 0 \quad \text{on } \operatorname{supp} \mathcal{V}_z.$$

□

Now, we prove the Assertion 3.3 for each case listed in (f_2).
Step 1. Suppose that $\mathcal{A}$ satisfy the condition (f_2)(a). We have

$$\int_\Omega |Du_k - Du|^p \, \mathrm{d}z \le c \int_\Omega \left(\mathcal{A}\left(z, u, Du_k\right) - \mathcal{A}(z, u, Du)\right) : \left(Du_k - Du\right) \, \mathrm{d}z.$$

We remark that the limit inferior of the right hand side of the above inequality is less than or equal to zero by Remark 3.5. Accordingly,

$$\liminf_{k\to\infty} \int_\Omega |Du_k - Du|^p \, dz = 0.$$

Let $E_{k,\epsilon} = \left\{x : |Du_k - Du| \ge \epsilon\right\}$. We have

$$\int_\Omega |Du_k - Du|^p \, \mathrm{d}z \ge \int_{E_{k,\epsilon}} |Du_k - Du|^p \, \mathrm{d}z \ge \epsilon^p \left|E_{k,\epsilon}\right|$$

which gives

$$\left|E_{k,\epsilon}\right| \le \frac{1}{\epsilon^p} \int_\Omega |Du_k - Du|^p \, \mathrm{d}z \to 0 \quad \text{as } k \to \infty.$$

As by Fatou Lemma

$$\int_\Omega \left(\frac{|Du_k - Du|}{\epsilon}\right)^p \mathrm{d}z \leqslant \lim_{k'\to\infty} \sup \int_\Omega \frac{|Du_{k'} - Du_k|}{\epsilon}\Bigg)^p \mathrm{d}z,$$

we have

$$\|Du_k - Du\|_{L^p(\Omega,\mathbb{R}^m)} \leqslant \sup_{k'} \left\{\|Du_{k'} - Du_k\|_{L^p(\Omega,\mathbb{R}^m)}\right\} < \varepsilon',$$

that is to say, $Du_k \to Du$ in $L^p(\Omega, \mathbb{R}^m)$. So that,

$$Du_k \to Du \quad \text{ in measure on } \Omega \text{ (for a subsequence).}$$

After extracting a suitable subsequence if necessary, we can infer that $Du_k \to Du$ for almost every $z \in \Omega$. Then $\mathcal{A}(z, u_k, Du_k) \to \mathcal{A}(z, u, Du)$ for almost every $z \in \Omega$, and in the measure. By the equiintegrability of $\mathcal{A}(z, u_k, Du_k) : Dv$, the Vitali theorem implies

$$\int_\Omega \mathcal{A}(z, u_k, Du_k) : Dv \,\mathrm{d}z \to \int_\Omega \mathcal{A}(z, u, Du) : Dv \,\mathrm{d}z \quad \text{ as } k \to \infty.$$

Step 2. For the case (f_2)(b), we argue as follows: We start by proving that for almost every $z \in \Omega$,

$$\operatorname{supp} \mathcal{V}_z \subset E_z = \Big\{\lambda \in \mathbb{M}^{m\times n} : Z(z, u, \lambda) = Z(z, u, Du) + \mathcal{A}(z, u, Du) : (\lambda - Du)\Big\}.$$

Let $\lambda \in \operatorname{supp} \mathcal{V}_z$, then by Lemma 3.6, we get

$$(1-\tau)(\mathcal{A}(z, u, \lambda) - \mathcal{A}(z, u, Du)) : (\lambda - Du) = 0, \quad \forall \tau \in [0, 1]. \tag{6}$$

On the other hand, by monotonicity, for $\tau \in [0, 1]$ we have

$$(1-\tau)(\mathcal{A}(z, u, Du + \tau(\lambda - Du)) - \mathcal{A}(z, u, \lambda)) : (Du - \lambda) \geq 0. \tag{7}$$

Subtracting (6) from (7), we get

$$(1-\tau)(\mathcal{A}(z, u, Du + \tau(\lambda - Du)) - \mathcal{A}(z, u, Du)) : (Du - \lambda) \geq 0 \tag{8}$$

for $\tau \in [0, 1]$. By monotonicity,

$$(\mathcal{A}(z, u, Du + \tau(\lambda - Du)) - \mathcal{A}(z, u, Du)) : \tau(\lambda - Du) \geq 0,$$

and since $\tau \in [0, 1]$, we have

$$(\mathcal{A}(z, u, Du + \tau(\lambda - Du)) - \mathcal{A}(z, u, Du)) : (1-\tau)(\lambda - Du) \geq 0.$$

The above inequality together with (8) implies

$$(\mathcal{A}(z, u, Du + \tau(\lambda - Du)) - \mathcal{A}(z, u, Du)) : (\lambda - Du) = 0 \quad \forall \tau \in [0, 1].$$

Integrating this equality over $[0, 1]$ and using the fact that

$$\mathcal{A}(z, u, Du + \tau(\lambda - Du)) : (\lambda - Du) = \frac{\partial Z}{\partial \tau}(z, u, Du + \tau(\lambda - Du)) : (\lambda - Du),$$

we conclude that

$$\begin{aligned} Z(z, u, \lambda) &= Z(z, u, Du) + \int_0^1 \mathcal{A}(z, u, Du + \tau(\lambda - Du)) : (\lambda - Du) d\tau \\ &= Z(z, u, Du) + \mathcal{A}(z, u, Du) : (\lambda - Du). \end{aligned}$$

Hence, $\lambda \in E_z$, i.e. $\operatorname{supp} \mathcal{V}_z \subset E_z$. In view of the convexity of Z, we have

$$Z(z, u, \lambda) \geq Z(z, u, Du) + \mathcal{A}(z, u, Du) : (\lambda - Du).$$

For all $\lambda \in E_z$, put $A(\lambda) = Z(z, u, \lambda)$ and $B(\lambda) = Z(z, u, Du) + \mathcal{A}(z, u, Du) : (\lambda - Du)$. Since $\lambda \mapsto A(\lambda)$ is continuous and differentiable, we obtain for all $S \in \mathbb{M}^{m\times n}$ and $\tau \in \mathbb{R}$

$$\begin{aligned} \frac{A(\lambda + \tau S) - A(\lambda)}{\tau} &\geq \frac{B(\lambda + \tau F) - B(\lambda)}{\tau} \text{ if } \tau > 0, \\ \frac{A(\lambda + \tau S) - A(\lambda)}{\tau} &\leq \frac{B(\lambda + \tau S) - B(\lambda)}{\tau} \quad \text{ if } \tau < 0. \end{aligned}$$

Thus, $DA = DB$ and therefore

$$\mathcal{A}(z,u,\lambda) = \mathcal{A}(z,u,Du) \quad \forall \lambda \in E_z \supset \operatorname{supp} \mathcal{V}_z. \tag{9}$$

The equiintegrability of $\mathcal{A}(z,u,Du_k)$ implies that its weak L^1-limit is given by

$$\begin{aligned}\bar{\mathcal{A}}(z) := \int_{\mathbb{M}^{m\times n}} \mathcal{A}(z,u,\lambda)\, \mathrm{d}v_z(\lambda) &= \int_{\operatorname{supp}\mathcal{V}_z} \mathcal{A}(z,u,\lambda)\, \mathrm{d}v_z(\lambda) \\ &= \int_{\operatorname{supp}\mathcal{V}_z} \mathcal{A}(z,u,Du)\, \mathrm{d}\mathcal{V}_z(\lambda) = \mathcal{A}(z,u,Du)\end{aligned} \tag{10}$$

where we have used (9) and $\|\mathcal{V}_z\|_{\mathcal{M}} = 1$. Now, consider the Carathéodory function

$$\omega(z,u,\lambda) = |\mathcal{A}(z,u,\lambda) - \bar{\mathcal{A}}(z)|, \quad \lambda \in \mathbb{M}^{m\times n}.$$

The sequence $\omega_k(z) := \omega(z,u_k,Du_k(z))$ is equiintegrable by that of $\mathcal{A}(z,u_k,Du_k(z))$, hence its weak L^1-limit is given by

$$\omega_k \rightharpoonup \bar{\omega} \text{ in } L^1(\Omega),$$

where

$$\begin{aligned}\bar{\omega}(z) &= \int_{\mathbb{M}^{m\times n}} |\mathcal{A}(z,u,\lambda) - \bar{\mathcal{A}}(z)|\, \mathrm{d}\mathcal{V}_z(\lambda) \\ &= \int_{\operatorname{supp} v_z} |\mathcal{A}(z,u,\lambda) - \bar{\mathcal{A}}(z)|\, d\mathcal{V}_z(\lambda) = 0 \quad \text{(by (10) and (9))}.\end{aligned}$$

Since $\omega_k \geq 0$, we deduce that $\omega_k \to 0$ in $L^1(\Omega)$ as $k \to \infty$.
Hence,

$$\int_\Omega \mathcal{A}(z,u_k,Du_k) : Dv\, \mathrm{d}z \to \int_\Omega \mathcal{A}(z,u,Du) : Dv\, \mathrm{d}x \quad \text{as} \quad k \to \infty.$$

Step 3. The last case $(f_2)(c)$, we claim that for a.e. $z \in \Omega$ and every $S \in \mathbb{M}^{m\times n}$

$$\mathcal{A}(z,u,\lambda) : S = \mathcal{A}(z,u,Du) : S + (\nabla\mathcal{A}(z,u,Du)) : (Du - S)$$

holds on $\operatorname{supp}\mathcal{V}_x$, where ∇ is the derivative with respect to the second variable of $\mathcal{A}$. The monotonicity of $\mathcal{A}$ implies that for $\tau \in \mathbb{R}$

$$(\mathcal{A}(z,u,\lambda) - \mathcal{A}(z,u,Du+\tau S)) : (\lambda - Du - \tau S) \geq 0$$

which implies

$$-\mathcal{A}(z,u,\lambda) : \tau S \geq -\mathcal{A}(z,u,\lambda) : (\lambda - Du) + \mathcal{A}(z,u,Du+\tau S) : (\lambda - Du - \tau S).$$

By virtue of Lemma 3.6, we get

$$-\mathcal{A}(z,u,\lambda) : \tau S \geq -\mathcal{A}(z,u,Du) : (\lambda - Du) + \mathcal{A}(z,u,Du+\tau S) : (\lambda - Du - \tau S).$$

Note that $\mathcal{A}(z,u,Du+\tau S) = \mathcal{A}(z,u,Du) + \nabla\mathcal{A}(z,u,Du)\tau S + o(\tau)$, then

$$\begin{aligned}&\mathcal{A}(z,u,Du+\tau S) : (\lambda - Du - \tau S) \\ &\quad = \mathcal{A}(z,u,Du+\tau S) : (\lambda - Du) - \mathcal{A}(z,u,Du+\tau S) : \tau S \\ &\quad = \mathcal{A}(z,u,Du) : (\lambda - Du) + \nabla\mathcal{A}(z,u,Du)\tau S : (\lambda - Du) - \mathcal{A}(z,u,Du) : \tau S \\ &\qquad - \nabla\mathcal{A}(z,u,Du)\tau S : \tau S + o(\tau) \\ &\quad = \mathcal{A}(z,u,Du) : (\lambda - Du) + \tau[\nabla\mathcal{A}(z,u,Du)S : (\lambda - Du) - \mathcal{A}(z,u,Du)] + o(\tau).\end{aligned}$$

Therefore,

$$-\mathcal{A}(z,u,\lambda) : \tau S \geq \tau[(\nabla\mathcal{A}(z,u,Du)S) : (\lambda - Du) - \mathcal{A}(z,u,Du) : S] + o(\tau).$$

Since τ is arbitrary in $\mathbb{R}$, then our claim follows. By the equiintegrability of $\mathcal{A}(z,u,Du_k)$, its weak L^1-limit is then given by

$$\bar{\mathcal{A}}(z) = \int_{\operatorname{supp} v_z} \mathcal{A}(z,u,\lambda)\,\mathrm{d}\mathcal{V}_z(\lambda) = \mathcal{A}(z,u,Du)$$

where we have used our claim and $Du(z) = \langle \mathcal{V}_z, id\rangle$. On the other hand, since $L^{p'}(\Omega;\mathbb{M}^{m\times n})$ is reflexive, the sequence $\{\mathcal{A}(z,u,Du_k)\}$ converges weakly in $L^{p'}(\Omega;\mathbb{M}^{m\times n})$ and its weak $L^{p'}$-limit is also $\mathcal{A}(z,u,Du)$. Then, we conclude that

$$\int_\Omega \mathcal{A}(z,u_k,Du_k) : Dv\,\mathrm{d}z \to \int_\Omega \mathcal{A}(z,u,Du) : Dv\,\mathrm{d}z \quad \text{as } k\to\infty.$$

Hence, $\int_\Omega \mathcal{A}(z,u_k,Du_k) : Dv\,\mathrm{d}z \to \int_\Omega \mathcal{A}(z,u,Du) : Dv\,\mathrm{d}z$ as $k\to\infty$ in the cases (a), (b) and (c).

- It is clear that

$$\int_\Omega \phi(u_k) : Dv\,dx \to \int_\Omega \phi(u) : Dv\,dx \quad \text{as } k\to\infty.$$

Next, we pass to the limit , we assert that

$$\begin{aligned}(\mathcal{L}u_k, v) &= \int_\Omega \mathcal{A}(z,u_k,Du_k) : Dv + \phi(u_k) : Dv\,\mathrm{d}x\\ &\to \int_\Omega \mathcal{A}(z,u,Du) : Dv + \phi(u) : Dv\,\mathrm{d}x\\ &=(\mathcal{L}u, v).\end{aligned}$$

This is the strong-weakly continuous of $\mathcal{L}$ on $\mathcal{K}_{\psi,\theta}$. This ends the proof of Assertion 3.3.

Now we can apply Theorem 2.2 and the above lemmas to obtain the existence. For this we conclude the existence of an element $u \in \mathcal{K}_{\psi,\theta}$ such that $\langle \mathcal{L}(u), v-u\rangle \geq 0$, i.e.

$$\int_\Omega \mathcal{A}(z,u,Du) : (Dv-Du) + \phi(u) : (Dv-Du)\,\mathrm{d}z \geq 0 \quad \text{for all } v \in \mathcal{K}_{\psi,\theta}.$$

4. Uniqueness of Weak Solutions to Problem

Uniqueness is obtained proving the following theorem:

Theorem 4.1. *Suppose $\mathcal{K}_{\psi,\theta} \neq \phi$. Under conditions $(f_1)-(f_2)(c)$, there exists a unique solution $u \in \mathcal{K}_{\psi,\theta}$ to the obstacle problem* (1).
That is to say, there exists a unique $u \in \mathcal{K}_{\psi,\theta}$ such that

$$\int_\Omega \mathcal{A}(z,u,Du) : (Dv-Du) + \phi(u) : (Dv-Du)\,\mathrm{d}z \geq 0 \quad \textit{for all } v \in \mathcal{K}_{\psi,\theta}.$$

Proof. The lemmas above lead to the immediate existence of two weak solutions $u_1, u_2 \in \mathcal{K}_{\psi,\theta}$ to the obstacle problem (1), then

$$\int_\Omega \mathcal{A}(z,u_1,Du_1) : (Du_2-Du_1)\,\mathrm{d}z + \phi(u_1) : (Du_2-Du_1)\,\mathrm{d}z \geq 0$$

and

$$-\int_{\Omega}\mathcal{A}(z,u_1,Du_1):(Du_2-Du_1)\,\mathrm{d}z+\phi(u_1):(Du_2-Du_1)\,\mathrm{d}z$$
$$=\int_{\Omega}\mathcal{A}(z,u_2,Du_2):(Du_1-Du_2)\,\mathrm{d}z+\phi(u_2):(Du_1-Du_2)\,\mathrm{d}z\geq 0.$$

Moreover,

$$\int_{\Omega}\mathcal{A}(z,u_1,Du_1)-\mathcal{A}(z,u_2,Du_2):(Du_1-Du_2)+\phi(u_1)-\phi(u_2):(Du_1-Du_2)\,\mathrm{d}z\leq 0.$$

By looking at (f_2)(c), it can be concluded that

$$\int_{\Omega}\mathcal{A}(z,u_1,Du_1)-\mathcal{A}(z,u_2,Du_2):(Du_1-Du_2)\,\mathrm{d}x=0 \text{ on } \Omega$$

and

$$\int_{\Omega}\phi(u_1)-\phi(u_2):(Du_1-Du_2)\,\mathrm{d}x=0.$$

We have now established that $u_1 = u_2$ almost everywhere on Ω, thus finishing the proof. □

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