Filomat 38:17 (2024), 5971–5982
https://doi.org/10.2298/FIL2417971R

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On a weighted Hermite-Hadamard inequality in operator variables with applications for weighted operator means

Mustapha Raïssouli[a], Lahcen Tarik[b], Mohamed Chergui[c]

[a] *Department of Mathematics, Faculty of Science, Moulay Ismail University, Meknes, Morocco.*
[b] *LAGA-Laboratory, Science Faculty, Ibn Tofail University, Kenitra, Morocco.*
[c] *Department of Mathematics, CRMEF-Rabat-Sale-Kenitra, EREAM Team, LaREAMI-Lab, Kenitra, Morocco.*

Abstract. This paper investigates a new weighted version of the standard Hermite-Hadamard inequalities for operator convex functions and highlights certain related properties. As an application, new weighted operator means have been pointed out and some refinements to several operator mean inequalities have been discussed.

1. Introduction and preliminaries

Let $(H, \langle ., .\rangle)$ be a complex Hilbert space. The notation $\mathcal{B}(H)$ refers to the C^*-algebra of all bounded linear operators acting on H. An operator $S \in \mathcal{B}(H)$ is positive if $\langle Sx, x\rangle \geq 0$ for all $x \in \mathbb{C}$. This induces a partial ordering on the sub-space of self-adjoint operators. Thus, for $S, T \in \mathcal{B}(H)$ self-adjoint, we write $S \leq T$ to mean that $T - S$ is positive. The notation $\mathcal{B}^+(H)$ stands for the closed cone of positive operators and $\mathcal{B}^{+*}(H)$ refers to the open cone of positive invertible operators of $\mathcal{B}(H)$.

With this regard, we recall that if f and g are two analytic real-valued functions defined on a nonempty interval $J \subset \mathbb{R}$ such that $f(t) \leq g(t)$ for all $t \in J$, then for any operators $S \in \mathcal{B}(H)$, with spectra in J, we have $f(S) \leq g(S)$, where $f(S)$ is defined using the functional calculus techniques as usual.

Let $\mathcal{C}_J(H)$ be the class of all self-adjoint operators with spectra in J. For $S, T \in \mathcal{C}_J(H)$, we define the segment $[S, T] := \Big\{(1-t)\,S + t\,T;\ \ t \in [0, 1]\Big\} \subset \mathcal{C}_J(H)$.

The function $f : J \to \mathbb{R}$ is said to be operator monotone if $S \leq T$ implies $f(S) \leq f(T)$, where $S, T \in \mathcal{C}_J(H)$. We say that f is operator convex (resp. operator concave) on J if the following inequality

$$f\Big((1-\lambda)S + \lambda T\Big) \leq (\geq)(1-\lambda)f(S) + \lambda f(T) \tag{1}$$

holds for all $S, T \in \mathcal{C}_J(H)$ and $\lambda \in [0, 1]$.

Otherwise, the following inequalities

$$f\left(\frac{S+T}{2}\right) \leq \int_0^1 f\Big((1-t)S + tT\Big)\,dt \leq \frac{f(S) + f(T)}{2} \tag{2}$$

2020 *Mathematics Subject Classification*. Primary 47A63, 47A60, 47A64
Keywords. Weighted Hermite-Hadamard inequalities, operator inequalities, operator means
Received: 05 September 2023; Revised: 02 November 2023; Accepted: 29 November 2023
Communicated by Dragan S. Djordjević
Email addresses: raissouli.mustapha@gmail.com (Mustapha Raïssouli), lahcen.tarik@uit.ac.ma (Lahcen Tarik), chergui_m@yahoo.fr (Mohamed Chergui)

hold, whenever $f : J \to \mathbb{R}$ is operator convex and $S, T \in C_J(H)$. If f is operator concave on J then (2) are reversed. Inequalities (2) represent an efficient tool in generalizing and refining several classical operator inequalities. See [2, 5, 6, 14] for instance and the related references cited therein.

An important result concerning convex operator functions is the extension of integral Jensen inequality pointed out in [8]. Namely, for $f : J \to \mathbb{R}$ convex, let $\left(\phi_t\right)_{t\in J}$ be the unital field of positive linear mappings from $\mathbb{A}$ to $\mathbb{B}$, two unital C^*-algebras, and ν be a bounded Radon measure. The following Jensen operator inequality

$$f\left(\int_J \phi_t(A_t)d\nu(t)\right) \leq \int_J \phi_t\left(f(A_t)\right)d\nu(t), \tag{3}$$

holds for any bounded field $(A_t)_{t\in J} \subset C_J(H)$ such that $t \longmapsto A_t$ is norm continuous on J.

Otherwise, if f is an operator convex function of class C^1 on J, then we have [7]

$$Df(S)(T-S) \leq f(T) - f(S) \leq Df(T)(T-S) \tag{4}$$

for any $S, T \in C_J(H)$, where $Df(S)(X)$ is the directional derivative of f at S in the direction $X \in \mathcal{B}(H)$, namely

$$Df(S)(X) := \lim_{\epsilon\to 0} \frac{f(S+\epsilon X) - f(S)}{\epsilon}.$$

The inequalities (2) are main tool for establishing many operator inequalities in mean-theory by using either the representative function of the evoked operator mean or its integral representation. In fact, every operator mean σ in the sense of Kubo-Ando [9], of two operators $S, T \in \mathcal{B}^{+*}(H)$ can be expressed via a unique positive operator monotone function φ_σ defined on the interval $(0, \infty)$ by the following relationship

$$S\sigma T = S^{1/2}\varphi_\sigma\left(S^{-1/2}TS^{-1/2}\right)S^{1/2}, \tag{5}$$

with $\varphi_\sigma(1) = 1$. Such φ_σ is called the representative function of the operator mean σ. The mean σ is symmetric if $\sigma(S,T) = \sigma(T,S)$ for any $S, T \in \mathcal{B}^{+*}(H)$, or equivalently, $\varphi_\sigma(x) = x\varphi_\sigma\left(x^{-1}\right)$ for all $x > 0$.

An operator mean σ is said to be λ-weighted [17] if its representative function φ_σ is derivable at 1 with the condition $\varphi_\sigma'(1) = \lambda$. Examples of some standard weighted operator means are the weighted arithmetic, harmonic and geometric operator means defined, respectively, for $S, T \in \mathcal{B}^{*+}(H)$ and $\lambda \in (0,1)$ as follows

$$S\nabla_\lambda T = (1-\lambda)S + \lambda T;\ S!_\lambda T = \left((1-\lambda)S^{-1} + \lambda T^{-1}\right)^{-1};\ S\sharp_\lambda T = S^{1/2}\left(S^{-1/2}TS^{-1/2}\right)^\lambda S^{1/2}.$$

These weighted operator means satisfy the following inequalities

$$S!_\lambda T \leq S\sharp_\lambda T \leq S\nabla_\lambda T, \tag{6}$$

and they are not symmetric unless $\lambda = 1/2$ case where they are simply denoted by $T\nabla S$, $T!S$ and $S\sharp T$, respectively.

Other symmetric operator means such as the logarithmic operator mean [1, 11, 13] and the (chaotic) identric operator mean [15] are defined, respectively, by

$$L(S,T) := \left(\int_0^1 S^{-1}!_t T^{-1}\,dt\right)^{-1} = \int_0^1 S\sharp_t T\,dt. \tag{7}$$

$$I(S,T) := \exp\left(\int_0^1 \log\left(S\nabla_t T\right)dt\right). \tag{8}$$

We notice that $L(S,T)$ is an operator mean in the sense of Kubo-Ando while $I(S,T)$ is not, see [15] for more details. A weighted version $L_\lambda(S,T)$ of (7), the weighted logarithmic operator mean, was introduced in [3, 10] by its representative function defined on $(0,+\infty)$ for any $\lambda \in (0,1)$ by

$$\phi_\lambda(x) = \frac{1}{\log x}\left(\frac{1-\lambda}{\lambda}\left(x^\lambda - 1\right) + \frac{\lambda}{1-\lambda}x^\lambda\left(x^{1-\lambda} - 1\right)\right). \tag{9}$$

For an equivalent writing under another point of view of $L_\lambda(S,T)$ see [13, 16]. Note that $L_{1/2}(S,T) = L(S,T)$ for any $S,T \in \mathcal{B}^{+*}(H)$. For a weighted version $I_\lambda(S,T)$ of the identric operator mean (8), we can consult [15].

The following chain of operator inequalities holds [10]

$$S!_\lambda T \leq S\sharp_\lambda T \leq L_\lambda(S,T) \leq S\nabla_\lambda T. \tag{10}$$

Now, let us observe the following remark which is of interest.

Remark 1.1. *(i) Let σ_λ, for $\lambda \in [0,1]$, be a given λ-weighted operator mean. Setting $m := \sigma_{1/2}$, we then say that σ_λ is the weighted m-operator mean.*
(ii) If σ_λ, $\lambda \in [0,1]$, is a weighted m-operator mean then $m := \sigma_{1/2}$ is of course unique. However, it is possible to have more than one weighted m-operator mean. Section 4 of this paper explains more this latter situation.

This manuscript will be organized as follows: in Section 2, we investigate a new weighted variant of (2). Section 3 is devoted to giving some refinements and reverses of this weighted inequality. Section 4 is focused on applying the findings of the previous sections to introduce new weighted logarithmic operator means. The comparison of these weighted logarithmic operator means with the above one $L_\lambda(S,T)$ is also discussed.

2. Weighted Hermite-Hadamard operator inequality

For the sake of simplicity, we extend the weighted operator arithmetic mean to operators in $\mathcal{B}(H)$ by stating

$$S\nabla_\lambda T := (1-\lambda)S + \lambda T, \quad \text{for any } S,T \in \mathcal{B}(H) \text{ and } \lambda \in [0,1].$$

Our first main result reads as follows.

Theorem 2.1. *Let $f : J \to \mathbb{R}$ be an operator convex function. For any $\lambda \in (0,1)$ and $S,T \in C_J(H)$, the following inequalities hold*

$$f\Big(S\nabla_\lambda T\Big) \leq \int_0^1 f\Big(S\nabla_t T\Big)d\nu_\lambda(t) \leq f(S)\nabla_\lambda f(T), \tag{11}$$

where ν_λ is the probability measure defined on $[0,1]$ by

$$d\nu_\lambda(t) = \Big((1-\lambda)(1-t)^{\frac{1-2\lambda}{\lambda}} + \lambda t^{\frac{2\lambda-1}{1-\lambda}}\Big)dt. \tag{12}$$

If $f : J \to \mathbb{R}$ is operator concave then (11) *are reversed.*

Proof. Applying Jensen operator integral inequality (3), with $A_t = S\nabla_t T$ and $J = [0,1]$, we get

$$f\left(\int_0^1 S\nabla_t T d\nu_\lambda(t)\right) \leq \int_0^1 f\left(S\nabla_t T\right)d\nu_\lambda(t).$$

Moreover, using (1), we obtain

$$\int_0^1 f\left(S\nabla_t T\right)d\nu_\lambda(t) \leq \int_0^1 f(S)\nabla_t f(T)d\nu_\lambda(t) = f(S)\nabla_\lambda f(T),$$

where the last equality holds by (12) and real-integration. Thus, the proof is completed. □

Remark 2.2. *(i) In what follows,* (11) *will be called the weighted Hermite-Hadamard operator inequalities,* (*WHHOI*) *in short. If* $\lambda = 1/2$, $d\nu_{1/2}(t) = dt$. *Thus,* (11) *coincides with* (2).
(ii) It is important to notice the following relation,

$$d\nu_{1-\lambda}(1-t) = d\nu_\lambda(t) \ \text{ for any } \ \lambda \in (0,1). \tag{13}$$

The (*WHHOI*) enable us to refine of the standard Hermite-Hadamard operator inequality (2) as recited in the following corollary.

Corollary 2.3. *Let* $f : J \to \mathbb{R}$ *be operator convex. For any* $\lambda \in (0,1)$ *and* $S, T \in C_J(H)$, *the following inequalities hold*

$$f\Big(S\nabla T\Big) \le \int_0^1 f\Big(S\nabla_t T\Big)dt \le J_f(S,T) \le f(S)\nabla f(T), \tag{14}$$

where we set

$$J_f(S,T) := \int_0^1 \int_0^1 f\Big(S\nabla_t T\Big)\, d\nu_\lambda(t) d\lambda.$$

If f is operator concave then (14) are reversed.

Proof. Integrating (11) with respect to $\lambda \in (0,1)$ and using the left inequality in (2), we get (14). □

Let $\lambda \in (0,1)$ and $S, T \in C_J(H)$. We put

$$\mathcal{M}_\lambda(f; S, T) := \int_0^1 f\Big(S\nabla_t T\Big)d\nu_\lambda(t). \tag{15}$$

The operator map $\lambda \mapsto \mathcal{M}_\lambda(f; S, T)$ can be extended on the whole interval $[0,1]$ as justifying by the following corollary.

Corollary 2.4. *Let* $f : J \to \mathbb{R}$ *be operator convex (resp. operator concave). For any* $S, T \in C_J(H)$, *there holds*

$$\lim_{\lambda\downarrow 0} \mathcal{M}_\lambda(f; S, T) = f(S), \ \lim_{\lambda\uparrow 1} \mathcal{M}_\lambda(f; S, T) = f(T).$$

Proof. The desired results follow from (11) by using the fact that if $f : C \to \mathbb{R}$ is operator convex on J then it is norm-continuous on J. □

3. Refinements and reverses of (*WHHOI*)

The following lemma, which provides a refinement and a reverse of (1), will be needed in the sequel. See [4, 12] for instance.

Lemma 3.1. *Let* $f : J \to \mathbb{R}$ *be operator convex. Then the following inequalities*

$$r(a,b)\Big(f(S)\nabla_a f(T) - f\Big(S\nabla_a T\Big)\Big) \le f(S)\nabla_b f(T) - f\Big(S\nabla_b T\Big) \le R(a,b)\Big(f(S)\nabla_a f(T) - f\Big(S\nabla_a T\Big)\Big), \tag{16}$$

hold for any $S, T \in C_J(H)$ *and* $a, b \in (0,1)$, *where we set*

$$r(a,b) := \min\left(\frac{b}{a}, \frac{1-b}{1-a}\right), \ \ R(a,b) := \max\left(\frac{b}{a}, \frac{1-b}{1-a}\right). \tag{17}$$

If f *is operator concave then* (16) *are reversed.*

We also need the following lemma.

Lemma 3.2. *For any $a, \lambda \in (0,1)$, the following relations hold*

$$\int_0^1 r(a,b)\, d\nu_\lambda(b) = \alpha_{a,1-\lambda} + \alpha_{1-a,\lambda} \tag{18}$$

$$\int_0^1 R(a,b)\, d\nu_\lambda(b) = \frac{\lambda}{a} + \frac{1-\lambda}{1-a} - \alpha_{a,1-\lambda} - \alpha_{1-a,\lambda}, \tag{19}$$

where, for $x, y > 0$, we set

$$\alpha_{x,y} := y^2 \frac{1 - x^{\frac{1-y}{y}}}{1-x}.$$

Proof. It is obvious that $0 \le b \le a$ if and only if $\frac{b}{a} \le \frac{1-b}{1-a}$. Thus we can write

$$\int_0^1 r(a,b)\, d\nu_\lambda(b) = \frac{1}{a}\int_0^a b.d\nu_\lambda(b) + \frac{1}{1-a}\int_a^1 (1-b).d\nu_\lambda(b).$$

The use of (12) leads to

$$b.d\nu_\lambda(b) = \Big((1-\lambda)(1-b)^{\frac{1-2\lambda}{\lambda}} - (1-\lambda)(1-b)^{\frac{1-\lambda}{\lambda}} + \lambda b^{\frac{\lambda}{1-\lambda}}\Big)\, db.$$

By (13), it is easy to see that $(1-b)d\nu_\lambda(b) = (1-b)d\nu_{1-\lambda}(1-b)$. After some simple integral calculations, we obtain

$$\int_0^a b.d\nu_\lambda(b) = -\lambda(1-a)^{\frac{1-\lambda}{\lambda}} + \lambda(1-\lambda)\Big[(1-a)^{\frac{1}{\lambda}} + a^{\frac{1}{1-\lambda}}\Big] + \lambda^2,$$

$$\begin{aligned}\int_a^1 (1-b).d\nu_\lambda(b) &= \int_a^1 (1-b).d\nu_{1-\lambda}(1-b)\\ &= \int_0^{1-a} t.d\nu_{1-\lambda}(t)\\ &= -(1-\lambda)a^{\frac{\lambda}{1-\lambda}} + \lambda(1-\lambda)\Big[(1-a)^{\frac{1}{\lambda}} + a^{\frac{1}{1-\lambda}}\Big] + (1-\lambda)^2.\end{aligned}$$

Combining these latter results, we deduce (18). The relation (19) can be proved by making some algebraic manipulations and using the following formula

$$R(a,b) + r(a,b) = \frac{1}{1-a} + \frac{1-2a}{a(1-a)}b.$$

The details are straightforward and therefore omitted here. □

We can now state the following result which provides a refinement and a reverse of the right inequality in (11).

Theorem 3.3. *Let $f : J \to \mathbb{R}$ be operator convex. For any $a, \lambda \in [0,1]$ and $S, T \in C_J(H)$ the following inequalities hold*

$$\begin{aligned} m(a,\lambda)\Big(f(S)\nabla_a f(T) - f\big(S\nabla_a T\big)\Big) &\le f(S)\nabla_\lambda f(T) - \int_0^1 f\big(S\nabla_t T\big)d\nu_\lambda(t)\\ &\le M(a,\lambda)\Big(f(S)\nabla_a f(T) - f\big(S\nabla_a T\big)\Big), \end{aligned}\tag{20}$$

where we set

$$m(a,\lambda) := (1-\lambda)^2 \frac{1-a^{\frac{\lambda}{1-\lambda}}}{1-a} + \lambda^2 \frac{1-(1-a)^{\frac{1-\lambda}{\lambda}}}{a},\ M(a,\lambda) := \frac{1-\lambda}{1-a} + \frac{\lambda}{a} - m(a,\lambda).$$

If f is operator concave then (20) *are reversed.*

Proof. Multiplying all sides of (16) by $d\nu_\lambda(b)$ and then integrating with respect to $b \in [0,1]$, we obtain the desired inequalities by the use of (18) and (19). The details are simple and therefore omitted here for the reader. □

Taking $a = 1/2$ in Theorem 3.3 and administering some computations, we get the following corollary.

Corollary 3.4. *For $f : J \to \mathbb{R}$ operator convex, $\lambda \in [0,1]$ and $S, T \in C_J(H)$, there hold*

$$l(\lambda)\Big(f(S)\nabla f(T) - f\big(S\nabla T\big)\Big) \le f(S)\nabla_\lambda f(T) - \int_0^1 f\big(S\nabla_t T\big) d\nu_\lambda(t) \le u(\lambda)\Big(f(S)\nabla f(T) - f\big(S\nabla T\big)\Big), \tag{21}$$

where,

$$l(\lambda) := 2\left[(1-\lambda)^2\left(1 - 2^{\frac{\lambda}{\lambda-1}}\right) + \lambda^2\left(1 - 2^{\frac{\lambda-1}{\lambda}}\right)\right] \text{ and } u(\lambda) := 2 - l(\lambda).$$

If f is operator concave then (21) *are reversed.*

The following results focuses on refinement and reverse of the left inequality in (11).

Theorem 3.5. *Let $f : J \to \mathbb{R}$ be operator convex, $S, T \in C_J(H)$ and $\lambda \in [0,1]$. Then there hold*

$$\begin{aligned} f(S\nabla_\lambda T) \le \int_0^1 f\Big((S\nabla_\lambda T)\nabla_\lambda (S\nabla_x T)\Big) d\nu_\lambda(x) &\le \int_0^1 \mathcal{M}_\lambda(f; S\nabla_\lambda T, S\nabla_x T)\, d\nu_\lambda(x) \\ &\le f\big(S\nabla_\lambda T\big)\nabla_\lambda \mathcal{M}_\lambda(f; S, T) \le \int_0^1 f\big(S\nabla_x T\big) d\nu_\lambda(x). \end{aligned} \tag{22}$$

If f is operator concave then (22) *are reversed.*

Proof. For any $S, T \in C_J(H)$ and $\lambda \in [0,1]$, we have

$$\begin{aligned} f(S\nabla_\lambda T) &= f\left((S\nabla_\lambda T)\nabla_\lambda (S\nabla_\lambda T)\right) \\ &= f\left((S\nabla_\lambda T)\nabla_\lambda\Big(\int_0^1 S\nabla_x T\, d\nu_\lambda(x)\Big)\right) \\ &= f\left(\int_0^1 (S\nabla_\lambda T)\nabla_\lambda (S\nabla_x T)\, d\nu_\lambda(x)\right) \\ &\le \int_0^1 f\Big((S\nabla_\lambda T)\nabla_\lambda (S\nabla_x T)\Big) d\nu_\lambda(x) \quad \text{by convexity of } f. \\ &\le \int_0^1 \left\{\int_0^1 f\left((S\nabla_\lambda T)\nabla_t (S\nabla_x T)\right) d\nu_\lambda(t)\right\} d\nu_\lambda(x) \quad \text{by (11)} \\ &= \int_0^1 \mathcal{M}_\lambda\left(f; S\nabla_\lambda T, S\nabla_x T\right) d\nu_\lambda(x) \\ &\le \int_0^1 \Big[f(S\nabla_\lambda T)\nabla_\lambda f(S\nabla_x T)\Big] d\nu_\lambda(x) \quad \text{by (11)} \\ &= f(S\nabla_\lambda T)\nabla_\lambda \int_0^1 f(S\nabla_x T)\, d\nu_\lambda(x) \\ &\le \int_0^1 f(S\nabla_x T)\, d\nu_\lambda(x) \quad \text{by (11).} \end{aligned}$$

Thus, the proof is completed. □

Theorem 3.6. *Let $f : J \longrightarrow \mathbb{R}$ be operator convex. Assume that f is differentiable on J then for any $S, T \in C_J(H)$ and $u \in [0,1]$, we have*

$$(1-u)\int_0^1 (\lambda-t)Df\Big((S\nabla_\lambda T)\nabla_u(S\nabla_t T)\Big)(S-T)\,d\nu_\lambda(t) \leq \int_0^1 f(S\nabla_t T)d\nu_\lambda(t) - \int_0^1 f\Big((S\nabla_\lambda T)\nabla_t(S\nabla_x T)\Big)d\nu_\lambda(x)$$
$$\leq (1-u)\int_0^1 (\lambda-t)Df(S\nabla_t T)(S-T)\,d\nu_\lambda(t). \quad (23)$$

Proof. According to (4) we can write

$$Df(A)(B-A) \leq f(B) - f(A) \leq Df(B)(B-A),$$

with $A = (S\nabla_\lambda T)\nabla_u(S\nabla_t T) \in [S,T]$ and $B = S\nabla_t T \in [S,T]$. Therefore, we have

$$(1-u)(\lambda-t)Df\Big((S\nabla_\lambda T)\nabla_u(S\nabla_t T)\Big)(S-T) \leq f(S\nabla_t T) - f\Big((S\nabla_\lambda T)\nabla_u(S\nabla_t T)\Big)$$
$$\leq (1-u)(\lambda-t)Df\Big(S\nabla_t T\Big)(S-T). \quad (24)$$

Multiplying all sides of (24) by $d\nu_\lambda(t)$ and then integrating with respect to $t \in [0,1]$, we get the desired result and the proof is achieved. □

Making $u = 0$ in (23) and noticing that $\int_0^1 (\lambda - t)d\nu_\lambda(t) = 0$, we may state the following corollary.

Corollary 3.7. *With the same assumptions as in Theorem 3.6, there hold*

$$0 \leq \int_0^1 f(S\nabla_t T)d\nu_\lambda(t) - f(S\nabla_\lambda T) \leq \int_0^1 (\lambda-t)Df(S\nabla_t T)(S-T)d\nu_\lambda(t). \quad (25)$$

4. Application: Two new weighted logarithmic operator means

As application of the previous results, we will introduce here some new weighted operator means and we establish some related properties. We begin by pointing out the following lemma which will be needed.

Lemma 4.1. *Let $\lambda \in [0,1]$. The functions*

$$x \mapsto f_\lambda(x) = \left(\int_0^1 \Big(1-t+tx\Big)^{-1} d\nu_\lambda(t)\right)^{-1}, \quad (26)$$

$$x \mapsto g_\lambda(x) = \int_0^1 x^t d\nu_\lambda(t) \quad (27)$$

are operator monotone on $(0,+\infty)$. Moreover, we have

$$f_\lambda(1) = g_\lambda(1) = 1 \ \textit{and} \ \frac{df_\lambda}{dx}(1) = \frac{dg_\lambda}{dx}(1) = \lambda. \quad (28)$$

Proof. The functions $u : x \mapsto x^{-1}$ and $v : x \mapsto \int_0^1 u(1-t+tx)\, d\nu_\lambda(t)$ are operator monotone decreasing on $(0,\infty)$, then $f_\lambda = uov$ is operator monotone. Also, the function $x \mapsto x^t$ is operator monotone on $(0,+\infty)$ for $0 \leq t \leq 1$ and then g_λ is as well.

The two first relations of (28) are immediate. We now prove the two other ones. For fixed $\lambda \in [0,1]$, consider the function $\psi(x) := \int_0^1 \Big(1-t+tx\Big)^{-1} d\nu_\lambda(t)$ for $x > 0$. By the $(WHHOI)$ applied in $H = \mathbb{R}$ for the convex function $f(x) = 1/x$ on $(0,\infty)$ we have $\left(1-\lambda+\lambda x\right)^{-1} \le \psi(x) \le 1-\lambda+\lambda x^{-1}$ for any $x>0$. This, with the fact that $\psi(1)=1$, implies that for any $x>1$ we have

$$\frac{\left(1-\lambda+\lambda x\right)^{-1}-1}{x-1} \le \frac{\psi(x)-\psi(1)}{x-1} \le -\frac{\lambda}{x}, \tag{29}$$

with reversed inequalities if $x<1$. It is easy to check that the two extremes sides of (29) both tend to $-\lambda$ when x goes to 1, and so $\psi'(1) = -\lambda$. Since $f_\lambda(x) = \left(\psi(x)\right)^{-1}$ we then deduce that $\frac{df_\lambda}{dx}(1) = \lambda$.

On another part, the standard arithmetic-geometric-harmonic mean inequality implies that, for any $x>0$ and $t\in[0,1]$, we have $\left(1-t+tx^{-1}\right)^{-1} \le x^t \le 1-t+tx$. It follows that

$$\psi(x^{-1}) = \int_0^1 \left(1-t+tx^{-1}\right)^{-1} d\nu_\lambda(t) \le g_\lambda(x) := \int_0^1 x^t d\nu_\lambda(t) \le \int_0^1 (1-t+tx) d\nu_\lambda(t) = 1-\lambda+\lambda x.$$

This, with $g_\lambda(1)=1$ and by the same way as previous, allows us to check that $\frac{dg_\lambda}{dx}(1)=\lambda$, so completing the proof. □

Proposition 4.2. *For $S,T \in \mathcal{B}^{+*}(H)$ and $\lambda\in[0,1]$, we set*

$$\mathcal{L}_\lambda(S,T) := \left(\int_0^1 \left(S\nabla_t T\right)^{-1} d\nu_\lambda(t)\right)^{-1}. \tag{30}$$

and

$$\mathscr{L}_\lambda(S,T) := \int_0^1 S\sharp_t T d\nu_\lambda(t). \tag{31}$$

Then $\mathcal{L}_\lambda$ and $\mathscr{L}_\lambda$ are λ-weighted operator means. Furthermore, we have

$$\mathcal{L}_{1/2}(S,T) = \mathscr{L}_{1/2}(S,T) = L(S,T), \tag{32}$$

where $L(S,T)$ is the logarithmic operator mean defined by (7).

Proof. It is not hard to see that

$$\mathcal{L}_\lambda(S,T) = S^{\frac12}\left[\int_0^1 \left((1-t)I + t\left(S^{-\frac12}TS^{-\frac12}\right)\right)^{-1} d\nu_\lambda(t)\right]^{-1} S^{\frac12} = S^{\frac12} f_\lambda\left(S^{-\frac12}TS^{-\frac12}\right)S^{\frac12} \tag{33}$$

and

$$\mathscr{L}_\lambda(S,T) = S^{\frac12}\left[\int_0^1 \left(S^{-\frac12}TS^{-\frac12}\right)^t d\nu_\lambda(t)\right] S^{\frac12} = S^{\frac12} g_\lambda\left(S^{-\frac12}TS^{-\frac12}\right)S^{\frac12}. \tag{34}$$

By Lemma 4.1, $\mathcal{L}_\lambda$ and $\mathscr{L}_\lambda$ are λ-weighted operator means, with representative functions f_λ and g_λ given, respectively, by (26) and (27). This concludes the proof. □

The following result gives a comparison between some of the previous weighted operator means.

Proposition 4.3. *For any $S,T\in\mathcal{B}^{+*}(H)$ and $\lambda\in[0,1]$, there hold*

$$S!_\lambda T \le \mathcal{L}_\lambda(S,T) \le S\nabla_\lambda T, \tag{35}$$

$$S!_\lambda T \le \mathcal{L}_\lambda^{-1}(S^{-1},T^{-1}) \le \mathscr{L}_\lambda(S,T) \le S\nabla_\lambda T. \tag{36}$$

Proof. By applying (11) for the operator convex function $x \mapsto 1/x$ on $(0, +\infty)$, we get

$$(S\nabla_\lambda T)^{-1} \leq \int_0^1 (S\nabla_t T)^{-1} d\nu_\lambda(t) \leq S^{-1}\nabla_\lambda T^{-1},$$

hence (35). By (6), we have for any $t \in [0,1]$, $S!_t T \leq S\sharp_t T \leq S\nabla_t T$. Multiplying all sides of this latter inequalities by $d\nu_\lambda(t)$ and then integrating with respect to $t \in [0,1]$, we obtain

$$\mathcal{L}_\lambda^{-1}(S^{-1}, T^{-1}) \leq \mathscr{L}_\lambda(S,T) \leq S\nabla_\lambda T.$$

For the left inequality in (36), it suffices to notice that

$$\mathcal{L}_\lambda^{-1}(S^{-1}, T^{-1}) \geq (S^{-1}\nabla_\lambda T^{-1})^{-1} = S!_\lambda T.$$

This ends the proof. □

We now notice the following remark which may be of interest for the reader.

Remark 4.4. *(i) By virtue of* (32), $\mathcal{L}_\lambda(S,T)$ *and* $\mathscr{L}_\lambda(S,T)$ *are also called the weighted logarithmic operator means.*
(ii) A question arises from the above: are $L_\lambda(S,T)$, $\mathcal{L}_\lambda(S,T)$ *and* $\mathscr{L}_\lambda(S,T)$ *equal or different? They are in fact different as confirmed by the example below.*

Example 4.5. *Let us take* $\lambda = 1/3$, $S = I$ *and* $T = 2I$, *where* I *is the identity operator of* H. *Using some simple real integration tools and some numerical computations, we find*

$$\mathcal{L}_{1/3}(S,T) = 3\left(\int_0^1 \frac{2-2t+t^{-1/2}}{1+t}dt\right)^{-1} I = \frac{6}{8\log 2 + \pi - 4} I \simeq 1.2801\, I;$$
$$\mathscr{L}_{1/3}(S,T) = \frac{1}{3}\int_0^1 \left(2-2t+t^{-1/2}\right)2^t dt\, I = \frac{4\log^{\frac{3}{2}}(2)\mathbf{D}(\sqrt{\log 2}) + 2 - \log(4)}{3\log^2(2)} I \simeq 1.2846\, I;$$
$$L_{1/3}(S,T) = \frac{3\sqrt[3]{2}-2}{2\log 2} I \simeq 1.2838\, I.$$

where $\mathbf{D}$ *stands for the Dawson's integral defined by* $\mathbf{D}(x) = \int_0^x \exp(t^2 - x^2)dt$. *Our claim is then confirmed.*

To give another result about the comparison of the previous weighted operator means we need to state the following lemma.

Lemma 4.6. *Let* $\lambda \in (0,1)$ *and consider the real function* F_λ *defined on* $(0,1)$ *by*

$$F_\lambda(t) = (1-\lambda)(1-t)^{\frac{1-2\lambda}{\lambda}} + \lambda t^{\frac{2\lambda-1}{1-\lambda}}.$$

Then, we have
(i) $F_{1/2}(t) = 1$ *for all* $t \in (0,1)$,
(ii) If $\lambda < 1/2$ *then* $t \mapsto F_\lambda(t)$ *is strictly decreasing on* $(0,1)$ *and* $F_\lambda(t) \geq \lambda$ *for all* $t \in (0,1)$,
(iii) If $\lambda > 1/2$ *then* $t \mapsto F_\lambda(t)$ *is strictly increasing on* $(0,1)$ *and* $F_\lambda(t) \geq 1-\lambda$ *for all* $t \in (0,1)$.

Proof. It is a simple exercise of Real Analysis when studying the variations of the function $t \mapsto F_\lambda(t)$ on $(0,1)$. We omit the details here. □

Proposition 4.7. *Let* $S,T \in \mathcal{B}^{+*}(H)$ *and* $\lambda \in [0,1]$. *The following assertions hold:*
(i) If $\lambda < 1/2$ *then* $\lambda^2\, \mathcal{L}_\lambda(S,T) \leq \lambda\, L(S,T) \leq \mathscr{L}_\lambda(S,T)$.
(ii) If $\lambda > 1/2$ *then* $(1-\lambda)^2\, \mathcal{L}_\lambda(S,T) \leq (1-\lambda)\, L(S,T) \leq \mathscr{L}_\lambda(S,T)$.

Proof. We use Lemma 4.6 with definitions of $L(S,T), \mathcal{L}_\lambda(S,T)$ and $\mathscr{L}_\lambda(S,T)$ given, respectively, by (7), (30) and (31). The details are straightforward and therefore omitted here for the reader. □

In what follows, we provide some refinements and estimations of the inequalities (35).

Proposition 4.8. *Let $S,T \in \mathcal{B}^{+*}(H)$. For any $a,\lambda \in [0,1]$, we have*

$$m(a,\lambda)\Big((S!_a T)^{-1} - (S\nabla_a T)^{-1}\Big) \le (S!_\lambda T)^{-1} - \mathcal{L}_\lambda^{-1}(S,T) \le M(a,\lambda)\Big((S!_a T)^{-1} - (S\nabla_a T)^{-1}\Big). \tag{37}$$

Proof. The proof is based on applying (20) to the operator convex function $f(x) = 1/x$ on $(0,+\infty)$. □

Corollary 4.9. *Let $S,T \in \mathcal{B}^{+*}(H)$. For any $\lambda \in [0,1]$ one has*

$$(S!_\lambda T)^{-1}\nabla_{M(\lambda,\lambda)}(S\nabla_\lambda T)^{-1} \le \mathcal{L}_\lambda^{-1}(S,T) \le (S!_\lambda T)^{-1}\nabla_{m(\lambda,\lambda)}(S\nabla_\lambda T)^{-1}. \tag{38}$$

Proof. We take $a=\lambda$ in (37), and by noticing that $m(\lambda,\lambda) = 1-(1-\lambda)\lambda^{\frac{\lambda}{1-\lambda}} - \lambda\Big(1-\lambda\Big)^{\frac{1-\lambda}{\lambda}} \le 1$, we obtain the desired result. □

Remark 4.10. *Using the right inequality of* (38)*, we get the following refinement of the inequalities proved in Proposition 4.3.*

$$S!_\lambda T \le (S!_\lambda T)!_{m(\lambda,\lambda)}(S\nabla_\lambda T) \le \mathcal{L}_\lambda(S,T)$$

Proposition 4.11. *For any $S,T \in \mathcal{B}^{+*}(H)$ and $\lambda \in [0,1]$, we have*

$$\mathcal{L}_\lambda(S,T) \le (S\nabla_\lambda T)!_\lambda\, \mathcal{L}_\lambda(S,T) \le \left[\int_0^1 \mathcal{L}_\lambda^{-1}(S\nabla_\lambda T, S\nabla_x T)d\nu_\lambda(x)\right]^{-1}$$
$$\le \left[\int_0^1 \Big((S\nabla_\lambda T)\nabla_\lambda (S\nabla_x T)\Big)^{-1} d\nu\lambda(x)\right]^{-1} \le S\nabla_\lambda T.$$

Proof. We apply (22) for the operator convex function $f(x) = 1/x$ on $(0,+\infty)$. □

We end this section by stating more inequalities involving some of the previous weighted operator means.

Theorem 4.12. *Let $S,T \in \mathcal{B}^{+*}(H)$ and $s,\lambda \in [0,1]$ the following inequalities hold*

$$S\nabla_\lambda\left(S\sharp_s T\right) \le \int_0^1 S\sharp_s (S\nabla_t T)d\nu_\lambda(t) \le S\sharp_s (S\nabla_\lambda T). \tag{39}$$

Proof. Applying Theorem 2.1 for the operator concave function $f(x) = x^s$ on $(0,+\infty)$ with $s \in [0,1]$, we get

$$A^s\nabla_\lambda B^s \le \int_0^1 (A\nabla_t B)\, d\nu_\lambda(t) \le (A\nabla_\lambda B)^s, \tag{40}$$

for all $A,B \in C_{(0,+\infty)}(H)$.
Since $I, S^{-\frac12}TS^{-\frac12} \in C_{(0,\infty)}(H)$, we can therefore replace in (40) A and B respectively by I and $S^{-\frac12}TS^{-\frac12}$. This leads to the following inequalities

$$I\nabla_\lambda\left(S^{-\frac12}TS^{-\frac12}\right)^s \le \int_0^1 \left(I\nabla_t\left(S^{-\frac12}TS^{-\frac12}\right)\right)d\nu_\lambda(t) \le \left(I\nabla_\lambda\left(S^{-\frac12}TS^{-\frac12}\right)\right)^s.$$

Multiplying the right and left sides of these inequalities by $S^{\frac12}$, we deduce (39). □

Corollary 4.13. *Let $S, T \in \mathcal{B}^{+*}(H)$ and $\lambda \in [0,1]$. Then the following inequalities hold*

$$S\nabla_\lambda \mathscr{L}_\lambda(S,T) \leq \int_0^1 \mathscr{L}(S, S\nabla_t T) d\nu_\lambda(t) \leq \mathscr{L}_\lambda(S, S\nabla_\lambda T).$$

Proof. Multiplying all sides of (39) by $d\nu_\lambda(s)$ and integrating with respect to $s \in [0,1]$, we obtain the desired inequalities. □

Theorem 4.14. *For $S, T \in \mathcal{B}^{+*}(H)$ and $s, a, \lambda \in [0,1]$ the following inequalities hold*

$$m(a,\lambda)\Big(S\nabla_a\left(S\sharp_s T\right) - S\sharp_s\big(S\nabla_a T\big)\Big) \leq S\nabla_\lambda\left(S\sharp_s T\right) - \int_0^1 S\sharp_s\,(S\nabla_t T) d\nu_\lambda(t)$$
$$\leq M(a,\lambda)\Big(S\nabla_a\left(S\sharp_s T\right) - S\sharp_s\big(S\nabla_a T\big)\Big). \quad (41)$$

Proof. Employing Theorem 3.5 for the operator concave function $f(x) = x^s$ on $(0,+\infty)$ with $s \in [0,1]$, we obtain

$$m(a,\lambda)\left(A^s\nabla_a B^s - (A\nabla_a B)^s\right) \leq A^s\nabla_\lambda B^s - \int_0^1 (A\nabla_t B)^s d\nu_\lambda(t) \leq M(a,\lambda)\left(A^s\nabla_a B^s - (A\nabla_a B)^s\right),$$

for all $A, B \in C_{(0,+\infty)}(H)$.
Noticing that $I, S^{-\frac{1}{2}}TS^{-\frac{1}{2}} \in C_{(0,\infty)}(H)$, we can therefore replace A and B respectively by I and $S^{-\frac{1}{2}}TS^{-\frac{1}{2}}$.
So, we get

$$m(a,\lambda)\left(I\nabla_a\left(S^{-\frac{1}{2}}TS^{-\frac{1}{2}}\right)^s - (I\nabla_a\left(S^{-\frac{1}{2}}TS^{-\frac{1}{2}}\right)^s\right) \leq I\nabla_\lambda\left(S^{-\frac{1}{2}}TS^{-\frac{1}{2}}\right)^s - \int_0^1 (I\nabla_t\left(S^{-\frac{1}{2}}TS^{-\frac{1}{2}}\right))^s d\nu_\lambda(t)$$
$$\leq M(a,\lambda)\left(I\nabla_a\left(S^{-\frac{1}{2}}TS^{-\frac{1}{2}}\right)^s - (I\nabla_a\left(S^{-\frac{1}{2}}TS^{-\frac{1}{2}}\right)^s\right).$$

By multiplying the right and left sides of these inequalities by $S^{\frac{1}{2}}$, we get the inequalities (41). □

Corollary 4.15. *Let $S, T \in \mathcal{B}^{+*}(H)$ and $a, \lambda \in [0,1]$. Then we have*

$$m(a,\lambda)\Big(S\nabla_a \mathscr{L}_\lambda(S,T) - \mathscr{L}_\lambda(S, S\nabla_\lambda T)\Big) \leq S\nabla_\lambda \mathscr{L}_\lambda(S,T) - \int_0^1 \mathscr{L}_\lambda(S, S\nabla_t T) d\nu_\lambda(t)$$
$$\leq M(a,\lambda)\Big(S\nabla_a \mathscr{L}_\lambda(S,T) - \mathscr{L}_\lambda(S, S\nabla_a T)\Big).$$

Proof. Multiplying all sides of (41) by $d\nu_\lambda(s)$ and integrating with respect to $s \in [0,1]$, we obtain the desired inequalities. □

Acknowledgment
The authors express their thanks to the anonymous reviewer(s) for the careful and thorough examination of this manuscript and for the insightful comments, which significantly enhanced the manuscript's quality.

References

[1] I. Al-Subaihi and M. Raïssouli, *Further inequalities involving the weighted geometric operator mean and the Heinz operator mean*, Linear Multilinear Algebra, **90/17** (2022), 4432-4454.
[2] V. Bacak and R. Türkmen, *Refinements of Hermite-Hadamard type inequality for operator convex functions*, J. Inequal. Appl. **2013**, 2013:262.
[3] A. Burqan, A. Abu-Snainah, and R. Saadeh, *Improvements of Logarithmic and Identric Mean Inequalities for Scalars and Operators*, J. Applied Mathematics, Hindawi, (2023), 1-7.
[4] S. S. Dragomir, *Bounds of the normalised Jensen's functional*, Bull. Aust. Math.Soc. **74/3** (2006), 471-478.

[5] S. S. Dragomir, *Hermite-Hadamard type inequalities for operator convex functions*, Appl. Math. Computation **218/3** (2011), 766-772.
[6] S. S. Dragomir, *Inequalities of Hermite-Hadamard type for functions of selfadjoint operators and matrices*, J. Math. Inequal. **11/1** (2017), 241-259.
[7] S. S. Dragomir, *Reverse Jensen integral inequalities for operator convex functions in terms of Fréchet derivative*, Bull. Iran. Math. Soc., **47**(2021), 1969-1987,.
[8] F. Hansen, J. Pečarić, and I. Perić, *Jensen's operator inequality and its converses*, Math. Scand, **100/1** (2007), 61-73.
[9] F. Kubo and T. Ando, *Means of positive linear operators*, Mathematische Annalen, **246/3** (1980), 205-224.
[10] R. Pal, M. Singh, M. S. Moslehian, and J. S. Aujla, *A new class of operator monotone functions via operator means*, Linear Multilinear Algebra, **64/12**(2016), 2463-2473.
[11] M. Raïssouli, *Logarithmic functional mean in convex analysis*, J. Inequal. Pure Appl. Math.,**10/4** (2009).
[12] M. Raïssouli, *Functional versions of some refined and reversed operator mean-inequalities*, Bull. Aust. Math. Soc. **96/3** (2017), 496-503.
[13] M. Raïssouli and S. Furuichi, *The logarithmic mean of two convex functionals*, Open Mathematics, **18/1** (2020), 1667-1684.
[14] M. Raïssouli, L. Tarik, M. Chergui and B. El Wahbi, *Further Hermite-Hadamard type inequalities involving operator h-convex functions*, Filomat, **36/18** (2022), 6333-6346.
[15] M. Raïssouli, *On the weighted chaotic identric mean of two accretive matrices*, Bull. Iran. Math. Soc. **48/6** (2022), 3855-3881.
[16] M. Raïssouli, *On the weighted logarithmic mean of accretive matrices*, Filomat, **36/12** (2022), 4185-4194.
[17] Y. Udagawa, T. Yamazaki, and M. Yanagida, *Some properties of weighted operator means and characterizations of interpolational means*, Linear Algebra Appl., **517** (2017), 217-234.