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A lower bound on the Mostar index of tricyclic graphs

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Abstract. For a graph G, the Mostar index of G is the sum of $|n_u - n_v|$ over all edges $e = uv$ of G, where n_u denotes the number of vertices of G that have a smaller distance in G to u than to v, and analogously for n_v. In this paper, we obtain a lower bound for the Mostar index on tricyclic graphs and identify those graphs that attain the lower bound.

1. Introduction

Let $G = (V, E)$ be a simple, connected and finite graph with vertex set $V(G)$ and edge set $E(G)$. The order and size of G are the cardinality of $V(G)$ and $E(G)$, respectively. For $v \in V(G)$, let $N_G(v)$ be the set of vertices that are adjacent to v in G. The degree of $v \in V(G)$, denoted by $d_G(v)$, is the cardinality of $N_G(v)$. A vertex with degree one is called a pendent vertex and an edge incident to a pendent vertex is called a pendent edge. The distance between u and v in G is the least length of the path connecting u and v and is denoted by $d_G(u, v)$. A graph G with n vertices is a tricyclic graph if $|E(G)| = n + 2$.

A molecular graph is a simple graph such that its vertices correspond to the atoms and the edges to the bonds of a molecule. A topological index of G is a real number related to G. They are widely used for characterizing molecular graphs, establishing relationships between the structure and properties of molecules, predicting the biological activity of chemical compounds, and making their chemical applications [9, 10, 20]. Let $e = uv \in E(G)$, and define three subsets of $V(G)$ as follows:

$$N_u(e) = \{x \in V(G) : d_G(u, x) < d_G(v, x)\},$$
$$N_v(e) = \{x \in V(G) : d_G(v, x) < d_G(u, x)\},$$
$$N_0(e) = \{x \in V(G) : d_G(v, x) = d_G(u, x)\}.$$

Let $n_i(e) = |N_i(e)|$ (or put $n_i := n_i(e)$ for short), for $i = u, v$. A graph G is distance-balanced if $n_u = n_v$ for each edge $uv \in E(G)$ [14].

Doslić et al. [6] introduced a bond-additive structural invariant as a quantitative refinement of the distance non-balancedness and also a measure of peripherality in graphs, named the Mostar index. For a graph G, the Mostar index is defined as

$$Mo(G) = \sum_{e=uv\in E(G)} \psi(uv),$$

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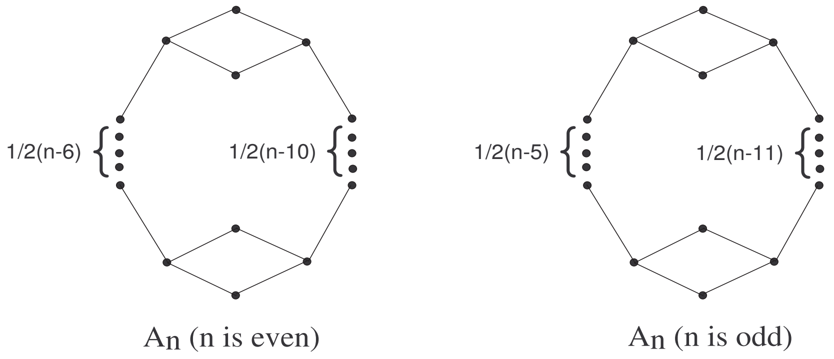

Figure 1: The Graphs for Theorem 1.1.

where $\psi(uv) = |n_u - n_v|$.

Doslić et al. [6] studied the Mostar index for acyclic and unicyclic graphs, and provided a cut method for calculating the Mostar index of benzenoid systems. The extremal Bicyclic graphs with respect to Mostar index has been studied by Tepeh [17]. Hayat and Zhou [11] obtained the upper bound for the Mostar index among cacti with fixed order and cycles, and characterized all the cacti that attain the bound. Hayat and Zhou [12] identified those trees with the largest and/or smallest Mostar index in the class of trees of order n with fixed parameters like diameter, maximum degree and the number of pendent vertices. Deng and Li [3] identified the trees with a fixed degree sequence having the largest Mostar index. Deng and Li [4] also studied the extremal problem for the Mostar index of trees with a given number of segments sequence. Gao et al. [7] studied the difference of Mostar index and irregularity of graphs. Ali and Doslić [1] gave more modifications to the Mostar index. One can refer [2, 5, 8, 13, 15, 16, 18, 19] for more studies about the Mostar index.

To have a full understanding of the relationship between the Mostar index and the structural properties of the graphs, in this paper, we consider the Mostar index among tricyclic graphs, and more precisely, we give a lower bound for the Mostar index on tricyclic graphs of order n, and identify those graphs that achieve the lower bound.

Theorem 1.1. *Let G be a tricyclic graph of order $n \geq 29$. Then*

$$Mo(G) \geq \begin{cases} 8, & \text{if } n \text{ is even}, \\ 10, & \text{if } n \text{ is odd} \end{cases}$$

with equality if and only if $G \cong A_n$, where A_n is depicted in Fig 1.

2. Proof of Theorem 1.1

Let $\mathcal{A}_n^1$ be the set of tricyclic graphs of order n with exactly one cut vertex. Let $\mathcal{A}_n^2$ be the set all tricyclic graphs of order n with connectivity of at least 2. Moreover, if $G \in \mathcal{A}_n^2$, then it must be one of the graphs $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ depicted in Fig 2. Set a_i $(i = 1, 2, \ldots, 6)$ as the lengths of the corresponding paths between vertices of degree at least 3. We mark these paths as $P(a_i)$ $(i = 1, 2, \ldots, 6)$.

By simple calculation, it is easy to check that

$$Mo(A_n) = \begin{cases} 8, & \text{if } n \text{ is even}, \\ 10, & \text{if } n \text{ is odd} \end{cases}$$

i.e., A_n satisfies the equality of Theorem 1.1.

To complete the proof it suffices to show that, for any graph G $(G \ncong A_n)$ of order $n \geq 29$, $Mo(A_n) < Mo(G)$.

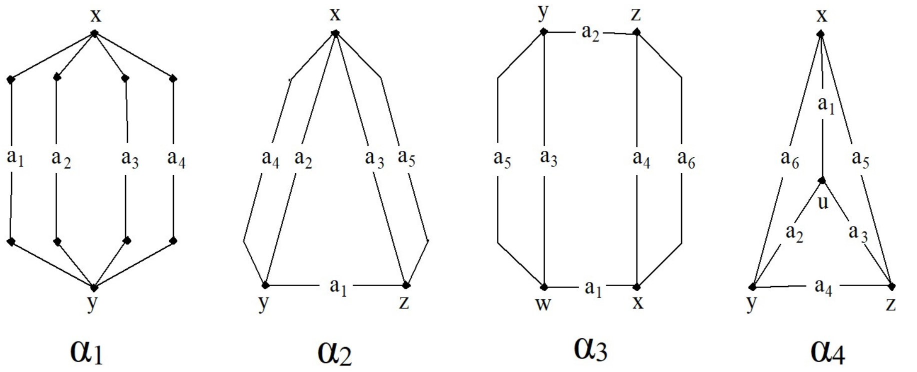

Figure 2: Graphs $\alpha_1, \alpha_2, \alpha_3$ and α_4.

Lemma 2.1. *Let $G \in \mathcal{A}_n^1$ of order $n \geq 13$ with at least one pendent edge. Then $Mo(G) > Mo(A_n)$.*

Proof. Let e' be a pendent edge in G. Then, for $n \geq 13$, we have $Mo(G) = \sum_{e \in E(G)} \psi(e) \geq \psi(e') = n - 2 > 10$, i.e., $Mo(G) > Mo(A_n)$. □

Lemma 2.2. *Let $G \in \mathcal{A}_n^1$ of order $n \geq 13$ such that G contains no pendent edge. Then $Mo(G) > Mo(A_n)$.*

Proof. Note that $G \in \mathcal{A}_n^1$ such that G contain no pendent edge, then there must be a cut vertex say w, that is to say, G is composed of a bicyclic graph H and a cycle U with a common vertex w. Clearly, the order of H is at least 4. If U is even, then for each $e \in U$, we have $\psi(e) = |V(H)| - 1 = n - |V(U)|$. Thus,

$$Mo(G) = \sum_{e \in E(G)} \psi(e) \geq \sum_{e \in E(U)} \psi(e) = |E(U)|(|V(H)| - 1) \geq 4 \times 3 > 10.$$

If U is odd, then for each $e \in U$ but the edge xy such that $d(w, x) = d(w, y)$, we have $\psi(e) = |V(H)| - 1 = n - |V(U)|$. Thus,

$$Mo(G) = \sum_{e \in E(G)} \psi(e) \geq \sum_{e \in E(U)} \psi(e) = (|E(U)| - 1)(|V(H)| - 1),$$

if the size of U is at leat 5, then $Mo(G) > 10$; if the size of U is 3, then $\psi(e) = |V(H)| - 1 = n - |V(U)| \geq 10$, we have $Mo(G) > 10$. Hence, $Mo(G) > Mo(A_n)$. □

Lemma 2.3. *Let $G \in \mathcal{A}_n^2$ such that $G \cong \alpha_1$ (see Fig 2), and $e = uv \in E(G)$. Then $\psi(uv) \leq 1$ if and only if e is the middle edge of an odd path of $P(a_i)$ $(i = 1, 2, 3, 4)$.*

Proof. Let $P(a_i)$ $(i = 1, 2, 3, 4)$ be the paths connecting x and y, and $e = uv \in P(a_i)$. We consider the following three possible cases.

Case 1. One of the x, y is in $N_u(e)$ and the other is in $N_v(e)$.

Let d_x (resp. d_y) be the distance between x (resp. y) and the edge e. Without loss of generality, we assume that $x \in N_u(e), y \in N_v(e)$ and $d_x > d_y$. Then $d_x - d_y$ vertices more in $N_u(e)$ than in $N_v(e)$ on the path $P(a_i)$, but on each path $P(a_j)$ $(j \neq i)$, $d_x - d_y$ vertices more in $N_v(e)$ than in $N_u(e)$. We have $\psi(uv) = |(d_x - d_y) - 3(d_x - d_y)| = 2|d_y - d_x|$.

Case 2. Both the x, y are in $N_u(e)$ or $N_v(e)$.

Assume that $x, y \in N_u(e)$. Then all vertices from the paths $P(a_j)$ $(j \neq i)$ are in $N_u(e)$. Let c be the length of the shortest cycle of G that contain e, we have $n_v(e) = \lfloor \frac{c}{2} \rfloor$, and $n_u(e) = \lfloor \frac{c}{2} \rfloor + |V(G)| - c$. Therefore, $\psi(uv) = |V(G)| - c \geq 2$.

Case 3. One of x, y is in $N_0(e)$.

Assume that $x \in N_u(e)$ and $y \in N_0(e)$. Then the shortest cycle C of G that contains e is odd. Let $z_i \in P(a_i)$ ($P(a_i) \nsubseteq C$) be the furthest vertex from e such that $z_i \in N_0(e)$. Let k be the length of the shortest path of the four paths $P(a_i)$ ($i = 1, 2, 3, 4$). Then $\psi(uv) = \sum_i (d(x, z_i) - 1) \geq \sum_i (k + d(y, z_i) - 1) \geq 2((k - 1)$, with equality if and only if two paths of $P(a_i)$ ($i = 1, 2, 3, 4$) have length k.

From the above three cases, we conclude that $\psi(uv) \leq 1$ if and only if either the two paths of $P(a_i)$ ($i = 1, 2, 3, 4$) have length one or $|d_y - d_x| = 0$. The former is impossible as G is a simple graph, the latter is possible only when e is the middle edge of an odd path of $P(a_i)$ ($i = 1, 2, 3, 4$). □

Lemma 2.4. *Let $G \in \mathcal{A}_n^2$ of order $n \geq 13$ such that $G \cong \alpha_1$ (see Fig 2). Then $Mo(G) > Mo(A_n)$.*

Proof. Assume that $a_1 \leq a_2 \leq a_3 \leq a_4$, then $a_2 \geq 2$. We take the six edges that are incident to x or y but not belongs to $P(a_1)$. Let e' be one of the six edges, then by Lemma 2.3, $\psi(e') \geq 2$. We have, $Mo(G) \geq 6 \times 2 > 10$. Hence, $Mo(G) > Mo(A_n)$. □

Lemma 2.5. *Let $G \in \mathcal{A}_n^2$ of order $n \geq 13$ such that $G \cong \alpha_2$ (see Fig 2). Then $Mo(G) > Mo(A_n)$.*

Proof. Assume that $a_2 \leq a_4, a_3 \leq a_5$. We proceed with the following four possible cases.

Case 1. $a_2 = a_4$, $a_3 = a_5$.

Subcase 1.1. $a_2 = a_4 = a_3 = a_5 \geq 2$.

Let e_1 be one of the four edges that are incident to x. Then $\psi(e_1) = 2(a_5 - 1)$. If $a_5 = 2$, then $a_1 \geq 7$, as $n \geq 13$. Let e_2 be one of the two edges which are incident to y or z in $P(a_1)$. Then $\psi(e_2) \geq 2$, implying that $Mo(G) \geq 4 \times 2 + 2 \times 2 > 10$. If $a_5 \geq 3$, then $Mo(G) \geq 4 \times 4 > 10$.

Subcase 1.2. $a_2 = a_4 > a_3 = a_5 \geq 2$.

Let e' (resp. e'') be the edges incident to x or y in the paths $P(a_2)$ (resp. $P(a_4)$). Then $\psi(e') = \psi(e'') = a_4 + a_5 - 2$. If $a_4 + a_5 \geq 6$, then $Mo(G) \geq 4 \times 4 > 10$. If $a_4 + a_5 = 5$, i.e., $a_4 = 3, a_5 = 2$, then $a_1 \geq 5$. Now consider $xx' \in P(a_3)$, then $\psi(xx') \geq 4$, implying that $Mo(G) > 10$.

Case 2. $a_2 + 1 = a_4$, $a_3 + 1 = a_5$.

Subcase 2.1. $a_1 + a_3 - 1 \geq a_2$.

We choose five edges xx' (resp. xx'', zz'') from $P(a_3)$ (resp. $P(a_5)$) and xx_1, yy_1 from $P(a_4)$. We have $\psi(xx') = a_4 - 1 + a_5 - 2 = a_2 + a_5 - 2$, $\psi(xx'') = \psi(zz'') = a_4 + a_2 - 1 = 2a_2$, $\psi(xx_1) = \psi(yy_1) \geq 4$. So $Mo(G) \geq 3a_2 + a_5 - 2 + 8 > 10$.

Subcase 2.2. $a_2 \geq a_1 + a_3 + 1$.

We consider four edges xx_1, yy_1 (resp. xx_2, zz_2) from $P(a_2)$ (resp. $P(a_5)$). We have $\psi(xx_1) = \psi(yy_1) = \psi(xx_2) = \psi(zz_2) \geq 4$. So $Mo(G) \geq 4 \times 4 > 10$.

Subcase 2.3. $a_2 = a_1 + a_3$

Let e be one of the two edges incident to x or z in $P(a_5)$. Then $\psi(e) = a_4 - 1 + a_2 - 1 = 2a_2 - 1$. Since $n \geq 13$, we get $b \geq 4$, implying that $Mo(G) \geq 2 \times 7 > 10$.

Case 3. $a_2 + 1 = a_4$, $a_3 = a_5 \geq 2$.

Subcase 3.1. $a_1 + a_3 - 1 \geq a_2$.

Let e be one of the four edges incident to x or z in $P(a_3)$ and $P(a_5)$. Then $\psi(e) = a_4 - 1 + a_5 - 1 = a_4 + a_5 - 2$. Since, $a_4 + a_5 \geq 4$. If $a_4 + a_5 \geq 5$, then $Mo(G) \geq 4 \times 3 > 10$.

If $a_4 + a_5 = 4$, we consider $xx' \in P(a_4)$. If $a_4 = 2, a_5 = 2$, then $a_2 = 1, a_3 = 2, a_1 \geq 8$, $\psi(xx') \geq 4$. Thus, $Mo(G) \geq 4 \times 2 + 4 > 10$.

Subcase 3.2. $a_2 > a_1 + a_3 - 1$.

Let e (resp. e′) be one of the four edges incident to x or z in $P(a_3)$ (resp. $P(a_5)$). Then $\psi(e) = \psi(e') = a_1 + 2a_3 - 2 = a_4 + a_5 - 2$. Since, $a_1 + 2a_3 \geq 5$, then $Mo(G) \geq 4 \times 3 > 10$.

Case 4. $a_4 \geq b_2 + 2$.

Let e be one of the two edges incident to x or y in $P(a_4)$. Then $\psi(e) \geq a_1 + a_3 + a_5 - 2$. Since $a_3 + a_5 \geq 3$, $a_1 + a_3 + a_5 \geq 4$. If $a_1 + a_3 + a_5 \geq 8$, then $\psi(e) \geq 6$. We have, $Mo(G) \geq 2 \times 6 > 10$.

If $6 \leq a_1 + a_3 + a_5 \leq 7$, then $a_5 \geq 2$. Let e' be one of the two edges incident to x or z in $P(a_5)$. Then $\psi(e') \geq 2$, we have $Mo(G) \geq 2 \times 4 + 2 \times 2 > 10$.

If $a_1 + a_3 + a_5 = 5$, $a_5 = 3$. Since $n \geq 13$, then $a_2 + a_4 - 1 \geq 9$, we have $\psi(e') \geq a_2 + a_4 - 1 \geq 9$, so $Mo(G) \geq 2 \times 9 > 10$.

If $a_1 + a_3 + a_5 = 4$, $a_5 = 2$, then $\psi(e') \geq 4$, i.e., $Mo(G) \geq 2 \times 2 + 2 \times 4 > 10$.

Hence, from the above cases, it follows that $Mo(G) > Mo(A_n)$. □

Lemma 2.6. *Let $G \in \mathcal{A}_n^2$ of order $n \geq 13$ such that $G \cong \alpha_3$ (see Fig 2). Then $Mo(G) > Mo(A_n)$.*

Proof. Without loss of generality, we assume that $a_4 \leq a_6$, $a_3 \leq a_5$. We proceed with the following four possible cases.

Case 1. $a_4 = a_6, a_3 = a_5$.

We may assume that $a_1 \leq a_2$.

Subcase 1.1. $a_4 = a_6 = a_3 = a_5 = 2$.

If $a_2 \geq a_1 + 4$. Let e_1 be one of the two edges incident to w in $P(a_3)$ and $P(a_5)$, e_2 be one of the two edges incident to x in $P(a_4)$ and $P(a_6)$, e_3 be one of the two edges incident to y or z in $P(a_2)$. Then $\psi(e_1) = \psi(e_2) = 2$, and $\psi(e_3) \geq 2$. Thus, $Mo(G) \geq 4 \times 2 + 2 \times 2 > 10$.

If $a_1 \leq a_2 \leq a_1 + 1$. Let e_4 be one of the two edges incident to w or x in $P(a_1)$. Then $\psi(e_1) = \psi(e_2) = 2$, $\psi(e_3) = \psi(e_4) \geq 1$. Hence, $Mo(G) \geq 4 \times 2 + 4 \times 1 > 10$.

Subcase 1.2. $a_4 = a_6 = a_3 = a_5 \geq 3$.

Let e be one of the four edges incident to w or y in $P(a_3), P(a_5)$. Then $\psi(e) = a_6 - 1 + a_3 - 1 = 2(a_3 - 1) \geq 4$. We have, $Mo(G) \geq 4 \times 4 > 10$.

Subcase 1.3. $2 \leq a_4 = a_6 < a_3 = a_5$.

Let e be one of the four edges incident to w or y in $P(a_3), P(a_5)$. Then $\psi(e) = a_6 - 1 + a_3 - 1 = a_3 + a_6 - 2$. Since $a_3 + a_6 \geq 5$, we have $Mo(G) \geq 4 \times 3 > 10$.

Case 2. $a_4 + 1 = a_6, a_3 + 1 = a_5$.

Subcase 2.1. $a_1 + a_3 - 1 \geq a_4 + a_6$.

Let e_1 be one of the two edges incident to w or y in $P(a_5)$. Then $\psi(e_1) \geq a_2 + a_4 + a_6 - 1 = a_2 + 2a_4$. If $a_4 \geq 2$ or $a_2 \geq 3$, then $Mo(G) > 10$. If $a_4 = 1, a_2 \leq 3$, then we consider two edges incident to z or x in $P(a_6)$, let e_2 be one of the two. We have, $\psi(e_2) \geq 3$, implying that, $Mo(G) \geq 2 \times 3 + 2 \times 3 > 10$.

Subcase 2.2. $a_1 + a_3 \leq a_4 + a_6 - 1$.

It is the same to Subcase 2.1.

Subcase 2.3. $a_1 + a_3 = a_4 + a_6$. We choose two edges $yy' \in P(a_5)$, $xx' \in P(a_6)$, then $\psi(yy') = a_2 + a_4 + a_6 - 2 = a_2 + 2a_4 - 1$, $\psi(xx') = a_1 + a_3 + a_5 - 2 = a_1 + 2a_3 - 1$. Since $n = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 \geq 13$, so $(a_1 + 2a_3 - 1) + (a_2 + 2a_4 - 1) \geq 11$, implying that $Mo(G) > 10$.

Case 3. $a_4 = a_6, a_3 + 1 = a_5$.

Subcase 3.1. $a_1 + a_4 - 1 \geq a_2 + a_5$.

Let e be one of the four edges incident to z or x in $P(a_4), P(a_6)$. Then $\psi(e) \geq a_5 - 1 + a_6 - 1 = a_3 + a_4 - 1 \geq 4$. Since $a_3 + a_4 \geq 3$. If $a_3 + a_4 \geq 4$, then $Mo(G) \geq 4 \times 3 > 10$. If $a_3 + a_4 = 3$, then we choose $yy' \in P(a_5)$, $\psi(yy') \geq 4$, so $Mo(G) \geq 4 \times 2 + 4 > 10$.

Subcase 3.2. $a_1 + a_4 \leq a_2 + a_5$.

Let e be one of the two edges incident to w or y in $P(a_5)$. Then $\psi(e) = a_1 + 2a_4 - 1$. Since $a_1 + 2a_4 \geq 5$. If $a_1 + 2a_4 \geq 7$, then $Mo(G) \geq 2 \times 6 > 10$. If $a_1 + 2a_4 \leq 6$, then we choose $yy' \in P(a_3)$, $\psi(yy') \geq 3$, so $Mo(G) \geq 2 \times 4 + 3 > 10$.

Case 4. $a_5 \geq a_3 + 2$.

Let e be one of the two edges incident to w or y in $P(a_5)$. Then $\psi(e) = a_1 + a_2 + a_4 + a_6 - 1$. Since $a_4 + a_6 \geq 3$, $a_1 + a_2 + a_4 + a_6 \geq 5$. If $a_1 + a_2 + a_4 + a_6 \geq 7$, then $Mo(G) \geq 2 \times 6 > 10$. If $5 \leq a_1 + a_2 + a_4 + a_6 \leq 6$, then we consider two edges incident to z in $P(a_4), P(a_6)$, let e' be one of the two edges. Then $\psi(e') \geq 2$, implying that, $Mo(G) \geq 2 \times 4 + 2 \times 2 > 10$. □

Lemma 2.7. *Let $G \in \mathcal{A}_n^2$ of order $n \geq 29$ such that $G \cong \alpha_4$ (see Fig 2). Then $Mo(G) > Mo(A_n)$.*

Proof. Without loss of generality, we assume that $a_1 = max\{a_1, a_2, a_3, a_4, a_5, a_6\}$. Since $n \geq 29$, then $a_1 \geq 6$. We consider the edge $uu' \in P(a_1)$. Since $d_G(z, u) \leq d_G(z, u')$ by the choice of a_1, we have $z \in N_u(uu')$ or

$z \in N_0(uu')$. Also, $z \in N_0(uu')$ if and only if $a_1 = a_3 \le a_2 + a_4$ and $a_5 = 1$. The same is true for y. Suppose C is the shortest cycle containing uu', and let $|C| = k$.

We consider the following three possible cases.

Case 1. $a_1 > \frac{k+1}{2}$.

In this case, $x, y, z \in N_u(uu')$. Let $e \in P(a_1)$ such that the distance between e and x or u is no more than one. Then, we have $\psi(e) = n - k$.

If $n - k \ge 4$, then $Mo(G) \ge 4 \times 4 > 10$.

If $n - k = 3$ and C is composed of paths $P(a_1), P(a_2)$ and $P(a_6)$, then $a_3 + a_4 + a_5 \le 5, a_2 + a_6 \le 4$. Let e be one of the four edges in $P(a_1)$ such that the distance between e and x or u is no more than one. Then we have $\psi(e) = 3$, implying that, $Mo(G) \ge 4 \times 3 > 10$.

If $n - k = 3$ and C is composed of paths $P(a_1), P(a_3), P(a_4)$ and $P(a_6)$, then either one of the two vertices is in $P(a_2)$, another two vertices are in $P(a_6)$, or one of the two vertices in $P(a_6)$ another two vertices are in $P(a_2)$. It is the case when C is composed of paths $P(a_1), P(a_2)$ and $P(a_6)$.

If $n - k = 2$ and C is composed of paths $P(a_1), P(a_2)$ and $P(a_6)$, then $a_3 + a_4 + a_5 \le 4$, $a_2 + a_6 \le 3$. Let e be one of the six edges in $P(a_1)$ such that the distance between e and x or u is no more than two. Then we have $\psi(e) = 2$, implying that $Mo(G) \ge 6 \times 2 > 10$.

If $n - k = 2$ and C is composed of paths $P(a_1), P(a_3), P(a_4)$ and $P(a_6)$, then one of the two vertices is in $P(a_2)$, another two vertices are in $P(a_6)$. It is the case when C is composed of paths $P(a_1), P(a_2)$ and $P(a_6)$.

If $n - k = 2$ and C is composed of paths $P(a_1), P(a_2)$ and $P(a_6)$, then $V(G) - V(C) = \{z\}$, and $a_3 = a_4 = a_5 = 1$. Since $P(a_1) \cup P(a_2) \cup P(a_6)$ is the shortest cycle, we have $a_2 = a_6 = 1$ and $a_1 \ge 26$, by $n \ge 29$. We consider all the edges in $P(a_1)$ except the middle one when a_1 is odd. Therefore, $\psi(e) = 1$, implying that $Mo(G) \ge a_1 - 1 > 10$.

If $n - k = 1$ and C is composed of paths $P(a_1), P(a_3), P(a_4)$ and $P(a_6)$, then it is not possible because such a cycle does not exist.

Case 2. $a_1 < \frac{k+1}{2}$.

Subcase 2.1. One of the y, z is in $N_0(uu')$.

Assume that $z \in N_0(uu')$, then $a_1 = a_3 \le a_2 + a_4$ and $a_5 = 1$.

If $z \in V(C)$, for $y \in N_u(uu')$, then $C = P(a_1) \cup P(a_2) \cup P(a_6)$. Let w be the furthest vertex in $P(a_6)$ such that $w \in N_u(uu')$, w' be the vertex adjacent to w but not in $N_u(uu')$. If the cycle $P(a_1) \cup P(a_2) \cup P(a_6)$ is even, then $d_G(w, y) + a_2 = d_G(w', x) + a_1 - 1$, i.e., $d_G(w, y) - d_G(w', x) = a_1 - b_2 - 1$. If the cycle $P(a_1) \cup P(a_2) \cup P(a_6)$ is odd, then $d_G(w, y) + a_2 + 1 = d_G(w', x) + a_1 - 1$, i.e., $d_G(w, y) - (d_G(w', x) - 1) = a_1 - b_2 - 1$. Let w_1 be the furthest vertex in $P(a_4)$ such that $w_1 \in N_u(uu')$, w_1' be the vertex adjacent to w_1 but not in $N_u(uu')$. If the cycle $P(a_1) \cup P(a_2) \cup P(a_6)$ is even, then $d_G(w_1, y) + a_2 = d_G(w_1', z) + a_3 = d_G(w_1', z) + a_1$, i.e., $d_G(w_1, y) = a_1 - b_2 + d_G(w_1', z)$. If the cycle $P(a_1) \cup P(a_2) \cup P(a_6)$ is odd, then $d_G(w_1, y) + a_2 + 1 = d_G(w_1', z) + a_1$, i.e., $d_G(w_1, y) = a_1 - b_2 - 1 + d_G(w_1', z)$. So $\psi(uu') = a_2 + 2(a_1 - a_2 - 1) \ge 2a_1 - a_2 - 2 \ge a_1 - 2 \ge 4$. A similar result can be obtained for the remaining edges in $P(a_1)$. Thus, $Mo(G) \ge a_1 \times 4 > 10$.

If $z \notin V(C)$, then $C = P(a_1) \cup P(a_2) \cup P(a_6)$. So $\psi(uu') = a_3 - 1 = a_1 - 1 \ge 5$. A similar result can be obtained for the remaining edges in $P(a_1)$. Thus, $Mo(G) \ge a_1 \times 5 > 10$.

Subcase 2.2. Both of y and z are in $N_0(uu')$.

We have $a_1 = a_2 = a_3$, $a_5 = a_6 = 1$. Therefore, $\psi(uu') = a_3 - 1 = a_1 - 1 \ge 5$. We can obtain a similar result for other edges in $P(a_1)$. Thus, $Mo(G) \ge a_1 \times 5 > 10$.

Subcase 2.3. Both of y and z are in $N_u(uu')$.

In this case, we have $\psi(uu') \ge a_1 + a_4 - 2 \ge a_1 - 1 \ge 5$. We can obtain a similar result for other edges in $P(a_1)$. Thus, $Mo(G) \ge a_1 \times 5 > 10$.

Case 3. $a_1 = \frac{k+1}{2}$.

Subcase 3.1. C is composed of the paths $P(a_1), P(a_3), P(a_4)$ and $P(a_6)$.

Clearly, $y, z \in N_u(uu')$ and $b_2 > a_3 + a_4$. Let w be the furthest vertex in $P(a_5)$ such that $w \in N_u(uu')$, w' be the vertex adjacent to w but not in $N_u(uu')$. Then $d_G(w, z) = a_1 - a_3 - 1 + d_G(x, w')$. We have $\psi(uu') \ge a_2 - 1 + d_G(w, z) \ge a_1 + a_2 - a_3 - 3 \ge a_1 - 1 \ge 5$. A similar result can be obtained for the remaining edges in $P(a_1)$. Thus, $Mo(G) \ge a_1 \times 5 > 10$.

Subcase 3.2. C is composed of the paths $P(a_1), P(a_2)$ and $P(a_6)$.

Clearly, $y \in N_u(uu')$ and $b_2 \le a_3 + a_4$.

If $z \in N_0(uu')$, then $a_1 = a_3 \leq a_2 + a_4$ and $a_5 = 1$. We have $\psi(uu') \geq a_3 - 1 = a_1 - 1 \geq 5$. We can obtain a similar result for other edges in $P(a_1)$. Thus, $Mo(G) \geq a_1 \times 5 > 10$.

If $z \in N_u(uu')$, similar to Subcase 3.1, we have $d_G(w,z) \geq a_1 - a_3 - 2$, if $a_3 \leq a_2 + a_4$; $d_G(w,z) \geq a_1 - (a_2 + a_4) - 2$, if $a_3 \geq a_2 + a_4$. So $\psi(uu') \geq a_4 - 1 + a_3 + d_G(w,z) \geq a_1 + a_4 - 3 \geq a_1 - 2 \geq 4$. A similar result can be obtained for the remaining edges in $P(a_1)$. Thus, $Mo(G) \geq a_1 \times 4 > 10$. □

The proof of Theorem 1.1 follows from Lemmas 2.1, 2.2, 2.3, 2.4, 2.5, 2.6 and 2.7.

3. Concluding remarks

We obtained a lower bound for the Mostar index on tricyclic graphs for $n \geq 29$ and identified the graphs that attain the lower bound. In fact, Theorem 1.1 can be improved to $n \geq 10$, which needs more details of the proof, therefore we put the following conjecture.

Conjecture 3.1. *Let G be a tricyclic graph of order $n \geq 10$. Then*

$$Mo(G) \geq \begin{cases} 8, & \text{if } n \text{ is even}, \\ 10, & \text{if } n \text{ is odd} \end{cases}$$

with equality if and only if $G \cong A_n$, where A_n is depicted in Fig 1.

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