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Applications of $\mathbb{Z}_2\mathbb{Z}_2[u]\mathbb{Z}_2[u^k]$-additive cyclic codes in the construction of optimal codes

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Abstract. Let $\mathbb{Z}_2 = \{0, 1\}$, $\mathfrak{R} = \mathbb{Z}_2 + u\mathbb{Z}_2$, where $u^2 = 0$ and $\mathfrak{R}_{u^t} = \mathbb{Z}_2 + u\mathbb{Z}_2 + \cdots + u^{k-1}\mathbb{Z}_2$, where $u^k = 0$. In this article, we study $\mathbb{Z}_2\mathfrak{R}\mathfrak{R}_{u^t}$-additive cyclic, additive dual codes and their structural properties. The additive cyclic codes are characterized as $\mathfrak{R}_{u^t}[y]$-submodules of the ring

$$\mathcal{S}_{\beta_1,\beta_2,\beta_3} = \mathbb{Z}_2[y]/\langle y^{\beta_1} - 1\rangle \times \mathfrak{R}[y]/\langle y^{\beta_2} - 1\rangle \times \mathfrak{R}_{u^t}[y]/\langle y^{\beta_3} - 1\rangle.$$

The extended Gray map is represented by $\Psi_1 : \mathbb{Z}_2^{\beta_1} \times \mathfrak{R}^{\beta_2} \times \mathfrak{R}_{u^t}{}^{\beta_3} \longrightarrow \mathbb{Z}_2^{\beta_1+2\beta_2+k\beta_3}$ and is utilized to construct the binary codes with good parameters. The minimal generating polynomials and smallest spanning sets of the above specified codes are obtained. We also establish the relationship between the minimal generating polynomials of additive cyclic codes and their duals. Further, we provide some examples that support our main results. Finally, the optimal binary codes are determined in Table.

1. Introduction

In recent years, cyclic codes are the most investigated class of codes. For instance, [17] and [25] both explored the algebraic structure and the generators of cyclic codes over $\mathbb{Z}_{p^m}$. Double cyclic codes over different rings are a relatively new idea in the literature of algebraic coding theory. A double cyclic code is one in which the set of coordinates may be divided into two subsets, each of which has its coordinates shifted cyclically such that the code remains invariant. Keep in mind that we acquire a cyclic code if one of these sets of coordinates is empty. Some examples of double cyclic codes over the rings $\mathbb{Z}_2$ and $\mathbb{Z}_4$ have been evaluated in [14, 23].

In 1973, additive codes were first defined by Delsarte [18, 19] in terms of association schemes. Generally, an additive code is defined as a subgroup of the underlying abelian group. In the special case of a binary Hamming scheme, when the underlying abelian group is of order 2^n, the structure for the abelian groups are those which are of the form $\mathbb{Z}_2^{\beta_1} \times \mathbb{Z}_4^{\beta_2}$ with $\beta_1 + 2\beta_2 = n$. Therefore, the subgroup C of $\mathbb{Z}_2^{\beta_1} \times \mathbb{Z}_4^{\beta_2}$ is the only additive code in a binary Hamming scheme. Borges et al. [16] developed the study $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic codes.

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In 2015, $\mathbb{Z}_2\mathbb{Z}_2[u]$-additive codes were introduced by Aydogdu et al. [7] as generalization of $\mathbb{Z}_2\mathbb{Z}_4$-additive codes and they determined the $\mathbb{Z}_2\mathbb{Z}_2[u]$-additive cyclic codes and also defined a mixed code consisting of the binary part and non-binary part from the ring $\mathbb{Z}_2 + u\mathbb{Z}_2$, where $u^2 = 0$. The $\mathbb{Z}_2\mathbb{Z}_4$-additive codes were further generalized to $\mathbb{Z}_2\mathbb{Z}_{2^s}$-additive codes by Aydogdu et al. [11]. Later on, Aydogdu et al. [12] generalized $\mathbb{Z}_2\mathbb{Z}_4$-additive codes and $\mathbb{Z}_2\mathbb{Z}_{2^s}$-additive codes to $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$-additive codes. In 2018, Borges et al. [15] described the structural properties of $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$-additive codes and obtained $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$-additive cyclic codes. Note that in $\mathbb{Z}_2\mathbb{Z}_4$-additive codes and $\mathbb{Z}_2\mathbb{Z}_{2^s}$-additive codes, $\mathbb{Z}_2$ is considered as $\mathbb{Z}_4$-algebra and $\mathbb{Z}_{2^s}$-algebra, respectively. Also in $\mathbb{Z}_2\mathbb{Z}_2[u]$-additive codes, $\mathbb{Z}_2$ is known as a $\mathbb{Z}_2[u]$-algebra and $\mathbb{Z}_{p^r}$ is a $\mathbb{Z}_{p^s}$-algebra in $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$-additive codes. In 2018, J. Gao et al. [20] gave the structural properties of additive cyclic codes over $\mathbb{Z}_p\mathbb{Z}_p[u]$, where $u^2 = 0$. They also found the minimal generating sets of additive cyclic codes. Moreover, they determined the relationship of generators between the additive codes and their dual codes.

In 2019, Minjia Shi et al. [29] described $\mathbb{Z}_2\mathbb{Z}_2[u, v]$-additive cyclic codes, where $u^2 = v^2 = 0$, $uv = vu$, which was the generalization of previously introduced $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic codes. Recently, Mahmoudi et al. [28] gave the structures of $\mathbb{Z}_2(\mathbb{Z}_2 + u\mathbb{Z}_2 + u^2\mathbb{Z}_2)$, $(\mathbb{Z}_2 + u\mathbb{Z}_2)(\mathbb{Z}_2 + u\mathbb{Z}_2 + u^2\mathbb{Z}_2)$, where $u^3 = 0$, $\mathbb{Z}_2(\mathbb{Z}_2 + u\mathbb{Z}_2 + v\mathbb{Z}_2)$, $u^2 = v^2 = uv = vu = 0$ and $(\mathbb{Z}_2 + u\mathbb{Z}_2)(\mathbb{Z}_2 + u\mathbb{Z}_2 + u^2\mathbb{Z}_2)$, where $u^2 = v^2 = uv = uv = 0$ and determined additive codes, dual additive codes and found singleton bound.

In 2018, Wu et al. [30] given the concept of $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive cyclic codes and obtained their generator polynomials along with their duals. They studied the structure of separable and non-separable $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive cyclic codes. In 2019, Aydogdu et al. [10] introduced the algebraic structure of $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-cyclic codes and studied the generator polynomials and minimal generating sets of this family of codes. Dinh et al. [22] discussed the structural properties of $\mathbb{F}_q\mathbf{RS}$-cyclic codes, where $\mathbf{R} = \mathbb{F}_q + u\mathbb{F}_q$, $u^2 = 1$ and $\mathbf{S} = \mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q + uv\mathbb{F}_q$, $u^2 = v^2 = 1$, $uv = vu$. Further, they applied these codes in the construction of quantum error-correcting codes. Later, Dinh et al. [21] studied $\mathbb{F}_2\mathbf{RS}$-cyclic codes and constructed several optimal and near-optimal codes, where $\mathbf{R} = \mathbb{F}_2 + u\mathbb{F}_2$, $u^2 = 0$ and $\mathbf{S} = \mathbb{F}_2 + u\mathbb{F}_2 + u^2\mathbb{F}_2$, $u^3 = 0$. Also, they presented some examples of optimal and near-optimal codes.

Motivated by aforementioned work, we consider two rings $\mathfrak{R} = \mathbb{Z}_2 + u\mathbb{Z}_2$, where $u^2 = 0$ and $\mathfrak{R}_{u^t} = \mathbb{Z}_2 + u\mathbb{Z}_2 + \cdots + u^{k-1}\mathbb{Z}_2$, where $u^k = 0$ with characteristic 2. $\mathfrak{R}_{u^t}$ is a local ring and the maximal ideal is principal. In this article, we determine $\mathbb{Z}_2\mathfrak{R}\mathfrak{R}_{u^t}$-additive cyclic codes and their duals. We also find the optimal binary images from $\mathbb{Z}_2\mathfrak{R}\mathfrak{R}_{u^t}$-additive codes. It is to be noted that the additive code of length $(\beta_1, \beta_2, \beta_3)$ is the subgroup of the commutative group $\mathbb{Z}_2^{\beta_1} \times \mathfrak{R}^{\beta_2} \times \mathfrak{R}_{u^t}{}^{\beta_3}$. The $\mathbb{Z}_2\mathfrak{R}\mathfrak{R}_{u^t}$-additive code is a linear code over $\mathbb{Z}_2$ if $\beta_2 = 0$ and $\beta_3 = 0$, over $\mathfrak{R}$ if $\beta_1 = 0$ and $\beta_3 = 0$ and over $\mathfrak{R}_{u^t}$ if $\beta_1 = 0$ and $\beta_2 = 0$. Clearly, we observe that it is the generalization of linear code over $\mathbb{Z}_2$, $\mathfrak{R}$ and $\mathfrak{R}_{u^t}$. Furthermore, we obtain the generator polynomials and minimal spanning sets for $\mathbb{Z}_2\mathfrak{R}\mathfrak{R}_{u^t}$-additive cyclic codes. Also, the relationship between the generators of C and its dual are established. Finally, these codes are classified as $\mathfrak{R}_{u^t}[y]$-submodules of the ring $\mathcal{S}_{\beta_1,\beta_2,\beta_3} = \mathbb{Z}_2[y]/\langle y^{\beta_1} - 1\rangle \times \mathfrak{R}[y]/\langle y^{\beta_2} - 1\rangle \times \mathfrak{R}_{u^t}[y]/\langle y^{\beta_3} - 1\rangle$.

This paper is organized as follows: In Section 2, we define some basic notions, Gray maps and the extensions of Gray maps. Section 3 contains the cyclic structures of the rings $\mathfrak{R} = \mathbb{Z}_2 + u\mathbb{Z}_2$, where $u^2 = 0$ and $\mathfrak{R}_{u^t} = \mathbb{Z}_2 + u\mathbb{Z}_2 + \cdots + u^{k-1}\mathbb{Z}_2$, where $u^k = 0$. In Section 4, we study $\mathbb{Z}_2\mathfrak{R}\mathfrak{R}_{u^t}$-additive cyclic codes and find the minimal generating sets when β_2 is odd and β_3 is even(or odd). In Section 5, we define the duality of $\mathbb{Z}_2\mathfrak{R}\mathfrak{R}_{u^t}$-additive cyclic codes and their results. Section 6 contains some examples and a table of binary optimal codes from $\mathbb{Z}_2\mathfrak{R}\mathfrak{R}_{u^t}$-additive cyclic codes. The last section concludes the article.

2. Preliminaries

Let $\mathfrak{R} = \mathbb{Z}_2 + u\mathbb{Z}_2$, $u^2 = 0$ and $\mathfrak{R}_{u^t} = \mathbb{Z}_2 + u\mathbb{Z}_2 + \cdots + u^{k-1}\mathbb{Z}_2$, $u^k = 0$ be two rings with characteristic 2. Any element $z \in \mathfrak{R}_{u^t}$ can be written as $z = a_0 + ua_1 + \cdots + u^{k-1}a_{k-1}$ for all $a_i \in \mathbb{Z}_2$, where $0 \le i \le k-1$. An element $z = a_0 + ua_1 + \cdots + u^{k-1}a_{k-1} \in \mathfrak{R}_{u^t}$ is a unit if and only if a_0 is unit. Let

$$\mathbb{Z}_2\mathfrak{R}\mathfrak{R}_{u^t} = \{(c,\ c',\ c'') \mid c \in \mathbb{Z}_2,\ c' \in \mathfrak{R},\ c'' \in \mathfrak{R}_{u^t}\}.$$

Define three maps $\theta_1 : \mathfrak{R}_{u^t} \longrightarrow \mathbb{Z}_2$, $\theta_2 : \mathfrak{R}_{u^t} \longrightarrow \mathfrak{R}$ and $\theta_3 : \mathfrak{R} \longrightarrow \mathbb{Z}_2$ such that $\theta_1(a_0 + ua_1 + \cdots + u^{k-1}a_{k-1}) = a_0$, $\theta_2(a_0 + ua_1 + \cdots + u^{k-1}a_{k-1}) = a_0 + ua_1$ and

$\theta_3(a_0 + ua_1) = a_1$, respectively. Clearly, θ_1, θ_2 and θ_3 are well-defined onto ring homomorphisms. Let $\mathbb{Z}_2^{\beta_1}$ be a β_1-tuples of $\mathbb{Z}_2$, $\Re^{\beta_2}$ be a β_2-tuples of $\Re$ and ${\Re_{u^t}}^{\beta_3}$ be a β_3-tuples of $\Re_{u^t}$, where β_1, β_2 and β_3 are positive integers. Let $\mathbf{y} = (y, y^{'}, y^{''}) \in \mathbb{Z}_2^{\beta_1} \times \Re^{\beta_2} \times {\Re_{u^t}}^{\beta_3}$ be a vector, where $y = (y_0, y_1, \dots, y_{\beta_1-1})$, $y^{'} = (y^{'}_0, y^{'}_1, \dots, y^{'}_{\beta_2-1})$ and $y^{''} = (y^{''}_0, y^{''}_1, \dots, y^{''}_{\beta_3-1})$. For any $z = a_0 + ua_1 + \cdots + u^{k-1}a_{k-1} \in \Re_{u^t}$, the $\Re_{u^t}$-scalar multiplication on $\mathbb{Z}_2^{\beta_1} \times \Re^{\beta_2} \times {\Re_{u^t}}^{\beta_3}$ is defined as follows:

$$z\mathbf{y} = (\theta_1(z)y_0, \dots, \theta_1(z)y_{\beta_1-1} | \theta_2(z)y^{'}_0, \dots, \theta_2(z)y^{'}_{\beta_2-1} | zy^{''}_0, \dots, zy^{''}_{\beta_3-1}) \in \mathbb{Z}_2^{\beta_1} \times \Re^{\beta_2} \times {\Re_{u^t}}^{\beta_3}, \tag{1}$$

where $\theta_1(z)y_i$, $\theta_2(z)y^{'}_j$ and $zy^{''}_\ell$ are performed mod2 for all $0 \leq i \leq \beta_1 - 1$, $0 \leq j \leq \beta_2 - 1$ and $0 \leq \ell \leq \beta_3 - 1$. The structure $\mathbb{Z}_2^{\beta_1} \times \Re^{\beta_2} \times {\Re_{u^t}}^{\beta_3}$ forms an $\Re_{u^t}$-module under the usual addition and multiplication defined in (1). Let

$$\mathcal{S}_{\beta_1,\beta_2,\beta_3} = \mathbb{Z}_2[y]/\langle y^{\beta_1} - 1\rangle \times \Re[y]/\langle y^{\beta_2} - 1\rangle \times \Re_{u^t}[y]/\langle y^{\beta_3} - 1\rangle.$$

Define a map

$$\Phi : \mathbb{Z}_2^{\beta_1} \times \Re^{\beta_2} \times {\Re_{u^t}}^{\beta_3} \longrightarrow \mathcal{S}_{\beta_1,\beta_2,\beta_3}$$

$$d = (f|g|h) \longmapsto d(y) = (f(y)|g(y)|h(y)),$$

where $(f|g|h) = (f_0, f_1, \dots, f_{\beta_1-1} | g_0, g_1, \dots, g_{\beta_2-1} | h_0, h_1, \dots, h_{\beta_3-1})$,
$f(y) = f_0 + f_1 y + \cdots + f_{\beta_1-1}y^{\beta_1-1}$, $g(y) = g_0 + g_1 y + \cdots + g_{\beta_2-1}y^{\beta_2-1}$ and
$h(y) = h_0 + h_1 y + \cdots + h_{\beta_3-1}y^{\beta_3-1}$. For any $\ell(y) = \ell_0 + \ell_1 y + \cdots + \ell_r y^r \in \Re_{u^t}[y]$ and $d(y) = (f(y)|g(y)|h(y)) \in \mathcal{S}_{\beta_1,\beta_2,\beta_3}$, define the $\Re_{u^t}[y]$-scalar multiplication

$$\ell(y) \cdot d(y) = (\theta_1(\ell(y))f(y) | \theta_2(\ell(y))g(y) | \ell(y)h(y)), \tag{2}$$

where $\theta_1(\ell(y)) = \theta_1(\ell_0) + \theta_1(\ell_1)y + \cdots + \theta_2(\ell_r)y^r$ and
$\theta_2(\ell(y)) = \theta_2(\ell_0) + \theta_2(\ell_1)y + \cdots + \theta_2(\ell_r)y^r$. Then $\mathcal{S}_{\beta_1,\beta_2,\beta_3}$ forms an $\Re_{u^t}[y]$-module under the usual addition and scalar multiplication of polynomials defined in (2).

Definition 2.1. *A non-empty subset C of $\mathbb{Z}_2^{\beta_1} \times \Re^{\beta_2} \times {\Re_{u^t}}^{\beta_3}$ is called an $\mathbb{Z}_2\Re\Re_{u^t}$-additive code if C is a subgroup of $\mathbb{Z}_2^{\beta_1} \times \Re^{\beta_2} \times {\Re_{u^t}}^{\beta_3}$, that is, C is isomorphic to*

$$\mathbb{Z}_2^{\ell_1} \times \mathbb{Z}_2^{2m_1} \times \mathbb{Z}_2^{m_2} \times \mathbb{Z}_2^{kn_1} \times \mathbb{Z}_2^{(k-1)n_2} \times \cdots \times \mathbb{Z}_2^{n_k},$$

for some positive integers ℓ_1, m_1, m_2, $n_1, n_2, \dots, n_k$.

Definition 2.2. *A non-empty subset C of $\mathbb{Z}_2^{\beta_1} \times \Re^{\beta_2} \times {\Re_{u^t}}^{\beta_3}$ is called an $\mathbb{Z}_2\Re\Re_{u^t}$-additive cyclic code if*

- *(i) C is an additive code.*
- *(ii) For any codeword $\mathbf{z} = (a_0, a_1, \dots, a_{\beta_1-1} | b_0, b_1, \dots, b_{\beta_2-1} | c_0, c_1, \dots, c_{\beta_3-1}) \in C$ its cyclic shift $T(z) = (a_{\beta_1-1}, a_0, \dots, a_{\beta_1-2} | b_{\beta_2-1}, b_0, \dots, b_{\beta_2-2} | c_{\beta_3-1}, c_0, \dots, c_{\beta_3-2}) \in C$.*

Let us define Gray maps as follows:

$$\phi_1 : \Re \longrightarrow \mathbb{Z}_2^2 \tag{3}$$

such that $\phi_1(e + uf) = (f, e + f)$ for all $e, f \in \mathbb{Z}_2$ and

$$\phi_2 : \Re_{u^t} \longrightarrow \mathbb{Z}_2^k \tag{4}$$

$$
\begin{array}{rcl}
0 & \longmapsto & (000\cdots 00) \\
1 & \longmapsto & (100\cdots 00) \\
u & \longmapsto & (110\cdots 00) \\
\vdots & & \\
u^{k-1} & \longmapsto & (\underbrace{111\cdots 11}_{k}),
\end{array}
$$

and for any $(a_0 + ua_1 + \cdots + u^{k-1}a_{k-1}) \in \mathfrak{R}_{\mathfrak{u}^{\mathfrak{t}}}$ such that

$$\phi_2(a_0 + ua_1 + \cdots + u^{k-1}a_{k-1}) = (a_0\phi_2(1) + a_1\phi_2(u) + \cdots + a_{k-1}\phi_2(u^{k-1}))$$

for all $a_0, a_1, \ldots, a_{k-1} \in \mathbb{Z}_2$. Using (3) and (4), we can define another Gray map

$$\Psi : \mathbb{Z}_2 \times \mathfrak{R} \times \mathfrak{R}_{\mathfrak{u}^{\mathfrak{t}}} \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2^2 \times \mathbb{Z}_2^k \tag{5}$$

as $\Psi(c|c^{'}|c^{''}) = (c, \phi_1(c^{'}), \phi_2(c^{''}))$. An extension of the map Ψ is defined by

$$\Psi_1 : \mathbb{Z}_2^{\beta_1} \times \mathfrak{R}^{\beta_2} \times \mathfrak{R}_{\mathfrak{u}^{\mathfrak{t}}}{}^{\beta_3} \longrightarrow \mathbb{Z}_2^{\beta_1+2\beta_2+k\beta_3} \tag{6}$$

such that $\Psi_1(\mathbf{y} = (y|y^{'}|y^{''})) = (y, \phi_1(y^{'}), \phi_2(y^{''}))$, where

$$\mathbf{y} = (y_0, y_1, \ldots, y_{\beta_1-1}|y^{'}_0, y^{'}_1, \ldots, y^{'}_{\beta_2-1}|y^{''}_0, y^{''}_1, \ldots, y^{''}_{\beta_3-1}) \in \mathbb{Z}_2^{\beta_1} \times \mathfrak{R}^{\beta_2} \times \mathfrak{R}_{\mathfrak{u}^{\mathfrak{t}}}{}^{\beta_3}.$$

Definition 2.3. *Let $\boldsymbol{y} = (y|y^{'}|y^{''}) \in \mathbb{Z}_2^{\beta_1} \times \mathfrak{R}^{\beta_2} \times \mathfrak{R}_{\mathfrak{u}^{\mathfrak{t}}}{}^{\beta_3}$, where $y \in \mathbb{Z}_2^{\beta_1}$, $y^{'} \in \mathfrak{R}^{\beta_2}$ and $y^{''} \in \mathfrak{R}_{\mathfrak{u}^{\mathfrak{t}}}{}^{\beta_3}$. Then the Gray weight of $\boldsymbol{y}$ is defined as*

$$w_G(\boldsymbol{y}) = w_H(y) + w_H(\phi_1(y^{'})) + w_H(\phi_2(y^{''})),$$

where w_H denotes the Hamming weight.

Definition 2.4. *Let $\boldsymbol{y}, \boldsymbol{z} \in \mathbb{Z}_2^{\beta_1} \times \mathfrak{R}^{\beta_2} \times \mathfrak{R}_{\mathfrak{u}^{\mathfrak{t}}}{}^{\beta_2}$. Then the Gray distance between $\boldsymbol{y}$ and $\boldsymbol{z}$ is defined as*

$$d_G(\boldsymbol{y}, \boldsymbol{z}) = w_G(\boldsymbol{y} - \boldsymbol{z}) = d_H((y|\phi_1(y^{'})|\phi_2(y^{''})), (z|\phi_1(z^{'})|\phi_2(z^{''}))).$$

3. Structure of cyclic codes over $\mathfrak{R}$ and $\mathfrak{R}_{\mathfrak{u}^{\mathfrak{t}}}$

In this section, we discuss the structural properties of $\mathfrak{R}$ and $\mathfrak{R}_{\mathfrak{u}^{\mathfrak{t}}}$. Further, we obtain generating sets of these structures for different lengths. The generating sets of $\mathfrak{R}$ and $\mathfrak{R}_{\mathfrak{u}^{\mathfrak{t}}}$ will be used in the subsequent sections.

Lemma 3.1. *A code C of length β_2 over $\mathfrak{R}$ is cyclic code if and only if C is an $\mathfrak{R}$-submodule of $\mathfrak{R}_{\beta_2} = \mathfrak{R}[y]/\langle y^{\beta_2} - 1\rangle$.*

Lemma 3.2. *[27, Theorem 12.13] If for any $f(y), g(y) \in \mathfrak{R}[y]$, where $g(y)$ has unit as its leading coefficient, then*

$$f(y) = g(y)q(y) + r(y),$$

for some $q(y),\ r(y) \in \mathfrak{R}[y]$, where $r(y) = 0$ or $\deg(r(y)) < \deg(g(y))$.

Proof. The proof is directly followed by [27, Theorem 12.13]. □

Let C_1 be a cyclic code in $\Re_{\beta_2}$. We can define a map $\theta_3 : \Re \longrightarrow \mathbb{Z}_2$ by $\theta_3(a + ub) = a$. Clearly, η is a ring homomorphism in $\Re$. The extension of θ_3 can be expressed by

$$\eta_1 : C_1 \longrightarrow \Re[y]/\langle y^{\beta_2} - 1\rangle$$

such that $\eta_1(a_0 + a_1 y + \cdots + a_{\beta_2-1} y^{\beta_2-1}) = \theta_3(a_0) + \theta_3(a_1)y + \cdots + \theta_3(a_{\beta_2-1})y^{\beta_2-1}$. Now, we can easily obtain the kernel of η_1 as

$$ker(\eta_1) = \{ub(y) \mid b(y) \in \mathbb{Z}_2[y]/\langle y^{\beta_2} - 1\rangle\} = \langle ub_1(y)\rangle,$$

where $b_1(y)|(y^{\beta_2} - 1)(\mathrm{mod}2)$. Since the image of η_1 is also an ideal in $\mathbb{Z}_2[y]/\langle y^{\beta_2} - 1\rangle\}$, a binary cyclic code is generated by $f(y)$ with $f(y)|y^{\beta_2} - 1$. Hence, $C_1 = \langle f(y) + up(y), ub_1(y)\rangle$, for some binary polynomial $p(y)$ and $b_1(y)|p(y)\frac{y^{\beta_2}-1}{f(y)}$. Obviously, $uf(y) \in ker(\eta_1)$. This implies that $b_1(y)|f(y)$.

Now, we state the following known lemmas which are essential in describing the proofs of various results in the subsequent sections:

Lemma 3.3. *[1] Let C_1 be a cyclic code in $\Re_{\beta_2} = \Re[y]/\langle y^{\beta_2} - 1\rangle$.*

(1) *If β_2 is odd, then $\Re_{\beta_2}$ is principal ideal ring and*

$$C_1 = \langle f(y), ub_1(y)\rangle = \langle f(y) + ub_1(y)\rangle,$$

where $f(y),\ b_1(y) \in \mathbb{Z}_2[y]/\langle y^{\beta_2} - 1\rangle$ and $b_1(y)|f(y)|(y^{\beta_2} - 1)$.

(2) *If β_2 is not odd, then*

  (i) *$C_1 = \langle f(y) + up(y)\rangle$, where $f(y)|(y^{\beta_2} - 1)(\mathrm{mod}2)$ and $f(y) + up(y)|(y^{\beta_2} - 1)$ in $\Re$.*

  (ii) *$C_1 = \langle f(y) + up(y), ub_1(y)\rangle$, where $f(y)$, $b_1(y)$ and $p(y)$ are binary polynomials such that $b_1(y)|f(y)|(y^{\beta_2} - 1)(\mathrm{mod}2)$, $b_1(y)|p(y)\frac{y^{\beta_2}-1}{f(y)}$ and $\deg(b_1(y)) > \deg(p(y))$.*

Lemma 3.4. *[32] Let C_2 be a cyclic code in $\Re_{u^t\beta_3} = \Re_{u^t}[y]/\langle y^{\beta_3} - 1\rangle$.*

(1) *If β_3 is odd, then $\Re_{u^t\beta_3}$ is principal ideal ring and*

$$C_2 = \langle g(y) + ua_1(y) + u^2 a_2(y) + \cdots + u^{k-1} a_{k-1}\rangle,$$

where $a_{k-1}(y)|a_{k-2}(y)\cdots a_2(y)|a_1(y)|g(y)|(y^{\beta_3} - 1)$.

(2) *If β_3 is not odd, then*

(i) *$C_2 = \langle g(y) + up_1(y) + \cdots + u^{k-1}p_{k-1}\rangle$, where $g(y)|(y^{\beta_3} - 1)(\mathrm{mod}2)$ and $g(y) + up_1(y) + \cdots + u^{k-1}p_{k-1}|(y^{\beta_3} - 1)$ over $\Re_{u^t}$, $\deg(p_i(y)) < \deg(p_{i-1}(y))$ for all $2 \le i \le k-1$. OR*

(ii) *$C_2 = \langle g(y) + up_1(y) + \cdots + u^{k-1}p_{k-1}, u^{k-1}a_{k-1}\rangle$, where $a_{k-1}(y)|g(y)|(y^{\beta_3} - 1)(\mathrm{mod}2)$ and $g(y) + up_1(y) + \cdots + u^{k-1}p_{k-1}|(y^{\beta_3} - 1)$ over $\Re_{u^t}$, $\deg(p_i(y)) < \deg(p_{i-1}(y))$ for all $2 \le i \le k-1$. OR*

(iii)

$$C_2 = \left\langle \begin{array}{c} g(y) + up_1(y) + \cdots + u^{k-1}p_{k-1}, \\ ua_1(y) + u^2 q_1(y) + \cdots + u^{k-1}q_{k-2}, \\ ua_2(y) + \cdots + u^{k-1}l_{k-3}, \ldots, u^{k-2}a_{k-2} + u^{k-1}t_1, \\ u^{k-1}a_{k-1} \end{array} \right\rangle,$$

where $a_{k-1}(y)|a_{k-2}(y)|\cdots|a_1(y)|g(y)|(y^{\beta_3} - 1)(\mathrm{mod}2)$ and $a_1(y)|p_{i_1}(y)\frac{y^{\beta_3}-1}{g(y)}$, $a_2(y)|q_{i_2}(y)\frac{y^{\beta_3}-1}{g(y)}, \ldots, a_{k-2}(y)|t_1(y)\frac{y^{\beta_3}-1}{a_{k-1}(y)}$ over $\Re_{u^t}$, where $1 \le i_1 \le k-1$, $1 \le i_2 \le k-2$ $\deg(p_i(y)) < \deg(p_{i-1}(y))$ for all $2 \le i \le k-1$.

4. $\mathbb{Z}_2\mathfrak{R}\mathfrak{R}_{u^t}$-additive cyclic codes

In this section, we obtain a set of generators for $\mathbb{Z}_2\mathfrak{R}\mathfrak{R}_{u^t}$-additive cyclic codes as $\mathfrak{R}_{u^t}[y]$-submodules of $\mathcal{S}_{\beta_1,\beta_2,\beta_3}$. Here, C will always denote an $\mathbb{Z}_2\mathfrak{R}\mathfrak{R}_{u^t}$-additive cyclic code. Since C and $\mathfrak{R}_{u^t}[y]/\langle y^{\beta_3}-1\rangle$ are $\mathfrak{R}_{u^t}[y]$-submodules of $\mathcal{S}_{\beta_1,\beta_2,\beta_3}$, we define a map

$$\eta : C \longrightarrow \mathfrak{R}_{u^t}[y]/\langle y^{\beta_3}-1\rangle,$$

by $\eta(f(y)|g(y)|h(y)) = h(y)$. Clearly, η is a module homomorphism whose image is $\mathfrak{R}_{u^t}[y]$-submodule in $\mathfrak{R}_{u^t}[y]/\langle y^{\beta_3}-1\rangle$ and $ker(\eta)$ is a submodule of C. Further, $\eta(C)$ can easily be identified as an ideal in the ring $\mathfrak{R}_{u^t}[y]/\langle y^{\beta_3}-1\rangle$. Firstly, we assume that β_3 is odd. Since $\eta(C)$ is an ideal in $\mathfrak{R}_{u^t}[y]/\langle y^{\beta_3}-1\rangle$,

$$\eta(C) = \langle h(y) + ua_1(y) + u^2a_2(y) + \cdots + u^{k-1}a_{k-1}\rangle$$

with $a_i(y) \mid h(y) \mid (y^{\beta_3}-1)(\text{mod}2)$, for $i = 1, 2, \ldots, k-1$. Let us define

$$ker(\eta) = \{(f(y)|g(y)|0) \in C | f(y) \in \mathbb{Z}_2[y]/\langle y^{\beta_1}-1\rangle, g(y) \in \mathfrak{R}[y]/\langle y^{\beta_2}-1\rangle\},$$

$$J = \{(f(y), g(y)) \in \mathbb{Z}_2[y]/\langle y^{\beta_1}-1\rangle \times \mathfrak{R}[y]/\langle y^{\beta_2}-1\rangle | (f(y)|g(y)|0) \in ker(\eta)\}.$$

It is clear that J is an ideal in the ring $\mathbb{Z}_2[y]/\langle y^{\beta_1}-1\rangle \times \mathfrak{R}[y]/\langle y^{\beta_2}-1\rangle$ and hence a cyclic code. Therefore, by the well-known result on generators of binary cyclic codes, we have $J = \langle f(y), g(y)\rangle$. Now, for any element $(f_1(y)|g_1(y)|0) \in ker(\eta)$, we get $(f_1(y), g_1(y)) \in J = \langle f(y), g(y)\rangle$ and it can be written as $(f_1(y), g_1(y)) = m_1(y)(f(y), g(y))$ for some polynomial $m_1(y) \in \mathfrak{R}[y]/\langle y^{\beta_2}-1\rangle$. Thus, $(f_1(y), g_1(y), 0) = (\theta_3(m_1(y))f(y), m_1(y)g(y), 0)$. This implies that $ker(\eta)$ is a submodule of C generated by an element of the form $(f(y), g(y), 0)$, where $f(y)|(y^{\beta_1}-1)(\text{mod}2)$ and $g(y)|(y^{\beta_2}-1)(\text{mod}2)$. By the first isomorphism theorem for rings, we have

$$\frac{C}{ker(\eta)} \cong \langle h(y) + ua_1(y) + u^2a_2(y) + \cdots + u^{k-1}a_{k-1}\rangle.$$

This implies that any $\mathbb{Z}_2\mathfrak{R}\mathfrak{R}_{u^t}$-additive cyclic code can be generated as a $\mathfrak{R}_{u^t}[y]$-submodule of $\mathcal{S}_{\beta_1,\beta_2,\beta_3}$ by $(f_1(y)|0|0)$, $(f_2(y)|g(y)+up_1(y)|0)$ and

$$(f_3(y)|\ell_1(y)|g(y) + ua_1(y) + u^2a_2(y) + \cdots + u^{k-1}a_{k-1}).$$

Hence, any element in $c(y) \in C$ can be expressed as

$$\begin{aligned} c(y) &= d_1(y) \times (f_1(y)|0|0) + d_2(y) \times (f_2(y)|g(y) + up_1(y)|0) + \\ &\quad d_3(y) \times (f_3(y)|\ell_1(y)|g(y) + ua_1(y) + u^2a_2(y) + \cdots + u^{k-1}a_{k-1}), \end{aligned}$$

where $d_1(y)$, $d_2(y)$ and $d_3(y)$ are polynomials in the ring $\mathfrak{R}_{u^t}[y]$. Similarly, if β_3 is even, then

$$\frac{C}{ker(\eta)} \cong \left\langle \begin{array}{c} h(y) + up_1(y) + \cdots + u^{k-1}p_{k-1}, ua_1(y) + u^2q_1(y) + \cdots + u^{k-1}q_{k-2}, \\ ua_2(y) + \cdots + u^{k-1}l_{k-3}, \ldots, u^{k-2}a_{k-2} + u^{k-1}t_1, u^{k-1}a_{k-1} \end{array} \right\rangle.$$

Definition 4.1. *A subset $C \subseteq \mathcal{S}_{\beta_1,\beta_2,\beta_3}$ is called an $\mathbb{Z}_2\mathfrak{R}\mathfrak{R}_{u^t}$-additive cyclic code if and only if C is a subgroup of $\mathcal{S}_{\beta_1,\beta_2,\beta_3}$ and for all*

$$\begin{aligned} d(y) &= (f(y)|g(y)|h(y)) \\ &= (f_0 + \cdots + f_{\beta_1-1}y^{\beta_1-1}|g_0 + \cdots + g_{\beta_2-1}y^{\beta_2-1}|h_0 + \cdots + h_{\beta_3-1}y^{\beta_3-1}) \in C, \end{aligned}$$

we have

$$\begin{aligned} &y \cdot d(y) = \\ &\quad (f_{\beta_1-1} + \cdots + f_{\beta_1-2}y^{\beta_1-1}|g_{\beta_2-1} + \cdots + g_{\beta_2-2}y^{\beta_2-1}|h_{\beta_3-1} + \cdots + h_{\beta_3-2}y^{\beta_3-1}) \in C. \end{aligned}$$

Theorem 4.2. *A code C is an $\mathbb{Z}_2\mathfrak{R}\mathfrak{R}_{u^t}$-additive cyclic code if and only if C is a $\mathfrak{R}_{u^t}[y]$-submodule of $\mathcal{S}_{\beta_1,\beta_2,\beta_3}$.*

Proof. Let C be an $\mathbb{Z}_2\mathfrak{R}\mathfrak{R}_{u^t}$-additive cyclic code. Then we have to show that for any $d(y) \in C$ and $l(y) \in \mathfrak{R}_{u^t}[y]$, $l(y)d(y) \in C$. Assume that $d(y) = (f(y)|g(y)|h(y)) \in C$, where $f(y) = (f_0 + f_1 y + \cdots + f_{\beta_1-1}y^{\beta_1-1})$, $g(y) = (g_0 + g_1 y + \cdots + g_{\beta_2-1}y^{\beta_2-1})$ and $h(y) = (h_0 + h_1 y + \cdots + h_{\beta_3-1}y^{\beta_3-1})$. The multiplication

$$\begin{aligned} y \cdot d(y) &= (f_{\beta_1-1} + f_0 y + \cdots + f_{\beta_1-2}y^{\beta_1-1}|g_{\beta_2-1} + g_0 y + \cdots + g_{\beta_2-2}y^{\beta_2-1}|h_{\beta_3-1} \\ &\quad + h_0 y + \cdots + h_{\beta_3-2}y^{\beta_3-1}) \end{aligned}$$

represents the cyclic shift $T(d(y))$ of $d(y)$. Since C is $\mathbb{Z}_2\mathfrak{R}\mathfrak{R}_{u^t}$- additive cyclic code, $y^i d(y) \in C$ for all $i \in \mathbb{N}$. It follows that $l(y) \cdot d(y) \in C$. This implies that C is $\mathfrak{R}_{u^t}[y]$- submodule of $\mathcal{S}_{\beta_1,\beta_2,\beta_3}$. The Converse part is directly followed by Definition 4.1. □

Theorem 4.3. *Let*

$$C = \left\langle \begin{array}{c} (f_1(y)|0|0), (f_2(y)|g(y) + up_1(y)|0), \\ (f_3(y)|\ell_1(y)|h(y) + ua_1(y) + u^2a_2(y) + \cdots + u^{k-1}a_{k-1} \end{array} \right\rangle$$

be an $\mathbb{Z}_2\mathfrak{R}\mathfrak{R}_{u^t}$-additive cyclic code. Then $\deg(f_i(y)) < \deg(f_1)$, *where* $i = 2, 3$ *and* $\deg(\ell_1(y)) < \deg(g(y))$.

Proof. Suppose that $\deg(f_i(y)) \geq \deg(f_1(y))$. For $i = 2$, using division algorithm

$$f_2(y) = f_1(y)q(y) + r(y),$$

for some polynomial $q(y),\ r(y) \in \mathfrak{R}_{\beta_1}$ and either $r(y) = 0$ or $\deg(r(y)) < \deg(f_1(y))$, w get

$$C = \left\langle \begin{array}{c} (f_1(y)|0|0), (f_1(y)q(y) + r(y)|g(y) + up_1(y)|0), \\ (f_3(y)|\ell_1(y)|h(y) + ua_1(y) + u^2a_2(y) + \cdots + u^{k-1}a_{k-1}) \end{array} \right\rangle.$$

Since $(f_1(y)q(y)|0|0) \in \langle q(y)(f_1(y)|0|0)\rangle$. Hence

$$C = \left\langle \begin{array}{c} (f_1(y), 0), (r(y)|g(y) + up_1(y)|0), \\ (f_3(y)|\ell_1(y)|h(y) + ua_1(y) + u^2a_2(y) + \cdots + u^{k-1}a_{k-1}) \end{array} \right\rangle.$$

This implies that $\deg(f_2(y)) < \deg(f_1)$. Similarly, other cases can be easily proved. □

Theorem 4.4. *Let*

$$C = \left\langle \begin{array}{c} (f_1(y)|0|0), (f_2(y)|g(y) + ub_1(y)|0), \\ (f_3(y)|\ell_1(y)|h(y) + ua_1(y) + u^2a_2(y) + \cdots + u^{k-1}a_{k-1}(y)) \end{array} \right\rangle$$

be an $\mathbb{Z}_2\mathfrak{R}\mathfrak{R}_{u^t}$-additive cyclic code of length $(\beta_1, \beta_2, \beta_3)$, where β_2 and β_3 are odd integers. If $m_g(y) = \frac{(y^{\beta_2}-1)}{g(y)+ub_1(y)}$ and $m_h(y) = \frac{(y^{\beta_3}-1)}{h(y)+ua_1(y)+u^2a_2(y)+\cdots+u^{k-1}a_{k-1})}$, then $f_1(y) \mid m_g(y)f_2(y)$ and $g(y) + ub_1(y) \mid m_h(y)\ell_1(y)$.

Proof. Let $\eta(m_g(y)(f_2(y)|g(y) + ub_1(y)|0)) = \eta(m_g(y)f_2(y)|0|0)$. It gives that $(m_g(y)f_2(y)|0|0) \in ker(\eta)$ and hence $f_1(y) \mid m_g(y)f_2(y)$. Similarly, we consider that

$$\eta(m_h(y)(f_3(y)|\ell_1(y)|h(y) + ua_1(y) + \cdots + u^{k-1}a_{k-1})) = \eta(m_h(y)f_2(y)|m_h(y)\ell_1(y)|0)),$$

we get $(m_h(y)f_2(y)|m_h(y)\ell_1(y)|0)) \in ker(\eta)$. Therefore, $g(y) + ub_1(y) \mid m_h(y)\ell_1(y)$. □

Theorem 4.5. *Let*

$$C = \left\langle \begin{array}{c} (f_1(y)|0|0), (f_2(y)|g(y) + ub_1(y)|0), \\ (f_3(y)|\ell_1(y)|h(y) + ua_1(y) + u^2a_2(y) + \cdots + u^{k-1}a_{k-1}(y)) \end{array} \right\rangle$$

be an $\mathbb{Z}_2\mathfrak{R}\mathfrak{R}_{u^t}$-additive cyclic code of length $(\beta_1,\beta_2,\beta_3)$, where β_2 and β_3 are odd integers and $b_1(y)|g(y)|(y^{\beta_2}-1)$, $a_{k-1}(y) \mid a_{k-2}(y) \mid a_{k-3}(y) \mid \cdots \mid a_2(y) \mid a_1(y) \mid h(y) \mid (y^{\beta_3}-1)$. Let $t_1(y) = \gcd\{m_g b_1, y^{\beta_2}-1\}$, $t_2(y) = \frac{y^{\beta_2}-1}{t_1(y)}$, $m_{a_1}(y) = \frac{h(y)}{a_1(y)}$ and $m_{a_i}(y) = \frac{a_{i-1}(y)}{a_i(y)}$, where $i = 2, 3, \ldots, k-1$. If

$$
\begin{aligned}
S_1 &= \bigcup_{i=0}^{\beta_1-\deg(f_1(y))-1} \{y^i \cdot (f_1(y)|0|0)\};\\
S_2 &= \bigcup_{i=0}^{\beta_2-\deg(g(y))-1} \{y^i \cdot (f_2(y)|g(y)+ub_1(y)|0)\};\\
S_3 &= \bigcup_{i=0}^{\deg(g(y))-\deg(b_1(y))-1} \{y^i \cdot (m_g(y)f_2(y)|um_g(y)b_1(y)|0)\};\\
S_4 &= \bigcup_{i=0}^{\beta_3-\deg(h(y))-1} \{y^i \cdot (f_3(y)|\ell_1(y)|h(y)+ua_1(y)+u^2a_2(y)+\cdots+u^{k-1}a_{k-1}(y))\};\\
S_5 &= \bigcup_{i=0}^{\deg(h(y)-\deg(a_1(y))-1} \{y^i \cdot (m_h(y)f_3(y)|m_h(y)\ell_1(y)|m_h(y)(ua_1(y)+\\
&\qquad u^2a_2(y)+\cdots+u^{k-1}a_{k-1}(y)))\};\\
S_6 &= \bigcup_{i=0}^{\deg(a_1(y)-\deg(a_2(y))-1} \{y^i \cdot (m_h(y)m_{a_1}(y)f_3(y)|m_h(y)m_{a_1}(y)\ell_1(y)|\\
&\qquad m_h(y)m_{a_1}(y)(u^2a_2(y)+\cdots+u^{k-1}a_{k-1}(y)))\};\\
&\vdots\\
S_{k+3} &= \bigcup_{i=0}^{\deg(a_{k-2}(y))-\deg(a_{k-1}(y))-1} \{y^i \cdot (m_h(y)M_{k-1}(y)f_3(y)|m_h(y)M_{k-1}(y)\ell_1(y)m_h|\\
&\qquad m_h(y)M_{k-1}(y)u^{k-1}a_{k-1}(y))\},
\end{aligned}
$$

where $M_{k-1}(y) = m_{a_1}(y)\cdots m_{a_{k-1}}(y)$ for $k \geq 1$, then

$$S = S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6 \cup \cdots \cup S_{k+3}$$

is a minimal generating set for the code C and

$$|C| = 2^{\beta_1-\deg(f_1)}2^{2\beta_2-\deg(g)-\deg(b_1)}2^{k\beta_3-\deg(h)-\deg(a_1)-\deg(a_2)-\cdots-\deg(a_{k-1})}.$$

Proof. Let $c(y) \in C$ be a codeword and $c_i(y) \in \mathfrak{R}_{u^t}[y]$, $1 \leq i \leq 3$. Then

$$
\begin{aligned}
c(y) &= c_1(y)\cdot(f_1(y)|0|0) + c_2(y)\cdot(f_2(y)|g(y)+ub_1(y)|0)\\
&\quad + c_3(y)\cdot(f_3(y)|\ell_1(y)|h(y)+ua_1(y)+u^2a_2(y)+\cdots+u^{k-1}a_{k-1}(y))\\
&= (\theta_1(c_1(y))f_1(y)|0|0) + c_2(y)\cdot(f_2(y)|g(y)+ub_1(y)|0)\\
&\quad + c_3(y)\cdot(f_3(y)|\ell_1(y)|h(y)+ua_1(y)+u^2a_2(y)+\cdots+u^{k-1}a_{k-1}(y)).
\end{aligned}
$$

If $\deg(\theta(c_1(y))) \leq \beta_1 - \deg(f_1) - 1$, then $(f_1(y)|0|0) \in \text{span}(S_1)$. Otherwise, by division algorithm, we have $(\theta(c_1(y)) = \frac{(y^{\beta_1}-1)}{f_1(y)}d_1(y) + e_1(y)$, where $\deg(e_1) \leq \beta_1 - \deg(f_1) - 1$. Therefore,

$$
\begin{aligned}
(\theta(c_1(y))(f_1(y)|0|0) &= ((\frac{(y^{\beta_1}-1)}{f_1(y)}d_1(y) + e_1(y))(f_1(y))|0|0)\\
&= (e_1(y)f_1(y)|0|0) = e_1(y)(f_1(y)|0|0).
\end{aligned}
$$

This implies that $(\theta(c_1(y))(f_1(y)), 0) \in \text{span}(S_1)$. Now, we have to show that $c_2(y) \cdot (f_2(y)|g(y) + ub_1|0) \in \text{span}(S_1 \cup S_2 \cup S_3) \subset \text{span}(S)$. Suppose that m_g divides $c_2(y)$, that is, $c_2(y) = d_2(y)m_g(y) + e_2(y)$, where $e_2(y) = 0$ or $\deg(e_2(y)) \leq \deg(m_g(y)) - 1$, we get

$$\begin{aligned}
&c_2(y) \cdot (f_2(y)|g(y) + ub_1(y)|0)\\
&= (d_2(y)m_g(y) + e_2(y) \cdot (f_2(y)|g(y) + ub_1(y)|0)\\
&= d_2(y)(m_g(y)f_2(y)|um_g(y)b_1(y)|0) + e_2(y)(f_2(y)|g(y) + ub_1(y)|0).
\end{aligned}$$

Cleraly, $e_2(y)(f_2(y)|g(y) + ub_1(y)|0) \in \text{span}(S_2)$. It remains to show that

$$d_2(y)(m_g(y)f_2(y)|um_g(y)b_1(y)|0) \in \text{span}(S_1 \cup S_3).$$

Since $t_1(y) \mid m_g(y)b_1(y)$, we obtain $m_g(y)b_1(y) = r_1(y)t_1(y)$. Hence, $m_g(y)b_1(y)t_2(y)$=0. By division algorithm, we have $d_2(y) = d_2^{'}(y)t_2(y) + e_2^{'}(y)$, where $e_2^{'}(y) = 0$ or
$\deg(e_2^{'}(y)) \leq \deg(t_2(y)) - 1$. The expression $d_2(y)(m_g(y)f_2(y)|um_g(y)b_1(y)|0)$ can be written as follows

$$\begin{aligned}
&d_2(y)(m_g(y)f_2(y)|um_g(y)b_1(y)|0)\\
&= (d_2^{'}(y)t_2(y) + e_2^{'}(y)(m_g(y)f_2(y)|um_g(y)b_1(y)|0)\\
&= d_2^{'}(y)(t_2(y)m_g(y)f_2(y)|ut_2(y)m_g(y)b_1(y)|0) + e_2^{'}(y)(m_g(y)f_2(y)|um_g(y)b_1(y)|0).\\
&= d_2^{'}(y)(t_2(y)m_g(y)f_2(y)|0|0) + e_2^{'}(y)(m_g(y)f_2(y)|um_g(y)b_1(y)|0).
\end{aligned}$$

This implies that $d_2^{'}(t_2m_gf_2|0|0) \in \text{span}(S_1)$. Since $(m_gf_2|um_gb_1(y)|0) \in \text{span}(S_3)$,

$$c_2(y) \cdot (f_2(y)|g(y) + ub_1|0) \in \text{span}(S_1 \cup S_2 \cup S_3).$$

Next, we need to show that

$$c_3(y) \cdot (f_3(y)|\ell_1(y)|h(y) + ua_1(y) + \cdots + u^{k-1}a_{k-1}(y)) \in \text{span}(S_4 \cup \cdots \cup S_{k+3}) \subset \text{span}(S).$$

Applying division algorithm, we get

$$c_3(y) = d_3(y)m_h(y) + e_3(y),$$

where $e_3(y) = 0$ or $\deg(e_3(y)) \leq \deg(m_h(y)) - 1$. Therefore,

$$\begin{aligned}
&c_3(y) \cdot (f_3(y)|\ell_1(y)|h(y) + ua_1(y) + u^2a_2(y) + \cdots + u^{k-1}a_{k-1}(y))\\
&= (d_3(y)m_h(y) + e_3(y)) \cdot (f_3(y)|\ell_1(y)|h(y) + ua_1(y) + u^2a_2(y) + \cdots + u^{k-1}a_{k-1}(y))\\
&= d_3(y)(m_h(y)f_3(y)|m_h(y)\ell_1(y)|m_h(y)(ua_1(y) + u^2a_2(y) + \cdots + u^{k-1}a_{k-1}(y)))\\
&\quad +e_3(y)(f_3(y)|\ell_1(y)|h(y) + ua_1(y) + u^2a_2(y) + \cdots + u^{k-1}a_{k-1}(y)).
\end{aligned}$$

Obviously, $e_3(y)(f_3(y)|\ell_1(y)|h(y) + ua_1(y) + u^2a_2(y) + \cdots u^{k-1}a_{k-1}(y)) \in \text{span}(S_4)$. Again, using the division algorithm, we have

$$d_3(y) = m_{a_1}(y)d_4(y) + e_4(y),$$

where $\deg(e_4(y)) < \deg(m_{a_1}(y))$ or $e_4(y) = 0$. Putting the value of $d_3(y)$ in the expression

$$d_3(y)(m_h(y)f_3(y)|m_h(y)\ell_1(y)|m_h(y)(ua_1(y) + u^2a_2(y) + \cdots + u^{k-1}a_{k-1}(y))),$$

we get

$$\begin{aligned}
&d_3(y)(m_h(y)f_3(y)|m_h(y)\ell_1(y)|m_h(y)(ua_1(y) + u^2a_2(y) + \cdots + u^{k-1}a_{k-1}(y)))\\
&= (m_{a_1}(y)d_4(y) + e_4(y))(m_h(y)f_3(y)|m_h(y)\ell_1(y)|m_h(y)(ua_1(y)\\
&\quad +u^2a_2(y) + \cdots + u^{k-1}a_{k-1}(y)))\\
&= d_4(m_h(y)m_{a_1}(y)f_3(y)|m_h(y)m_{a_1}(y)\ell_1(y)|m_h(y)m_{a_1}(y)(u^2a_2(y) + \cdots + u^{k-1}a_{k-1}(y)))\\
&\quad +(e_4(y))(m_h(y)f_3(y)|m_h(y)\ell_1(y)|m_h(y)(ua_1(y) + u^2a_2(y) + \cdots + u^{k-1}a_{k-1}(y))).
\end{aligned}$$

It is observed that

$$(e_4(y))(m_h(y)f_3(y)|m_h(y)\ell_1(y)|m_h(y)(ua_1(y)+u^2a_2(y)+\cdots+u^{k-1}a_{k-1}(y)))\in \mathrm{span}(S_5).$$

Continuing in the same process, we get

$$c_3(y)\cdot(f_3(y)|\ell_1(y)|h(y)+ua_1(y)+\cdots+u^{k-1}a_{k-1}(y))\in \mathrm{span}(S_4\cup\cdots\cup S_{k+3})\subset \mathrm{span}(S).$$

Finally, $c(y)\in \mathrm{span}(S)$. Hence, $S=S_1\cup S_2\cup S_3\cup S_4\cup S_5\cup S_6\cup\cdots\cup S_{k+3}$ is the minimal spanning set for C and

$$|C| = 2^{\beta_1-\deg(f_1)}2^{2\beta_2-\deg(g)-\deg(b_1)}2^{k\beta_3-\deg(h)-\deg(a_1)-\deg(a_2)-\cdots-\deg(a_{k-1})}.$$

□

Corollary 4.6. *Let*

$$C = \left\langle\ (f_3(y)|\ell_1(y)|h(y)+ua_1(y)+u^2a_2(y)+\cdots+u^{k-1}a_{k-1}(y))\ \right\rangle$$

be an $\mathbb{Z}_2\mathfrak{R}\mathfrak{R}_{u^t}$-additive cyclic code of length $(\beta_1,\beta_2,\beta_3)$, where β_2 and β_3 are odd integers and $b_1(y)|g(y)|(y^{\beta_2}-1)$, $a_{k-1}(y)\mid a_{k-2}(y)\mid a_{k-3}(y)\mid\cdots\mid a_2(y)\mid a_1(y)\mid h(y)\mid(y^{\beta_3}-1)$. Let $t_1(y)=\gcd\{m_g(y)b_1(y),y^{\beta_2}-1\}$, $t_2(y)=\frac{y^{\beta_2}-1}{t_1(y)}$, $m_{a_1}(y)=\frac{h(y)}{a_1(y)}$ and $m_{a_i}(y)=\frac{a_{i-1}(y)}{a_i(y)}$, where $i=2,3,\dots,k-1$. If

$$\begin{aligned}
S_1 &= \bigcup_{i=0}^{\beta_3-\deg(h(y))-1}\{y^i\cdot(f_1(y)|\ell_1(y)|h(y)+ua_1(y)+u^2a_2(y)+\cdots+u^{k-1}a_{k-1}(y))\};\\
S_2 &= \bigcup_{i=0}^{\deg(h(y)-\deg(a_1(y))-1}\{y^i\cdot(m_h(y)f_1(y)|m_h(y)\ell_1(y)|m_h(y)(ua_1(y)+\\
&\qquad u^2a_2(y)+\cdots+u^{k-1}a_{k-1}(y)))\};\\
S_3 &= \bigcup_{i=0}^{\deg(a_1(y)-\deg(a_2(y))-1}\{y^i\cdot(m_h(y)m_{a_1}(y)f_1(y)|m_h(y)m_{a_1}(y)\ell_1(y)|\\
&\qquad m_h(y)m_{a_1}(y)(u^2a_2(y)+\cdots+u^{k-1}a_{k-1}(y)))\},\\
&\ \vdots\\
S_k &= \bigcup_{i=0}^{\deg(a_{k-2}(y))-\deg(a_{k-1}(y))-1}\{y^i\cdot(m_h(y)M_{k-1}(y)f_1(y)|m_h(y)M_{k-1}(y)\ell_1(y)m_h|\\
&\qquad m_h(y)M_{k-1}(y)u^{k-1}a_{k-1}(y))\},
\end{aligned}$$

where $M_{k-1}(y)=m_{a_1}(y)\cdots m_{a_{k-1}}(y)$ for $k\geq 1$, then $S=S_1\cup S_2\cup S_3\cup\cdots\cup S_k$ is a minimal generating set for the code C and

$$|C| = 2^{k\beta_3-\deg(h)-\deg(a_1)-\deg(a_2)-\cdots-\deg(a_{k-1})}.$$

Theorem 4.7. *Let*

$$C = \left\langle \begin{array}{c} (f_1(y)|0|0),(f_2(y)|g(y)+ub_1(y)|0),\\ (f_3(y)|\ell_1(y)|h(y)+up_1(y)+u^2p_2(y)+\cdots+u^{k-1}p_{k-1}(y)),\\ (f_4(y)|l_2(y)|ua_1(y)+u^2q_1(y)+\cdots+u^{k-1}q_{k-2}(y)),\dots,\\ (f_{k+1}(y)|l_{k-1}(y)|u^{k-2}a_2(y)+u^{k-1}t(y)),(f_{k+2}(y)|l_k(y)|u^{k-1}a_{k-1}(y)) \end{array}\right\rangle$$

be an $\mathbb{Z}_2\mathfrak{R}\mathfrak{R}_{u^t}$-additive cyclic code of length $(\beta_1,\beta_2,\beta_3)$, where β_2 and β_3 are odd and even integers, respectively. Also, let $b_1(y)\mid g(y)\mid(y^{\beta_2}-1)$, $a_{k-1}(y)\mid a_{k-2}(y)\mid\cdots\mid a_1(y)\mid h(y)\mid(y^{\beta_3}-1)$, $m_g(y)=\frac{y^{\beta_2}-1}{g(y)}$, $t_1(y)=\gcd\{m_g(y)b_1(y),y^{\beta_2}-1\}$,

$t_2(y) = \frac{y^{\beta_2}-1}{t_1(y)}$, $m_h(y) = \frac{y^{\beta_3}-1}{h(y)}$, $m(y) = \gcd\{m_h(y)p_1(y), \ldots, m_h p_{k-1}, y^{\beta_3} - 1\}$, $\hat{m}(y) = \frac{y^{\beta_3}-1}{m(y)}$, $m_{a_1}(y) = \frac{y^{\beta_3}-1}{a_1(y)}$, $m_1(y) = \gcd\{m_{a_1}(y)q_1(y), \ldots, m_{a_1}(y)q_{k-2}(y), y^{\beta_3} - 1\}$, $\hat{m}_1(y) = \frac{y^{\beta_3}-1}{m_1(y)}$. *Let*

$$\begin{aligned}
S_1 &= \bigcup_{i=0}^{\beta_1-\deg(f_1(y))-1} \{y^i \cdot (f_1(y)|0|0)\};\\
S_2 &= \bigcup_{i=0}^{\beta_2-\deg(g(y))-1} \{y^i \cdot (f_2(y)|g(y) + ub_1(y)|0)\};\\
S_3 &= \bigcup_{i=0}^{\deg(g(y))-\deg(b_1(y))-1} \{y^i \cdot (m_g(y)f_2(y)|um_g(y)b_1(y)|0)\},\\
S_4 &= \bigcup_{i=0}^{\beta_3-\deg(h(y))-1} \{y^i \cdot (f_3(y)|\ell_1(y)|h(y) + up_1(y) + u^2p_2(y) + \cdots + u^{k-1}p_{k-1}(y))\};\\
S_5 &= \bigcup_{i=0}^{\deg(m(y))-1} \{y^i \cdot (m_h(y)f_3(y)|m_h(y)\ell_1(y)|m_h(y)(up_1(y) +\\
&\qquad u^2p_2(y) + \cdots + u^{k-1}p_{k-1}(y)))\};\\
S_6 &= \bigcup_{i=0}^{\deg(h(y)-\deg(a_1(y))-1} \{y^i \cdot (f_4(y)|\ell_2(y)|ua_1(y) + u^2q_1(y) + \cdots + u^{k-1}q_{k-2}(y)))\};\\
S_7 &= \bigcup_{i=0}^{\deg(m_1(y))-1} \{y^i \cdot (m_{a_1}(y)f_4(y)|m_{a_1}(y)\ell_2(y)|m_{a_1}(y)(u^2q_1(y) + \cdots + u^{k-1}q_{k-2}(y)))\};\\
&\vdots\\
S_{2k+3} &= \bigcup_{i=0}^{\deg(a_{k-2}(y)-\deg(a_{k-1}(y)-1} \{y^i \cdot (f_{k+2}(y)|\ell_k(y)|u^{k-1}a_{k-1}(y)))\},
\end{aligned}$$

then $S = S_1 \cup S_2 \cup S_3 \cup \cdots \cup S_{2k+3}$ *is a minimal spanning set for the code* C.

Proof. Let $c(y) \in C$ be a codeword and $c_i(y) \in \Re_{u^t}[y]$, $1 \leq i \leq k+2$. Then

$$\begin{aligned}
c(y) &= c_1(y) \cdot (f_1(y)|0|0) + c_2(y) \cdot (f_2(y)|g(y) + ub_1(y)|0)\\
&\quad + c_3(y) \cdot (f_3(y)|\ell_1(y)|h(y) + up_1(y) + u^2p_2(y) + \cdots + u^{k-1}p_{k-1}(y))\\
&\quad + c_4(y) \cdot (f_4(y)|l_2(y)|ua_1(y) + u^2q_1(y) + \cdots + u^{k-1}q_{k-2}(y))\\
&\quad + \cdots + c_{k+1}(y) \cdot (f_{k+1}(y)|l_{k-1}(y)|u^{k-2}a_2(y) + u^{k-1}t(y))\\
&\quad + c_{k+2}(y) \cdot (f_{k+2}(y)|l_k(y)|u^{k-1}a_{k-1}(y))\\
&= (\theta_1(c_1(y))f_1(y)|0|0) + c_2(y) \cdot (f_2(y)|g(y) + ub_1(y)|0)\\
&\quad + c_3(y) \cdot (f_3(y)|\ell_1(y)|h(y) + up_1(y) + u^2p_2(y) + \cdots + u^{k-1}p_{k-1}(y))\\
&\quad + c_4(y) \cdot (f_4(y)|l_2(y)|ua_1(y) + u^2q_1(y) + \cdots + u^{k-1}q_{k-2}(y))\\
&\quad + \cdots + c_{k+1}(y) \cdot (f_{k+1}(y)|l_{k-1}(y)|u^{k-2}a_2(y) + u^{k-1}t(y))\\
&\quad + c_{k+2}(y) \cdot (f_{k+2}(y)|l_k(y)|u^{k-1}a_{k-1}(y)).
\end{aligned}$$

If $\deg(\theta(c_1(y))) \leq \beta_1 - \deg(f_1(y)) - 1$, then $(f_1(y)|0|0) \in \text{span}(S_1)$. Otherwise, by division algorithm, we have

$(\theta(c_1(y)) = \frac{(y^{\beta_1}-1)}{f_1(y)} d_1(y) + e_1(y)$, where $\deg(e_1(y)) \leq \beta_1 - \deg(f_1(y)) - 1$. Therefore,

$$\begin{aligned}(\theta(c_1)(f_1(y)|0|0) &= ((\frac{(y^{\beta_1}-1)}{f_1(y)} d_1 + e_1)(f_1(y))|0|0) \\ &= (e_1(f_1(y))|0|0) = e_1(f_1(y)|0|0).\end{aligned}$$

This implies that $(\theta(c_1)(f_1(y)), 0) \in \text{span}(S_1)$. Now, we have to show that

$$c_2(y) \cdot (f_2(y)|g(y) + ub_1|0) \in \text{span}(S_1 \cup S_2 \cup S_3) \subset \text{span}(S).$$

Suppose that $m_g(y)$ divides $c_2(y)$, that is,

$$c_2(y) = d_2(y)m_g(y) + e_2(y),$$

where $e_2(y) = 0$ or $\deg(e_2(y)) \leq \deg(m_g(y)) - 1$, we get

$$\begin{aligned}c_2(y) \cdot (f_2(y)|g(y) + ub_1(y)|0) &= (d_2(y)m_g(y) + e_2(y) \cdot (f_2(y)|g(y) + ub_1(y)|0) \\ &= d_2(y)(m_g(y)f_2(y|um_g(y)b_1(y)|0) \\ &\quad + e_2(y)(f_2(y)|g(y) + ub_1(y)|0).\end{aligned}$$

Cleraly, $e_2(y)(f_2|g(y) + ub_1(y)|0) \in \text{span}(S_2)$. It remains to show that

$$d_2(y)(m_g(y)f_2(y), um_g(y)b_1(y)|0) \in \text{span}(S_1 \cup S_3).$$

Since $t_1(y) \mid m_g(y)b_1(y)$, we obtain $m_g(y)b_1(y) = r_1(y)t_1(y)$. Hence, $m_g(y)b_1(y)t_2(y) = 0$. By division algorithm, we have

$$d_2(y) = d_3(y)t_2(y) + e_3(y),$$

where $e_3(y) = 0$ or $\deg(e_3(y)) \leq \deg(t_2(y)) - 1$. The expression $d_2(y)(m_g(y)f_2(y)|um_g(y)b_1(y)|0)$ can be written as

$$\begin{aligned}&d_2(y)(m_g(y)f_2(y)|um_g(y)b_1(y)|0) \\ &= (d_3(y)t_2(y) + e_3(y)(m_g(y)f_2(y)|um_g(y)b_1(y)|0) \\ &= d_3(y)(t_2(y)m_g(y)f_2(y)|ut_2(y)m_g(y)b_1(y)|0) + e_3(y)(m_g(y)f_2(y)|um_g(y)b_1(y)|0). \\ &= d_3(y)(t_2(y)m_g(y)f_2(y)|0|0) + e_3(y)(m_g(y)f_2(y)|um_g(y)b_1(y)|0).\end{aligned}$$

The above expression shows that $d_3(t_2m_gf_2|0|0) \in \text{span}(S_1)$ and $(m_gf_2, um_gb_1(y)|0) \in \text{span}(S_3)$. Therefore,

$$c_2(y) \cdot (f_2(y)|g(y) + ub_1|0) \in \text{span}(S_1 \cup S_2 \cup S_3).$$

Next, we show that

$$c_3(y) \cdot (f_3(y)|\ell_1(y)|h(y) + up_1(y) + \cdots + u^{k-1}p_{k-1}(y)) \in \text{span}(S_2 \cup S_4 \cup S_5) \subset \text{span}(S).$$

Applying division algorithm, we get

$$c_3(y) = d_4(y)m_h(y) + e_4(y),$$

where $e_4(y) = 0$ or $\deg(e_4(y)) \leq \deg(m_h(y)) - 1$. Therefore,

$$\begin{aligned}&c_3(y) \cdot (f_3(y)|\ell_1(y)|h(y) + up_1(y) + u^2p_2(y) + \cdots + u^{k-1}p_{k-1}(y)) \\ &= (d_4(y)m_h(y) + e_4(y)) \cdot (f_3(y)|\ell_1(y)|h(y) + up_1(y) + u^2p_2(y) + \cdots + u^{k-1}p_{k-1}(y)) \\ &= d_4(y)(m_h(y)f_3(y)|m_h(y)\ell_1(y)|m_h(y)(up_1(y) + u^2p_2(y) + \cdots + u^{k-1}p_{k-1}(y))) \\ &\quad + e_4(y)(f_3(y)|\ell_1(y)|h(y) + up_1(y) + u^2p_2(y) + \cdots + u^{k-1}p_{k-1}(y)).\end{aligned}$$

Obviously, $e_4(y)(f_3(y)|\ell_1(y)|h(y) + up_1(y) + u^2p_2(y) + \cdots + u^{k-1}p_{k-1}(y)) \in \text{span}(S_4)$. Again, using the division algorithm, we have

$$d_4(y) = \hat{m}(y)d_5(y) + e_5(y),$$

where $\deg(e_5(y)) < \deg(\hat{m}(y))$ or $e_5(y) = 0$. Putting the value of $d_4(y)$ in the expression

$$d_4(y)(m_h(y)f_3(y)|m_h(y)\ell_1(y)|m_h(y)(up_1(y) + u^2p_2(y) + \cdots + u^{k-1}p_{k-1}(y))),$$

we get

$$\begin{aligned}
&d_4(y)(m_h(y)f_3(y)|m_h(y)\ell_1(y)|m_h(y)(up_1(y) + u^2p_2(y) + \cdots + u^{k-1}p_{k-1}(y)))\\
&= (\hat{m}(y)d_5(y) + e_5(y))(m_h(y)f_3(y)|m_h(y)\ell_1(y)|m_h(y)(up_1(y) + \cdots + u^{k-1}p_{k-1}(y)))\\
&= d_5(m_h(y)\hat{m}(y)f_3(y)|m_h(y)\hat{m}(y)\ell_1(y)|m_h(y)\hat{m}(y)(up_1(y) + \cdots + u^{k-1}p_{k-1}(y)))\\
&\quad +(e_5(y))(m_h(y)f_3(y)|m_h(y)\ell_1(y)|m_h(y)(up_1(y) + \cdots + u^{k-1}p_{k-1}(y)))\\
&= d_5(m_h(y)\hat{m}(y)f_3(y)|m_h(y)\hat{m}(y)\ell_1(y)|0)\\
&\quad +(e_5(y))(m_h(y)f_3(y)|m_h(y)\ell_1(y)|m_h(y)(up_1(y) + \cdots + u^{k-1}p_{k-1}(y))).
\end{aligned}$$

It is observed that $d_5(m_h(y)\hat{m}(y)f_3(y)|m_h(y)\hat{m}(y)\ell_1(y)|0) \in \text{span}(S_2)$ and

$$(e_5(y))(m_h(y)f_3(y)|m_h(y)\ell_1(y)|m_h(y)(up_1(y) + \cdots + u^{k-1}p_{k-1}(y))) \in \text{span}(S_5).$$

This implies that

$$c_3(y) \cdot (f_3(y)|\ell_1(y)|h(y) + up_1(y) + \cdots + u^{k-1}p_{k-1}(y)) \in \text{span}(S_2 \cup S_4 \cup S_5) \subset \text{span}(S).$$

Following the same process, it is required to show that

$$c_{k+2}(y) \cdot (f_{k+2}(y)|l_k(y)|u^{k-1}a_{k-1}(y)) \in \text{span}(S_2 \cup S_{2k+3}).$$

Now, we assume that $c_{k+2}(y) = d_{2k+3}(y)a_{k-1}(y) + e_{2k+3}(y)$,

$$\begin{aligned}
&c_{k+2}(y) \cdot (f_{k+2}(y)|l_k(y)|u^{k-1}a_{k-1}(y))\\
&= (d_{2k+3}(y)a_{k-1}(y) + e_{2k+3}(y)) \cdot (f_{k+2}(y)|l_k(y)|u^{k-1}a_{k-1}(y))\\
&= d_{2k+3}(y)(a_{k-1}f_{k+2}(y)|a_{k-1}l_k(y)|0) + e_{2k+3}(y) \cdot (f_{k+2}(y)|l_k(y)|u^{k-1}a_{k-1}(y))
\end{aligned}$$

Clearly, $d_{2k+3}(y)(a_{k-1}(y)f_{k+2}(y)|a_{k-1}(y)l_k(y)|0) \in \text{span}(S_2)$ and $e_{2k+3}(y) \cdot (f_{k+2}(y)|l_k(y)|u^{k-1}a_{k-1}(y)) \in S_{2k+3}$. This implies that

$$c_{k+2}(y) \cdot (f_{k+2}(y)|l_k(y)|u^{k-1}a_{k-1}(y)) \in \text{span}(S_2 \cup S_{2k+3}).$$

$c(y) \in \text{span}(S)$. Finally, we reach the required conclusion. And hence

$$S = S_1 \cup S_2 \cup \cdots \cup S_{2k+3}$$

is the minimal spanning set for C. □

From Theorem 4.6, the following results follow immediately.

Corollary 4.8. *Let $C = \langle (f_1(y)(y)|0|0) \rangle$ be an $\mathbb{Z}_2\mathfrak{R}\mathfrak{R}_{u^t}$-additive cyclic code of length $(\beta_1, \beta_2, \beta_3)$ and $f_1(y) \mid y^{\beta_1} - 1$. If*

$$S_1 = \bigcup_{i=0}^{\beta_1-\deg(f_1(y))-1} \{y^i \cdot (f_1(y)|0|0)\},$$

then S_1 forms a minimal spanning set for C with $|C| = 2^{(\beta_1-\deg(f_1(y)))}$.

Corollary 4.9. *Let* $C = \langle (f_1(y)|0|0), (f_2|g(y) + ub_1(y)|0)\rangle$ *be an* $\mathbb{Z}_2\Re\Re_{u^t}$*-additive cyclic code of length* $(\beta_1, \beta_2, \beta_3)$*, where* β_2 *is an odd integer and* $g(y) + ub_1(y)) \mid y^{\beta_2} - 1$*. If*

$$\begin{aligned} S_1 &= \bigcup_{i=0}^{\beta_1-\deg(f_1(y))-1} \{y^i \cdot (f_1(y)|0|0)\}; \\ S_2 &= \bigcup_{i=0}^{\beta_2-\deg(g(y))-1} \{y^i \cdot (f_2(y)|g(y) + ub_1(y)|0)\}; \\ S_3 &= \bigcup_{i=0}^{\deg(g(y))-\deg(b_1(y))-1} \{y^i \cdot (m_g(y)f_2(y)|um_g(y)b_1(y)|0)\}, \end{aligned}$$

then $S_1 \cup S_2 \cup S_3$ *forms a minimal spanning set for C with* $|C| = 2^{(\beta_1-\deg(f_1(y)))}2^{(2\beta_2-\deg(g_1(y))-\deg(b_1(y)))}$.

Corollary 4.10. *Let*

$$C = \left\langle \begin{array}{c} (f_1(y)|0|0), (f_2(y)|g(y) + ub_1(y)|0), \\ (f_3(y)|\ell_1(y)|h(y) + up_1(y) + u^2p_2(y) + \cdots + u^{k-1}p_{k-1}(y)) \end{array} \right\rangle$$

be an $\mathbb{Z}_2\Re\Re_{u^t}$*-additive cyclic code of length* $(\beta_1, \beta_2, \beta_3)$*, where* β_2 *and* β_3 *are odd and even integers, respectively. Also, let* $b_1(y) \mid g(y) \mid (y^{\beta_2} - 1)$, $a_{k-1}(y) \mid a_{k-2}(y) \mid \cdots \mid a_1(y) \mid h(y) \mid (y^{\beta_3} - 1)$, $m_g(y) = \frac{y^{\beta_2}-1}{g(y)}$, $t_1(y) = \gcd\{m_g(y)b_1(y), y^{\beta_2} - 1\}$, $t_2(y) = \frac{y^{\beta_2}-1}{t_1(y)}$, $m_h(y) = \frac{y^{\beta_3}-1}{h(y)}$, $m(y) = \gcd\{m_h(y)p_1(y), \ldots, m_h(y)p_{k-1}(y), y^{\beta_3} - 1\}$, $\hat{m}(y) = \frac{y^{\beta_3}-1}{m(y)}$*, Let*

$$\begin{aligned} S_1 &= \bigcup_{i=0}^{\beta_1-\deg(f_1(y))-1} \{y^i \cdot (f_1(y)|0|0)\}; \\ S_2 &= \bigcup_{i=0}^{\beta_2-\deg(g(y))-1} \{y^i \cdot (f_2(y)|g(y) + ub_1(y)|0)\}; \\ S_3 &= \bigcup_{i=0}^{\deg(g(y))-\deg(b_1(y))-1} \{y^i \cdot (m_g(y)f_2(y)|um_g(y)b_1(y)|0)\}, \\ S_4 &= \bigcup_{i=0}^{\beta_3-\deg(h(y))-1} \{y^i \cdot (f_3(y)|\ell_1(y)|h(y) + up_1(y) + u^2p_2(y) + \cdots + u^{k-1}p_{k-1}(y))\}; \\ S_5 &= \bigcup_{i=0}^{\deg(m(y))-1} \{y^i \cdot (m_h(y)f_3(y)|m_h(y)\ell_1(y)|m_h(y)(up_1(y) + \\ &\qquad u^2p_2(y) + \cdots + u^{k-1}p_{k-1}(y)))\}; \end{aligned}$$

then $S = S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5$ *is a minimal spanning set for the code C.*

5. De21uality of $\mathbb{Z}_2\Re\Re_{u^t}$-additive cyclic codes

In this section, we give the relationship between the generator polynomial of C and its dual code. Let $f(y) \in \Re_{u^t}[y]$ and $\deg(f(y)) = t$. Then its reciprocal polynomial can be defined as $f^*(y) = y^{\deg(f(y))} f(\frac{1}{y})$.

To study the dual of $\mathbb{Z}_2\Re\Re_{u^t}$-additive cyclic codes, we need to define a new inner product on $\mathbb{Z}_2^{\beta_1} \times \Re^{\beta_2} \times {\Re_{u^t}}^{\beta_3}$ as

$$\mathbf{z}_1 \cdot \mathbf{z}_2 = (u^{k-1} \sum_{i=0}^{\beta_1-1} a_i a_i^{'} + u \sum_{j=0}^{\beta_2-1} b_j b_j^{'} + \sum_{k=0}^{\beta_3-1} c_k c_k^{'})(\text{mod}2), \tag{7}$$

where $\mathbf{z_1} = (a_0, a_1, \ldots, a_{\beta_1 - 1}|b_0, b_1, \ldots, b_{\beta_2 - 1}|c_0, c_1, \ldots, c_{\beta_3 - 1})$ and
$\mathbf{z_2} = (a_0^{'}, a_1^{'}, \ldots, a_{\beta_1-1}^{'}|b_0^{'}, b_1^{'}, \ldots, b_{\beta_2-1}^{'}|c_0^{'}, c_1^{'}, \ldots, c_{\beta_3-1}^{'})$.

Definition 5.1. *Let C be any $\mathbb{Z}_2\mathfrak{R}\mathfrak{R}_{u^t}$-additive cyclic code. Then the dual code of C with respect to the inner product defined in 7 is defined as*

$$C^{\perp} = \{\mathbf{z_2} \in \mathbb{Z}_2^{\beta_1} \times \mathfrak{R}^{\beta_2} \times \mathfrak{R}_{u^t}{}^{\beta_3} | \ \mathbf{z_1} \cdot \mathbf{z_2} = 0 \, for \ \ all \ \mathbf{z_1} \in C\}.$$

Theorem 5.2. *Let C be any $\mathbb{Z}_2\mathfrak{R}\mathfrak{R}_{u^t}$-additive cyclic code. Then $C^{\perp}$ is also cyclic.*

Proof. Let C be any $\mathbb{Z}_2\mathfrak{R}\mathfrak{R}_{u^t}$-additive cyclic code and

$$z_2 = (a_0^{'}, a_1^{'}, \ldots, a_{\beta_1-1}^{'}|b_0^{'}, b_1^{'}, \ldots, b_{\beta_2-1}^{'}|c_0^{'}, c_1^{'}, \ldots, c_{\beta_3-1}^{'}) \in C^{\perp}.$$

We have to prove that $z_1 \cdot T(z_2) = 0$. Since C is cyclic, we have $T^{\ell}(z_1)$ also in C, where $\ell = \mathrm{lcm}(\beta_1, \beta_2, \beta_3)$. Now, we can write

$$\begin{aligned}
0 &= T^{\ell-1}(z_1) \cdot z_2 \\
&= (a_1, \ldots, a_{\beta_1-1}, a_0|b_1, \ldots, b_{\beta_2-1}, b_0|c_1, \ldots, c_{\beta_3-1}, c_0) \cdot (a_0^{'}, a_1^{'}, \ldots, a_{\beta_1-1}^{'}|b_0^{'}, b_1^{'}, \ldots, b_{\beta_2-1}^{'}| \\
&\quad c_0^{'}, c_1^{'}, \ldots, c_{\beta_3-1}^{'}) \\
&= u^{k-1}(a_1a_0^{'} + a_2a_1^{'} + \cdots + a_0a_{\beta_1-1}^{'}) + u(b_1b_0^{'} + b_2b_1^{'} + \cdots + b_0b_{\beta_2-1}^{'}) + (c_1c_0^{'} + c_2c_1^{'} \\
&\quad + \cdots + c_0c_{\beta_3-1}^{'}) \\
&= u^{k-1}(a_0a_{\beta_1-1}^{'} + a_1a_0^{'} + \cdots + a_{\beta_1-1}a_{\beta_1-2}^{'}) + u(b_0b_{\beta_2-1}^{'} + b_1b_0^{'} + \cdots + b_{\beta_2-1}b_{\beta_2-2}^{'}) + \\
&\quad (c_0c_{\beta_3-1}^{'} + c_1c_0^{'} + \cdots + c_{\beta_3-1}c_{\beta_3-2}^{'}) \\
&= z_1 \cdot T(z_2).
\end{aligned}$$

This implies that $T(z_2) \in C^{\perp}$. Hence, $C^{\perp}$ is $\mathbb{Z}_2\mathfrak{R}\mathfrak{R}_{u^t}$-additive cyclic code. □

Assume that $\omega_m(y) = \sum_{i=0}^{m-1} y^i$ is a polynomial. Now, let $m = \mathrm{lcm}\{\beta_1, \beta_2, \beta_3\}$ and
$\mathbf{f}(y) = (f(y)|f^{'}(y)|f^{''}(y)), \mathbf{g}(y) = (g(y)|g^{'}(y)|g^{''}(y) \in \mathcal{S}_{\beta_1,\beta_2,\beta_3}$. Define a map

$$\zeta : \mathcal{S}_{\beta_1,\beta_2,\beta_3} \times \mathcal{S}_{\beta_1,\beta_2,\beta_3} \longrightarrow \frac{\mathfrak{R}_{u^t}[y]}{\langle y^m - 1\rangle}$$

such that

$$\begin{aligned}
\zeta(\mathbf{f}(y), \mathbf{g}(y)) &= u^k f(y)\omega_{\frac{m}{\beta_1}}(y^{\beta_1})y^{m-1-\deg(g(y))}g^*(y) \\
&\quad + uf^{'}(y)\omega_{\frac{m}{\beta_2}}(y^{\beta_2})y^{m-1-\deg(g^{'}(y))}g^{'*}(y) \\
&\quad + f^{''}(y)\omega_{\frac{m}{\beta_3}}(y^{\beta_3})y^{m-1-\deg(g^{''}(y))}g^{''*}(y).
\end{aligned}$$

Now, we state the relevant lemmas that will be used to demonstrate the continuing results.

Lemma 5.3. *Let $n_1, n_2 \in \mathbb{N}$. Then*

$$y^{n_1n_2} - 1 = (y^{n_1} - 1)\omega_{n_2}(y^{n_1}).$$

Proof. Let $x^{n_2} - 1 = (x-1)(x^{n_2-1} + x^{n_2-2} + \cdots + x + 1) = (x-1)\omega_{n_2}(x)$. Putting $x = y^{n_1}$, we get the desired result. □

Lemma 5.4. *[21, Lemma 6.5] Let $\boldsymbol{f}, \boldsymbol{g} \in \mathbb{Z}_2^{\beta_1} \times \mathfrak{R}^{\beta_2} \times \mathfrak{R}^{\beta_3}$ with associated polynomials*
$\boldsymbol{f}(y) = (f(y)|f^{'}(y)|f^{''}(y)), \boldsymbol{g}(y) = (g(y)|g^{'}(y)|g^{''}(y)) \in \mathcal{S}_{\beta_1,\beta_2,\beta_3}$. Then $\boldsymbol{f}$ is orthogonal to $\boldsymbol{g}$ and all its shifts if and only if

$$\zeta(\boldsymbol{f}(y), \boldsymbol{g}(y)) = 0.$$

Theorem 5.5. *Let* $\boldsymbol{f}(y) = (f(y)|f^{'}(y)|f^{''}(y)), \boldsymbol{g}(y) = (g(y)|g^{'}(y)|g^{''}(y)) \in \mathcal{S}_{\beta_1,\beta_2,\beta_3}$ *such that* $\zeta(\boldsymbol{f}(y), \boldsymbol{g}(y)) = 0$.

(*i*) *If* $f^{'}(y) = 0$ *or* $g^{'}(y) = 0$ *and* $f^{''}(y) = 0$ *or* $g^{''}(y) = 0$*, then* $f(y)g^{*}(y) = 0 \bmod (y^{\beta_1} - 1)$.

(*ii*) *If* $f(y) = 0$ *or* $g(y) = 0$ *and* $f^{''}(y) = 0$ *or* $g^{''}(y) = 0$*, then* $f^{'}(y)g^{'*}(y) = 0 \bmod (y^{\beta_2} - 1)$.

(*iii*) *If* $f(y) = 0$ *or* $g(y) = 0$ *and* $f^{'}(y) = 0$ *or* $g^{'}(y) = 0$*, then* $f^{''}(y)g^{''*}(y) = 0 \bmod (y^{\beta_3} - 1)$.

Proof. (*i*) Suppose that either $f^{'}(y) = 0$ or $g^{'}(y) = 0$ and $f^{''}(y) = 0$ or $g^{''}(y) = 0$. Then we need to show that $f(y)g^{*}(y) = 0 \bmod (y^{\beta_2} - 1)$. Since

$$\begin{aligned} 0 &= \zeta(\mathbf{f}(y), \mathbf{g}(y)) \\ &= f(y)\omega_{\frac{m}{\beta_1}}(y^{\beta_1})y^{m-1-\deg(g(y))}g^{*}(y) \mathrm{mod}(\mathrm{y^m} - 1) \end{aligned}$$

This implies that there exist a polynomial $h(y) \in \mathbb{Z}_2[y]$ such that

$$\begin{aligned} f(y)\omega_{\frac{m}{\beta_1}}(y^{\beta_1})y^{m-1-\deg(g(y))}g^{*}(y) &= h(y)\mathrm{mod}(\mathrm{y^m} - 1) \\ &= h(y)(y^m - 1). \end{aligned}$$

By Proposition 5.1, $y^m - 1 = (y^{\beta_1} - 1)\omega_{\frac{m}{\beta_1}}(y^{\beta_1})$, we get

$$\begin{aligned} f(y)y^m g^{*}(y) &= h^{'}(y)(y^{\beta_1} - 1) \\ f(y)g^{*}(y) &= 0 \bmod (y^{\beta_1} - 1). \end{aligned}$$

Similarly, we can prove other cases. □

Theorem 5.6. *Let*

$$C = \left\langle \begin{array}{c} (f_1(y)|0|0), (f_2(y)|g(y) + ub_1(y)|0), \\ (f_3(y)|\ell_1(y)|h(y) + ua_1(y) + u^2a_2(y) + \cdots + u^{k-1}a_{k-1}(y)) \end{array} \right\rangle$$

be an $\mathbb{Z}_2\mathfrak{R}\mathfrak{R}_{u^t}$*-additive cyclic code of length* $(\beta_1, \beta_2, \beta_3)$*, where* β_2 *and* β_3 *are odd integers and* $b_1(y)|g(y)|(y^{\beta_2} - 1)$*,* $a_{k-1}(y) \mid a_{k-2}(y) \mid \cdots \mid a_2(y) \mid a_1(y) \mid h(y) \mid (y^{\beta_3} - 1)$*. If*

$$C^{\perp} = \left\langle \begin{array}{c} (\bar{f_1}(y)|0|0), (\bar{f_2}(y)|\bar{g}(y) + u\bar{b}_1(y)|0), \\ (\bar{f_3}(y)|\bar{\ell}_1(y)|\bar{h}(y) + u\bar{a}_1(y) + u^2\bar{a}_2(y) + \cdots + u^{k-1}\bar{a}_{k-1}(y)) \end{array} \right\rangle$$

is a dual of C, then

(*i*) $\bar{f_1}^{*}(y) \gcd(f_1(y), f_2(y), f_3(y)) = h_1(y)(y^{\beta_1} - 1)$,

(*ii*) $\frac{f_1(y)\ell_1(y)(uf_3(y)g(y) + u^{k-1}f_2(y)\ell_1(y))}{\gcd(f_1(y), f_2(y), f_3(y), \ell_1(y))} \cdot (\bar{g}(y) + u\bar{b}_1(y))^{*} = h_2(y)(y^{\beta_2} - 1)$.

Proof. (*i*) Since $(f_1(y)|0|0), (f_2(y)|g(y)+ub_1(y)|0), (f_3(y)|\ell_1(y)|h(y)+ua_1(y)+\cdots+u^{k-1}a_{k-1}(y)) \in C$ and $(\bar{f_1}(y)|0|0) \in C^{\perp}$, by Lemma 5.2, we get

$$\zeta((f_1(y)|0|0), (\bar{f_1}(y)|0|0)) = 0,$$

$$\zeta((f_2(y)|g(y) + ub_1(y)|0), (\bar{f_1}(y)|0|0)) = 0$$

and

$$\zeta((f_3(y)|\ell_1(y)|h(y) + ua_1(y) + \cdots + u^{k-1}a_{k-1}(y), (\bar{f_1}(y)|0|0)) = 0.$$

Using Theorem 5.1, we obtain $f_1(y)\bar{f}_1^{*}(y) = 0$, $f_2(y)\bar{f}_1^{*}(y) = 0$ and $f_3(y)\bar{f}_1^{*}(y) = 0$. It is obvious that $\bar{f}_1^{*}(y) \gcd(f_1(y), f_2(y), f_3(y)) = 0 \bmod (y^{\beta_1} - 1)$. This implies that there exists a polynomial $h_1(y) \in \mathbb{Z}_2[y]$ such that

$$\bar{f}_1^{*}(y) \gcd(f_1(y), f_2(y), f_2(y)) = h_1(y)(y^{\beta_1} - 1).$$

(ii) We know that

$$(f_1(y)|0|0), (f_2(y)|g(y)+ub_1(y)|0), (f_3(y)|\ell_1(y)|h(y)+ua_1(y)+\cdots+u^{k-1}a_{k-1}(y)) \in C.$$

Then any element $c(y) \in C$ can be expressed as

$$\begin{aligned} c(y) &= \frac{f_2(y)f_3(y)\ell_1(y)}{\gcd(f_1(y),f_2(y),f_3(y),\ell_1(y))} \times (f_1(y)|0|0) \\ &\quad + u\frac{f_1(y)f_3(y)\ell_1(y)}{\gcd(f_1(y),f_2(y),f_3(y),\ell_1(y))} \times (f_2(y)|g(y)+ub_1(y)|0) \\ &\quad + u^{k-1}\frac{f_1(y)f_2(y)\ell_1(y)}{\gcd(f_1(y),f_2(y),f_3(y),\ell_1(y))} \times (f_3(y)|\ell_1(y)|h(y)+ua_1+\cdots+u^{k-1}a_{k-1}) \\ &= (0|u\frac{f_1(y)f_3(y)\ell_1(y)}{\gcd(f_1(y),f_2(y),f_3(y),\ell_1(y))}g(y)|0) \\ &\quad +(0|u^{k-1}\frac{f_1(y)f_2(y)\ell_1^2(y)}{\gcd(f_1(y),f_2(y),f_3(y),\ell_1(y))}|u^{k-1}\frac{f_1(y)f_2(y)\ell_1(y)}{\gcd(f_1(y),f_2(y),f_3(y),\ell_1(y))}h(y)) \\ &= (0|\frac{f_1(y)\ell_1(y)(uf_3(y)g(y)+u^{k-1}f_2(y)\ell_1(y))}{\gcd(f_1(y),f_2(y),f_3(y),\ell_1(y))}|\frac{u^{k-1}f_1(y)f_2(y)\ell_1(y)}{\gcd(f_1(y),f_2(y),f_3(y),\ell_1(y))}h(y)). \end{aligned}$$

This implies that

$$\zeta((0|\frac{W_1(y)}{W(y)}|\frac{u^{k-1}f_1(y)f_2(y)\ell_1(y)}{W(y)}), (\bar{f_2}|\bar{g}(y)+u\bar{b}_1(y)|0)) = 0,$$

where $W_1(y) = f_1(y)\ell_1(y)(uf_3(y)g(y)+u^{k-1}f_2(y)\ell_1(y))$ and
$W(y) = \gcd(f_1(y), f_2(y), f_3(y), \ell_1(y)h(y))$. By Theorem 5.1, we get

$$\frac{f_1(y)\ell_1(y)(uf_3(y)g(y)+u^{k-1}f_2(y)\ell_1(y))}{\gcd(f_1(y),f_2(y),f_3(y),\ell_1(y))} \cdot (\bar{g}(y)+u\bar{b}_1(y))^* = 0.$$

This means that there exists a polynomial $h_2(y) \in \Re[y]$ such that

$$\frac{f_1(y)\ell_1(y)(uf_3(y)g(y)+u^{k-1}f_2(y)\ell_1(y))}{\gcd(f_1(y),f_2(y),f_3(y),\ell_1(y))} \cdot (\bar{g}(y)+u\bar{b}_1(y))^* = h_2(y)(y^{\beta_2}-1).$$

□

6. Examples & Table

In this section, we discuss some examples of additive cyclic codes of different lengths. Also, the generator polynomials and the minimal spanning sets are determined.

Example 6.1. *Let C be a $\mathbb{Z}_2\Re\Re_u$-additive cyclic code of length $(3,3,5)$. Then C is a $\Re_u$-submodule of $\mathcal{S}_{3,3,5} = \mathbb{Z}_3[y]/\langle y^3-1\rangle \times \Re[y]/\langle y^3-1\rangle \times \Re_u[y]/\langle y^5-1\rangle$. By Theorem 4.5, suppose that $f_1(y) = y-1$, $f_2(y) = 1$, $g(y) = b_1(y) = y-1$, $f_3(y) = y+1$, $\ell_1(y) = y^2+y+1$, $h(y) = y-1$, $a_1(y) = a_2(y) = y-1 = a_3(y)$. If*

$$\begin{aligned} S_1 &= \{(y+1|0|0),\ y\cdot(y+1|0|0)\}; \\ S_2 &= \{(1|(1+u)(y+1)|0),\ y\cdot(1|(1+u)(y+1)|0)\}; \\ S_3 &= \bigcup_{i=0}^{3}\{y^i\cdot(y+1|y^2+y+1|(y+1)(1+u+u^2+u^3))\}, \end{aligned}$$

then $S_1 \cup S_2 \cup S_3$ forms a minimal spanning set for C.

Example 6.2. *Let C be a $\mathbb{Z}_2\Re\Re_u$-additive cyclic code of length* $(2,3,4)$*. Then C is a $\Re_u$-submodule of $\mathcal{S}_{2,3,4} = \mathbb{Z}_2[y]/\langle y^2-1\rangle \times \Re[y]/\langle y^3-1\rangle \times \Re_u[y]/\langle y^4-1\rangle$. By Theorem 4.6, Suppose that $f_1(y) = f_2(y) = f_3(y) = f_4(y) = f_5(y) = y+1$, $g(y) = b_1(y) = y-1$, $\ell_1(y) = \ell_2(y) = \ell_3(y) = y^2+y+1$, $h(y) = (y+1)^3$, $a_1(y) = (y+1)^2$, $a_2(y) = y+1$, $p_1(y) = p_2(y) = q_1(y) = 1$, $m_g(y) = \frac{y^2-1}{g(y)} = y+1$, $m_h(y) = \frac{y^4-1}{h(y)} = y+1$, $m_{a_1}(y) = \frac{y^4-1}{a_1(y)} = (y+1)^2$, $m_{a_2}(y) = \frac{y^4-1}{a_2(y)} = (y+1)^3$. If*

$$
\begin{aligned}
S_1 &= \{(f_1(y)|0|0)\};\\
S_2 &= \{(f_2(y)|g(y)+ub_1(y)|0),\ y(f_2(y)|g(y)+ub_1(y)|0)\};\\
S_3 &= \{(f_3(y)|\ell_1(y)|h(y)+up_1(y)+u^2p_2(y))\};\\
S_4 &= \bigcup_{i=0}^{2}\{y^i\cdot(m_h(y)f_3(y)|m_h(y)\ell_1(y)|m_h(y)(up_1(y)+u^2p_2(y))\};\\
S_5 &= \bigcup_{i=0}^{1}\{y^i\cdot(f_4(y)|\ell_2(y)|ua_1(y)+u^2q_1(y))\};\\
S_6 &= \bigcup_{i=0}^{1}\{y^i\cdot(m_{a_1}(y)f_4(y)|m_{a_1}(y)\ell_2(y)|u^2m_{a_1}(y)q_1(y))\};\\
S_7 &= \{(f_5(y)|\ell_3(y)|u^2a_2(y))\},
\end{aligned}
$$

then $S_1\cup S_2\cup S_3\cup S_4\cup S_5\cup S_6\cup S_7$ forms a minimal spanning set for C.

Table: Optimal binary images from $\mathbb{Z}_2\Re\Re_{u^t}$-additive cyclic codes.

k	$(\beta_1,\beta_2,\beta_3)$	Generators	Binary Image
4	$(1,1,3)$	$f_1(y)=y-1$, $f_2(y)=0$, $g(y)=b_1(y)=y-1$, $f_3(y)=\ell_1(y)=0$, $h(y)=y^3-1$, $a_1(y)=a_2(y))=a_3(y)=y-1$	$[15,6,6]$
5	$(1,1,3)$	$f_1(y)=y-1$, $f_2(y)=0$, $g(y)=b_1(y)=y-1$, $h(y)=y^3-1=a_1(y)=a_2(y)=a_3(y)$, $f_3(y)=\ell_1(y)=1$, $a_4(y)=y-1$	$[18,2,12]$
5	$(1,1,3)$	$f_1(y)=y-1$, $f_2(y)=0$, $g(y)=b_1(y)=y-1$, $f_3(y)=\ell_1(y)=0$, $h(y)=y^3-1=a_1(y)$, $a_2(y))=a_3(y)=a_4(y)=y-1$	$[18,4,8]$
3	$(1,1,6)$	$f_1(y)=y-1$, $f_2(y)=0$, $g(y)=b_1(y)=y-1$, $f_3(y)=0$, $\ell_1(y)=0$, $l_2(y)=1$, $h(y)=y^6-1$, $a_1(y)=y^4+y^3+y+1$, $a_2(y)=y^2+1$ $f_4(y)=1$, $f_5(y)=1$, $l_3(y)=1$	$[21,7,8]$
5	$(2,3,3)$	$f_1(y)=y^2-1$, $f_2(y)=0$, $g(y)=b_1(y)=y^3-1$, $h(y)=y^2+y+1=a_1(y)=a_2(y)=a_3(y)=a_4(y)$, $f_3(y)=\ell_1(y)=1$	$[23,5,11]$
4	$(2,3,5)$	$f_1(y)=y^2-1$, $f_2(y)=0$, $g(y)=b_1(y)=y^3-1$, $h(y)=a_1(y)=a_2(y)=y^5-1$, $f_3(y)=\ell_1(y)=1$, $a_3(y)=y^4+y^3+y^2+y+1$	$[28,4,13]$

k	$(\beta_1, \beta_2, \beta_3)$	Generators	Binary Image
3	(1,5,6)	$f_1(y) = y - 1$, $f_2(y) = 0$, $g(y) = y^5 - 1 = b_1(y)$, $f_3(y) = 0$, $\ell_1(y) = 0$, $h(y) = y^6 - 1$, $f_4(y) = 1$, $l_2(y) = y^4 + y^3 + y^2 + y + 1$, $a_1(y) = y^3 + y^2 + y + 1$, $a_2(y) = y^2 + 1$, $f_5(y) = 1$, $l_3(y) = y^4 + y^3 + y^2 + y + 1$	[29, 7, 12]
3	(9,5,4)	$f_1(y) = y^9 - 1$, $f_2(y) = 0$, $g(y) = y^5 - 1 = b_1(y)$, $f_3(y) = 0$, $\ell_1(y) = 0$, $h(y) = y^4 - 1$, $f_4(y) = y^7 + y^6 + y^4 + y^3 + y + 1$, $l_2(y) = y^4 + y^3 + y^2 + y + 1$, $a_1(y) = y^4 - 1$, $a_2(y) = y - 1$, $f_5(y) = y^7 + y^6 + y^4 + y^3 + y + 1$, $l_3(y) = y^4 + y^3 + y^2 + y + 1$	[31, 3, 17]
4	(2,1,7)	$f_1(y) = y^2 - 1$, $f_2(y) = 0$, $g(y) = b_1(y) = y - 1$, $f_3(y) = 1$, $\ell_1(y) = 0$, $h(y) = y^7 - 1$, $a_1(y) = a_2(y) = a_3(y) = y^3 + y^2 + 1$	[32, 12, 10]
4	(2,1,7)	$f_1(y) = y^2 - 1$, $f_2(y) = 0$, $g(y) = b_1(y) = y - 1$, $f_3(y) = 1$, $\ell_1(y) = 1$, $h(y) = y^7 - 1$, $a_1(y) = a_2(y) = a_3(y) = y^4 + y^2 + y + 1$	[32, 9, 14]
3	(1,7,6)	$f_1(y) = y - 1$, $f_2(y) = 0$, $g(y) = y^7 - 1 = b_1(y)$, $f_3(y) = 0$, $\ell_1(y) = 0$, $h(y) = y^6 - 1$, $f_4(y) = 1$, $l_2(y) = y^6 + y^5 + y^4 + y^3 + y^2 + y + 1$, $a_1(y) = y^3 + y^2 + y + 1$, $a_2(y) = y^2 + 1$, $f_5(y) = 1$, $l_3(y) = y^6 + y^5 + y^4 + y^3 + y^2 + y + 1$	[33, 7, 14]
5	(1,1,7)	$f_1(y) = y - 1$, $f_2(y) = 0$, $g(y) = b_1(y) = y - 1$, $f_3(y) = 1$, $\ell_1(y) = 1$, $h(y) = a_1(y) = y^7 - 1$, $a_2(y) = a_3(y) = a_4(y) = y^3 + y^2 + 1$	[38, 12, 14]
5	(2,1,7)	$f_1(y) = y^2 - 1$, $f_2(y) = 0$, $g(y) = b_1(y) = y - 1$, $f_3(y) = 0$, $\ell_1(y) = 0$, $h(y) = a_1(y) = y^7 - 1$, $a_2(y) = a_3(y) = a_4(y) = y^4 + y^2 + y + 1$	[39, 9, 16]
5	(1,1,9)	$f_1(y) = y - 1$, $f_2(y) = 0$, $g(y) = b_1(y) = y - 1$, $f_3(y) = 1$, $\ell_1(y) = 1$, $h(y) = a_1(y) = y^9 - 1$, $a_2(y) = a_3(y) = y^9 - 1$, $a_4(y) = y^7 + y^6 + y^4 + y^3 + y + 1$	[48, 2, 32]

7. CONCLUSION

In this article, we have described the structures of rings $\Re = \mathbb{Z}_2 + u\mathbb{Z}_2$, where $u^2 = 0$ and $\Re_{u^t} = \mathbb{Z}_2 + u\mathbb{Z}_2 + \cdots + u^{k-1}\mathbb{Z}_2$, where $u^k = 0$ with characteristic 2. The characterization of $\mathbb{Z}_2\Re\Re_{u^t}$-additive cyclic codes and their duality have been presented. The structural attributes of $\mathbb{Z}_2\Re\Re_{u^t}$-additive codes have been studied. We have also established the relationship between the minimal generating polynomials of additive cyclic codes and their duals. Furthermore, the minimal generating sets for even and odd lengths of $\mathbb{Z}_2\Re\Re_{u^t}$-additive cyclic codes have been determined. We have also obtained optimal binary images from $\mathbb{Z}_2\Re\Re_{u^t}$-additive cyclic codes that have a number of advantages over linear codes, including the fact that they are more efficient. In future work, it would be an interesting problem to generalize this over the ring $\mathbb{Z}_2\mathbb{Z}_2[u^3]\mathbb{Z}_2[u^k]$, where $u^3 = 0$ and $u^k = 0$, respectively.

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