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Wiener index of the cozero-divisor graph of a finite commutative ring

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Abstract. Let R be a ring with unity. The cozero-divisor graph $\Gamma'(R)$ of a ring R is an undirected simple graph whose vertices are the set of all non-zero and non-unit elements of R, and two distinct vertices x and y are adjacent if and only if $x \notin Ry$ and $y \notin Rx$. To extend the corresponding results of the ring $\mathbb{Z}_n$ of integer modulo n, in this article, we derive a closed-form formula of the Wiener index of the cozero-divisor graph of a finite commutative ring R. As applications, we compute the Wiener index of $\Gamma'(R)$, when either R is the product of ring of integers modulo n or a reduced ring. At the final part of this paper, we provide a SageMath code to compute the Wiener index of the cozero-divisor graph of these classes of rings including the ring $\mathbb{Z}_n$ of integers modulo n.

1. Introduction and Preliminaries

The Wiener index is one of the most frequently used topological indices in chemistry as a molecular shape descriptor. This was first used by H. Wiener in 1947 and then the formal definition of the Wiener index was introduced by Hosoya [17]. The *Wiener index* of a graph is defined as the sum of the lengths of the shortest paths between all pairs of vertices in a graph. Other than the chemistry, the Wiener index was used to find various applications in quantitative structure-property relationships (see [19]). The Wiener index was also employed in crystallography, communication theory, facility location, cryptography, etc. (see [13, 16, 26]). An application of the Wiener index has been established in the water pipeline network, which is essential for water supply management (see [14]). Other utilization of the Wiener index can be found in [15, 18, 31, 32] and reference therein.

The association of the graphs to rings was introduced by Beck [12] and he was mainly interested in the coloring of a graph associated to a commutative ring. Then Anderson and Livingston [8] studied a subgraph of the graph introduced by Beck and named as the zero divisor graph. Further, various aspects of the zero divisor graphs have been explored, see [7, 20, 21, 24] and reference therein. Moreover, various other graphs, namely: inclusion ideal graph, total graph, annihilating-ideal graph, co-maximal graph and cozero-divisor graphs of rings have been introduced and studied extensively. Afkhami *et al.* [1] introduced the cozero-divisor graph of a commutative ring and studied its basic graph-theoretic properties including

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completeness, girth and clique number etc. They also investigated the relations between the zero-divisor graph and the cozero-divisor graph. The *cozero-divisor* graph $\Gamma'(R)$ of the ring R with unity is an undirected simple graph whose vertex set is the set of all non-zero and non-unit elements of R, and two distinct vertices x, y are adjacent if and only if $x \notin Ry$ and $y \notin Rx$. The complement of the cozero-divisor graph and the characterization of the commutative rings with forest, star, or unicyclic cozero-divisor graphs have been investigated in [2]. Akbari *et al.* [5] studied the cozero-divisor graph associated to the polynomial ring and the ring of power series. Some of the work on the cozero-divisor graphs of rings can be found in [3, 4, 6, 11, 22, 23, 25].

Over the recent years, the Wiener index of certain graphs associated with rings has been studied by various authors. The Wiener index of the zero divisor graph of the ring $\mathbb{Z}_n$ of integers modulo n has been studied in [9, 29]. Recently, Selvakumar *et al.* [28] calculated the Wiener index of the zero divisor graph for a finite commutative ring with unity. The Wiener index of the unit graph associated with commutative rings has been investigated in [10]. The Wiener index of the cozero-divisor graph of the ring $\mathbb{Z}_n$ has been obtained in [23]. The aim of this manuscript is to extend the results of [23] to an arbitrary ring. In this connection, we study the Wiener index of the cozero-divisor graph of a finite commutative ring with unity. First, we provide the necessary results and notations used throughout the paper. The remaining paper is arranged as follows: In Section 2, a closed-form formula of the Wiener index of the cozero-divisor graph of a finite commutative ring with unity is presented. Using this formula, in Section 3, we obtain the Wiener index of the cozero-divisor graph of the ring R, where R is the product of ring of integers modulo n. In Section 4, we compute the Wiener index of the cozero-divisor graph of a finite commutative reduced ring. In Section 5, we derive SageMath codes to compute the Wiener index of the cozero-divisor graph of various classes of rings.

Now we recall the necessary definitions, results and notations of graph theory from [30]. A *graph* Γ is a pair $\Gamma = (V, E)$, where $V = V(\Gamma)$ and $E = E(\Gamma)$ are the set of vertices and edges of Γ, respectively. Let Γ be a graph. Two distinct vertices x and y of Γ are *adjacent*, denoted by $x \sim y$, if there is an edge between x and y. Otherwise, we denote it by $x \nsim y$. A *subgraph* Γ' of a graph Γ is a graph such that $V(\Gamma') \subseteq V(\Gamma)$ and $E(\Gamma') \subseteq E(\Gamma)$. If $U \subseteq V(\Gamma)$, then the subgraph $\Gamma(U)$ of Γ induced by U is the graph with vertex set U and two vertices of $\Gamma(U)$ are adjacent if and only if they are adjacent in Γ. The *complement* $\overline{\Gamma}$ of Γ is a graph with the same vertex set as Γ and distinct vertices x, y are adjacent in $\overline{\Gamma}$ if they are not adjacent in Γ. A graph Γ is said to be *complete* if any two distinct vertices are adjacent. The complete graph on n vertices is denoted by K_n. A *path* in a graph is a sequence of distinct vertices with the property that each vertex in the sequence is adjacent to the next vertex of it. The graph Γ is said to be *connected* if there is a path between every pair of vertices. The *distance* between any two vertices x and y of Γ, denoted by $d(x, y)$ (or $d_\Gamma(x, y)$), is the number of edges in a shortest path between x and y. The *Wiener index* $W(\Gamma)$ of the connected graph Γ is defined as the sum of all the distances between every pair of vertices, that is

$$W(\Gamma) = \frac{1}{2} \sum_{u \in V(\Gamma)} \sum_{v \in V(\Gamma)} d(u, v)$$

Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$ be k pairwise disjoint graphs. Then the *generalised join graph* $\Gamma[\Gamma_1, \Gamma_2, \ldots, \Gamma_k]$ of $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$ is the graph formed by replacing each vertex u_i of Γ by Γ_i and then joining each vertex of Γ_i to every vertex of Γ_j whenever $u_i \sim u_j$ in Γ (cf. [27]).

Let R be a ring. An element a of R is said to be a *zero-divisor* if there exists a non-zero element $x \in R$ such that $ax = xa = 0$. An element u of a ring R with unity 1, is said to be a *unit* if there exists $v \in R$ such that $uv = vu = 1$. The set of zero-divisors and the set of units of the ring R are denoted by $Z(R)$ and $U(R)$, respectively. The set of all non-zero zero-divisors of the ring R is denoted by $Z(R)^*$. For $x \in R$, the principal ideal generated by x is denoted by (x). For a positive integer k, we write $[k] = \{1, 2, \ldots, k\}$.

2. Formulae for the Wiener index of the cozero-divisor graph of a finite commutative ring

The purpose of this section is to provide a closed-form formula of the Wiener index of the cozero-divisor graph of a finite commutative ring. Let R be a finite commutative ring with unity. Define a relation $\equiv$ on $V(\Gamma'(R))$ such that

$$x \equiv y \text{ if and only if } (x) = (y).$$

Note that $\equiv$ is an equivalence relation. Let $x_1, x_2, \ldots, x_k$ be the representatives of the equivalence classes of $X_1, X_2, \ldots, X_k$, respectively under the relation $\equiv$. We begin with the following lemma.

Lemma 2.1. *In the cozero-divisor graph $\Gamma'(R)$, an element of X_i is adjacent to an element of X_j if and only if $(x_i) \nsubseteq (x_j)$ and $(x_j) \nsubseteq (x_i)$.*

Proof. Suppose $a \in X_i$ and $b \in X_j$. Then $(a) = (x_i)$ and $(b) = (x_j)$ in R. If $a \sim b$ in $\Gamma'(R)$, then $(a) \nsubseteq (b)$ and $(b) \nsubseteq (a)$. It follows that $(x_i) \nsubseteq (x_j)$ and $(x_j) \nsubseteq (x_i)$. The converse holds by the definition of $\Gamma'(R)$. □

Corollary 2.2. *(i) For $i \in [k]$, the induced subgraph $\Gamma'(X_i)$ of $\Gamma'(R)$ is isomorphic to $\overline{K}_{|X_i|}$.*

(ii) For distinct $i, j \in [k]$, an element of X_i is adjacent to either all or none of the elements of X_j.

Define a subgraph $\Upsilon'(R)$ (or Υ') induced by the set $\{x_1, x_2, \ldots, x_k\}$ of representatives of the respective equivalence classes $X_1, X_2, \ldots, X_k$ under the relation $\equiv$.

In view of Corollary 2.2 and Υ' (defined above), we have the following proposition.

Proposition 2.3. *For each $i \in [k]$, let Γ'_i be the subgraph of $\Gamma'(R)$ induced by the set X_i. Then*

$$\Gamma'(R) = \Upsilon'[\Gamma'_1, \Gamma'_2, \ldots, \Gamma'_k].$$

Lemma 2.4. *Let $\Upsilon'(R)$ be a subgraph of $\Gamma'(R)$ defined above and let $\Upsilon'(R)$ contains at least two vertices. Then $\Upsilon'(R)$ is connected if and only if the cozero-divisor graph $\Gamma'(R)$ is connected. Moreover, for a connected graph $\Gamma'(R)$ and $a, b \in V(\Gamma'(R))$, we have*

$$d_{\Gamma'(R)}(a,b) = \begin{cases} 2 & \text{if } a, b \in X_i \text{ for some } i, \\ d_{\Upsilon'(R)}(x_i, x_j) & \text{if } a \in X_i, b \in X_j \text{ and } i \neq j. \end{cases}$$

Proof. First suppose that the graph $\Upsilon'(R)$ is connected. Let a, b be two arbitrary vertices of $\Gamma'(R)$. Suppose that $a \in X_i$ and $b \in X_j$. If $i = j$, then $a \nsim b$ in $\Gamma'(R)$. Since $\Upsilon'(R)$ is connected, we have $x_t \in X_t$ such that $x_i \sim x_t$ in $\Gamma'(R)$. Consequently, $a \sim x_t \sim b$ in $\Gamma'(R)$ and $d_{\Gamma'(R)}(a,b) = 2$. We may now suppose that $i \neq j$. If $a \sim b$, then there is nothing to prove. Further, suppose that $a \nsim b$ in $\Gamma'(R)$. Connectedness of $\Upsilon'(R)$ implies that there exists a path $x_i \sim x_{i_1} \sim x_{i_2} \sim \cdots \sim x_{i_t} \sim x_j$, where $i \neq j$. It follows that $a \sim x_{i_1} \sim x_{i_2} \sim \cdots \sim x_{i_t} \sim b$ in $\Gamma'(R)$ and $d_{\Gamma'(R)}(a,b) = d_{\Upsilon'(R)}(x_i, x_j)$. Therefore, $\Gamma'(R)$ is connected. The converse is straightforward. □

Let R be a finite commutative ring with unity. As a consequence of Proposition 2.3 and Lemma 2.4, we have the following theorem.

Theorem 2.5. *The Wiener index of the cozero-divisor graph $\Gamma'(R)$ of a finite commutative ring with unity is given by*

$$W(\Gamma'(R)) = 2\sum \binom{|X_i|}{2} + \sum_{1 \leq i < j \leq k} |X_i||X_j| d_{\Upsilon'(R)}(x_i, x_j),$$

where x_i is a representative of the equivalence class X_i under the relation $\equiv$.

In the subsequent sections, we use Theorem 2.5 to derive the Wiener index of the cozero-divisor graph $\Gamma'(R)$ of various classes of rings.

3. Wiener index of the cozero-divisor graph of the product of ring of integers modulo n

First note that the cozero-divisor graph $\Gamma'(\mathbb{Z}_4)$ is a graph with exactly one vertex. Observe that the graph $\Gamma'(\mathbb{Z}_p)$, where p is a prime, is a graph without any vertices. Moreover, for $\alpha \geq 2$, the cozero-divisor graph of the ring $\mathbb{Z}_{p^\alpha}$, where $p^\alpha \neq 4$, is a graph with $p^{\alpha-1}-1$ vertices without any edge. Consequently, except the ring $\mathbb{Z}_{p^\alpha}$, we obtain the Wiener index of the cozero-divisor graph of the ring R such that $R \cong \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k}$ or $R \cong \mathbb{Z}_{p_1^{m_1}} \times \mathbb{Z}_{p_2^{m_2}} \times \cdots \times \mathbb{Z}_{p_k^{m_k}}$. In this connection, first we obtain all the possible distances $d(x, y)$ of any two vertices x, y of the graph $\Gamma'(R)$. We begin with the following lemma.

Lemma 3.1. *Let $R \cong \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k}$ and let $x = (x_1, x_2, \ldots, x_r, \ldots, x_k)$, $y = (y_1, y_2, \ldots, y_r, \ldots, y_k) \in R$. Define $S_r = \{(x, y) : \ x_r, y_r \in Z(\mathbb{Z}_{n_r})^* \text{ and } (x_r) \subseteq (y_r), x_i = 0, y_i \in U(\mathbb{Z}_{n_i}) \text{ for each } i \neq r\}$. Then*

$$d_{\Gamma'(R)}(x, y) = \begin{cases} 1 & \text{if } x \sim y, \\ 2 & \text{if } x \nsim y \text{ and } (x, y) \notin S_r \text{ for all } r, \\ 3 & \text{if } (x, y) \in S_r \text{ for some } r. \end{cases}$$

Proof. Recall that all the elements of the ring $\mathbb{Z}_{n_i}$ are either units or zero-divisors. For $R \cong \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k}$, the definition of $\Gamma'(R)$ implies that for $x = (x_1, x_2, \ldots, x_k) \in V(\Gamma'(R))$, all x_i's $(1 \leq i \leq k)$ are neither units nor zero elements. Clearly, if $x \sim y$, then $d(x, y) = 1$. Consequently, to compute the distances $d(x, y)$ between any two non-adjacent vertices x and y of $\Gamma'(R)$, we consider the following cases on the possibilities of x_i's such that $x \in V(\Gamma'(R))$, that is, x is a non-unit and non-zero element of the ring R.

Case-1. *$x_i \in Z(\mathbb{Z}_{n_i})^*$ for each $i \in [k]$.* First suppose that $y_i \in Z(\mathbb{Z}_{n_i})^*$ for each $i \in [k]$. Then consider $z = (1, 0, \ldots, 0) \in R$. Note that $(x) \nsubseteq (z)$ and $(z) \nsubseteq (x)$. Similarly, $(y) \nsubseteq (z)$ and $(z) \nsubseteq (y)$. It follows that $x \sim z \sim y$ in $\Gamma'(R)$ and so $d(x, y) = 2$. Now let $y_j = 0$ for some $j \in [k]$ and $y_i \in Z(\mathbb{Z}_{n_i})^*$ for remaining $i \in [k]$. Since $x \nsim y$, it implies that $(y_i) \subseteq (x_i)$ for each $i \in [k]$. Choose $z = (z_1, z_2, \ldots, z_k) \in R$ such that $z_i = 0$ if $y_i \in Z(\mathbb{Z}_{n_i})^*$, and $z_j \in U(\mathbb{Z}_{n_j})$ if $y_j = 0$. Note that $(z) \nsubseteq (x)$ and $(z) \nsubseteq (y)$. Also, $(x) \nsubseteq (z)$ and $(y) \nsubseteq (z)$. Therefore, $x \sim z \sim y$ and so $d(x, y) = 2$. Further, note that if $y_j = 0$ for some $j \in [k]$ and $y_i \in U(\mathbb{Z}_{n_i})$ for remaining $i \in [k]$, then one can observe that $x \sim y$ and so $d(x, y) = 1$. Now, let $y_i \in U(\mathbb{Z}_{n_i})$ for some $i \in [k]$ and $y_j \in Z(\mathbb{Z}_{n_j})^*$ for remaining $j \in [k]$. Since $x \nsim y$, we obtain $(x_i) \subseteq (y_i)$ for each $i \in [k]$. Thus, consider $z = (z_1, z_2, \ldots, z_k) \in R$ such that $z_i = 0$ if $y_i \in U(\mathbb{Z}_{n_i})$, and $z_j \in U(\mathbb{Z}_{n_j})$ if $y_j \in Z(\mathbb{Z}_{n_j})^*$. It follows that $x \sim z \sim y$ in $\Gamma'(R)$ and so $d(x, y) = 2$. Finally, assume that $y_j \in Z(\mathbb{Z}_{n_j})^*$ and $y_l = 0$ for some $j, l \in [k]$, and $y_i \in U(\mathbb{Z}_{n_i})$ for remaining $i \in [k]$. With this possibility, we have $(x) \nsubseteq (y)$ and $(y) \nsubseteq (x)$. Thus, $d(x, y) = 1$.

Case-2. *$x_j = 0$ for some $j \in [k]$ and $x_i \in U(\mathbb{Z}_{n_i})$ for remaining $i \in [k]$.* Since $d(x, y) = d(y, x)$, then by **Case-1**, for $y = (y_1, y_2, \ldots, y_k)$ such that $y_i \in Z(\mathbb{Z}_{n_i})^*$ for each $i \in [k]$, we get $d(x, y) = 1$. Now let $y_j = 0$ for some $j \in [k]$ and $y_i \in U(\mathbb{Z}_{n_i})$ for remaining $i \in [k]$. For $x \nsim y$, consider $z = (z_1, z_2, \ldots, z_k) \in R$ such that

$$z_i = \begin{cases} 1 & \text{when both } x_i = y_i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

With this choice of z, note that $(x) \nsubseteq (z)$ and $(z) \nsubseteq (x)$. Similarly, $(y) \nsubseteq (z)$ and $(z) \nsubseteq (y)$. It follows that $x \sim z \sim y$ and so $d(x, y) = 2$. Further, suppose that $y_j = 0$ for some $j \in [k]$ and $y_i \in Z(\mathbb{Z}_{n_i})^*$ for remaining $i \in [k]$. Since $x \nsim y$, we obtain $(y_i) \subseteq (x_i)$ for each $i \in [k]$. Consider $z = (z_1, z_2, \ldots, z_k)$ such that $z_i \in U(\mathbb{Z}_{n_i})$ if $y_i = 0$, and $z_j = 0$ if $y_j \in Z(\mathbb{Z}_{n_j})^*$. It follows that $x \sim z \sim y$. Consequently, $d(x, y) = 2$. Next, let $y_i \in U(\mathbb{Z}_{n_i})$ for some $i \in [k]$ and $y_j \in Z(\mathbb{Z}_{n_j})^*$ for remaining $j \in [k]$. Since $x \nsim y$, we have $(x_i) \subseteq (y_i)$ for each $i \in [k]$. Then consider $z = (z_1, z_2, \ldots, z_k) \in R$ such that

$$z_i = \begin{cases} 0 & \text{if } y_i \in U(\mathbb{Z}_{n_i}), \\ 1 & \text{if } y_i \in Z(\mathbb{Z}_{n_i})^*. \end{cases}$$

It follows that $x \sim z \sim y$ in $\Gamma'(R)$ and so $d(x, y) = 2$. Finally, assume that $y_j \in Z(\mathbb{Z}_{n_j})^*$ and $y_l = 0$ for some $j, l \in [k]$, and $y_i \in U(\mathbb{Z}_{n_i})$ for remaining $i \in [k]$. For $x \nsim y$, consider $z = (z_1, z_2, \ldots, z_k) \in R$ such that $z_i \in U(\mathbb{Z}_{n_i})$ if $x_i = 0$, and $z_j = 0$ if $x_j \in U(\mathbb{Z}_{n_j})$. Consequently, $x \sim z \sim y$ and so $d(x, y) = 2$.

Case-3. *$x_j = 0$ for some $j \in [k]$ and $x_i \in Z(\mathbb{Z}_{n_i})^*$ for remaining $i \in [k]$.* If $y = (y_1, y_2, \ldots, y_k)$ is such that $y_i \in Z(\mathbb{Z}_{n_i})^*$ for each $i \in [k]$, then by **Case-1** and the fact $d(x, y) = d(y, x)$, we have $d(x, y) = 2$. Similarly, if $y_j = 0$ for some $j \in [k]$ and $y_i \in U(\mathbb{Z}_{n_i})$ for remaining $i \in [k]$, then by **Case-2**, we obtain $d(x, y) = 2$.

Now let $y_j = 0$ for some $j \in [k]$ and $y_i \in Z(\mathbb{Z}_{n_i})^*$ for remaining $i \in [k]$. Then consider $z = (z_1, z_2, \ldots, z_k) \in R$ such that $z_i = 0$ if $x_i \in Z(\mathbb{Z}_{n_i})^*$, and $z_j = 1$ if $x_j = 0$. It follows that $(x) \nsubseteq (z)$ and $(z) \nsubseteq (x)$. Also, $(y) \nsubseteq (z)$ and $(z) \nsubseteq (y)$. Consequently, $x \sim z \sim y$ and so $d(x, y) = 2$. Next, assume that $y_j \in Z(\mathbb{Z}_{n_j})^*$ and $y_l = 0$ for some $j, l \in [k]$, and $y_i \in U(\mathbb{Z}_{n_i})$ for remaining $i \in [k]$. Since $x \nsim y$, it implies that $(x_i) \subseteq (y_i)$ for each $i \in [k]$. Consider $z = (z_1, z_2, \ldots, z_k)$ such that $z_i = 1$ if $x_i = 0$, and $z_j = 0$ if $x_j \in Z(\mathbb{Z}_{n_j})^*$. Consequently, we have $x \sim z \sim y$ in $\Gamma'(R)$. Therefore, $d(x, y) = 2$. Further, let $y_i \in U(\mathbb{Z}_{n_i})$ for some $i \in [k]$ and $y_j \in Z(\mathbb{Z}_{n_j})^*$ for remaining $j \in [k]$. Since $x \nsim y$ in $\Gamma'(R)$, we get $(x_i) \subseteq (y_i)$ for each $i \in [k]$. Suppose that there exists exactly one $r \in [k]$ such that $x_r \in Z(\mathbb{Z}_{n_r})^*$ and $x_i = 0$ for each $i \in [k] \setminus \{r\}$. Also, $y_r \in Z(\mathbb{Z}_{n_r})^*$ and $y_i \in U(\mathbb{Z}_{n_i})$ for each $i \in [k] \setminus \{r\}$. Then $(x_r) \subseteq (y_r)$. It follows that $(x, y) \in S_r$. Let $a = (a_1, a_2, \ldots, a_r, \ldots, a_k) \in V(\Gamma'(R))$ such that $a \sim y$. Then $(y_r) \subsetneq (a_r)$ and it follows that $(x_r) \subseteq (y_r) \subsetneq (a_r)$. Consequently, $(x_i) \subseteq (a_i)$ for each $i \in [k]$ and so $a \nsim x$ in $\Gamma'(R)$. Therefore, $d(x, y) > 2$. Consider $z = (z_1, z_2, \ldots, z_k)$ and $z' = (z'_1, z'_2, \ldots, z'_k) \in R$ such that

$$z_i = \begin{cases} 1 & \text{if } x_i = 0, \\ 0 & \text{if } x_i \in Z(\mathbb{Z}_{n_i})^* \end{cases}$$

and

$$z'_i = \begin{cases} 0 & \text{if } y_i \in U(\mathbb{Z}_{n_i}), \\ 1 & \text{if } y_i \in Z(\mathbb{Z}_{n_i})^*. \end{cases}$$

Then note that $(z) \nsubseteq (z')$ and $(z') \nsubseteq (z)$. Also, $(x) \nsubseteq (z)$ and $(z) \nsubseteq (x)$. Moreover, $(y) \nsubseteq (z')$ and $(z') \nsubseteq (y)$. It follows that $x \sim z \sim z' \sim y$ in $\Gamma'(R)$. Therefore, $d(x, y) = 3$.

Next, we claim that if there exist t and $r \in [k]$ such that $x_t \in Z(\mathbb{Z}_{n_t})^*$ and $x_r \in Z(\mathbb{Z}_{n_r})^*$, then $d(x, y) \leq 2$. Since $x \nsim y$, we obtain $(x) \subsetneq (y)$. If there exists $i_1 \in [k]$ such that $x_{i_1}, y_{i_1} \in Z(\mathbb{Z}_{n_{i_1}})^*$, then take $r = i_1$. Now consider $z = (z_1, z_2, \ldots, z_k) \in R$ such that

$$z_i = \begin{cases} 0 & \text{if } i = t, \\ 1 & \text{if } i = r, \\ 0 & \text{if } y_i \in U(\mathbb{Z}_{n_i}) \text{ and } i \neq \{t, r\}, \\ 1 & \text{if } y_i \in Z(\mathbb{Z}_{n_i})^* \text{ and } i \neq \{t, r\}. \end{cases}$$

Then note that $x \sim z \sim y$ in $\Gamma'(R)$. Therefore, $d(x, y) = 2$.

Case-4. *$x_i \in U(\mathbb{Z}_{n_i})$ for some $i \in [k]$ and $x_j \in Z(\mathbb{Z}_{n_j})^*$ for remaining $j \in [k]$.* Let $y_j \in U(\mathbb{Z}_{n_j})$ for some $j \in [k]$ and $y_i \in Z(\mathbb{Z}_{n_i})^*$ for remaining $i \in [k]$. For $x \nsim y$ in $\Gamma'(R)$, consider $z = (z_1, z_2, \ldots, z_k) \in R$ such that $z_i = 0$ if $x_i \in U(\mathbb{Z}_{n_i})$, and $z_j = 1$ if $x_j \in Z(\mathbb{Z}_{n_j})^*$. Note that $(z) \nsubseteq (x)$ and $(z) \nsubseteq (y)$. Also, $(x) \nsubseteq (z)$ and $(y) \nsubseteq (z)$. Therefore, $x \sim z \sim y$ and so $d(x, y) = 2$. Next, let $y_j \in Z(\mathbb{Z}_{n_j})^*$ and $y_l = 0$ for some $j, l \in [k]$, and $y_i \in U(\mathbb{Z}_{n_i})$ for remaining $i \in [k]$. Since $x \nsim y$ in $\Gamma'(R)$, we get $(y_i) \subseteq (x_i)$ for each $i \in [k]$. Thus, consider $z = (z_1, z_2, \ldots, z_k)$ such that $z_i = 1$ if $x_i \in Z(\mathbb{Z}_{n_i})^*$, and $z_j = 0$ if $x_j \in U(\mathbb{Z}_{n_j})$. Notice that $(x) \nsubseteq (z)$ and $(z) \nsubseteq (x)$. Similarly, $(y) \nsubseteq (z)$ and $(z) \nsubseteq (y)$. Therefore, $x \sim z \sim y$ and so $d(x, y) = 2$.

Case-5. *$x_j \in Z(\mathbb{Z}_{n_j})^*$ and $x_l = 0$ for some $j, l \in [k]$, and $x_i \in U(\mathbb{Z}_{n_i})$ for remaining $i \in [k]$.* In view of the earlier cases and $d(x, y) = d(y, x)$, in this case we require to compute $d(x, y)$ only when $y = (y_1, y_2, \ldots, y_k) \in R$ such that $y_j \in Z(\mathbb{Z}_{n_j})^*$ and $y_l = 0$ for some $j, l \in [k]$, and $y_i \in U(\mathbb{Z}_{n_i})$ for remaining $i \in [k]$. For non-adjacent vertices x and y, choose $z = (z_1, z_2, \ldots, z_k) \in R$ such that

$$z_i = \begin{cases} 0 & \text{if } x_i \in Z(\mathbb{Z}_{n_i})^* \text{ and } x_i \in U(\mathbb{Z}_{n_i}), \\ 1 & \text{if } x_i = 0. \end{cases}$$

Then observe that $(x) \nsubseteq (z)$ and $(z) \nsubseteq (x)$. Moreover, $(y) \nsubseteq (z)$ and $(z) \nsubseteq (y)$. Consequently, $x \sim z \sim y$. It follows that $d(x, y) = 2$.

Thus, the result holds. □

Now for a positive integer n, let $d_1, d_2, \ldots, d_t$ be all the proper divisors of n. Then define

$$\mathcal{A}_{d_i} = \{x \in \mathbb{Z}_n : \gcd(x, n) = d_i\}.$$

We shall use the following facts and observations without referring them explicitly.

(i) $|\mathcal{A}_{d_i}| = \phi(\frac{n}{d_i})$ (cf. [33]), where ϕ is the Euler-totient function.

(ii) For each d_i, where $1 \le i \le t$, $\mathcal{A}_{d_i} \subsetneq Z(\mathbb{Z}_n)^*$.

(iii) For $u, v \in \mathcal{A}_{d_i}$, we have $(u) = (v) = (d_i)$.

Let $R \cong \mathbb{Z}_{p_1^{m_1}} \times \mathbb{Z}_{p_2^{m_2}} \times \cdots \times \mathbb{Z}_{p_k^{m_k}}$ $(k \ge 2)$, where each p_i is a prime. For each $i \in [k]$, consider $X_i^0 = \{0\}$, $X_i^1 = U(\mathbb{Z}_{p_i^{m_i}})$ and $X_i^j = \mathcal{A}_{p_i^{j-1}}$ for $2 \le j \le m_i$. Then

$$|X_i^j| = \begin{cases} 1 & \text{if } j = 0, \\ p_i^{m_i} - p_i^{m_i-1} & \text{if } j = 1, \\ p_i^{m_i-j+1} - p_i^{m_i-j} & \text{if } 2 \le j \le m_i. \end{cases}$$

Let $x = (x_1, x_2, \ldots, x_k)$ and $y = (y_1, y_2, \ldots, y_k) \in R$. One can observe that $(x) = (y)$ if and only if $(x_i) = (y_i)$ for each i. Consequently, each equivalence class of $V(\Gamma'(R))$, under the relation $\equiv$, is of the form $X_1^{j_1} \times X_2^{j_2} \times \cdots \times X_k^{j_k}$, where $0 \le j_r \le m_r$. Moreover, $|X_1^{j_1} \times X_2^{j_2} \times \cdots \times X_k^{j_k}| = \prod_{i=1}^k |X_i^{j_i}|$.

In view of Lemma 3.1, now we calculate the Wiener index of $\Gamma'(R)$, where $R \cong \mathbb{Z}_{p_1^{m_1}} \times \mathbb{Z}_{p_2^{m_2}} \times \cdots \times \mathbb{Z}_{p_k^{m_k}}$ and $k \ge 2$. Let $x = (x_1^{j_1}, x_2^{j_2}, \ldots, x_k^{j_k})$ and $y = (y_1^{l_1}, y_2^{l_2}, \ldots, y_k^{l_k})$ be the representatives of two distinct equivalence classes $X_1^{j_1} \times X_2^{j_2} \times \cdots \times X_k^{j_k}$ and $X_1^{l_1} \times X_2^{l_2} \times \cdots \times X_k^{l_k}$, respectively.

Theorem 3.2. *Let $R \cong \mathbb{Z}_{p_1^{m_1}} \times \mathbb{Z}_{p_2^{m_2}} \times \cdots \times \mathbb{Z}_{p_k^{m_k}}$ and S_r be the set defined in Lemma 3.1. Then the Wiener index of the cozero-divisor graph $\Gamma'(R)$ is given by*

$$\begin{aligned} W(\Gamma'(R)) = {} & 2 \sum_{(x_1^{j_1}, x_2^{j_2}, \ldots x_k^{j_k}) \in \Upsilon'} \binom{\prod_{\substack{i=1 \\ j_i \ge 1}}^k (p_i^{m_i - j_i + 1} - p_i^{m_i - j_i})}{2} + \sum_{x \sim y} \left(\prod_{\substack{i=1 \\ j_i \ge 1}}^k (p_i^{m_i - j_i + 1} - p_i^{m_i - j_i}) \right) \left(\prod_{\substack{i=1 \\ l_i \ge 1}}^k (p_i^{m_i - l_i + 1} - p_i^{m_i - l_i}) \right) \\ & + 2 \sum_{\substack{x \nsim y \\ (x,y) \notin S_r \\ (y,x) \notin S_r}} \left(\prod_{\substack{i=1 \\ j_i \ge 1}}^k (p_i^{m_i - j_i + 1} - p_i^{m_i - j_i}) \right) \left(\prod_{\substack{i=1 \\ l_i \ge 1}}^k (p_i^{m_i - l_i + 1} - p_i^{m_i - l_i}) \right) \\ & + 3 \sum_{(x,y) \in S_r} \left(\prod_{\substack{i=1 \\ j_i \ge 1}}^k (p_i^{m_i - j_i + 1} - p_i^{n_i - j_i}) \right) \left(\prod_{\substack{i=1 \\ l_i \ge 1}}^k (p_i^{m_i - l_i + 1} - p_i^{m_i - l_i}) \right). \end{aligned}$$

Example 3.3. *Let $R \cong \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_9$. Then note that $X_1^0 = X_2^0 = X_3^0 = \{0\}$, $X_1^1 = U(\mathbb{Z}_2) = \{1\}$, $X_2^1 = U(\mathbb{Z}_4) = \{1, 3\}$, $X_3^1 = U(\mathbb{Z}_9) = \{1, 2, 4, 5, 7, 8\}$, $X_2^2 = \mathcal{A}_2 = \{2\}$ and $X_3^2 = \mathcal{A}_3 = \{3, 6\}$. Thus, the set of all equivalence classes of $V(\Gamma'(R))$ is $\{X_1^{j_1} \times X_2^{j_2} \times X_3^{j_3} : 0 \le j_1 \le 1, 0 \le j_2, j_3 \le 2\} \setminus \{X_1^0 \times X_2^0 \times X_3^0, X_1^1 \times X_2^1 \times X_3^1\}$. Consequently, we have* *the following* 16 *equivalence classes of* $V(\Gamma'(R))$*, viz.* $Y_1 = X_1^0 \times X_2^0 \times X_3^1$, $Y_2 = X_1^0 \times X_2^0 \times X_3^2$, $Y_3 = X_1^0 \times X_2^1 \times X_3^0$, $Y_4 = X_1^0 \times X_2^2 \times X_3^0$, $Y_5 = X_1^0 \times X_2^1 \times X_3^1$, $Y_6 = X_1^0 \times X_2^2 \times X_3^1$, $Y_7 = X_1^0 \times X_2^1 \times X_3^2$, $Y_8 = X_1^0 \times X_2^2 \times X_3^2$, $Y_9 = X_1^1 \times X_2^0 \times X_3^0$, $Y_{10} = X_1^1 \times X_2^0 \times X_3^1$, $Y_{11} = X_1^1 \times X_2^0 \times X_3^2$, $Y_{12} = X_1^1 \times X_2^1 \times X_3^0$, $Y_{13} = X_1^1 \times X_2^1 \times X_3^2$, $Y_{14} = X_1^1 \times X_2^2 \times X_3^0$, $Y_{15} = X_1^1 \times X_2^2 \times X_3^1$ *and* $Y_{16} = X_1^1 \times X_2^2 \times X_3^2$*. Let* y_i *be the representative of the equivalence class* Y_i*, where* $1 \leq i \leq 16$*. Note that* S_1 *is an empty set and* $S_2 = \{(y_4, y_{15})\}$*,* $S_3 = \{(y_2, y_{13})\}$*. Now, the pair of equivalence classes whose elements are at distance two in* $\Gamma'(R)$ *are*

$\{\{Y_1,Y_2\}, \{Y_1,Y_5\}, \{Y_1,Y_6\}, \{Y_1,Y_{10}\}, \{Y_1,Y_{15}\}, \{Y_2,Y_5\}, \{Y_2,Y_6\}, \{Y_2,Y_7\}, \{Y_2,Y_8\}, \{Y_2,Y_{10}\}, \{Y_2,Y_{11}\}, \{Y_2,Y_{15}\}, \{Y_2,Y_{16}\}, \{Y_3,Y_4\}, \{Y_3,Y_5\}, \{Y_3,Y_7\}, \{Y_3,Y_{12}\}, \{Y_3,Y_{13}\}, \{Y_4,Y_5\}, \{Y_4,Y_6\}, \{Y_4,Y_7\}, \{Y_4,Y_8\}, \{Y_4,Y_{12}\}, \{Y_4,Y_{13}\}, \{Y_4,Y_{14}\}, \{Y_4,Y_{16}\}, \{Y_5,Y_6\}, \{Y_5,Y_7\}, \{Y_5,Y_8\}, \{Y_6,Y_8\}, \{Y_6,Y_{15}\}, \{Y_7,Y_8\}, \{Y_7,Y_{13}\}, \{Y_8,Y_{13}\}, \{Y_8,Y_{15}\}, \{Y_8,Y_{16}\}, \{Y_9,Y_{10}\}, \{Y_9,Y_{11}\}, \{Y_9,Y_{12}\}, \{Y_9,Y_{13}\}, \{Y_9,Y_{14}\}, \{Y_9,Y_{15}\}, \{Y_9,Y_{16}\}, \{Y_{10},Y_{11}\}, \{Y_{10},Y_{15}\}, \{Y_{11},Y_{13}\}, \{Y_{11},Y_{15}\}, \{Y_{11},Y_{16}\}, \{Y_{12},Y_{13}\}, \{Y_{12},Y_{14}\}, \{Y_{13},Y_{14}\}, \{Y_{13},Y_{16}\}, \{Y_{14},Y_{15}\}, \{Y_{14},Y_{16}\}, \{Y_{15},Y_{16}\}\}$.

Thus, the Wiener index of the cozero-divisor graph of the ring $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_9$ *is given by*

$$\begin{aligned}
W(\Gamma'(\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_9)) = {} & 2 \times \frac{1}{2}[30+2+2+0+132+30+12+2+0+30+2+2+12+0+30+2] \\
& + [6(2+1+4+2+1+2+2+4+1+2) + 2(2+1+1+2+1) \\
& + 2(6+2+1+6+2+1+6+2) + (1+6+2) + 12(1+6+2+2+4+1+6+2) \\
& + 6(4+1+6+2+2+4+1+2) + 4(1+6+2+2+1+6+2) + 2(1+6+2+2+1) \\
& + (0) + 6(2+4+1+2) + 2(2+1) + 2(6+2) + 4(6) + (0)] \\
& + 2[6(2+12+6+6+6) + 2(12+6+4+2+6+2+6+2) + 2(1+12+4+2+4) \\
& + (12+6+4+2+2+4+1+2) + 12(6+4+2) + 6(2+6) + 4(2+4) + 2(4+6+2) \\
& + (6+2+2+4+1+6+2) + 6(2+6) + 2(4+6+2) + 2(4+1) + 4(1+2) + (6+2) + 6(2)] \\
& + 3[(1 \times 6) + (2 \times 4)] \\
= {} & 2611.
\end{aligned}$$

4. The Wiener index of the cozero-divisor graph of reduced ring

In this section, we obtain the Wiener index of the cozero-divisor graph of a finite commutative reduced ring. Let R be a reduced ring i.e. $R \cong F_{q_1} \times F_{q_2} \times \cdots \times F_{q_k}$ with $k \geq 2$, where F_{q_i} is a finite field with q_i elements. Notice that, for $x = (x_1, x_2, \ldots, x_k)$ and $y = (y_1, y_2, \ldots, y_k) \in R$ such that $(x) = (y)$, we have $x_i = 0$ if and only if $y_i = 0$ for each i. For $i_1, i_2, \ldots, i_r \in [k]$, define

$$X_{\{i_1,i_2,\ldots,i_r\}} = \{(x_1, x_2, \ldots, x_k) \in R : \text{only } x_{i_1}, x_{i_2}, \ldots, x_{i_r} \text{ are non-zero}\}.$$

Note that the sets X_A, where A is a non-empty proper subset of $[k]$, are the equivalence classes of $V(\Gamma'(R))$ under the relation $\equiv$. Let x_A be the representative of equivalence class X_A. Now we obtain the possible distances between the vertices of $\Upsilon'(R)$.

Lemma 4.1. *For the distinct vertices* x_A *and* x_B *of* $\Upsilon'(R)$*, we have*

$$d_{\Upsilon'(R)}(x_A, x_B) = \begin{cases} 1 & \text{if } A \nsubseteq B \text{ and } B \nsubseteq A, \\ 2 & \text{otherwise.} \end{cases}$$

Proof. First assume that $A \nsubseteq B$ and $B \nsubseteq A$. Then $(x_A) \nsubseteq (x_B)$ and $(x_B) \nsubseteq (x_A)$. It follows that $d_{\Upsilon'(R)}(x_A, x_B) = 1$. Now, without loss of generality, let $A \subsetneq B$. Then there exists $i \in [k]$ such that $i \notin B$ and so $i \notin A$. By Lemma 2.1, we have $x_A \sim x_{\{i\}} \sim x_B$. Thus, $d_{\Upsilon'(R)}(x_A, x_B) = 2$. □

For distinct subsets $A, B \subsetneq [k]$, define $D_1 = \{\{A, B\} : A \nsubseteq B\}$ and $D_2 = \{\{A, B\} : A \subsetneq B\}$. Using Theorem 2.5 and the sets D_1 and D_2, we obtain the Wiener index of the cozero-divisor $\Gamma'(R)$ of a reduced ring R in the following theorem.

Theorem 4.2. *The Wiener index of the cozero-divisor graph of a finite commutative reduced ring $R \cong F_{q_1} \times F_{q_2} \times \cdots \times F_{q_k}$, where $k \geq 2$, is given by*

$$W(\Gamma'(R)) = 2\sum_{A \subsetneq [k]} \binom{\prod_{i\in A}(q_i - 1)}{2} + \sum_{\{A,B\}\in D_1} \left(\prod_{i\in A}(q_i - 1)\right)\left(\prod_{j\in B}(q_j - 1)\right) + 2\sum_{\{A,B\}\in D_2} \left(\prod_{i\in A}(q_i - 1)\right)\left(\prod_{j\in B}(q_j - 1)\right).$$

Proof. The proof follows from Lemma 4.1. □

Example 4.3. *[23, Corollary 6.2] Let $R = \mathbb{Z}_{pq} \cong \mathbb{Z}_p \times \mathbb{Z}_q$, where p and q are distinct primes. Then we have two distinct equivalence classes, $X_{\{1\}} = \{(a, 0) : a \in \mathbb{Z}_p \setminus \{0\}\}$ and $X_{\{2\}} = \{(0, b) : b \in \mathbb{Z}_q \setminus \{0\}\}$, of the equivalence relation $\equiv$. Moreover, $D_1 = \{\{\{1\}, \{2\}\}\}$ and D_2 is an empty set. Note that $|X_{\{1\}}| = p - 1$ and $|X_{\{2\}}| = q - 1$. Consequently, by Theorem 4.2, we get $W(\Gamma'(\mathbb{Z}_{pq})) = (p - 1)(p - 2) + (q - 1)(q - 2) + (p - 1)(q - 1) = p^2 + q^2 - 4p - 4q + pq + 5$.*

Example 4.4. *Let $R = \mathbb{Z}_{pqr} \cong \mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r$, where p, q, r are distinct primes. For $a \in \mathbb{Z}_p \setminus \{0\}$, $b \in \mathbb{Z}_q \setminus \{0\}$ and $c \in \mathbb{Z}_r \setminus \{0\}$, we have the equivalence classes : $X_{\{1\}} = \{(a, 0, 0)\}$, $X_{\{2\}} = \{(0, b, 0)\}$, $X_{\{3\}} = \{(0, 0, c)\}$, $X_{\{1,2\}} = \{(a, b, 0)\}$, $X_{\{1,3\}} = \{(a, 0, c)\}$, $X_{\{2,3\}} = \{(0, b, c)\}$. Moreover, $D_1 = \Big\{\{\{1\}, \{2\}\}, \{\{1\}, \{3\}\}, \{\{2\}, \{3\}\}, \{\{1, 2\}, \{1, 3\}\}, \{\{1, 2\}, \{2, 3\}\}, \{\{1, 3\}, \{2, 3\}\}, \{\{1\}, \{2, 3\}\}, \{\{2\}, \{1, 3\}\}, \{\{3\}, \{1, 2\}\}\Big\}$ and $D_2 = \Big\{\{\{1\}, \{1, 2\}\}, \{\{1\}, \{1, 3\}\}, \{\{2\}, \{1, 2\}\}, \{\{2\}, \{2, 3\}\}, \{\{3\}, \{1, 3\}\}, \{\{3\}, \{2, 3\}\}\Big\}$. Also, $|X_{\{1\}}| = p - 1$, $|X_{\{2\}}| = q - 1$, $|X_{\{3\}}| = r - 1$, $|X_{\{1,2\}}| = (p - 1)(q - 1)$, $|X_{\{1,3\}}| = (p - 1)(r - 1)$, $|X_{\{2,3\}}| = (q - 1)(r - 1)$. Then, by Theorem 4.2, the Wiener index of $\Gamma'(R)$ is given by*

$$\begin{aligned} W(\Gamma'(\mathbb{Z}_{pqr})) = {} & 2\binom{p-1}{2} + 2\binom{q-1}{2} + 2\binom{r-1}{2} + 2\binom{(p-1)(q-1)}{2} + 2\binom{(p-1)(r-1)}{2} + 2\binom{(q-1)(r-1)}{2} \\ & + (p-1)(q-1) + (p-1)(r-1) + (q-1)(r-1) + (p-1)(q-1)(p-1)(r-1) \\ & + (p-1)(q-1)(q-1)(r-1) + (p-1)(r-1)(q-1)(r-1) + (p-1)(q-1)(r-1) \\ & + (q-1)(p-1)(r-1) + (r-1)(p-1)(q-1) + 2(p-1)\left[(p-1)(q-1)\right] \\ & + 2(p-1)\left[(p-1)(r-1)\right] + 2(q-1)\left[(p-1)(q-1)\right] + 2(q-1)\left[(q-1)(r-1)\right] \\ & + 2(r-1)\left[(p-1)(r-1)\right] + 2(r-1)\left[(q-1)(r-1)\right]. \end{aligned}$$

Simplifying this expression, we get

$$W(\Gamma'(\mathbb{Z}_{pqr})) = pqr(p+q+r-3) + p^2q^2 + p^2r^2 + q^2r^2 - p^2(q+r) - q^2(p+r) - r^2(p+q) - 2(pq+pr+qr) + 4(p+q+r) - 3.$$

Let $\tau(n)$ be the number of divisors of n and let $D = \{d_1, d_2, \ldots, d_{\tau(n)-2}\}$ be the set of all proper divisors of $n = p_1^{n_1} p_2^{n_2} \cdots p_r^{n_r} \cdots p_k^{n_k}$, where $k \geq 2$. If $d_i \mid d_j$, then define

$$A = \{(d_i, d_j) \in D \times D \mid d_i \neq p_r^s\};$$

$$B = \{(d_i, d_j) \in D \times D \mid d_i = p_r^s \text{ and } \frac{n}{d_j} \neq p_r^t\};$$

$$C = \{(d_i, d_j) \in D \times D \mid d_i = p_r^s \text{ and } \frac{n}{d_j} = p_r^t\}.$$

Using the notations defined above, the following theorem which was proved in [23], by using different approach, can also be obtained by using Lemma 3.1 and Theorem 3.2.

Theorem 4.5. *[23, Theorem 6.3] For $n = p_1^{n_1} p_2^{n_2} \cdots p_r^{n_r} \cdots p_k^{n_k}$ with $k \geq 2$ and p_i's are distinct primes, we have*

$$W(\Gamma'(\mathbb{Z}_n)) = \sum_{i=1}^{\tau(n)-2} \phi\left(\frac{n}{d_i}\right)\left(\phi\left(\frac{n}{d_i}\right) - 1\right) + \frac{1}{2}\sum_{\substack{d_i \nmid d_j \\ d_j \nmid d_i}} \phi\left(\frac{n}{d_i}\right)\phi\left(\frac{n}{d_j}\right) + 2\sum_{(d_i,d_j)\in A} \phi\left(\frac{n}{d_i}\right)\phi\left(\frac{n}{d_j}\right)$$
$$+ 2\sum_{(d_i,d_j)\in B} \phi\left(\frac{n}{d_i}\right)\phi\left(\frac{n}{d_j}\right) + 3\sum_{(d_i,d_j)\in C} \phi\left(\frac{n}{d_i}\right)\phi\left(\frac{n}{d_j}\right).$$

5. SageMath Code

In this section, we produce a SAGE code to compute the Wiener index of the cozero-divisor graph of ring classes considered in this paper including the ring $\mathbb{Z}_n$ of integers modulo n. On providing the value of integer n, the following SAGE code computes the Wiener index of the graph $\Gamma'(\mathbb{Z}_n)$.

```
cozero_divisor_graph=Graph()
E=[]
n=72

for i in range(n):
    for j in range(n):
        if(i!=j):
            p=gcd(i,n)
            q=gcd(j,n)
            if (p%q!=0 and q%p!=0):
                E.append((i,j))

cozero_divisor_graph.add_edges(E)

if(E==[]):
    V=[]
    for i in range(1,n):
        if (gcd(i,n)!=1):
            V.append(i)
    cozero_divisor_graph.add_vertices(V)


W=cozero_divisor_graph.wiener_index();

if (W==oo):
    print("Wiener Index undefined for Null Graph")
else :
    print("Wiener Index:", W)
```

Using the given code, in Table 1, we obtain the Wiener index of $\Gamma'(\mathbb{Z}_n)$ for some values of n.

n	100	500	1000	1500	2000	2500
$W(\Gamma'(\mathbb{Z}_n))$	2954	77174	306202	930248	1222530	1946274

Table 1: Wiener index of $\Gamma'(\mathbb{Z}_n)$

Let R be a reduced ring i.e. $R \cong F_{q_1} \times F_{q_2} \times \cdots \times F_{q_n}$, where F_{q_i} is a field with q_i elements. The following code determines the Wiener index of $\Gamma'(R)$ by providing the values of the field size q_i $(1 \leq i \leq n)$.

```
field_orders=[3,5,7]
P=Subsets(range(len(field_orders)))[1:-1]
P=[Set(i) for i in P]

D1=[]
D2=[]
for i in P:
    for j in P:
        if (not(i.issubset(j) or j.issubset(i)) and P.index(i) > P.index(j)):
            D1.append([i,j])
        if (i.issubset(j) and i!=j):
            D2.append([i,j])

partial_sum=0
for i in P:
    sum_pp=1
    for j in i:
        sum_pp *= field_orders[j]-1
    partial_sum +=((sum_pp*(sum_pp-1))/2)

D1_sum=0
for i in D1:
    D1_pp=1
    for j in i[0]:
        D1_pp *= field_orders[j]-1
    for k in i[1]:
        D1_pp *= field_orders[k]-1
    D1_sum += D1_pp

D2_sum=0
for i in D2:
    D2_pp=1
    for j in i[0]:
        D2_pp *= field_orders[j]-1
    for k in i[1]:
        D2_pp *= field_orders[k]-1
    D2_sum += D2_pp

W = 2*partial_sum + D1_sum + 2*D2_sum
print("Wiener Index:" , W)
```

Using the given code, in the following tables, we obtain the Wiener index of the cozero-divisor graphs of the reduced rings $F_{q_1} \times F_{q_2}$ (see Table 2) and $F_{q_1} \times F_{q_2} \times F_{q_3}$ (see Table 3), respectively.

(q_1, q_2)	$(9, 25)$	$(49, 81)$	$(101, 121)$	$(125, 139)$	$(163, 169)$	$(289, 343)$
$W(\Gamma'(F_{q_1} \times F_{q_2}))$	800	12416	36180	51270	81354	297774

Table 2: Wiener index of $\Gamma'(F_{q_1} \times F_{q_2})$

(q_1, q_2, q_3)	$(7, 8, 13)$	$(9, 25, 49)$	$(53, 64, 81)$	$(83, 101, 121)$	$(125, 131, 169)$	$(289, 343, 361)$
$W(\Gamma'(F_{q_1} \times F_{q_2} \times F_{q_3}))$	35196	2500400	108637254	620456582	2355211790	71251552134

Table 3: Wiener index of $\Gamma'(F_{q_1} \times F_{q_2} \times F_{q_3})$

Let $R \cong \mathbb{Z}_{p_1^{m_1}} \times \mathbb{Z}_{p_2^{m_2}} \times \cdots \times \mathbb{Z}_{p_k^{m_k}}$. Then the following SAGE code gives the value of $W(\Gamma'(R))$ after providing the values of $p_i^{m_i} (1 \leq i \leq k)$, where each p_i is a prime.

```
orders = [2,4,9]
A = cartesian_product([range(i) for i in orders]).list()
units = [{i for i in range(1,j) if gcd(i,j) == 1} for j in orders]

def contQ(lst1, lst2):
    flag = True
    for i in range(len(orders)):
        p=gcd(lst1[i],orders[i])
        q=gcd(lst2[i],orders[i])
        if(not(lst1[i]==0 or {lst2[i]}.issubset(units[i]) or p%q==0)):
            flag = False
    return flag

E=[]
for i in A:
    for j in A:
        if(not(contQ(i,j) or contQ(j,i))and A.index(i) > A.index(j)):
            E.append([i,j])

G = Graph()
G.add_edges(E)
W=G.wiener_index()
print("Wiener_Index:", W)
```

Using the given code, we obtain the Wiener index of the cozero-divisor graph of the ring $R \cong \mathbb{Z}_{p_1^{m_1}} \times \mathbb{Z}_{p_2^{m_2}} \times \cdots \times \mathbb{Z}_{p_k^{m_k}}$, where $k \geq 2$, in Table 4.

R	$W(\Gamma'(R))$
$\mathbb{Z}_4 \times \mathbb{Z}_9$	420
$\mathbb{Z}_9 \times \mathbb{Z}_{25}$	8808
$\mathbb{Z}_{16} \times \mathbb{Z}_{25}$	48870
$\mathbb{Z}_{27} \times \mathbb{Z}_{49}$	268022
$\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_4$	521
$\mathbb{Z}_5 \times \mathbb{Z}_7 \times \mathbb{Z}_{11}$	14948
$\mathbb{Z}_8 \times \mathbb{Z}_9 \times \mathbb{Z}_{16}$	167769
$\mathbb{Z}_4 \times \mathbb{Z}_9 \times \mathbb{Z}_{25}$	327394
$\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_9 \times \mathbb{Z}_9$	232937
$\mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_8 \times \mathbb{Z}_8$	333963

Table 4: Wiener index of $\Gamma'(R)$

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