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Ulam-Hyers stability of fractional stochastic differential equations with time-delays and non-instantaneous impulses

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Abstract. In this article, a class of fractional stochastic differential equations (FSDEs) with time-delays and non-instantaneous impulses is investigated. With the help of the Krasnoselskii's fixed point theorem and contraction mapping principle, we derive the existence and uniqueness of the solutions to the purposed system. Subsequently, by virtue of the stochastic analysis techniques and generalized Grönwall inequality, the Ulam-Hyers stability (U-Hs) of the addressed system is established. At last, we present an example to illustrate the theoretical results.

1. Introduction

Fractional differential equations (FDEs) are widely applied in all aspects of applied mathematics in light of its good memory, and a large number of scholars have made great contributions to this subject. For example, in Physics [12], Biology [24], and Engineering [29]. For a systematic knowledge of the fractional differential equations, the reader can see [22, 23, 38] and references therein.

In recent years, many experts have gained a strong interest in fractional impulsive differential equations (FIDEs), it's due to the sudden change of the state of the system at some moments. The pulses were classified as instantaneous and non-instantaneous impulses based on the duration of the mutation or perturbed process. Up to now, some scholars have studied such equations and obtained many interesting results. See also [8–10, 16, 17, 21, 25, 26, 30, 31, 35] and the references therein.

Moreover, we find articles about FIDEs, a huge amount of papers focus on solving the fixed moment impulsive problem. However, in real phenomena, mutations do not always occur at fixed points, normally at random points. The solution of the stochastic FIDEs is a stochastic process, which is fundamentally different from deterministic FIDEs. Currently, some properties [1, 3, 5, 11, 13, 19, 36] of solutions of FSDEs with instantaneous impulses are studied. However, there are few relevant properties of the solutions of fractional stochastic differential systems involving non-instantaneous impulses. Therefore, the research on this topic has great research prospects.

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The research of the existence of fractional differential system solutions is an eternal topic [2–4, 6, 9, 16, 17, 19, 21, 27, 35, 37]. In literature [37], the equations in the following form are investigated thoroughly:

$$\begin{cases} {}_0\mathbb{D}_t^q\mathcal{T}(t) = \sigma(t,\mathcal{T}(t)), t \in U = [0,T], t \neq t_j, j = 1,2,\ldots,m, \\ \Delta\mathcal{T}|_{t=t_j} = I_j(\mathcal{T}(t_j^-)), \quad j = 1,2,\ldots,m, \\ \mathcal{T}(0) = \mathcal{T}_0, \end{cases} \tag{1.1}$$

where ${}^C\mathbb{D}_{0^+}^q$ is the Caputo differential derivative, $q \in (0,1)$, and $\sigma : U \times \mathbb{R} \to \mathbb{R}$ is appropriate continuous function to be specified later. $I_j : \mathbb{R} \to \mathbb{R}(j = 1,2,\ldots,m)$ are appropriate functions, and $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = T$, let $U_j = \left(t_j, t_{j+1}\right], j = 1,2,\ldots,m$, and $U_0 = [0,t_1]$, and $\Delta\mathcal{T}|_{t=t_j} = I_j(\mathcal{T}(t_j^-))$, where $\mathcal{T}(t_j^-)$ and $\mathcal{T}(t_j^+)$ represent the left and right limits of $\mathcal{T}(t)$ at t_j. The existence results of the solutions were obtained by the authors through using the Krasnoselskii's fixed point theorem. Indeed, instantaneous impulses cannot describe certain dynamics of the evolving process. For instance, when treating uremic patients in pharmacy dynamics, the natural accumulation of toxins in the body and the dialysis process can be approximated as a non-instantaneous impulse process.

We notice that Luo et al. studied the following Hyers-Ulam stability of solutions for the FDEs with time-varying delays and non-instantaneous impulses in [17]:

$$\begin{cases} {}^C\mathbb{D}_t^q\mathcal{T}(t) = A\mathcal{T}(t) + B\mathcal{T}(t-\tau(t)) + \sigma(t,\mathcal{T}(t),\mathcal{T}(t-\tau(t))), \\ \qquad\qquad t \in [0,t_1] \cup \left(s_j, t_{j+1}\right], j = 1,2,\ldots,p, \\ \mathcal{T}(t) = \phi_j\left(t,\mathcal{T}(t),\mathcal{T}\left(t_j^-\right)\right), \quad t \in \left(t_j, s_j\right], j = 1,2,\ldots,p, \\ \mathcal{T}\left(s_j^+\right) = \mathcal{T}\left(s_j^-\right) = \mathcal{T}\left(t_j\right), \quad j = 1,2,\ldots,p, \\ \mathcal{T}(t) = \chi(t), \quad t \in [-\tau,0] \end{cases} \tag{1.2}$$

where ${}^C\mathbb{D}_{0^+}^q$ is the Caputo differential derivative, $0 < q < 1$, $U = [-\tau,T]$, let two increasing finite sequences of $\{t_j\}$ and $\{s_j\}$ satisfy the relation $s_0 = 0 < t_1 < s_1 < t_2 < \cdots < s_p < s_{p+1} = T$, for $t \in U$, $0 \leq \tau(t) \leq \tau$, A and B are bounded operators defined on $\mathbb{R}$. Let $\sigma : U \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $\phi_j : U \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be continuous non-instantaneous impulsive functions. Let $\chi(t) \in C([-\tau,0],\mathbb{R})$ and $\|\chi\| := \sup\{|\chi(t)| : -\tau \leq t \leq 0\}$. By utilizing the Krasnoselskii's fixed point theorem, the authors derived the existence conclusions for the solutions of the consider system.

Luo and Zhu [19] investigated the following stochastic FDEs with instantaneous impulses:

$$\begin{cases} {}^C\mathbb{D}_{0^+}^q\mathcal{T}(t) = A\mathcal{T}(t) + \sigma(t,\mathcal{T}(t),\mathcal{T}(t-h)) \\ \qquad\qquad + \varphi(t,\mathcal{T}(t),\mathcal{T}(t-h))\dfrac{dW(t)}{dt}, t \in U = [0,T], t \neq t_j, \\ \mathcal{T}\left(t_j^+\right) = \mathcal{T}\left(t_j^-\right) + I_j\left(\mathcal{T}\left(t_j\right)\right), j = 1,2,\cdots,p, \\ \mathcal{T}(t) = \phi(t), \quad t \in [-h,0], \end{cases} \tag{1.3}$$

where ${}^C\mathbb{D}_{0^+}^q$ is the Caputo differential derivative, $\frac{1}{2} < q < 1$, $A \in \mathbb{R}^m \times \mathbb{R}^m$, $h \in \mathbb{R}^+$ denotes the delay, $\sigma : U \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$, $\varphi : U \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^{m\times d}$ are continuous functions, $W(t)$ is a d-dimensional standard Wiener process, and $\mathcal{T}(t)$ is a $\mathbb{R}^m$-valued random variable represented as $\mathcal{T}(t) = (\mathcal{T}_1(t),\mathcal{T}_2(t),\ldots,\mathcal{T}_m(t))^{\mathrm{T}}$, $\mathcal{T}(t_j^-)$ and $\mathcal{T}(t_j^+)$ are the left and right limits of $\mathcal{T}(t)$ at time t_j, respectively. $\mathcal{T}(t)$ is left continuous at $t = t_j$, let $0 = t_0 < t_1 < t_2 < \cdots < t_p < t_{p+1} = T$ and $\phi(t)$ is history function satisfying $\mathbb{E}(\left|\phi_0\right|^2) < \infty$, where $\phi_0 = \phi(0)$. The authors obtained the existence conclusions of the solutions by using contraction mapping principle and some assumptions.

It is well known that the stability analysis of the solutions is crucial in differential systems. Nevertheless, the analysis of the U-Hs is one of the most interesting themes for fractional systems, see [3, 4, 6, 11, 14, 15, 17, 18, 27, 28, 32]. In detail, in [15, 17, 18], the authors employed the generalized Grönwall's inequality to obtain U-Hs of FDEs. In [11] the authors derived the U-Hs of the Riemann-Liouville fractional neutral functional stochastic differential equation with impulses based on assumptions. In [14, 32], the authors made a thorough inquiry of Ulam-Hyers-Mittag-Leffler stability of fractional delay differential system, by virtue of the Picard operator method and the Grönwall's inequality, obtained the stability results of the considered system. In [28], the U-Hs of the fractional functional differential equations were explored by the authors, and according to the Banach contraction principle, deriving the U-Hs criterion of this equation.

Inspired by the previous analyses, in this paper, let two increasing finite sequences of $\{t_k\}$ and $\{s_k\}$ satisfy the relation $t_0 = s_0 = 0 < t_1 < s_1 < t_2 < \cdots < s_m < t_{m+1} = T$, $m \in \mathbb{N}$. We shall discuss the following FSDEs with time-varying delays and non-instantaneous impulses:

$$\begin{cases} {}^C\mathbb{D}^{\xi}_{0^+} z(v) = Az(v) + \sigma(v, z(v), z(v-h(v))) \\ \quad +\rho(v, z(v), z(v-h(v)))\frac{dW(v)}{dv}, & v \in [0, t_1] \cup (s_k, t_{k+1}], k = 1, 2, \ldots, m, \\ z(v) = \phi_k\left(v, z(v), z(t_k^-)\right), & v \in (t_k, s_k], k = 1, 2, \ldots, m, \\ z\left(s_k^+\right) = z\left(s_k^-\right) = z(s_k), & k = 1, 2, \ldots, m, \\ z(v) = \phi(v), & v \in [-h, 0], \end{cases} \tag{1.4}$$

where ${}^C\mathbb{D}^{\xi}_{0^+}(\cdot)$ is the Caputo differential derivative, $\xi \in (\frac{1}{2}, 1]$, $A \in \mathbb{R}^d \times \mathbb{R}^d$, $K = [0, T]$, for $v \in [-h, T]$, $0 \le h(v) \le h$. Let $\sigma : K \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, $\rho : K \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^{d\times n}$ are continuous measurable functions and $\phi_k \in C(K \times \mathbb{R}^d \times \mathbb{R}^d, \mathbb{R}^d)$ be non-instantaneous impulsive functions, and $z(v)$ is a $\mathbb{R}^d$-valued random variable denoted as $z(v) = (z_1(v), z_2(v), \ldots, z_d(v))^{\mathrm{T}}$, $W(v)$ is a n-dimensional standard Wiener process on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let history function $\phi(v) \in C([-h, 0], \mathbb{R}^d)$ satisfying $E(\left\|\phi_0\right\|^2) < \infty$ (here $E(\cdot)$ denotes the expectation operator) and $z(0) = \phi(0) = \phi_0$.

Remark 1.1. *If $t_k = s_k$, $k = 1, 2, \ldots, m$, then the system (1.4) degenerates into an instantaneous impulsive stochastic differential equation. Under the circumstances, at any point of instantaneous impulse t_k the amount of jump of the solution $z(v)$ is provided by $\Delta_k = \phi_k(v, z(v), z(t_k^-))$.*

Compared with [9, 17, 19, 34, 37], the contributions of this paper are primarily embodied in the following two respects:

(1) In the literature [17, 34, 37], authors discussed the existence and stability of the Caputo type FDEs without random effects, while the system (1.4) has random term. The existence of random effects can more veritably reflect the objective process of change along with it is more generally model.

(2) On account of diverse systems, although the methods employed in the derivation of existence are similar to [17]. But there exist distinctions in the proving process. We utilize the techniques of stochastic analysis to deal with the stochastic part of the system under consideration.

The structure of the article is organized as follows: In Section 2, we shall introduce certain useful definitions, lemmas and theorems for our considerations. In Section 3, we establish the existence and uniqueness results for the purposed Eq. (1.4). Subsequently, in Section 4, the U-Hs result of the consider system is discussed. As an application, we give an illustrative example in Section 5.

2. Preliminaries and solution representation

For any $v \in [-h, T]$, let $\mathbb{L}^2(\Omega) := \mathbb{L}^2(\Omega, \mathcal{F}, \mathbb{P})$ represent the space of all $\mathcal{F}_v$-measurable together with mean square integrable functions $z : \Omega \to \mathbb{R}^d$ with $\|z(v)\|_{\mathbb{L}} := \sqrt{E\|z(v)\|^2} = \sqrt{\sum_{i=1}^d E|z_i(v)|^2}$. The matrix norm is defined as $\|A\| = \sqrt{\lambda_{max}(A^{\mathrm{T}}A)}$. A process $z : [-h, T] \to \mathbb{L}^2(\Omega)$ is referred to as be $\mathcal{F}_v$-adapted if $z(v) \in \mathbb{L}^2(\Omega)$.

Definition 2.1. *([16, 19]) Mittag-Leffler function with two parameter is given by the series:*

$$M_{\mu,\nu}(\zeta) = \sum_{i=0}^{\infty} \frac{\zeta^i}{\Gamma(\mu i + \nu)},$$

where $\zeta \in \mathbb{C}$, $\mathbb{R}(\zeta) > 0$ and $\Gamma(\zeta)$ is the gamma function defined as

$$\Gamma(\zeta) = \int_0^{\infty} e^{-v} v^{\zeta-1} dv,$$

$\mathbb{R}(\zeta) > 0$. *In particular, if $\nu = 1$, two parameter will reduce to one parameter function, in other words, $M_{\mu,1}(\zeta) = M_{\mu}(\zeta)$.*

Definition 2.2. *([30]) The Riemann-Liouville fractional integral operator of order $\xi > 0$ of a continuous function $\sigma : \mathbb{R}^+ \to \mathbb{R}^d$ is given by*

$$\left(\mathbb{I}_{0^+}^{\xi}\sigma\right)(v) = \frac{1}{\Gamma(\xi)} \int_0^v (v-s)^{\xi-1}\sigma(s)ds, \quad v > 0.$$

Definition 2.3. *([19, 30]) The Laplace transform of a function $\sigma(\zeta)$ is defined as*

$$L[\sigma(\zeta)] = \sigma_L(\zeta) = \int_0^{+\infty} e^{-s\zeta}\sigma(\zeta)d\zeta, \quad \zeta \in \mathbb{C},$$

Moreover, Convolution formula of two functions is given by

$$\rho(\zeta) * \sigma(\zeta) = \sigma(\zeta) * \rho(\zeta) = \int_0^{\zeta} \sigma(\zeta - \nu)\rho(\nu)d\nu,$$

where $\rho(\zeta)$ and $\sigma(\zeta)$ are two piecewise continuous functions.

Definition 2.4. *([30]) The Caputo fractional derivative of order $\xi > 0$, $n-1 < \xi < n (n \in \mathbb{N}^+)$ is defined as*

$$\left({}^{C}\mathbb{D}_{0^+}^{\xi}\sigma\right)(v) = \frac{1}{\Gamma(n-\xi)} \int_0^v (v-s)^{n-\xi-1}\sigma^{(n)}(s)ds.$$

Furthermore, when $0 < \xi < 1$, we obtain

$$\mathbb{I}_{0^+}^{\xi}\, {}^{C}\mathbb{D}_{0^+}^{\xi}\sigma(v) = \sigma(v) - \sigma(0).$$

Remark 2.5. *([19]) Laplace transformation of the Caputo fractional differential operator is*

$$L\left\{{}^{C}\mathbb{D}_{0^+}^{\xi}\sigma(\zeta); s\right\} = s^{\xi}L[\sigma(\zeta)] - \sum_{i=0}^{n-1} s^{\xi-i-1} f^{(i)}(0), \quad (n-1 < \xi < n).$$

Lemma 2.6. *([30]) Let $\mathbb{C}$ be complex plane, for any $\mu > 0$, $\nu > 0$ and $B \in \mathbb{C}^{d\times d}$ then*

$$L[t^{\nu-1}M_{\mu,\nu}(Bt^{\mu})] = s^{\mu-\nu}\left(s^{\mu}I - B\right)^{-1}, \quad \mathbb{R}(s) > \|A\|^{\frac{1}{\mu}},$$

holds, where I denotes the identity matrix and $\mathbb{R}(s)$ represents the real part of the complex numbers.

Lemma 2.7. *([20]) (Jensen's Inequality) Let $p \in \mathbb{N}$ and $c_1, c_2, \cdots, c_p$ be nonnegative real numbers, then*

$$\left(\sum_{i=1}^{p} c_i\right)^k \le p^{k-1}\sum_{i=1}^{p} c_i^k, \quad \text{for } k > 1.$$

Lemma 2.8. *([17]) (Generalized Grönwall inequality) Assume that $\beta > 0$, $c \in \mathbb{R}$, $\vartheta(v)$ is a non-negative locally integrable function on $[c, U)$, $\Upsilon(v)$ is a nondecreasing, non-negative continuous function defined on $[c, U)$, $\eta(v)$ is a non-negative locally integrable function on $[c, U)$ satisfying the inequality*

$$\eta(v) \le \vartheta(v) + \Upsilon(v)\int_c^v (v-s)^{\beta-1}\eta(s)ds,$$

then

$$\eta(v) \leq \vartheta(v) + \int_c^v \left[\sum_{k=1}^{\infty} \frac{(\Upsilon(v)\Gamma(\beta))^k}{\Gamma(\beta k)} (v-s)^{\beta k-1} \vartheta(s) \right] ds, \quad \forall t \in [c, U).$$

Furthermore, if $\vartheta(v)$ *is a nondecreasing function on* $[c, U)$*, then*

$$\eta(v) \leq \vartheta(v) M_\beta \left(\Upsilon(v)\Gamma(\beta)(v-c)^\beta \right),$$

where $M_\beta(\cdot)$ *is the Mittag-Leffler function defined by* ***Definition 2.1****.*

Lemma 2.9. *([9]) Let* $\sigma : J \to \mathbb{R}$ *be continuous function and* $0 < \xi < 1$*. A function* $z \in C(J, \mathbb{R})$ *is a solution of the fractional integral equation*

$$z(v) = z_b - \frac{1}{\Gamma(\xi)} \int_0^b (b-s)^{\xi-1} \sigma(s) ds + \frac{1}{\Gamma(\xi)} \int_0^v (v-s)^{\xi-1} \sigma(s) ds,$$

if and only if z *is a solution of the following fractional Cauchy problems*

$$\begin{cases} {}^c\mathbb{D}_{0^+}^{\xi} z(v) = \sigma(v), \quad v \in J, \\ z(b) = z_b, b > 0. \end{cases}$$

In this paper, we define the space as follows:

$U := PC([-h, T], \mathbb{L}^2(\Omega)) = \Big\{ z : [-h, T] \to \mathbb{L}^2(\Omega) : z \in C((t_k, t_{k+1}], \mathbb{L}^2(\Omega)), k = 0, 1, \cdots, m$, and there exist $z\left(t_k^+\right)$ and $z\left(t_k^-\right)$ with $z(t_k) = z\left(t_k^-\right), k = 1, 2, \cdots, m,\ z(v) = \phi(v), v \in [-h, 0] \Big\}$, and $\|\cdot\|_U$ defined by

$$\|z\|_U = \sup_{v \in [-h, T]} \left(\mathrm{E}\|z(v)\|^2 \right) < \infty, z \in U.$$

Clearly, $(U, \|\cdot\|_U)$ is a Banach space.

Lemma 2.10. *An* $\mathcal{F}_v$*-adapted and* $\mathbb{R}^d$*-valued stochastic process* $z(v)$*,* $v \in [-h, T]$*, is a solution of system (1.4) is equivalent to* $z(v)$ *is the solution of the following integral system:*

$$z(v) = \begin{cases} \phi(v), \quad v \in [-h, 0], \\ \phi_0 + \int_0^v (v-s)^{\xi-1} M_{\xi,\xi}(A(v-s)^\xi)[A\phi_0 + \sigma(s, z(s), z(s-h(s)))]ds \\ \quad + \int_0^v (v-s)^{\xi-1} M_{\xi,\xi}(A(v-s)^\xi)\rho(s, z(s), z(s-h(s)))dW(s), v \in [0, t_1], \\ \phi_k(v, z(v), z(t_k^-)), \quad v \in (t_k, s_k], k = 1, 2, \ldots, m, \\ \phi_k(s_k, z(s_k), z(t_k^-)) + \int_0^v (v-s)^{\xi-1} M_{\xi,\xi}(A(v-s)^\xi)[A\phi_0 \\ \quad + \sigma(s, z(s), z(s-h(s)))]ds + \int_0^v (v-s)^{\xi-1} M_{\xi,\xi}(A(v-s)^\xi) \\ \quad \rho(s, z(s), z(s-h(s)))dW(s) - \int_0^{s_k} (s_k-s)^{\xi-1} M_{\xi,\xi}(A(s_k-s)^\xi)[A\phi_0 \\ \quad + \sigma(s, z(s), z(s-h(s)))]ds - \int_0^{s_k} (s_k-s)^{\xi-1} M_{\xi,\xi}(A(s_k-s)^\xi) \\ \quad \rho(s, z(s), z(s-h(s)))dW(s), \quad v \in \bigcup_{k=1}^m (s_k, t_{k+1}]. \end{cases} \tag{2.1}$$

Proof. When $-h < v < 0$, from the problem (1.4), we can easily verify $z(v) = \phi(v)$, for $v \in (-h, 0)$.
Assume z satisfies system (1.4). When $v \in [0, t_1]$, we consider

$$\begin{aligned} {}^C\mathbb{D}_{0^+}^{\xi} z(v) =& Az(v) + \sigma(v, z(v), z(v-h(v))) \\ & + \rho(v, z(v), z(v-h(v))) \frac{dW(v)}{dv}, \quad v \in [0, t_1], \\ \text{with} \quad z(0) =& \phi_0. \end{aligned} \tag{2.2}$$

By employing Laplace transformation in both sides of equation (2.2) with respect to v, we get

$$s^\xi L[z(v)] - s^{\xi-1}[z(0)] = AL[z(v)] + \Delta(s) + \Sigma(s),$$

where $L[z(v)]$, $\Delta(s)$ and $\Sigma(s)$ denotes the Laplace transformation of $z(v)$, $\sigma(v,z(v),z(v-h(v)))$ and $\rho(v,z(v),z(v-h(v)))\frac{dW(v)}{dv}$, respectively. Therefore

$$L[z(v)] = (s^{\xi}I-A)^{-1}s^{\xi-1}[z(0)] + (s^{\xi}I-A)^{-1}\Delta(s) + (s^{\xi}I-A)^{-1}\Sigma(s).$$

From the observation of **Lemma 2.6** and **Definition 2.3**, equation (2.2) is equivalent to

$$\begin{aligned} L[z(v)] =&(s^{\xi}I-A)^{-1}s^{\xi}L[z(0)] + (s^{\xi}I-A)^{-1}\Delta(s) + (s^{\xi}I-A)^{-1}\Sigma(s) \\ =&L[z(0)] + (s^{\xi}I-A)^{-1}(L[Az(0)] + \Delta(s)) + (s^{\xi}I-A)^{-1}\Sigma(s) \\ =&L[z(0)] + L[v^{\xi-1}M_{\xi,\xi}(Av^{\xi})](L[Az(0)] + \Delta(s)) \\ &+ L[v^{\xi-1}M_{\xi,\xi}(Av^{\xi})]\Sigma(s). \end{aligned} \tag{2.3}$$

The convolution theorem of the Laplace transform applied to equation (2.3) gives the form

$$\begin{aligned} L[z(v)] =&L[z(0)] + v^{\xi-1}M_{\xi,\xi}(Av^{\xi}) * [Az(0) + \sigma(v,z(v),z(v-h(v)))] \\ &+ v^{\xi-1}M_{\xi,\xi}(Av^{\xi}) * \rho(v,z(v),z(v-h(v)))\frac{dW(v)}{dv}. \end{aligned}$$

Subsequently, taking inverse Laplace transformation, we get

$$\begin{aligned} z(v) =&z(0) + \int_0^v (v-s)^{\xi-1}M_{\xi,\xi}(A(v-s)^{\xi})\rho(s,z(s),z(s-h(s)))dW(s) \\ &+ \int_0^v (v-s)^{\xi-1}M_{\xi,\xi}(A(v-s)^{\xi})[Az(0) + \sigma(s,z(s),z(s-h(s)))]ds, \end{aligned}$$

when $v \in (t_1, s_1]$, $z(v) = \phi_1(v, z(v), z(t_1^-))$. In addition, when $v \in (s_1, t_2]$, we consider

$$\begin{aligned} {}^{C}\mathbb{D}_{0^+}^{\xi} z(v) =&Az(v) + \sigma(v,z(v),z(v-h(v))) \\ &+ \rho(v,z(v),z(v-h(v)))\frac{dW(v)}{dv}, \quad v \in (s_1, t_2], \\ \text{with} \quad z(s_1) =&\phi_1(s_1, z(s_1), z(t_1^-)). \end{aligned}$$

By **Lemma 2.9** and using the above procedure, we obtain

$$\begin{aligned} z(v) =&\phi_1(s_1, z(s_1), z(t_1^-)) + \int_0^v (v-s)^{\xi-1}M_{\xi,\xi}(A(v-s)^{\xi})[Az(0) \\ &+ \sigma(s,z(s),z(s-h(s)))]ds + \int_0^v (v-s)^{\xi-1}M_{\xi,\xi}(A(v-s)^{\xi}) \\ &\rho(s,z(s),z(s-h(s)))dW(s) - \int_0^{s_1} (s_1-s)^{\xi-1}M_{\xi,\xi}(A(s_1-s)^{\xi})[Az(0) \\ &+ \sigma(s,z(s),z(s-h(s)))]ds - \int_0^{s_1} (s_1-s)^{\xi-1}M_{\xi,\xi}(A(s_1-s)^{\xi}) \\ &\rho(s,z(s),z(s-h(s)))dW(s), \end{aligned}$$

when $v \in (t_2, s_2]$, $z(v) = \phi_2(v, z(v), z(t_2^-))$. When $v \in (s_2, t_3]$, we consider

$$\begin{aligned} {}^{C}\mathbb{D}_{0^+}^{\xi} z(v) =&Az(v) + \sigma(v,z(v),z(v-h(v))) \\ &+ \rho(v,z(v),z(v-h(v)))\frac{dW(v)}{dv}, \quad v \in (s_2, t_3], \\ \text{with} \quad z(s_2) =&\phi_2(s_2, z(s_2), z(t_2^-)). \end{aligned}$$

By **Lemma 2.9** and using the above procedure, we get

$$\begin{aligned} z(v) =&\phi_2(s_2, z(s_2), z(t_2^-)) + \int_0^v (v-s)^{\xi-1} M_{\xi,\xi}(A(v-s)^{\xi})[Az(0) \\ &+ \sigma(s, z(s), z(s-h(s)))]ds + \int_0^v (v-s)^{\xi-1} M_{\xi,\xi}(A(v-s)^{\xi}) \\ &\rho(s, z(s), z(s-h(s)))dW(s) - \int_0^{s_2} (s_2-s)^{\xi-1} M_{\xi,\xi}(A(s_2-s)^{\xi})[Az(0) \\ &+ \sigma(s, z(s), z(s-h(s)))]ds - \int_0^{s_2} (s_2-s)^{\xi-1} M_{\xi,\xi}(A(s_2-s)^{\xi}) \\ &\rho(s, z(s), z(s-h(s)))dW(s), \end{aligned}$$

In general, proceeding like this, when $v \in (s_k, t_{k+1}]$, we consider

$$\begin{aligned} {}^{C}\mathbb{D}_{0^+}^{\xi} z(v) =& Az(v) + \sigma(v, z(v), z(v-h(v))) \\ &+ \rho(v, z(v), z(v-h(v)))\frac{dW(v)}{dv}, \quad v \in (s_k, t_{k+1}], \\ \text{with} \quad z(s_k) =& \phi_k(s_k, z(s_k), z(t_k^-)). \end{aligned}$$

By **Lemma 2.9** and using the above procedure, we have

$$\begin{aligned} z(v) =&\phi_k(s_k, z(s_k), z(t_k^-)) + \int_0^v (v-s)^{\xi-1} M_{\xi,\xi}(A(v-s)^{\xi})[Az(0) \\ &+ \sigma(s, z(s), z(s-h(s)))]ds + \int_0^v (v-s)^{\xi-1} M_{\xi,\xi}(A(v-s)^{\xi}) \\ &\rho(s, z(s), z(s-h(s)))dW(s) - \int_0^{s_k} (s_k-s)^{\xi-1} M_{\xi,\xi}(A(s_k-s)^{\xi})[Az(0) \\ &+ \sigma(s, z(s), z(s-h(s)))]ds - \int_0^{s_k} (s_k-s)^{\xi-1} M_{\xi,\xi}(A(s_k-s)^{\xi}) \\ &\rho(s, z(s), z(s-h(s)))dW(s), \end{aligned}$$

Conversely, by proceeding the steps the proof is similar. Hence, the proof of the **Lemma 2.10** is completed. □

Lemma 2.11. *([33]) Let $\mu > \nu \geq 0$ and $0 < a < 1$, we can get*

$$\mu^a - \nu^a \leq (\mu - \nu)^a.$$

Definition 2.12. *Equation (1.4) is Ulam-Hyers stable with respect to ε if there exists a constant $N > 0$, such that for each $\varepsilon > 0$ and $\chi \in U$ of the following inequality:*

$$\begin{cases} E\|{}^{C}\mathbb{D}_{0^+}^{\xi}\chi(v) - A\chi(v) - \sigma(v, \chi(v), \chi(v-h(v))) \\ \qquad -\rho(v, \chi(v), \chi(v-h(v)))\frac{dW(v)}{dv}\|^2 \leq \varepsilon, & v \in [0, t_1] \cup \bigcup_{k=1}^{m} (s_k, t_{k+1}], \\ E\|\chi(v) - \phi_k(v, \chi(v), \chi(t_k^-))\|^2 \leq \varepsilon, & v \in \bigcup_{k=1}^{m} (t_k, s_k], \\ E\|\chi(v) - \phi(v)\|^2 \leq \varepsilon, & v \in [-h, 0], \end{cases} \tag{2.4}$$

and there is a solution $z \in U$ of system (1.4), satisfies

$$E\|\chi(v) - z(v)\|^2 \leq N\varepsilon, \quad \forall v \in K.$$

Remark 2.13. *A function $\chi(v)$ satisfies (2.4) if and only if there exist functions $H(v)$, $G(v)$ and a sequence $I_k(v)$ such that $E\|H(v)\|^2 \leq \frac{1}{4}\varepsilon$, $E\|G(v)\|^2 \leq \frac{1}{4}\varepsilon$ and $E\|I_k(v)\|^2 \leq \frac{1}{2}\varepsilon$ for all $k = 1, 2, \ldots, m$, $v \in [-h, T]$ and*

$$\begin{cases} {}^{C}\mathbb{D}^{\xi}_{0^+}\chi(v) = A\chi(v) + \sigma(v, \chi(v), \chi(v - h(v))) \\ \qquad\qquad +\rho(v, \chi(v), \chi(v - h(v)))\frac{dW(v)}{dv} + H(v), & v \in [0, t_1] \cup \bigcup\limits_{k=1}^{m} (s_k, t_{k+1}], \\ \chi(v) = \phi_k\left(v, \chi(v), \chi(t_k^-)\right) + I_k(v), & v \in \bigcup\limits_{k=1}^{m} (t_k, s_k], \\ \chi(v) = \phi(v) + G(v), & v \in [-h, 0], \end{cases} \tag{2.5}$$

Theorem 2.14. *([7]) (Krasnoselskii's fixed point theorem) Let $\mathbb{S}$ be a nonempty and closed convex subset of a Banach space $\mathbb{Y}$. Let B, D be operators such that*

(*a*) *$Bx + Dy \in \mathbb{S}$ whenever $x, y \in \mathbb{S}$,*

(*b*) *B is continuous and compact,*

(*c*) *D is a contraction mapping.*

Then there exists $z \in \mathbb{S}$ such that $z = Bz + Dz$.

3. Existence and uniqueness of solutions

In this section, we derive the existence and uniqueness of the solutions to the consider system (1.4) based on the Krasnoselskii's fixed point theorem and contraction mapping principle.

In order that our work to proceed smoothly, we make the following assumptions:

$\diamond$ (A_0) For σ, ρ in system (1.4), for all $\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4 \in U$ and $v \in [-h, T]$, there exists a positive continuous function $l(v)$ satisfying

$$\begin{aligned} &\|\sigma(v, \vartheta_1, \vartheta_2) - \sigma(v, \vartheta_3, \vartheta_4)\|^2 \vee \|\rho(v, \vartheta_1, \vartheta_2) - \rho(v, \vartheta_3, \vartheta_4)\|^2 \\ &\qquad\qquad \leq l(v)(\|\vartheta_1 - \vartheta_3\|^2 + \|\vartheta_2 - \vartheta_4\|^2), \end{aligned}$$

where $\sup\limits_{v\in[-h,T]} l(v) = \Lambda$ and $\sigma(v, 0, 0) = \rho(v, 0, 0) = 0$, $k = 1, 2, \ldots, m$.

$\diamond$ (A_1) For impulsive functions $\phi_k(v)(k = 1, 2, \ldots, m)$ and for all $\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4 \in U$, there is a positive sequence λ_k such that

$$\|\phi_k(v, \vartheta_1, \vartheta_2) - \phi_k(v, \vartheta_3, \vartheta_4)\|^2 \leq \lambda_k(\|\vartheta_1 - \vartheta_3\|^2 + \|\vartheta_2 - \vartheta_4\|^2),$$

and for all $\vartheta_1, \vartheta_2 \in U$, $\|\phi_k(v, \vartheta_1, \vartheta_2)\|^2 \leq d_k$ holds, where d_k is a positive constant.

$\diamond$ (A_2) History function ϕ satisfy Lipschitz condition. Namely, for all $v_1, v_2 \in [-h, 0]$, there exists a positive constant L such that

$$\|\phi(v_1) - \phi(v_2)\|^2 \leq L\|v_1 - v_2\|^2.$$

In this paper, we suppose that

$$\lambda = \max\{\lambda_k : k = 1, 2, \ldots, m\},$$

and

$$M_1 = \max_{v\in[-h,T]} |M_{\xi,\xi}(Av^{\xi})|.$$

Theorem 3.1. *Suppose that assumptions $(A_0) - (A_2)$ hold, $\lambda \in (0, \frac{1}{2})$ and for each $k = 1, 2, \ldots, m$, then system (1.4) has at least one solution in $B_q := \{z \in U : \|z\|_U \leq q\} \subseteq U$ provided that*

$$q \geq \max\left\{S_1, S_2, \|\phi\|_U\right\} \tag{3.1}$$

holds, where

$$S_1 = \frac{4((2\xi-1)+\|A\|^2M_1^2T^{2\xi})\|\phi_0\|_U}{(2\xi-1)-8\Lambda(T+1)T^{2\xi-1}M_1^2},$$

and

$$S_2 = \frac{7(2\xi-1)d_k+14\|A\|^2M_1^2T^{2\xi}\|\phi_0\|_U}{(2\xi-1)-28\Lambda(T+1)T^{2\xi-1}M_1^2}.$$

Proof. For a positive number q, it is easy to verify that B_q is a closed, convex and bounded set of U. Define the operators Ω on B_q as

$$(\Omega z)(v) = \begin{cases} \phi(v), \quad v\in[-h,0], \\ \phi_0+\int_0^v (v-s)^{\xi-1}M_{\xi,\xi}(A(v-s)^\xi)[A\phi_0+\sigma(s,z(s),z(s-h(s)))]ds \\ \quad +\int_0^v (v-s)^{\xi-1}M_{\xi,\xi}(A(v-s)^\xi)\rho(s,z(s),z(s-h(s)))dW(s), v\in[0,t_1], \\ \phi_k(v,z(v),z(t_k^-)), \quad v\in(t_k,s_k], k=1,2,\dots,m, \\ \phi_k(s_k,z(s_k),z(t_k^-))+\int_0^v (v-s)^{\xi-1}M_{\xi,\xi}(A(v-s)^\xi)[A\phi_0 \\ \quad +\sigma(s,z(s),z(s-h(s)))]ds+\int_0^v (v-s)^{\xi-1}M_{\xi,\xi}(A(v-s)^\xi) \\ \quad \rho(s,z(s),z(s-h(s)))dW(s)-\int_0^{s_k}(s_k-s)^{\xi-1}M_{\xi,\xi}(A(s_k-s)^\xi)[A\phi_0 \\ \quad +\sigma(s,z(s),z(s-h(s)))]ds-\int_0^{s_k}(s_k-s)^{\xi-1}M_{\xi,\xi}(A(s_k-s)^\xi) \\ \quad \rho(s,z(s),z(s-h(s)))dW(s), \quad v\in(s_k,t_{k+1}], k=1,2,\dots,m. \end{cases} \tag{3.2}$$

We have divided the proof into 4 steps:
Step 1: We shall show that Ω maps B_q into B_q. For this, let $z\in B_q$.

Case 1. For any $v\in[-h,0]$, from (3.1) and (3.2), yields

$$\|\Omega z\|_U = E\|(\Omega z)(v)\|^2 = E\|\phi\|^2 = \|\phi\|_U.$$

Case 2. For $v\in[0,t_1]$, by means of the **Lemma 2.7**, Hölder inequality, Itô's isometry and Assumption (A_0), we consider

$$\begin{aligned} & E\|(\Omega z)(v)\|^2 \\ \le\ & 4E\|\phi_0\|^2+4E\left\|\int_0^v (v-s)^{\xi-1}M_{\xi,\xi}(A(v-s)^\xi)A\phi_0 ds\right\|^2 \\ & +4E\left\|\int_0^v (v-s)^{\xi-1}M_{\xi,\xi}(A(v-s)^\xi)\sigma(s,z(s),z(s-h(s)))ds\right\|^2 \\ & +4E\left\|\int_0^v (v-s)^{\xi-1}M_{\xi,\xi}(A(v-s)^\xi)\rho(s,z(s),z(s-h(s)))dW(s)\right\|^2 \\ \le\ & 4\|\phi_0\|^2+\frac{4T\|A\|^2M_1^2\|\phi_0\|^2v^{2\xi-1}}{2\xi-1} \\ & +4M_1^2T\int_0^v (v-s)^{2\xi-2}(l(s)(E\|z(s)\|^2+E\|z(s-h(s))\|^2)ds \\ & +4M_1^2\int_0^v (v-s)^{2\xi-2}(l(s)(E\|z(s)\|^2+E\|z(s-h(s))\|^2)ds \\ \le\ & 4\left(1+\frac{\|A\|^2M_1^2T^{2\xi}}{2\xi-1}\right)\|\phi_0\|^2+\frac{8\Lambda M_1^2(T+1)T^{2\xi-1}}{2\xi-1}q. \end{aligned}$$

Taking supremum norm in both sides of the above inequality, by virtue of (3.1), we get

$$\|\Omega z\|_U \leq 4\left(1+\frac{\|A\|^2 M_1^2 T^{2\xi}}{2\xi-1}\right)\|\phi_0\|_U+\frac{8\Lambda M_1^2(T+1)T^{2\xi-1}}{2\xi-1}q \leq q.$$

Case 3. For $v \in (t_k, s_k]$, $k = 1, 2, \ldots, m$. By (A_1) and (3.1), we obtain

$$\|\Omega z\|_U = \|\phi_k(v, z(v), z(t_k^-))\|_U \leq d_k \leq q.$$

Case 4. For $v \in (s_k, t_{k+1}]$, $k = 1, 2, \ldots, m$. By virtue of the **Lemma 2.7**, Cauchy-Schwarz inequality, Itô's isometry and Assumption $(A_0) - (A_1)$, we consider

$$\begin{aligned} E\|(\Omega z)(v)\|^2 \leq& 7d_k + \frac{14T\|A\|^2 M_1^2 \|\phi_0\|^2 v^{2\xi-1}}{2\xi-1} + \frac{28\Lambda(T+1)M_1^2 v^{2\xi-1} q}{2\xi-1} \\ \leq& 7d_k + \frac{14\|A\|^2 M_1^2 \|\phi_0\|^2 T^{2\xi}}{2\xi-1} + \frac{28\Lambda(T+1)M_1^2 T^{2\xi-1}}{2\xi-1} q. \end{aligned}$$

Taking supremum norm in both sides of the above inequality, by virtue of (3.1), we have

$$\|\Omega z\|_U \leq 7d_k + \frac{14\|A\|^2 M_1^2 T^{2\xi}}{2\xi-1}\|\phi_0\|_U + \frac{28\Lambda(T+1)M_1^2 T^{2\xi-1}}{2\xi-1} q \leq q.$$

Hence, for any $v \in [-h, T]$, we can figure out that

$$\|\Omega z\|_U \leq q.$$

Consequently, Ω maps B_q into B_q.

Now, define the operators Ξ and Π as follows

$$(\Xi z)(v) = \begin{cases} 0, & v \in [-h, 0], \\ \phi_0, & v \in [0, t_1], \\ \phi_k(v, z(v), z(t_k^-)), & v \in (t_k, s_k], k = 1, 2, \ldots, m, \\ \phi_k(s_k, z(s_k), z(t_k^-)), & v \in (s_k, t_{k+1}], k = 1, 2, \ldots, m. \end{cases}$$

and

$$(\Pi z)(v) \\ = \begin{cases} \phi(v), \quad v \in [-h, 0], \\ \int_0^v (v-s)^{\xi-1} M_{\xi,\xi}(A(v-s)^\xi)[A\phi_0 + \sigma(s, z(s), z(s-h(s)))]ds \\ \quad + \int_0^v (v-s)^{\xi-1} M_{\xi,\xi}(A(v-s)^\xi)\rho(s, z(s), z(s-h(s)))dW(s), v \in [0, t_1], \\ 0, \quad v \in (t_k, s_k], k = 1, 2, \ldots, m, \\ \int_0^v (v-s)^{\xi-1} M_{\xi,\xi}(A(v-s)^\xi)[A\phi_0 + \sigma(s, z(s), z(s-h(s)))]ds \\ \quad + \int_0^v (v-s)^{\xi-1} M_{\xi,\xi}(A(v-s)^\xi)\rho(s, z(s), z(s-h(s)))dW(s) \\ \quad - \int_0^{s_k} (s_k-s)^{\xi-1} M_{\xi,\xi}(A(s_k-s)^\xi)[\sigma(s, z(s), z(s-h(s))) \\ \quad + A\phi_0]ds - \int_0^{s_k} (s_k-s)^{\xi-1} M_{\xi,\xi}(A(s_k-s)^\xi) \\ \quad \rho(s, z(s), z(s-h(s)))dW(s), \quad v \in (s_k, t_{k+1}], k = 1, 2, \ldots, m. \end{cases}$$

Obviously, we can know that $\Omega = \Xi + \Pi$.

Step 2: Ξ is a contracting map on B_q. For this, let $z \in B_q$.

Case 1. For any $v \in [-h, 0]$, for each $z_1, z_2 \in B_q$, we obtain

$$\|\Xi z_1 - \Xi z_2\|_U = 0.$$

Case 2. For any $v \in [0, t_1]$, for each $z_1, z_2 \in B_q$, one can get

$$\|\Xi z_1 - \Xi z_2\|_U = 0.$$

Case 3. For any $v \in (t_k, s_k]$, $k = 1, 2, \ldots, m$, for each $z_1, z_2 \in B_q$, and by (A_1), we have

$$\|\Xi z_1 - \Xi z_2\|_U \leq 2\lambda_k \|z_1 - z_2\|_U \leq 2\lambda \|z_1 - z_2\|_U.$$

Case 4. For any $v \in (s_k, t_{k+1}]$, $k = 1, 2, \ldots, m$, for each $z_1, z_2 \in B_q$, and by (A_1), we get

$$\|\Xi z_1 - \Xi z_2\|_U \leq 2\lambda_k \|z_1 - z_2\|_U \leq 2\lambda \|z_1 - z_2\|_U.$$

Hence, for any $v \in [-h, T]$, we can conclude that

$$\|\Xi z_1 - \Xi z_2\|_U \leq 2\lambda \|z_1 - z_2\|_U.$$

Since, $0 < \lambda < \frac{1}{2}$, which implies that Ξ is a contracting operator.

Step 3: Obviously, from the **step 1** we can figure out that Π is uniformly bounded on B_q. Next, we prove that Π is continuous. For this let $\{z_n\}$ be a sequence in B_q satisfying $z_n \to z$ $(n \to +\infty)$ in U.

Case 1. For any $v \in [-h, 0]$, one can obtain

$$\|\Pi z_n - \Pi z\|_U = 0.$$

Case 2. For any $v \in [0, t_1]$, with the help of the **Lemma 2.7**, Hölder inequality, Itô's isometry and Assumption (A_0), we consider

$$\begin{aligned}
&E\|(\Pi z_n)(v) - (\Pi z)(v)\|^2 \\
\leq &2E\left\|\int_0^v (v-s)^{\xi-1} M_{\xi,\xi}(A(v-s)^\xi)\,[\sigma(s, z_n(s), z_n(s-h(s))) \right.\\
&\left. - \sigma(s, z(s), z(s-h(s)))]ds\right\|^2 + 2E\left\|\int_0^v (v-s)^{\xi-1} M_{\xi,\xi}(A(v-s)^\xi) \right.\\
&\left. [\rho(s, z_n(s), z_n(s-h(s))) - \rho(s, z(s), z(s-h(s)))]dW(s)\right\|^2 \\
\leq &2\Lambda M_1^2 (T+1)\int_0^v (v-s)^{2\xi-2}(E\|z_n(s) - z(s)\|^2 \\
&+ E\|z_n(s-h(s)) - z(s-h(s))\|^2)ds \\
\leq &4\Lambda M_1^2 (T+1)\|z_n - z\|_U \int_0^v (v-s)^{2\xi-2} ds \\
\leq &\frac{4\Lambda M_1^2 (T+1) T^{2\xi-1}}{2\xi - 1}\|z_n - z\|_U.
\end{aligned}$$

Therefore, we have

$$\|\Pi z_n - \Pi z\|_U \leq \frac{4\Lambda M_1^2 (T+1) T^{2\xi-1}}{2\xi - 1}\|z_n - z\|_U \to 0.$$

Case 3. For $v \in (t_k, s_k]$, $k = 1, 2, \ldots, m$, we get

$$\|\Pi z_n - \Pi z\|_U = 0.$$

Case 4. For $v \in (s_k, t_{k+1}]$, $k = 1, 2, \ldots, m$. Similar to case 2 of **Step 3**, we consider

$$E\|(\Pi z_n)(v) - (\Pi z)(v)\|^2$$

$$\leq 4E\left\|\int_0^v (v-s)^{\xi-1}M_{\xi,\xi}(A(v-s)^{\xi})\,[\sigma(s,z_n(s),z_n(s-h(s)))$$
$$-\sigma(s,z(s),z(s-h(s)))]ds\right\|^2 + 4E\left\|\int_0^v (v-s)^{\xi-1}M_{\xi,\xi}(A(v-s)^{\xi})$$
$$[\rho(s,z_n(s),z_n(s-h(s)))-\rho(s,z(s),z(s-h(s)))]dW(s)\right\|^2$$
$$+ 4E\left\|\int_0^{s_k} (s_k-s)^{\xi-1}M_{\xi,\xi}(A(s_k-s)^{\xi})\,[\sigma(s,z_n(s),z_n(s-h(s)))$$
$$-\sigma(s,z(s),z(s-h(s)))]ds\right\|^2 + 4E\left\|\int_0^{s_k} (s_k-s)^{\xi-1}M_{\xi,\xi}(A(s_k-s)^{\xi})$$
$$[\rho(s,z_n(s),z_n(s-h(s)))-\rho(s,z(s),z(s-h(s)))]dW(s)\right\|^2$$
$$\leq 16\Lambda M_1^2(T+1)\|z_n-z\|_U\int_0^v (v-s)^{2\xi-2}ds$$
$$\leq \frac{16\Lambda M_1^2(T+1)T^{2\xi-1}}{2\xi-1}\|z_n-z\|_U.$$

Thus, we get

$$\|\Pi z_n-\Pi z\|_U \leq \frac{16\Lambda M_1^2(T+1)T^{2\xi-1}}{2\xi-1}\|z_n-z\|_U \to 0.$$

Here, we can figure out that $\|\Pi z_n-\Pi z\|_U \to 0$ as $n\to+\infty$.

Step 4: we shall show that Π maps bounded set into equicontinuous set of B_q.

Case 1. For each $z(v)\in B_q$, v_1, $v_2\in[-h,0]$ with $v_1<v_2$, and by assumption (A_2), one can get

$$\|(\Pi z)(v_2)-(\Pi z)(v_1)\|_U \leq L\|v_2-v_1\|^2 \to 0, \quad v_2\to v_1.$$

Case 2. For each $z(v)\in B_q$, v_1, $v_2\in[0,t_1]$ with $v_1<v_2$, by means of the **Lemma 2.7**, Cauchy-Schwarz inequality, It$\hat{o}$'s isometry, Assumption (A_0) and **Lemma 2.11**, we consider

$$E\|(\Pi z)(v_2)-(\Pi z)(v_1)\|^2$$
$$\leq 6E\left\|\int_0^{v_1} M_{\xi,\xi}(A(v-s)^{\xi})A\phi_0[(v_2-s)^{\xi-1}-(v_1-s)^{\xi-1}]ds\right\|^2$$
$$+ 6E\left\|\int_{v_1}^{v_2} (v_2-s)^{\xi-1}M_{\xi,\xi}(A(v-s)^{\xi})A\phi_0 ds\right\|^2$$
$$+ 6E\left\|\int_0^{v_1} M_{\xi,\xi}(A(v-s)^{\xi})\sigma(s,z(s),z(s-h(s)))\right.$$
$$\left.\times[(v_2-s)^{\xi-1}-(v_1-s)^{\xi-1}]ds\right\|^2$$
$$+ 6E\left\|\int_{v_1}^{v_2} (v_2-s)^{\xi-1}M_{\xi,\xi}(A(v-s)^{\xi})\sigma(s,z(s),z(s-h(s)))ds\right\|^2$$
$$+ 6E\left\|\int_0^{v_1} M_{\xi,\xi}(A(v-s)^{\xi})\rho(s,z(s),z(s-h(s)))\right.$$
$$\left.\times[(v_2-s)^{\xi-1}-(v_1-s)^{\xi-1}]dW(s)\right\|^2$$
$$+ 6E\left\|\int_{v_1}^{v_2} (v_2-s)^{\xi-1}M_{\xi,\xi}(A(v-s)^{\xi})\rho(s,z(s),z(s-h(s)))dW(s)\right\|^2$$

$$
\begin{aligned}
&\leq 6M_1^2T\|A\|^2\|\phi_0\|_U\int_0^{v_1}[(v_2-s)^{\xi-1}-(v_1-s)^{\xi-1}]^2ds\\
&\quad+\frac{6M_1^2T\|A\|^2\|\phi_0\|_U}{2\xi-1}(v_2-v_1)^{2\xi-1}\\
&\quad+\frac{12\Lambda M_1^2(T+1)q}{2\xi-1}(v_2-v_1)^{2\xi-1}\\
&\quad+12\Lambda M_1^2(T+1)q\int_0^{v_1}[(v_2-s)^{\xi-1}-(v_1-s)^{\xi-1}]^2ds\\
&\leq\frac{6M_1^2T\|A\|^2\|\phi_0\|_U}{2\xi-1}\left(v_2^{2\xi-1}-v_1^{2\xi-1}-(v_2-v_1)^{2\xi-1}\right)\\
&\quad+\frac{6M_1^2T\|A\|^2\|\phi_0\|_U}{2\xi-1}(v_2-v_1)^{2\xi-1}\\
&\quad+\frac{12\Lambda M_1^2(T+1)q}{2\xi-1}(v_2-v_1)^{2\xi-1}\\
&\quad+\frac{12\Lambda M_1^2(T+1)q}{2\xi-1}\left(v_2^{2\xi-1}-v_1^{2\xi-1}-(v_2-v_1)^{2\xi-1}\right)\\
&\leq\frac{6M_1^2(T\|A\|^2\|\phi_0\|_U+2\Lambda q+2T\Lambda q)}{2\xi-1}(v_2-v_1)^{2\xi-1}.
\end{aligned}
$$

Hence, we obtain

$$
\begin{aligned}
&\|(\Pi z)(v_2)-(\Pi z)(v_1)\|_U\\
&\leq\frac{6M_1^2(T\|A\|^2\|\phi_0\|_U+2\Lambda q+2T\Lambda q)}{2\xi-1}(v_2-v_1)^{2\xi-1}\to 0, v_2\to v_1.
\end{aligned}
$$

Case 3. For each $z(v)\in B_q$, $v_1, v_2\in(t_k,s_k](k=1,2,\ldots,m)$ with $v_1<v_2$, we get

$$\|(\Pi z)(v_2)-(\Pi z)(v_1)\|_U=0.$$

Case 4. For each $z(v)\in B_q$, $v_1, v_2\in(s_k,t_{k+1}](k=1,2,\ldots,m)$ with $v_1<v_2$. Same result to Case 2 of **Step 4**. Therefore, we have

$$
\begin{aligned}
&\|(\Pi z)(v_2)-(\Pi z)(v_1)\|_U\\
&\leq\frac{6M_1^2(T\|A\|^2\|\phi_0\|_U+2\Lambda q+2T\Lambda q)}{2\xi-1}(v_2-v_1)^{2\xi-1}\to 0, v_2\to v_1.
\end{aligned}
$$

Clearly, Π is equicontinuous.

By means of the Arzela-Ascoli Theorem together with collecting the **Step 3** and **4**, it is easy to know that Π is compact and continuous. According to **Theorem 2.14**, Namely, Krasnoselskii's fixed point theorem, hence, Ω has at least one fixed point. Therefore, the Theorem is proved. □

Theorem 3.2. *Assume that assumptions* $(A_0)-(A_1)$ *holds and* $\lambda\in(0,\frac{1}{2})$, *then the problem* (1.4) *has a unique solution in U provided that*

$$0\leq 10\lambda+\frac{20\Lambda M_1^2(T+1)T^{2\xi-1}}{2\xi-1}<1$$

Proof. Let $z(v), w(v)\in U$ and $z(v)=w(v)=\phi(v)$, $v\in[-h,0]$, we consider

Case 1. For each $v\in[-h,0]$, from the operator expression (3.2), yields

$$\|\Omega z-\Omega w\|_U=0,$$

which indicates that Ω is the contraction operator.

Case 2. For any $v \in [0, t_1]$, we consider

$$\begin{aligned}
&E\|(\Omega z)(v) - (\Omega w)(v)\|^2 \\
\leq& 2E\left\|\int_0^v (v-s)^{\xi-1}M_{\xi,\xi}(A(v-s)^{\xi})\left[\sigma(s, z(s), z(s-h(s)))\right.\right. \\
&\left.\left. - \sigma(s, w(s), w(s-h(s)))\right]ds\right\|^2 + 2E\left\|\int_0^v (v-s)^{\xi-1}M_{\xi,\xi}(A(v-s)^{\xi})\right. \\
&\left.[\rho(s, z(s), z(s-h(s))) - \rho(s, w(s), w(s-h(s)))]dW(s)\right\|^2 \\
\leq& 2\Lambda M_1^2(T+1)\int_0^v (v-s)^{2\xi-2}(E\|z(s)-w(s)\|^2 \\
&+ E\|z(s-h(s)) - w(s-h(s))\|^2)ds \\
\leq& \frac{4\Lambda M_1^2(T+1)T^{2\xi-1}}{2\xi-1}\|z-w\|_U.
\end{aligned}$$

Hence, we have

$$\|\Omega z - \Omega w\|_U \leq \frac{4\Lambda M_1^2(T+1)T^{2\xi-1}}{2\xi-1}\|z-w\|_U < \|z-w\|_U,$$

we can readily know Ω is the contraction operator.

Case 3. For $v \in (t_k, s_k]$, $k = 1, 2, \ldots, m$, we get

$$\|\Omega z - \Omega w\|_U \leq 2\lambda\|z-w\|_U < \|z-w\|_U,$$

we can figure out that Ω is the contraction operator.

Case 4. For $v \in (s_k, t_{k+1}]$, $k = 1, 2, \ldots, m$, we consider

$$\begin{aligned}
&E\|(\Omega z)(v) - (\Omega w)(v)\|^2 \\
\leq& 5E\|\phi_k(s_k, z(s_k), z(t_k^-)) - \phi_k(s_k, w(s_k), w(t_k^-))\|^2 \\
&+ 5E\left\|\int_0^v (v-s)^{\xi-1}M_{\xi,\xi}(A(v-s)^{\xi})\left[\sigma(s, z(s), z(s-h(s)))\right.\right. \\
&\left.\left. - \sigma(s, w(s), w(s-h(s)))\right]ds\right\|^2 + 5E\left\|\int_0^v (v-s)^{\xi-1}M_{\xi,\xi}(A(v-s)^{\xi})\right. \\
&\left.[\rho(s, z(s), z(s-h(s))) - \rho(s, w(s), w(s-h(s)))]dW(s)\right\|^2 \\
&+ 5E\left\|\int_0^{s_k} (s_k-s)^{\xi-1}M_{\xi,\xi}(A(s_k-s)^{\xi})\left[\sigma(s, z(s), z(s-h(s)))\right.\right. \\
&\left.\left. - \sigma(s, w(s), w(s-h(s)))\right]ds\right\|^2 + 5E\left\|\int_0^{s_k} (s_k-s)^{\xi-1}M_{\xi,\xi}(A(s_k-s)^{\xi})\right. \\
&\left.[\rho(s, z(s), z(s-h(s))) - \rho(s, w(s), w(s-h(s)))]dW(s)\right\|^2 \\
\leq& 5\lambda_k\left(E\|z(s_k) - w(s_k)\|^2 + E\|z(t_k^-) - w(t_k^-)\|^2\right) \\
&+ 10\Lambda M_1^2(T+1)\int_0^v (v-s)^{2\xi-2}E\|z(s)-w(s)\|^2 ds \\
&+ 10\Lambda M_1^2(T+1)\int_0^{s_k} (s_k-s)^{2\xi-2}E\|z(s)-w(s)\|^2 ds
\end{aligned}$$

$$\leq\left(10\lambda+\frac{20\Lambda M_1^2(T+1)T^{2\xi-1}}{2\xi-1}\right)E\|z(v)-w(v)\|^2.$$

Hence,we obtain

$$\|\Omega z-\Omega w\|_U\leq\left(10\lambda+\frac{20\Lambda M_1^2(T+1)T^{2\xi-1}}{2\xi-1}\right)\|z-w\|_U<\|z-w\|_U,$$

we can easily verify that Ω is the contraction operator.

Clearly, we can conclude that the operator Ω has a unique fixed point $z(v)\in U$ based on the Banach contraction principle. Thus, the Theorem is proved. □

4. Ulam-Hyers stability results

In the section, we study the U-Hs of the solution for the problem (1.4). We can derive that $\chi(v)$ is the solution of (2.5) if $\chi(v)$ satisfies the following

$$\chi(v)=\begin{cases}\phi(v)+G(v), \quad v\in[-h,0],\\ \phi_0+\int_0^v(v-s)^{\xi-1}M_{\xi,\xi}(A(v-s)^{\xi})[A\phi_0+H(s)+\sigma(s,z(s),z(s-h(s)))]ds\\ \quad+\int_0^v(v-s)^{\xi-1}M_{\xi,\xi}(A(v-s)^{\xi})\rho(s,z(s),z(s-h(s)))dW(s), v\in[0,t_1],\\ \phi_k(v,z(v),z(t_k^-))+I_k(v), \quad v\in(t_k,s_k], k=1,2,\dots,m,\\ \phi_k(s_k,z(s_k),z(t_k^-))+I_k(s_k)+\int_0^v(v-s)^{\xi-1}M_{\xi,\xi}(A(v-s)^{\xi})[A\phi_0\\ \quad+H(s)+\sigma(s,z(s),z(s-h(s)))]ds+\int_0^v(v-s)^{\xi-1}M_{\xi,\xi}(A(v-s)^{\xi})\\ \quad+\rho(s,z(s),z(s-h(s)))dW(s)-\int_0^{s_k}(s_k-s)^{\xi-1}M_{\xi,\xi}(A(s_k-s)^{\xi})[A\phi_0\\ \quad+H(s)+\sigma(s,z(s),z(s-h(s)))]ds-\int_0^{s_k}(s_k-s)^{\xi-1}M_{\xi,\xi}(A(s_k-s)^{\xi})\\ \quad\rho(s,z(s),z(s-h(s)))dW(s), \quad v\in(s_k,t_{k+1}], k=1,2,\dots,m.\end{cases} \tag{4.1}$$

Theorem 4.1. *Assume the validity of conditions $0<\lambda<\frac{1}{2}$ and $(A_0)-(A_1)$, then the addressed systems (1.4) is Ulam-Hyers stable on K.*

Proof. By equation (2.1) and equation (4.1), we derive

Case 1. For each $v\in[-h,0]$, one can get

$$E\|\chi(v)-z(v)\|^2=E\|G(v)\|^2\leq\frac{1}{4}\varepsilon<\varepsilon.$$

which indicates that the system (1.4) is U-H stable.

Case 2. For each $v\in[0,t_1]$, and based on assumption (A_0), we have

$$\begin{aligned}&E\|\chi(v)-z(v)\|^2\\ \leq&4E\|G(0)\|^2+4M_1^2T\int_0^v(v-s)^{2\xi-2}E\|H(s)\|^2ds\\ &+4M_1^2(T+1)\int_0^v(v-s)^{2\xi-2}l(s)\Big(E\|\chi(s)-z(s)\|^2\\ &+E\|\chi(s-h(s))-z(v-h(s))\|^2\Big)ds\\ \leq&\varepsilon+\frac{M_1^2Tv^{2\xi-1}}{2\xi-1}\varepsilon+4M_1^2(T+1)\int_0^v(v-s)^{2\xi-2}l(s)\Big(E\|\chi(s)-z(s)\|^2\\ &+E\|\chi(s-h(s))-z(v-h(s))\|^2\Big)ds.\end{aligned}$$

Let us set $\Theta(v) = E\|\chi(v) - z(v)\|^2$, then we obtain

$$\Theta(v) \le \varepsilon + \frac{M_1^2 T v^{2\xi-1}}{2\xi - 1}\varepsilon + 4\Lambda M_1^2(T+1)\int_0^v (v-s)^{2\xi-2}\left(\Theta(s) + \Theta(s-h(s))\right)ds. \tag{4.2}$$

We let $\eta(v) = \sup_{\theta\in[-h,v]} \Theta(\theta)$, $\forall v \in [0, t_1]$, then $\Theta(s) \le \eta(s)$ and $\Theta(s - h(s)) \le \eta(s)$, $\forall s \in [0, v]$. According to (4.2), for $\xi \in (\frac{1}{2}, 1)$, we have

$$\begin{aligned}\Theta(v) &\le \varepsilon + \frac{M_1^2 T v^{2\xi-1}}{2\xi - 1}\varepsilon + 8\Lambda M_1^2(T+1)\int_0^v (v-s)^{2\xi-2}\eta(s)ds \\ &= \varepsilon + \frac{M_1^2 T v^{2\xi-1}}{2\xi - 1}\varepsilon + 8\Lambda M_1^2(T+1)\int_0^v s^{2\xi-2}\eta(v-s)ds.\end{aligned}$$

Note that $\forall\theta \in [0, v]$, we have

$$\begin{aligned}\Theta(\theta) &\le \varepsilon + \frac{M_1^2 T \theta^{2\xi-1}}{2\xi - 1}\varepsilon + 8\Lambda M_1^2(T+1)\int_0^\theta s^{2\xi-2}\eta(\theta-s)ds \\ &\le \varepsilon + \frac{M_1^2 T v^{2\xi-1}}{2\xi - 1}\varepsilon + 8\Lambda M_1^2(T+1)\int_0^v s^{2\xi-2}\eta(v-s)ds.\end{aligned}$$

Thus, one can obtain

$$\begin{aligned}\eta(v) = \sup_{\theta\in[-h,v]}\Theta(\theta) &\le \max\left\{\sup_{\theta\in[-h,0]}\Theta(\theta), \sup_{\theta\in[0,v]}\Theta(\theta)\right\} \\ &\le \max\Bigg\{E\|G(0)\|^2, \varepsilon + \frac{M_1^2 T v^{2\xi-1}}{2\xi - 1}\varepsilon \\ &\qquad + 8\Lambda M_1^2(T+1)\int_0^v s^{2\xi-2}\eta(v-s)ds\Bigg\} \\ &\le \varepsilon + \frac{M_1^2 T^{2\xi}}{2\xi - 1}\varepsilon + 8\Lambda M_1^2(T+1)\int_0^v (v-s)^{2\xi-2}\eta(s)ds,\end{aligned}$$

where $\varepsilon + \frac{M_1^2 T^{2\xi}}{2\xi}\varepsilon$ and $8\Lambda M_1^2(T+1)$ are nondecreasing along with nonnegative, then applying **Lemma 2.8** (Namely, Generalized Grönwall inequality) we obtain

$$\begin{aligned}\eta(v) &\le \left(1 + \frac{M_1^2 T^{2\xi}}{2\xi - 1}\right) M_{2\xi-1}\left(8\Lambda M_1^2(T+1)\Gamma(2\xi-1)T^{2\xi-1}\right)\varepsilon \\ &= N\varepsilon,\end{aligned}$$

and letting $N = \left(1 + \frac{M_1^2 T^{2\xi}}{2\xi-1}\right) M_{2\xi-1}\left(8\Lambda M_1^2(T+1)\Gamma(2\xi-1)T^{2\xi-1}\right)$. Hence, we can readily obtain

$$E\|\chi(v) - z(v)\|^2 = \Theta(v) \le \eta(v) \le N\varepsilon,$$

Clearly, the system (1.4) is U-H stable.

Case 3. For each $v \in (t_k, s_k]$, $k = 1, 2, \ldots, m$, and via assumption (A_1), we derive

$$E\|\chi(v) - z(v)\|^2 \le 2E\|I_k(v)\|^2$$

$$+ 2\lambda_k \left(E\|\chi(v) - z(v))\|^2 + E\|\chi(t_k^-) - z(t_k^-))\|^2\right)$$
$$\leq \varepsilon + 2\lambda E\|\chi(v) - z(v)\|^2$$

Owing to $\lambda \in (0, \frac{1}{2})$, we can figure out that $E\|\chi(v) - z(v)\|^2 \leq \frac{\varepsilon}{1-2\lambda}$, which means that Eq. (1.4) is U-H stable.

Case 4. For $v \in (s_k, t_{k+1}]$, $k = 1, 2, \ldots, m$, we derive

$$\begin{aligned} E\|\chi(v) - z(v)\|^2 \leq & 4\varepsilon + 16\lambda E\|\chi(v) - z(v)\|^2 + \frac{4M_1^2 T v^{2\xi-1}\varepsilon}{2\xi - 1} \\ & + 8M_1^2(T + 1)\int_0^v (v - s)^{2\xi-2} l(s)\left(E\|\chi(s) - z(s)\|^2\right. \\ & \left. + E\|\chi(s - h(s)) - z(v - h(s))\|^2\right) ds \\ & + 8M_1^2(T + 1)\int_0^{s_k} (s_k - s)^{2\xi-2} l(s)\left(E\|\chi(s) - z(s)\|^2\right. \\ & \left. + E\|\chi(s - h(s)) - z(v - h(s))\|^2\right) ds. \end{aligned}$$

We also set $\Theta(v) = E\|\chi(v) - z(v)\|^2$, and let $\eta(v) = \sup_{\theta\in[-h,v]} \Theta(\theta)$, $\forall v \in (s_k, t_{k+1}]$, $k = 1, 2, \ldots, m$, then $\Theta(s) \leq \eta(s)$ and $\Theta(s - h(s)) \leq \eta(s)$, $\forall s \in [0, v]$, then we obtain

$$\begin{aligned} E\|\chi(v) - z(v)\|^2 \leq & \frac{4\varepsilon}{1 - 16\lambda} + \frac{4M_1^2 T v^{2\xi-1}\varepsilon}{(1 - 16\lambda)(2\xi - 1)} \\ & + \frac{32\Lambda M_1^2(1 + T)}{1 - 16\lambda}\int_0^v (v - s)^{2\xi-2}\eta(s)ds. \end{aligned}$$

Using the above procedure, we get

$$\begin{aligned} & E\|\chi(v) - z(v)\|^2 = \Theta(v) \leq \eta(v) \\ & \leq \varepsilon \cdot \frac{4}{1 - 16\lambda}\left(1 + \frac{M_1^2 T^{2\xi}}{2\xi - 1}\right) M_{2\xi-1}\left(\frac{32\Lambda M_1^2(1 + T)}{1 - 16\lambda}\Gamma(2\xi - 1)T^{2\xi-1}\right) \\ & = \varepsilon \cdot N, \end{aligned}$$

where $N = \frac{4}{1-16\lambda}\left(1 + \frac{M_1^2 T^{2\xi}}{2\xi-1}\right) M_{2\xi-1}\left(\frac{32\Lambda M_1^2(1+T)}{1-16\lambda}\Gamma(2\xi - 1)T^{2\xi-1}\right)$ is a constant, which means that Eq. (1.4) is U-H stable. □

5. Example

In this section, an example is presented to verify our prime results. Let us consider the following FSDEs with time-delays and non-instantaneous impulses:

$$\begin{cases} {}^C\mathbb{D}_{0^+}^{0.85} z(v) = 10^{-4}z(v) + 10^{-2}z(v) - 10^{-2}z(v - \sqrt{v}) + [10^{-2}cosz(v) \\ \qquad -10^{-2}sinz(v - \sqrt{v})]\frac{dW(v)}{dv}, \qquad v \in [0, 9] \setminus (4, 7], k = 0, 1, \\ \phi_k\left(v, z(v), z(t_k^-)\right) = 0.1e^{-|z(v)|} + 0.1cosz(t_k - 0), \qquad v \in (4, 7], k = 1, \\ \phi(v) = v + 2, \qquad v \in [-3, 0], \end{cases} \tag{5.1}$$

where $A = 10^{-4}I$, let I represent the identity matrix, $h(v) = \sqrt{v}$, $K = [0, 9]$, $\sigma(v, z(v), z(v - h(v))) = 10^{-2}z(v) - 10^{-2}z(v - \sqrt{v})$, $\rho(v, z(v), z(v - h(v))) = 10^{-2}cosz(v) - 10^{-2}sinz(v - \sqrt{v})$.

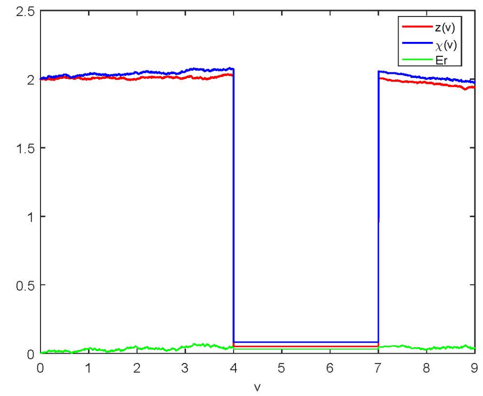

Figure 1: The stable solution for system (5.1) with $\varepsilon = 0.0015$.

Let $\lambda = 0.02$, $\Lambda = 2 \times 10^{-4}$, $d_k = 0.04$, $\|A\| = 0.01$, $L = 1$, by applying Matlab software, we have $M_1 \approx 0.9733$,

$$10\lambda + \frac{20\Lambda M_1^2(T+1)T^{2\xi-1}}{2\xi - 1} \approx 0.4520 < 1,$$

and

$$q \geq \max\left\{S_1, S_2, \|\phi\|_U\right\} \approx 4.4736.$$

Clearly, we can figure out that all conditions in **Theorem 3.1** and **Theorem 3.2** are satisfied, which means that Eq. (5.1) has a unique solution in B_q, here $q \geq 4.4736$. Meanwhile, system (5.1) satisfies the conditions of **Theorem 4.1**, thus, system (5.1) is U-H stable on $[0, 9]$ and the U-H stability constant $N = 651.9177$. Then from Figure 1 we can draw a conclusion.

6. Conclusion

Existence, uniqueness and stability of mild solution of FSDEs with time-delays and non-instantaneous impulses is considered in this work. The existence theorem is established by utilizing Krasnoselskii's fixed point theorem. Subsequently, the uniqueness theorem is obtained based on contraction mapping principle. Moreover, by virtue of the generalized Grönwall inequality and stochastic analysis techniques, we get the U-Hs result of the solution to the present system. Further, we discuss a representative example to illustrate the validity of the article's conclusions.

One interesting research direction is to discuss the stability of fuzzy FSDEs. With the introduction of fuzziness, the original system becomes complex and its stability may change. For future research, it will be interesting to ask what conditions we can add to restore stability.

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