Filomat 38:17 (2024), 6299–6310
https://doi.org/10.2298/FIL2417299B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A note on the generalized maximal numerical range of operators

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Abstract. This study explores the A-maximal numerical range of operators, represented as $W_{\max}^{A}(\cdot)$, where A is a positive bounded linear operator on a complex Hilbert space $\mathcal{H}$. The research provides new insights into the properties and characterizations of A-normaloid operators, including an extension of a recent result by Spitkovsky in [A note on the maximal numerical range, Oper. Matrices **13** (2019), 601–605]. Specifically, it is demonstrated that an A-bounded linear operator T on $\mathcal{H}$ is A-normaloid if and only if $W_{\max}^{A}(T) \cap \partial W_A(T) \neq \emptyset$, where $\partial W_A(T)$ denotes the boundary of the A-numerical range of T. Furthermore, novel A-numerical radius inequalities are introduced that generalize and enhance prior well-known results.

1. Introduction and Preliminaries

The numerical range and radius of a bounded linear operator on a Hilbert space have been extensively studied in operator theory for many decades. They provide essential geometric and analytic information about the operator and have a wide range of applications in various areas of mathematics and physics. Recently, the A-numerical range, which is a natural generalization of the classical numerical range, has been introduced in [5] for a positive bounded linear operator A on a Hilbert space. The A-numerical range has been studied extensively, and its supremum modulus is known as the A-numerical radius. For more details on these concepts, consult the recent book by Bhunia et al. [9].

Despite its importance in operator theory, the A-maximal numerical range has received less attention in the literature. In this study, we aim to provide new insights into the properties and characterizations of A-normaloid operators by exploring the A-maximal numerical range. We will introduce novel A-numerical radius inequalities that generalize and enhance prior well-known results. The results of this study will contribute to the understanding of the A-maximal numerical range and provide a foundation for further research in this area.

To achieve the goals of this study, we consider a non-trivial complex Hilbert space $\mathcal{H}$ with inner product $\langle\cdot,\cdot\rangle$ and associated norm $\|\cdot\|$. We use the notation $\mathbb{B}(\mathcal{H})$ to denote the C^*-algebra of all bounded linear operators on $\mathcal{H}$, with the identity operator denoted by $I_{\mathcal{H}}$ or simply I when no confusion arises. Throughout

2020 *Mathematics Subject Classification*. 47B20, 47A12, 46C05, 47A10.
Keywords. Positive operator, A-maximal numerical range, A-normaloid operator.
Received: 07 September 2023; Revised: 01 January 2024; Accepted: 12 January 2024
Communicated by Fuad Kittaneh
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this paper, we focus on operators in $\mathbb{B}(\mathcal{H})$, and we use the notation T^*, $\mathcal{R}(T)$, and $\mathcal{N}(T)$ to denote the adjoint, range, and null space of an operator T, respectively.

The following facts will be useful for the remainder of this article. An operator T is considered positive if $\langle Tx, x\rangle \geq 0$ for every $x \in \mathcal{H}$. We denote the cone of positive (semi-definite) operators as $\mathbb{B}(\mathcal{H})^+$, given by

$$\mathbb{B}(\mathcal{H})^+ = \Big\{T \in \mathbb{B}(\mathcal{H}) \,:\, \langle Tx, x\rangle \geq 0 \text{ for all } x \in \mathcal{H}\Big\}.$$

Throughout the rest of this article, $A \in \mathbb{B}(\mathcal{H})^+$ is a nonzero operator that defines a positive semidefinite sesquilinear form in the following manner:

$$\langle\cdot,\cdot\rangle_A : \mathcal{H}\times\mathcal{H} \longrightarrow \mathbb{C},\ (x,y) \longmapsto \langle x, y\rangle_A := \langle Ax, y\rangle = \langle A^{1/2}x, A^{1/2}y\rangle,$$

where $A^{1/2}$ represents the square root of A. We denote by $\|\cdot\|_A$ the seminorm induced by $\langle\cdot,\cdot\rangle_A$ which is given by $\|x\|_A = \sqrt{\langle x, x\rangle_A} = \|A^{1/2}x\|$ for every $x \in \mathcal{H}$. It can be checked that $\|x\|_A = 0$ if and only if $x \in \mathcal{N}(A)$. So, $\|\cdot\|_A$ is a norm on $\mathcal{H}$ if and only if A is one-to-one. Furthermore, one may verify that the semi-Hilbert space $(\mathcal{H}, \|\cdot\|_A)$ is complete if and only if $\mathcal{R}(A)$ is closed in $(\mathcal{H}, \|\cdot\|)$. For a given $T \in \mathbb{B}(\mathcal{H})$, if there exists $c > 0$ such that $\|Tx\|_A \leq c\|x\|_A$ for all $x \in \overline{\mathcal{R}(A)}$, then it holds:

$$\|T\|_A := \sup_{\substack{x\in\overline{\mathcal{R}(A)},\\ x\neq 0}} \frac{\|Tx\|_A}{\|x\|_A} = \sup_{\substack{x\in\overline{\mathcal{R}(A)},\\ \|x\|_A=1}} \|Tx\|_A < \infty.$$

If $A = I$, we get the classical norm of an operator T which will be denoted by $\|T\|$. From now on, we denote $\mathbb{B}^A(\mathcal{H}) := \{T \in \mathbb{B}(\mathcal{H}) \,:\, \|T\|_A < \infty\}$. It is important to note that $\mathbb{B}^A(\mathcal{H})$ is not generally a subalgebra of $\mathbb{B}(\mathcal{H})$ (see [15]). Further, it is not difficult to check that $\|T\|_A = 0$ if and only if $ATA = 0$. Recently, there are many papers that study operators defined on a semi-Hilbert space $(\mathcal{H}, \|\cdot\|_A)$. One may see [5–7, 9, 17, 19, 20] and their references.

Let $T \in \mathbb{B}(\mathcal{H})$. An operator $S \in \mathbb{B}(\mathcal{H})$ is called an A-adjoint operator of T if $\langle Tx, y\rangle_A = \langle x, Sy\rangle_A$ for all $x, y \in \mathcal{H}$ (see [1]). Clearly, S is an A-adjoint of T if and only if $AS = T^*A$, i.e., S is a solution in $\mathbb{B}(\mathcal{H})$ of the equation $AX = T^*A$. We mention here that this type of operator equations can be studied by using the following famous theorem due to Douglas (for its proof see [12]).

Theorem 1.1. *If $T, U \in \mathbb{B}(\mathcal{H})$, then the following statements are equivalent:*

(1) $\mathcal{R}(U) \subseteq \mathcal{R}(T)$,

(2) $TS = U$ *for some* $S \in \mathbb{B}(\mathcal{H})$,

(3) *there exists* $\lambda > 0$ *such that* $\|U^*x\| \leq \lambda\|T^*x\|$ *for all* $x \in \mathcal{H}$.

If one of these conditions holds, then there exists a unique solution of the operator equation $TX = U$, denoted by Q, such that $\mathcal{R}(Q) \subseteq \overline{\mathcal{R}(T^*)}$. Such Q is called the reduced solution of $TX = U$.

Let $\mathbb{B}_{A^{1/2}}(\mathcal{H})$ denote the set of all operators that admit $A^{1/2}$-adjoints. An application of Theorem 1.1 shows that

$$\mathbb{B}_{A^{1/2}}(\mathcal{H}) = \{T \in \mathbb{B}(\mathcal{H}) \,:\, \text{there exists } \lambda > 0 \text{ such that } \|Tx\|_A \leq \lambda\|x\|_A \text{ for all } x \in \mathcal{H}\}.$$

If $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$, then T is said A-bounded. It can be observed that if $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$, then $T(\mathcal{N}(A)) \subseteq \mathcal{N}(A)$. Further, the following property $\|TS\|_A \leq \|T\|_A\|S\|_A$ holds for all $T, S \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$. Also, if $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$, then the authors of [14] showed that

$$\begin{aligned}\|T\|_A &= \sup\{\|Tx\|_A \,:\, x \in \mathcal{H},\ \|x\|_A = 1\}\\ &= \sup\{|\langle Tx, y\rangle_A| \,:\, x, y \in \mathcal{H},\ \|x\|_A = \|y\|_A = 1\}.\end{aligned}$$

To obtain additional information on the category of A-bounded operators, we suggest that the reader consult [3, 15, 20] and the sources cited within those works. Note that $\mathbb{B}_{A^{1/2}}(\mathcal{H})$ is a subalgebra of $\mathbb{B}(\mathcal{H})$ which is neither closed nor dense in $\mathbb{B}(\mathcal{H})$. Moreover, the following inclusions:

$$\mathbb{B}_{A^{1/2}}(\mathcal{H}) \subseteq \mathbb{B}^A(\mathcal{H}) \subseteq \mathbb{B}(\mathcal{H}) \tag{1}$$

hold. It should be noted that typically, the inclusions specified in (1) are strict, which means that there are elements that belong to one set but not to the other. However, if A is an injective operator, then obviously $\mathbb{B}_{A^{1/2}}(\mathcal{H}) = \mathbb{B}^A(\mathcal{H})$. Further, if A has a closed range in $\mathcal{H}$, then it can be seen that $\mathbb{B}^A(\mathcal{H}) = \mathbb{B}(\mathcal{H})$. So, the inclusions in (1) remain equalities if A is injective and has a closed range. We refer to [1–3, 15] and the references therein for an account of results related the theory of semi-Hilbert spaces.

Baklouti et al. introduced the concept of the maximal numerical range induced by a positive operator A in their publication [5]. To be more precise, the definition is as follows.

Definition 1.2. *Let $T \in \mathbb{B}^A(\mathcal{H})$. The A-maximal numerical range of T, denoted by $W^A_{\max}(T)$, is defined as*

$$W^A_{\max}(T) = \Big\{\lambda \in \mathbb{C} \,:\, \text{there exists } (x_n) \subseteq \mathcal{H}\,,\ \|x_n\|_A = 1,\ \lim_{n\to+\infty} \langle Tx_n, x_n\rangle_A = \lambda,\ \text{and } \lim_{n\to+\infty} \|Tx_n\|_A = \|T\|_A\Big\}.$$

For every $T \in \mathbb{B}(\mathcal{H})$, it was shown in [5] that $W^A_{\max}(T)$ is non-empty, convex and compact subset of $\mathbb{C}$. Notice that the notion of the maximal numerical range of an operator $T \in \mathbb{B}(\mathcal{H})$, denoted by $W_{\max}(T)$ (that is when $A = I$; the identity operator), was first introduced by Stampfli in [22], in order to determine the norm of the inner derivation acting on $\mathbb{B}(\mathcal{H})$. Recall that the inner derivation δ_T associated with $T \in \mathbb{B}(\mathcal{H})$ is defined by

$$\delta_T : \mathbb{B}(\mathcal{H}) \longrightarrow \mathbb{B}(\mathcal{H}),\ X \longmapsto TX - XT.$$

For this, in the same paper [22], the author first established the following.

Theorem 1.3. *Let $T \in \mathbb{B}(\mathcal{H})$. Then the following conditions are equivalent:*

(1) $0 \in W_{\max}(T)$,

(2) $\|T\|^2 + |\lambda|^2 \leq \|T + \lambda\|^2$ *for any* $\lambda \in \mathbb{C}$,

(3) $\|T\| \leq \|T + \lambda\|$ *for any* $\lambda \in \mathbb{C}$.

Here $T + \lambda$ is denoted to be $T + \lambda I$ for any $\lambda \in \mathbb{C}$.

Corollary 1.4. *Let $T \in \mathbb{B}(\mathcal{H})$. Then there is a unique scalar c_T such that*

$$\|T - c_T\|^2 + |\lambda|^2 \leq \|(T - c_T) - \lambda\|^2, \text{ for all } \lambda \in \mathbb{C}.$$

Moreover, $0 \in W_{\max}(T)$ *if and only if* $c_T = 0$.

The scalar c_T is called the center of mass of T. It is worth noting that this scalar is the only one that satisfies the following:

$$\|T - c_T\| = \inf_{\lambda\in\mathbb{C}} \|T - \lambda\|.$$

The scalar $\|T - c_T\|$ is denoted by $d(T)$ and is called the distance of T to scalars. The author in [22] proved also that for any $T \in \mathbb{B}(\mathcal{H})$

$$\|\delta_T\| = 2d(T).$$

Recall that an operator $T \in \mathbb{B}(\mathcal{H})$ is said to be normaloid if $\omega(T) = \|T\|$, where $\omega(T)$ is denoted to be the numerical radius of T which is given by

$$\omega(T) = \sup\{|\lambda| \,:\, \lambda \in W(T)\}.$$

Here $W(T)$ is denoted to be the numerical range of T and it is defined by Toeplitz in [23] as

$$W(T) := \{\langle Tx, x\rangle :\ x \in \mathcal{H} \ \text{ with } \|x\| = 1\}.$$

Equivalent condition is $r(T) = \|T\|$, see, [18]. Here, $r(T)$ is the spectral radius of T. Recently, Spitkovsky in [21] gave the following characterization of a normaloid operator.

Theorem 1.5. *Let $T \in \mathbb{B}(\mathcal{H})$. Then the following conditions are equivalent:*

(1) *T is a normaloid operator,*

(2) $W_{\max}(T) \cap \partial W(T) \neq \emptyset$.

Here ∂L stands for the boundary of a subset L in the complex plane.

Notions of the numerical range and numerical radius are generalized in [5] as follows.

Definition 1.6. *Let $T \in \mathbb{B}(\mathcal{H})$. The A-numerical range and the A-numerical radius of T are respectively given by*

$$W_A(T) := \{\langle Tx, x\rangle_A \ : \ x \in \mathcal{H} \ \ with \ \|x\|_A = 1\},$$

and

$$\omega_A(T) := \sup\{|\lambda| \ : \ \lambda \in W_A(T)\}.$$

It is important to mention that $\omega_A(T)$ may be equal to $+\infty$ for some $T \in \mathcal{B}(\mathcal{H})$ (see [15]). However, $\omega_A(\cdot)$ defines a seminorm on $\mathbb{B}_{A^{1/2}}(\mathcal{H})$ which is equivalent to $\|T\|_A$. More precisely, for any $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$, we have

$$\frac{1}{2}\|T\|_A \leq \omega_A(T) \leq \|T\|_A\,, \tag{2}$$

see [5].
Recently, the concept of A-normaloid operators is introduced by the third author in [15] as follows.

Definition 1.7. *An operator $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$ is said to be A-normaloid if $r_A(T) = \|T\|_A$, where*

$$r_A(T) = \lim_{n\to+\infty} \|T^n\|_A^{\frac{1}{n}}.$$

Some characterizations of A-normaloid operators are proved in [15]. In particular, we have the following proposition.

Proposition 1.8 ([15]). *Let $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$. Then the following assertions are equivalent:*

(1) *T is A-normaloid,*

(2) $\|T^n\|_A = \|T\|_A^n$ *for all positive integer n,*

(3) $\omega_A(T) = \|T\|_A$,

(4) *there exists a sequence $(x_n) \subseteq \mathcal{H}$ such that $\|x_n\|_A = 1$, $\lim_{n\to+\infty} \|Tx_n\|_A = \|T\|_A$ and $\lim_{n\to+\infty} |\langle Tx_n, x_n\rangle_A| = \omega_A(T)$.*

The purpose of our work is to provide new characterizations of A-normaloid operators. Our approach is to study the operator range $\mathcal{R}(A^{1/2})$ equipped with its canonical Hilbertian structure, denoted by $\mathbf{R}(A^{1/2})$, and utilizing the connection between A-bounded operators and operators acting on the Hilbert space $\mathbf{R}(A^{1/2})$. We extend Theorem 1.5 to the context of semi-Hilbert spaces and establish several new properties related to the A-maximal numerical range of A-bounded operators. Our primary objective is to generalize Theorem 1.3 for $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$, and we also provide a sufficient and necessary condition for the A-center of mass of an operator $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$ to belong to $W_{\max}^A(T)$. Additionally, we investigate other properties of A-bounded operators.

For the remainder of this paper, we will use the notation $\Gamma_A(T)$ to denote the set defined as:

$$\Gamma_A(T) := \Big\{z \in \mathbb{C} \ : \ |z| = \|T\|_A\Big\}.$$

2. Main Results

In this section, we present our main results, beginning with a theorem that provides another useful characterization of A-normaloid operators. We use the notation $\overline{L}$ to denote the closure of any subset L in the complex plane.

Theorem 2.1. *Let $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$. Then the following conditions are equivalent:*

(1) *T is A-normaloid,*

(2) $\Gamma_A(T) \cap \overline{W_A(T)} \neq \emptyset$.

Proof. (1) ⇒ (2): Assume that T is A-normaloid. Then by Proposition 1.8 we have $\omega_A(T) = \|T\|_A$. So, there exists a sequence $(z_n) \subseteq W_A(T)$ such that $\lim\limits_{n\to+\infty} |z_n| = \|T\|_A$. By the compactness of $\overline{W_A(T)}$ we can, taking a subsequence of (z_n) if needed, assume that (z_n) converges to some $z \in \overline{W_A(T)}$. Therefore, $|z| = \|T\|_A$, so $z \in \Gamma_A(T) \cap \overline{W_A(T)}$.
(2) ⇒ (1): Let $z \in \Gamma_A(T) \cap \overline{W_A(T)}$. We have $\omega_A(T) \geq |z| = \|T\|_A$. From Inequalities (2), we deduce that $\omega_A(T) = \|T\|_A$. That is, T is A-normaloid. □

Our next objective is to generalize Theorems 1.3 and 1.5 for $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$. To achieve this, we need to recall some facts from [3]. Let $X = \mathcal{H}/\mathcal{N}(A)$ be the quotient space of $\mathcal{H}$ by $\mathcal{N}(A)$. It can be observed that $\langle\cdot,\cdot\rangle_A$ induces on X the following inner product:

$$[\overline{x}, \overline{y}] = \langle x, y\rangle_A = \langle Ax, y\rangle,$$

for every $\overline{x}, \overline{y} \in X$. We note that $(X, [\cdot,\cdot])$ is not complete unless $\mathcal{R}(A)$ is a closed subspace in $\mathcal{H}$. However, de Branges et al. proved in [11] (see also [16]) that the completion of X under the inner product $[\cdot,\cdot]$ is isomorphic to the Hilbert space $\mathcal{R}(A^{1/2})$ endowed with the following inner product:

$$(A^{1/2}x, A^{1/2}y) := \langle Px, Py\rangle, \ \forall\, x, y \in \mathcal{H},$$

where P stands for the orthogonal projection of $\mathcal{H}$ onto the closure of $\mathcal{R}(A)$.
Starting now, we will use the shorthand notation $\mathcal{R}(A^{1/2})$ for the Hilbert space $\big(\mathcal{R}(A^{1/2}), (\cdot,\cdot)\big)$. Moreover, the norm induced by $(\cdot,\cdot)$ on $\mathcal{R}(A^{1/2})$ will be denoted by $\|\cdot\|_{\mathcal{R}(A^{1/2})}$. It is important to highlight that $\mathcal{R}(A)$ is dense in $\mathcal{R}(A^{1/2})$ (as shown in [15]). As $\mathcal{R}(A) \subseteq \mathcal{R}(A^{1/2})$, we observe that

$$(Ax, Ay) = (A^{1/2}A^{1/2}x, A^{1/2}A^{1/2}y) = \langle PA^{1/2}x, PA^{1/2}y\rangle = \langle x, y\rangle_A, \tag{3}$$

for any $x, y \in \mathcal{H}$ and so

$$\|Ax\|_{\mathcal{R}(A^{1/2})} = \|x\|_A, \quad \text{for any } x \in \mathcal{H}. \tag{4}$$

To learn more about the Hilbert space $\mathcal{R}(A^{1/2})$, we refer the interested reader to [3].
Let us consider now the operator Z_A defined by:

$$Z_A : \mathcal{H} \longrightarrow \mathcal{R}(A^{1/2}),\ x \longmapsto Z_Ax = Ax.$$

Further, the following useful proposition is stated in [3].

Proposition 2.2. *Let $T \in \mathbb{B}(\mathcal{H})$. Then $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$ if and only if there exists a unique $\widehat{T} \in \mathbb{B}(\mathcal{R}(A^{1/2}))$ such that $Z_AT = \widehat{T}Z_A$.*

Before we move on, it is important to state the following lemmas. The proof of the first one can be found in [15].

Lemma 2.3. *Let $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$. Then*

(1) $\|T\|_A = \|\widehat{T}\|_{\mathbb{B}(\mathcal{R}(A^{1/2}))}$.

(2) $\omega_A(T) = \omega(\widehat{T})$.

Lemma 2.4. *Let $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$. Then*

$$W^A_{\max}(T) = W_{\max}(\widehat{T}),$$

where $\widehat{T}$ is the operator given by Proposition 2.2.

Proof. We have $Z_A T = \widehat{T} Z_A$, that is, $ATx = \widehat{T}Ax$ for all $x \in \mathcal{H}$. Now, let $\lambda \in W^A_{\max}(T)$, then there exists $(x_n) \subseteq \mathcal{H}$ such that $\|x_n\|_A = 1$,

$$\lim_{n\to+\infty} \langle Tx_n, x_n\rangle_A = \lambda, \text{ and } \lim_{n\to+\infty} \|Tx_n\|_A = \|T\|_A.$$

Set $y_n = Ax_n \in \mathcal{R}(A^{1/2})$. By using (3) together with (4), we have $\|y_n\|_{\mathcal{R}(A^{1/2})} = \|x_n\|_A = 1$ and

$$\langle Tx_n, x_n\rangle_A = (ATx_n, Ax_n) = (\widehat{T}y_n, y_n).$$

Again, by (4), we infer that

$$\|Tx_n\|_A = \|ATx_n\|_{\mathcal{R}(A^{1/2})} = \|\widehat{T}y_n\|_{\mathcal{R}(A^{1/2})}.$$

On the other hand, by Lemma 2.3 we have $\|T\|_A = \|\widehat{T}\|_{\mathbb{B}(\mathcal{R}(A^{1/2}))}$. This implies that $\lambda \in W_{\max}(\widehat{T})$ and so $W^A_{\max}(T) \subseteq W_{\max}(\widehat{T})$. Conversely, let $\lambda \in W_{\max}(\widehat{T})$, then there exists $(y_n) \subseteq \mathcal{R}(A^{1/2})$ such that $\|y_n\|_{\mathcal{R}(A^{1/2})} = 1$,

$$\lim_{n\to+\infty} (\widehat{T}y_n, y_n) = \lambda, \text{ and } \lim_{n\to+\infty} \|\widehat{T}y_n\|_{\mathcal{R}(A^{1/2})} = \|\widehat{T}\|_{\mathbb{B}(\mathcal{R}(A^{1/2}))} = \|T\|_A.$$

Since $(y_n) \subseteq \mathcal{R}(A^{1/2})$ for all n, there exists $(x_n) \subseteq \mathcal{H}$ such that $y_n = A^{1/2}x_n$. So, $\|A^{1/2}x_n\|_{\mathcal{R}(A^{1/2})} = 1$,

$$\lim_{n\to+\infty} (\widehat{T}A^{1/2}x_n, A^{1/2}x_n) = \lambda \text{ and } \lim_{n\to+\infty} \|\widehat{T}A^{1/2}x_n\|_{\mathcal{R}(A^{1/2})} = \|T\|_A. \tag{5}$$

On the other hand, since $\mathcal{R}(A)$ is dense in $\mathcal{R}(A^{1/2})$, then for any $n \in \mathbb{N}$, there exists $(x_{n,k}) \subseteq \mathcal{H}$ such that

$$\lim_{k\to+\infty} \|Ax_{n,k} - A^{1/2}x_n\|_{\mathcal{R}(A^{1/2})} = 0.$$

This gives

$$\lim_{k\to+\infty} \|Ax_{n,k}\|_{\mathcal{R}(A^{1/2})} = 1. \tag{6}$$

Moreover, by (5) we have

$$\lim_{n,k\to+\infty} (\widehat{T}Ax_{n,k}, Ax_{n,k}) = \lambda \text{ and } \lim_{n,k\to+\infty} \|\widehat{T}Ax_{n,k}\|_{\mathcal{R}(A^{1/2})} = \|T\|_A.$$

Let $z_k = \dfrac{x_{p_k,q_k}}{\|Ax_{p_k,q_k}\|_{\mathcal{R}(A^{1/2})}}$, where (p_k) and (q_k) are suitable strictly increasing sequences. So, by using (6), we obtain

$$\lim_{k\to+\infty} (\widehat{T}Az_k, Az_k) = \lambda \text{ and } \lim_{k\to+\infty} \|\widehat{T}Az_k\|_{\mathcal{R}(A^{1/2})} = \|T\|_A.$$

On the other hand, we have

$$(\widehat{T}Az_k, Az_k) = (ATz_k, Az_k) \text{ and } \|\widehat{T}Az_k\|_{\mathcal{R}(A^{1/2})} = \|ATz_k\|_{\mathcal{R}(A^{1/2})}.$$

So, by applying (3) together with (4), we infer that

$$\lim_{k\to+\infty} \langle Tz_k, z_k\rangle_A = \lambda \text{ and } \lim_{k\to+\infty} \|Tz_k\|_A = \|T\|_A.$$

Furthermore, $\|Az_k\|_{\mathcal{R}(A^{1/2})} = \|z_k\|_A = 1$. So, we deduce that $\lambda \in W^A_{\max}(T)$. Hence the proof is complete. □

At this point, we have the ability to demonstrate the following three theorems. Although the first theorem has already been established in [5], we can derive the same outcome directly from Lemma 2.4 and [22, Lemma 2].

Theorem 2.5. *Let $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$. Then $W^A_{\max}(T)$ is convex.*

Theorem 2.6. *Let $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$. Then the following conditions are equivalent:*

(1) $0 \in W^A_{\max}(T)$.

(2) $\|T\|_A^2 + |\lambda|^2 \leq \|T + \lambda\|_A^2$ *for any* $\lambda \in \mathbb{C}$.

(3) $\|T\|_A \leq \|T + \lambda\|_A$ *for any* $\lambda \in \mathbb{C}$.

Proof. To begin with, it is important to note that Theorem 1.3 enables us to establish the equivalence between the following statements:

(i) $0 \in W_{\max}(\widehat{T})$.

(ii) $\|\widehat{T}\|^2_{\mathbb{B}(\mathcal{R}(A^{1/2}))} + |\lambda|^2 \leq \|\widehat{T} + \lambda\|^2_{\mathbb{B}(\mathcal{R}(A^{1/2}))}$ for any $\lambda \in \mathbb{C}$.

(iii) $\|\widehat{T}\|_{\mathbb{B}(\mathcal{R}(A^{1/2}))} \leq \|\widehat{T} + \lambda\|_{\mathbb{B}(\mathcal{R}(A^{1/2}))}$ for any $\lambda \in \mathbb{C}$.

On the other hand, by Lemma 2.4, we have $W^A_{\max}(T) = W_{\max}(\widehat{T})$. Moreover, by Lemma 2.3, we have $\|T\|_A = \|\widehat{T}\|_{\mathbb{B}(\mathcal{R}(A^{1/2}))}$. Also, notice that $T + \lambda \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$ for any $\lambda \in \mathbb{C}$ since $\mathbb{B}_{A^{1/2}}(\mathcal{H})$ is a subalgebra of $\mathbb{B}(\mathcal{H})$. Then from Proposition 2.2, for any $\lambda \in \mathbb{C}$ there exists a unique $\widehat{T+\lambda} \in \mathbb{B}(\mathcal{R}(A^{1/2}))$ such that $Z_A(T + \lambda) = \widehat{T+\lambda} Z_A$. So, all what remains to prove is that $\|T + \lambda\|_A = \|\widehat{T} + \lambda\|_{\mathbb{B}(\mathcal{R}(A^{1/2}))}$ for any $\lambda \in \mathbb{C}$. But the above equality follows by applying Lemma 2.3 (1) together with the fact that $\widehat{T+\lambda} = \widehat{T} + \lambda$ (see [16]). □

In order to formulate the third theorem, which extends Theorem 1.5 to A-bounded operators, we need to introduce the following lemma.

Lemma 2.7. *Let $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$. Then*

$$\Gamma_A(T) \cap W^A_{\max}(T) = \Gamma_A(T) \cap \overline{W_A(T)}.$$

Proof. Since $W^A_{\max}(T) \subseteq \overline{W_A(T)}$ then the first inclusion holds. Now, let $\lambda \in \Gamma_A(T) \cap \overline{W_A(T)}$. Then $\lambda = \|T\|_A$ and there exists a sequence $(\lambda_n) \subseteq W_A(T)$ such that $\lambda = \lim_{n\to+\infty} \lambda_n$. So, there is a sequence $(x_n) \subseteq \mathcal{H}$ such that $\|x_n\|_A = 1$ and $\lambda_n = \langle Tx_n, x_n\rangle_A$ for all n. By applying the Cauchy-Schwarz inequality, we get

$$\begin{aligned} |\langle Tx_n, x_n\rangle_A| &= |\langle A^{1/2}Tx_n, A^{1/2}x_n\rangle| \\ &\leq \|Tx_n\|_A \|x_n\|_A \\ &= \|Tx_n\|_A \\ &\leq \|T\|_A. \end{aligned}$$

So, $\lim_{n\to+\infty} \|Tx_n\|_A = \|T\|_A$. Hence $\lambda \in \Gamma_A(T) \cap W^A_{\max}(T)$. □

One of the main results of this article can now be presented. The interior of any subset L in the complex plane will be denoted by $\overset{\circ}{L}$.

Theorem 2.8. *Let $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$. Then the following statements are equivalent:*

(1) *T is an A-normaloid operator,*

(2) $W^A_{\max}(T) \cap \partial W_A(T) \neq \emptyset$.

Proof. (1) ⇒ (2): Assume that T is an A-normaloid operator. Then by applying Theorem 2.1 together with Lemma 2.7, we get

$$\Gamma_A(T) \cap \overline{W_A(T)} = \Gamma_A(T) \cap W^A_{\max}(T) \neq \emptyset.$$

So, there exists $z \in \Gamma_A(T) \cap \overline{W_A(T)}$. Thus z must lie on the boundary of $W_A(T)$. Since z is also in $W^A_{\max}(T)$, $W^A_{\max}(T) \cap \partial W_A(T) \neq \emptyset$ as required.
(2) ⇒ (1): Assume that $W^A_{\max}(T) \cap \partial W_A(T) \neq \emptyset$. Notice that in view of Lemma 2.3 we have T is A-normaloid if and only if $\widehat{T}$ is a normaloid operator on the Hilbert space $\mathcal{R}(A^{1/2})$. So, in order to prove (1), it suffices to show that

$$W_{\max}(\widehat{T}) \cap \partial W(\widehat{T}) \neq \emptyset.$$

It was shown in [15] that $\overline{W(\widehat{T})} = \overline{W_A(T)}$. Hence $\partial\overline{W(\widehat{T})} = \partial\overline{W_A(T)}$. It is well known that if C is a convex subset in the complex plane, then $\overset{\circ}{C} = \overset{\circ}{\overline{C}}$. Thus $\partial C = \overline{C} \setminus \overset{\circ}{C} = \overline{\overline{C}} \setminus \overset{\circ}{\overline{C}} = \partial\overline{C}$. Therefore, since both of $W(\widehat{T})$ and $W_A(T)$ are convex, the equality $\partial\overline{W(\widehat{T})} = \partial\overline{W_A(T)}$ implies $\partial W(\widehat{T}) = \partial W_A(T)$. Moreover, $W^A_{\max}(T) = W_{\max}(\widehat{T})$ by Lemma 2.4. We deduce that $W_{\max}(\widehat{T}) \cap \partial W(\widehat{T}) \neq \emptyset$. This completes the proof. □

Remark 2.9. *The authors in* [10] *provided a characterization of normaloid operators in terms of their numerical radius. Specifically, an operator $T \in \mathbb{B}(\mathcal{H})$ is normaloid if and only if its numerical radius $\omega(T)$ equals its maximal numerical radius $\omega_{\max}(T)$, where $\omega_{\max}(T)$ is defined as*

$$\omega_{\max}(T) := \sup\left\{|\lambda| \,:\, \lambda \in W_{\max}(T)\right\}.$$

Using Lemmas 2.3 and 2.4, we can obtain an analogous characterization of A-normaloid operators. It is worth noting that this characterization was also established by the third author in [15], *but our approach here differs from that used in* [15].

Theorem 2.10. *Let $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$. Then the following statements are equivalent:*

(1) *T is an A-normaloid operator,*

(2) $\omega_A(T) = \omega^A_{\max}(T)$,

where $\omega^A_{\max}(T)$ is the A-maximal numerical radius defined by

$$\omega^A_{\max}(T) := \sup\left\{|\lambda| \,:\, \lambda \in W^A_{\max}(T)\right\}.$$

On the other hand, similarly to the argument presented in the proof of Theorem 2.6 and with the aid of Corollary 1.4, we can deduce the following corollary.

Corollary 2.11. *Let $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$. Then there is a unique scalar $c_A(T)$ such that*

$$\|T - c_A(T)\|_A^2 + |\lambda|^2 \leq \|(T - c_A(T)) - \lambda\|_A^2, \text{ for all } \lambda \in \mathbb{C}. \tag{7}$$

Moreover, $0 \in W^A_{\max}(T)$ if and only if $c_A(T) = 0$.

Note that $c_A(T) = c_{\widehat{T}}$; the center of mass of $\widehat{T}$. We call $c_A(T)$ the A-center of mass of T and we denote $d_A(T) = \|T - c_A(T)\|_A$ that we call the A-distance of T to scalars. Clearly, $c_A(T)$ is the unique scalar satisfying

$$d_A(T) = \inf_{\lambda\in\mathbb{C}} \|T - \lambda\|_A\,.$$

The theorem below gives a formula for $d_A(T)$, where $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$.

Theorem 2.12. *Let $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$. Then*

$$d_A^2(T) = \sup_{\|x\|_A=1} \left\{ \|Tx\|_A^2 - |\langle Tx, x\rangle_A|^2 \right\}.$$

Proof. For any $x \in \mathcal{H}$ with $\|x\|_A = 1$, we have

$$\begin{aligned} d_A^2(T) = \|T - c_A(T)\|_A^2 &\geq \|(T - c_A(T))x\|_A^2 \\ &= \|Tx\|_A^2 + |c_A(T)|^2 - 2Re(\overline{c_A(T)}\langle Tx, x\rangle_A) \\ &\geq \|Tx\|_A^2 - |\langle Tx, x\rangle_A|^2 + |c_A(T) - \langle Tx, x\rangle_A)|^2 \\ &\geq \|Tx\|_A^2 - |\langle Tx, x\rangle_A|^2 . \end{aligned}$$

Whence

$$d_A^2(T) \geq \sup_{\|x\|_A=1} \left\{ \|Tx\|_A^2 - |\langle Tx, x\rangle_A|^2 \right\}.$$

Conversely,

$$\|T - c_A(T)\|_A = \inf_{\lambda\in\mathbb{C}} \|T - \lambda\|_A = \inf_{\lambda\in\mathbb{C}} \|(T - c_A(T)) - \lambda\|_A .$$

Then $\|T - c_A(T)\|_A \leq \|(T - c_A(T)) - \lambda\|_A$ for any $\lambda \in \mathbb{C}$. Since $T - c_A(T) \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$, from Theorem 2.6 we get $0 \in W^A_{\max}(T - c_A(T))$. So, there exists a sequence $(x_n) \subseteq \mathcal{H}$ with $\|x_n\|_A = 1$ such that

$$\lim_{n\to+\infty} \langle (T - c_A(T))x_n, x_n\rangle_A = 0 \quad \text{and} \quad \lim_{n\to+\infty} \|(T - c_A(T))x_n\|_A = \|T - c_A(T)\|_A .$$

Then $\lim_{n\to+\infty} \langle Tx_n, x_n\rangle_A = c_A(T)$ and

$$\begin{aligned} \|T - c_A(T)\|_A^2 &= \lim_{n\to+\infty} \|(T - c_A(T))x_n\|_A^2 \\ &= \lim_{n\to+\infty} \|Tx_n\|_A^2 - |\langle Tx_n, x_n\rangle_A|^2 + |c_A(T) - \langle Tx_n, x_n\rangle_A)|^2 \\ &= \lim_{n\to+\infty} \|Tx_n\|_A^2 - |\langle Tx_n, x_n\rangle_A|^2 \\ &\leq \sup_{\|x\|_A=1} \left\{ \|Tx\|_A^2 - |\langle Tx, x\rangle_A|^2 \right\}. \end{aligned}$$

Consequently,

$$d_A^2(T) = \sup_{\|x\|_A=1} \left\{ \|Tx\|_A^2 - |\langle Tx, x\rangle_A|^2 \right\}.$$

The proof is complete. □

Remark 2.13. *Let $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$, there is a sequence $(x_n) \subseteq \mathcal{H}$ with $\|x_n\|_A = 1$ such that $\lim_{n\to+\infty} \langle Tx_n, x_n\rangle_A = c_A(T)$, we derive that $c_A(T) \in \overline{W_A(T)}$. However, $c_A(T)$ need not be contained in $W^A_{\max}(T)$. Indeed, in $\mathbb{C}^2$ let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $T = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$. According to* [8], *the A-center of mass of T is $c_A(T) = 2$, while $W^A_{\max}(T) = \{3\}$ (see* [4]*).*

The following corollary gives a sufficient and necessary condition to have $c_A(T) \in W^A_{\max}(T)$.

Corollary 2.14 (Pythagorean Relation). *Let $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$. Then the following statements are equivalent:*

(1) $c_A(T) \in W^A_{\max}(T)$,

(2) $d_A^2(T) + |c_A(T)|^2 = \|T\|_A^2$.

Proof. (1) ⇒ (2): Assume that $c_A(T) \in W^A_{\max}(T)$. There is a sequence $(x_n) \subseteq \mathcal{H}$ with $\|x_n\|_A = 1$ such that

$$\lim_{n\to+\infty} \langle Tx_n, x_n\rangle_A = c_A(T) \quad \text{and} \quad \lim_{n\to+\infty} \|Tx_n\|_A = \|T\|_A .$$

As above, we have

$$\begin{aligned}\|T - c_A(T)\|_A^2 &\geq \lim_{n\to+\infty} \|(T - c_A(T))x_n\|_A^2 \\ &= \lim_{n\to+\infty} \|Tx_n\|_A^2 - |\langle Tx_n, x_n\rangle_A|^2 \\ &= \|T\|_A^2 - |c_A(T)|^2 .\end{aligned}$$

Hence

$$\|T - c_A(T)\|_A^2 + |c_A(T)|^2 \geq \|T\|_A^2 .$$

Taking $\lambda = -c_A(T)$ in (7), we obtain

$$\|T - c_A(T)\|_A^2 + |c_A(T)|^2 \leq \|T\|_A^2 . \tag{8}$$

Hence

$$\|T - c_A(T)\|_A^2 + |c_A(T)|^2 = \|T\|_A^2 .$$

(2) ⇒ (1): Assume that $d_A^2(T) + |c_A(T)|^2 = \|T\|_A^2$. From the proof of Theorem 2.12, there is a sequence $(x_n) \subseteq \mathcal{H}$ with $\|x_n\|_A = 1$ such that $\lim\limits_{n\to+\infty} \langle Tx_n, x_n\rangle_A = c_A(T)$ and

$$\begin{aligned}d_A^2(T) = \|T - c_A(T)\|_A^2 &= \lim_{n\to+\infty} \|Tx_n\|_A^2 - |\langle Tx_n, x_n\rangle_A|^2 \\ &= \lim_{n\to+\infty} \|Tx_n\|_A^2 - |c_A(T)|^2 .\end{aligned}$$

Remembering the hypothesis, we infer that $\lim\limits_{n\to+\infty} \|Tx_n\|_A = \|T\|_A$. Consequently, $c_A(T) \in W^A_{\max}(T)$. □

Remark 2.15. *Let* $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$. *From Remark* 2.13, $c_A(T) \in \overline{W_A(T)}$. So, $|c_A(T)| \leq \omega_A(T)$. *We know that* $W^A_{\max}(T) \subseteq \overline{W_A(T)}$, *the following question arises: what about* $|c_A(T)|$ *and* $\omega^A_{\max}(T)$?

For any $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$, define

$$m^A_{\max}(T) := \inf\Big\{|\lambda| : \lambda \in W^A_{\max}(T)\Big\}.$$

The following answers the previous question.

Theorem 2.16. *Let* $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$. *Then*

$$|c_A(T)| \leq m^A_{\max}(T).$$

In particular,

$$|c_A(T)| \leq \omega^A_{\max}(T).$$

Proof. By an argument of compactness, there exists $\alpha \in W^A_{\max}(T)$ such that $|\alpha| = m^A_{\max}(T)$. Hence there is a sequence $(x_n) \subseteq \mathcal{H}$ with $\|x_n\|_A = 1$ satisfying

$$\alpha = \lim_{n\to+\infty} \langle Tx_n, x_n\rangle_A \quad \text{and} \quad \lim_{n\to+\infty} \|Tx_n\|_A = \|T\|_A .$$

Therefore, we have

$$\begin{aligned}\|T - c_A(T)\|_A^2 &\geq \|(T - c_A(T))x_n\|_A^2 \\ &= \|Tx_n\|_A^2 + |c_A(T)|^2 - 2Re(\overline{c_A(T)}\langle Tx_n, x_n\rangle_A) \\ &\geq \|Tx_n\|_A^2 + |c_A(T)|^2 - 2\,|c_A(T)|\,|\langle Tx_n, x_n\rangle_A| .\end{aligned}$$

It results that

$$\begin{aligned} \|T - c_A(T)\|_A^2 &\geq \|T\|_A^2 + |c_A(T)|^2 - 2\,|c_A(T)|\, m_{\max}^A(T) \\ &= \|T\|_A^2 - (m_{\max}^A(T))^2 + (m_{\max}^A(T) - |c_A(T)|)^2. \end{aligned} \tag{9}$$

Thus

$$\|T - c_A(T)\|_A^2 + (m_{\max}^A(T))^2 \geq \|T\|_A^2 + (m_{\max}^A(T) - |c_A(T)|)^2.$$

We see that

$$\|T - c_A(T)\|_A^2 + (m_{\max}^A(T))^2 \geq \|T\|_A^2 \tag{10}$$

and from (8), we get $m_{\max}^A(T) \geq |c_A(T)|$. □

Remark 2.17. *In* [13], *it is proved that*

$$\|T\|^2 \leq d^2(T) + \omega^2(T) \tag{11}$$

for any $T \in \mathbb{B}(\mathcal{H})$. *From* (10), *we have*

$$\|T\|_A^2 \leq d_A^2(T) + (m_{\max}^A(T))^2 \leq d_A^2(T) + \omega_A^2(T). \tag{12}$$

Note that (12) *is a refinement of* (11) *if we take* $A = I$. *Moreover,* (9) *yields:*

$$\|T\|_A^2 + |c_A(T)|^2 \leq d_A^2(T) + 2\,|c_A(T)|\, m_{\max}^A(T).$$

Then

$$2\,\|T\|_A\,|c_A(T)| \leq d_A^2(T) + 2\,|c_A(T)|\, m_{\max}^A(T).$$

Consequently, if $c_A(T) \neq 0$ (*i.e.,* $0 \notin W_{\max}^A(T)$), *then*

$$\|T\|_A \leq m_{\max}^A(T) + \frac{1}{2}\frac{d_A^2(T)}{|c_A(T)|}. \tag{13}$$

Therefore, if $c_A(T) \neq 0$, *we get from* (12) *and* (13)

$$\|T\|_A \leq \min\left\{\left(d_A^2(T) + (m_{\max}^A(T))^2\right)^{1/2},\; m_{\max}^A(T) + \frac{1}{2}\frac{d_A^2(T)}{|c_A(T)|}\right\}.$$

Note that if $c_A(T) = 0$, *then* $\|T\|_A = d_A(T)$.

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