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Coefficient bounds for q-close-to-convex functions associated with vertical strip domain

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Abstract. Let $\mathcal{A}$ denote the class of functions f which are analytic in the open unit disk $\mathbb{U}$ and given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (z \in \mathbb{U}).$$

In a very recent paper, Alqahtani *et al.* [AIMS Mathematics 8 (4) (2023), 9385–9399] defined a new subclass $\mathcal{S}_q^*(\alpha, \beta)$ $(0 \leq \alpha < 1 < \beta,\ 0 < q < 1)$ consists of functions $f \in \mathcal{A}$ satisfying the following condition:

$$\alpha < \Re\left(\frac{z\partial_q f(z)}{f(z)}\right) < \beta \qquad (z \in \mathbb{U}),$$

where $\partial_q f$ is Jackson's q-derivative of f.

In this study, we introduce a new subclass $\mathcal{S}_{q,g}(\alpha, \beta)$ of analytic and q-close-to-convex functions satisfying

$$\alpha < \Re\left(\frac{z\partial_q f(z)}{g(z)}\right) < \beta \qquad (z \in \mathbb{U}),$$

where $0 \leq \alpha < 1 < \beta$ and $g \in \mathcal{S}_q^*(\delta, \beta)$ with $0 \leq \delta < 1 < \beta$.

The main purpose of this paper is to determine some coefficient bounds for functions $f \in \mathcal{A}$ satisfying the non-homogenous Cauchy-Euler fractional q-differential equation associated with analytic functions belong to the class $\mathcal{S}_{q,g}(\alpha, \beta)$.

1. Introduction

The sets of real numbers, complex numbers and positive integers will be denoted by

$$\mathbb{R} = (-\infty, \infty), \quad \mathbb{C} = \mathbb{C}^* \cup \{0\} \quad \text{and} \quad \mathbb{N} = \{1, 2, 3, \ldots\} = \mathbb{N}_0 \setminus \{0\},$$

respectively. Assume that $\mathcal{H}$ is the class of analytic functions in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\},$$

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and let the class $\mathcal{P}$ be defined by

$$\mathcal{P} = \{ p \in \mathcal{H} : p(0) = 1 \quad \text{and} \quad \Re\left(p(z)\right) > 0 \;\; (z \in \mathbb{U}) \}.$$

For two functions $f, g \in \mathcal{H}$, we say that the function f is subordinate to g in $\mathbb{U}$, and write

$$f(z) \prec g(z) \qquad (z \in \mathbb{U}),$$

if there exists a Schwarz function ω, analytic in $\mathbb{U}$, with

$$\omega(0) = 0 \qquad \text{and} \qquad |\omega(z)| < 1 \qquad (z \in \mathbb{U})$$

such that

$$f(z) = g(\omega(z)) \qquad (z \in \mathbb{U}).$$

Indeed, it is known that

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Rightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Quantum calculus is ordinary classical calculus without the notion of limits. It defines q-calculus and h-calculus. Here h ostensibly stands for Planck's constant, while q stands for quantum. Recently, the area of q-calculus has attracted the serious attention of researchers. This great interest is due to its application in various branches of mathematics and physics. The application of q-calculus was initiated by Jackson [11, 12]. He was the first to develop q-integral and q-derivative in a systematic way. Later, geometrical interpretation of q-analysis has been recognized through studies on quantum groups. It also suggests a relation between integrable systems and q-analysis.

Many applications of the q-calculus can be found in the field of mathematics such as numerical analysis, fractional calculus, special polynomials and analytic number theory. Mathematicians, who have introduced and investigated of new analytic function subclasses in the field of geometric function theory (GFT), have extensively used the q-calculus especially, lately. One of the first contributions of using the q-calculus in GFT is given by Ismail *et al*. [10] who established generalized version of the starlike functions namely q-starlike functions.

In recent years, many applications of q-calculus associated with various families of analytic functions can be found in the literature. For example,the problem obtaining of the upper bound for both initial coefficients and second Hankel determinant for functions belong to the some subclasses of q-starlike functions are investigated by Çağlar *et al*. [8] and Srivastava *et al*. [21]. On the other hand, Al-Shbeil *et al*. [3], Srivastava *et al*. [22], Srivastava and El-Deeb [24] and Srivastava *et al*. [25] introduced the classes of analytic and bi-univalent functions defined by means of q-calculus and obtained some results for coefficient bounds of functions belonging to the these classes defined by authors (see also [9]). Further Ali *et al*. [1], by defining an interesting subclass of analytic and multivalent functions by means of the q-derivative operator, have investigated the coefficient bounds, distortion results, convex linear combinations, and the radii of starlikeness, convexity and close-to-convexity for functions belong to this class.

We need following basic definitions of the q-calculus which are used in this paper (see, for details, [11, 12]).

For $0 < q < 1$, the q-number and the q-factorial are defined by

$$[n]_q = \begin{cases} \frac{1-q^n}{1-q} & , \quad n \in \mathbb{C} \\ 1 + q + q^2 + \cdots q^{n-1} & , \quad n \in \mathbb{N} \end{cases}$$

and

$$[n]_q! = \begin{cases} 1 & , \quad n = 0 \\ \prod_{r=1}^{n} [r]_q & , \quad n \in \mathbb{N} \end{cases},$$

respectively. As $q \to 1^-$, $[n]_q \to n$ and $[n]_q! \to n!$.

For a function f defined on a subset of $\mathbb{C}$, Jackson's q-derivative $\partial_q f$ is defined by (see [11, 12])

$$\partial_q f(z) = \begin{cases} \frac{f(z)-f(qz)}{(1-q)z} & , \quad z \neq 0 \\ f'(0) & , \quad z = 0 \end{cases} \tag{1}$$

provided that $f'(0)$ exists. Then a function $g(z) = z^n$, we have

$$\partial_q (z^n) = [n]_q z^{n-1},$$

$$\lim_{q \to 1^-} \left(\partial_q (z^n) \right) = n z^{n-1} = g'(z),$$

where g' is the ordinary derivative.

Jackson [12] introduced the q-integral by

$$\int_0^z f(t) d_q t = z(1-q) \sum_{k=0}^{\infty} q^k f\left(zq^k\right),$$

as long as the series converges. Then a function $g(z) = z^n$, we obtain

$$\int_0^z g(t)\, d_q t = \int_0^z t^n d_q t = \frac{1}{[n+1]_q} z^{n+1} \qquad (n \neq -1)$$

and

$$\lim_{q \to 1^-} \int_0^z g(t)\, d_q t = \int_0^z g(t)\, dt,$$

where $\int_0^z g(t)\, dt$ is the ordinary integral.

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{2}$$

which are analytic in the open unit disk $\mathbb{U}$. We also denote by $\mathcal{S}$ the class of all functions in the normalized analytic function class $\mathcal{A}$ which are univalent in $\mathbb{U}$. We note that if the function $f \in \mathcal{A}$ is of the form (2), then we obtain

$$\partial_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1} \qquad (z \in \mathbb{U}). \tag{3}$$

For a function $f \in \mathcal{A}$ given by (2), from (1), we deduce that

$$\begin{aligned} \partial_q^{(1)} f(z) &= \sum_{n=1}^{\infty} [n]_q a_n z^{n-1} =: \partial_q f(z) \\ \partial_q^{(2)} f(z) &= \sum_{n=2}^{\infty} [n]_q [n-1]_q a_n z^{n-2} \\ &\vdots \\ \partial_q^{(j)} f(z) &= \sum_{n=j}^{\infty} \frac{[n]_q!}{[n-j]_q!} a_n z^{n-j} \end{aligned}$$

where $\partial_q^{(j)} f(z)$ is the j^{th} q-derivative of $f(z)$.

By means of the q-derivative, Alqahtani *et al.* [2] introduced the following class of functions associated with vertical strip domain as follows:

Definition 1.1. *[2] Let $\alpha, \beta \in \mathbb{R}$ such that $0 \le \alpha < 1 < \beta$ and $f \in \mathcal{A}$ be defined by* (2)*. Then*

$$f \in \mathcal{S}_q^*(\alpha, \beta) \Leftrightarrow \alpha < \Re\left(\frac{z\partial_q f(z)}{f(z)}\right) < \beta \qquad (z \in \mathbb{U}).$$

The class $\mathcal{S}_q^*(\alpha, \beta)$ is non-empty. For example, the function $f \in \mathcal{A}$ given by

$$f(z) = z \exp\left\{\frac{\beta - \alpha}{\pi} i \int_0^z \frac{1}{t} \log\left(\frac{1 - q e^{2\pi i \frac{1-\alpha}{\beta-\alpha}} t}{1 - qt}\right) d_q t\right\}$$

is in the class $\mathcal{S}_q^*(\alpha, \beta)$, (see [2]).

Remark 1.2. *In Definition* 1.1*,*
(i) *if we let $q \to 1^-$, then we have the class $\mathcal{S}(\alpha, \beta)$ which consists of functions $f \in \mathcal{A}$ satisfying*

$$\alpha < \Re\left(\frac{zf'(z)}{f(z)}\right) < \beta \qquad (z \in \mathbb{U})$$

introduced by Kuroki and Owa [13];
(ii) *if we let $\beta \to \infty$, then the class $\mathcal{S}_q^*(\alpha, \beta)$ reduces to the class $\mathcal{S}_q^*(\alpha)$ of q-starlike functions of order α $(0 \le \alpha < 1)$, which consists of functions $f \in \mathcal{A}$ satisfying*

$$\Re\left(\frac{z\partial_q f(z)}{f(z)}\right) > \alpha \qquad (z \in \mathbb{U})$$

studied by Seoudy and Aouf [18].

Lemma 1.3. *[2] Let $f \in \mathcal{A}$ be defined by* (2)*, $0 \le \alpha < 1 < \beta$ and $0 < q < 1$. Then*

$$f \in \mathcal{S}_q^*(\alpha, \beta) \Leftrightarrow \frac{z\partial_q f(z)}{f(z)} \prec 1 + \frac{\beta - \alpha}{\pi} i \log\left(\frac{1 - q e^{2\pi i \frac{1-\alpha}{\beta-\alpha}} z}{1 - qz}\right) \qquad (z \in \mathbb{U}).$$

Lemma 1.3 means that the function $f_{q,\alpha,\beta} : \mathbb{U} \to \mathbb{C}$ defined by

$$f_{q,\alpha,\beta}(z) = 1 + \frac{\beta - \alpha}{\pi} i \log\left(\frac{1 - q e^{2\pi i \frac{1-\alpha}{\beta-\alpha}} z}{1 - qz}\right) \tag{4}$$

is analytic in $\mathbb{U}$ with $f_{q,\alpha,\beta}(0) = 1$ and maps the unit disk $\mathbb{U}$ onto the vertical strip domain

$$\Omega_{\alpha,\beta} = \{w \in \mathbb{C} : \alpha < \Re(w) < \beta\}. \tag{5}$$

Also it has the form, [2],

$$f_{q,\alpha,\beta}(z) = 1 + \sum_{n=1}^{\infty} B_n z^n \in \mathcal{P}, \tag{6}$$

where

$$B_n = \frac{\beta - \alpha}{n\pi} q^n i \left(1 - e^{2n\pi i \frac{1-\alpha}{\beta-\alpha}}\right) \qquad (n \in \mathbb{N}). \tag{7}$$

Here, in our present sequel to some of the aforecited work of Alqahtani *et al.* [2], we first introduce the following subclasses of analytic functions.

Definition 1.4. *Let $\alpha, \beta \in \mathbb{R}$ such that $0 \leq \alpha < 1 < \beta$. We denote by $\mathcal{S}_{q,g}(\alpha,\beta)$ the class of functions $f \in \mathcal{A}$ satisfying*

$$\alpha < \Re\left(\frac{z\partial_q f(z)}{g(z)}\right) < \beta \qquad (z \in \mathbb{U}),$$

where $g \in \mathcal{S}_q^*(\delta,\beta)$ with $0 \leq \delta < 1 < \beta$.

Note that for given $0 \leq \alpha, \delta < 1 < \beta$, $f \in \mathcal{S}_{q,g}(\alpha,\beta)$ if and only if the following two subordination equations are satisfied:

$$\frac{z\partial_q f(z)}{g(z)} \prec \frac{1 + (1 - (1+q)\alpha) z}{1 - qz} \qquad \text{and} \qquad \frac{z\partial_q f(z)}{g(z)} \prec \frac{1 - (1 - (1+q)\beta) z}{1 + qz}.$$

Remark 1.5. *In Definition* 1.4,
(*i*) *if we let $q \to 1^-$, then we have the class $\mathcal{S}_g(\alpha,\beta)$ which consists of functions $f \in \mathcal{A}$ satisfying*

$$\alpha < \Re\left(\frac{zf'(z)}{g(z)}\right) < \beta \qquad (g \in \mathcal{S}(\delta,\beta), \quad z \in \mathbb{U})$$

introduced by Bulut [6];
(*ii*) *If we let $\beta \to \infty$, then the class $\mathcal{S}_{q,g}(\alpha,\beta)$ reduces to the class $C_q(\delta,\alpha)$ of q-close-to-convex functions of order δ and type α, which consists of functions $f \in \mathcal{A}$ satisfying*

$$\Re\left(\frac{z\partial_q f(z)}{g(z)}\right) > \alpha \qquad \left(g \in \mathcal{S}_q^*(\delta), \quad z \in \mathbb{U}\right)$$

introduced by Bulut [7].

Using (5) and by the principle of subordination, we can immediately obtain Lemma 1.6.

Lemma 1.6. *Let α, β and δ be real numbers such that $0 \leq \alpha, \delta < 1 < \beta$ and let the function $f \in \mathcal{A}$ be defined by* (2). *Then $f \in \mathcal{S}_{q,g}(\alpha,\beta)$ if and only if*

$$\frac{z\partial_q f(z)}{g(z)} \prec f_{q,\alpha,\beta}(z) \qquad (z \in \mathbb{U})$$

where $f_{q,\alpha,\beta}(z)$ is defined by (4).

The coefficient problem for close-to-convex functions studied many authors in recent years, (see, for example [4, 5, 16, 23, 26–29]). Upon inspiration from the recent work of Alqahtani *et al*. [2], we obtain coefficient bounds for the Taylor-Maclaurin coefficients for functions in the function class $\mathcal{S}_{q,g}(\alpha,\beta)$ of analytic functions which we have introduced here.

The main purpose of this paper is to obtain some coefficient bounds for functions belong to the subclass $\mathcal{B}_{q,g}(\alpha,\beta;u)$, which consists of functions $f\in\mathcal{A}$ satisfying the following non-homogenous Cauchy-Euler fractional q-differential equation:

$$z^2\partial_q^{(2)}f(z)+2(1+u)\,z\partial_q^{(1)}f(z)+u(1+u)f(z)=(1+u)(2+u)\varphi(z) \tag{8}$$

$$\left(f\in\mathcal{A};\ \varphi\in\mathcal{S}_{q,g}(\alpha,\beta);\ u\in\mathbb{R}\setminus(-\infty,-1]\right).$$

Lemma 1.7. *[17] Let the function* $\mathfrak{g}$ *given by*

$$\mathfrak{g}(z)=\sum_{k=1}^{\infty}\mathfrak{b}_k z^k \qquad (z\in\mathbb{U})$$

be convex in $\mathbb{U}$*. Also let the function* $\mathfrak{f}$ *given by*

$$\mathfrak{f}(z)=\sum_{k=1}^{\infty}\mathfrak{a}_k z^k \qquad (z\in\mathbb{U})$$

be holomorphic in $\mathbb{U}$*. If*

$$\mathfrak{f}(z)\prec\mathfrak{g}(z) \qquad (z\in\mathbb{U}),$$

then

$$|\mathfrak{a}_k|\leq|\mathfrak{b}_1| \qquad (k\in\mathbb{N}).$$

Lemma 1.8. *[15] Let* $p\in\mathcal{P}$ *with* $p(z)=1+c_1z+c_2z^2+\cdots$. *Then for any complex number* ν,

$$\left|c_2-\nu c_1^2\right|\leq 2\max\{1,\ |2\nu-1|\},$$

and the result is sharp for the functions given by

$$p(z)=\frac{1+z^2}{1-z^2} \quad \text{and} \quad p(z)=\frac{1+z}{1-z}.$$

2. Coefficient inequalities for the class $\mathcal{S}_q^*(\alpha,\beta)$

Theorem 2.1. *Let* α *and* β *be real numbers such that* $0\leq\alpha<1<\beta$ *and let the function* $f\in\mathcal{A}$ *be defined by* (2). *If* $f\in\mathcal{S}_q^*(\alpha,\beta)$*, then*

$$|a_n|\leq\frac{\prod_{k=2}^{n}\left([k-2]_q+\frac{2(\beta-\alpha)}{\pi}\sin\frac{\pi(1-\alpha)}{\beta-\alpha}\right)}{[n-1]_q!} \qquad (n=2,3,\ldots).$$

Proof. Let the function $f\in\mathcal{S}_q^*(\alpha,\beta)$ be of the form (2). Let us define the function $p(z)$ by

$$p(z)=\frac{z\partial_q f(z)}{f(z)} \qquad (z\in\mathbb{U}). \tag{9}$$

Then according to the assertion of Lemma 1.3, we get

$$p(z) \prec f_{q,\alpha,\beta}(z), \tag{10}$$

where $f_{q,\alpha,\beta}(z)$ is defined by (4). Hence, using Lemma 1.7, we obtain

$$\left|\frac{p^{(m)}(0)}{m!}\right| = |c_m| \leq |B_1| \qquad (m \in \mathbb{N}), \tag{11}$$

where

$$p(z) = 1 + c_1 z + c_2 z^2 + \cdots \qquad (z \in \mathbb{U}) \tag{12}$$

and (by (7))

$$|B_1| = \left|\frac{\beta-\alpha}{\pi} qi\left(1 - e^{2\pi i \frac{1-\alpha}{\beta-\alpha}}\right)\right| = \frac{2q(\beta-\alpha)}{\pi} \sin\frac{\pi(1-\alpha)}{\beta-\alpha}. \tag{13}$$

Also from (9), we find

$$z\partial_q f(z) = p(z) f(z) \qquad (z \in \mathbb{U}). \tag{14}$$

Since $a_1 = 1$, in view of (14), we obtain

$$\left([n]_q - 1\right) a_n = c_{n-1} + c_{n-2}a_2 + \cdots + c_1 a_{n-1} = \sum_{j=1}^{n-1} c_j a_{n-j} \qquad (n = 2, 3, \ldots). \tag{15}$$

Applying (11) into (15), we get

$$\left([n]_q - 1\right)|a_n| \leq |B_1| \sum_{j=1}^{n-1} \left|a_{n-j}\right| \qquad (n = 2, 3, \ldots).$$

For $n = 2, 3, 4$, we have

$$\begin{aligned}
|a_2| &\leq \frac{|B_1|}{\left([2]_q - 1\right)}, \\
|a_3| &\leq \frac{|B_1|}{\left([3]_q - 1\right)}(1 + |a_2|) \leq \frac{|B_1|}{\left([3]_q - 1\right)}\left(1 + \frac{|B_1|}{\left([2]_q - 1\right)}\right), \\
|a_4| &\leq \frac{|B_1|}{\left([4]_q - 1\right)}(1 + |a_2| + |a_3|) \leq \frac{|B_1|\left([2]_q - 1 + |B_1|\right)\left([3]_q - 1 + |B_1|\right)}{\left([4]_q - 1\right)\left([3]_q - 1\right)\left([2]_q - 1\right)},
\end{aligned}$$

respectively. Using the principle of mathematical induction and the fact that

$$[k]_q - 1 = q[k-1]_q \qquad (k \geq 2),$$

and the equality (13), we obtain

$$\begin{aligned}
|a_n| &\leq \prod_{k=2}^{n} \frac{[k-1]_q - 1 + |B_1|}{[k]_q - 1} \\
&= \frac{\prod_{k=2}^{n}\left(q[k-2]_q + |B_1|\right)}{q^{n-1}[n-1]_q!} \\
&= \frac{\prod_{k=2}^{n}\left([k-2]_q + \frac{2(\beta-\alpha)}{\pi}\sin\frac{\pi(1-\alpha)}{\beta-\alpha}\right)}{[n-1]_q!} \qquad (n = 2, 3, \ldots).
\end{aligned}$$

This evidently completes the proof of Theorem 2.1. □

Letting $\beta \to \infty$ in Theorem 2.1, we have the following result.

Corollary 2.2. *Let α be a real number such that $0 \leq \alpha < 1$ and let the function $f \in \mathcal{A}$ be defined by* (2). *If $f \in \mathcal{S}_q^*(\alpha)$, then*

$$|a_n| \leq \frac{\prod_{k=2}^{n}\left([k-2]_q + 2(1-\alpha)\right)}{[n-1]_q!} \qquad (n = 2, 3, \ldots).$$

Remark 2.3. *If we let $q \to 1^-$ in Theorem 2.1, then we have [13, Theorem 2.1].*

Theorem 2.4. *Let α and β be real numbers such that $0 \leq \alpha < 1 < \beta$ and let the function $f \in \mathcal{A}$ be defined by* (2). *If $f \in \mathcal{S}_q^*(\alpha, \beta)$, then for any $\mu \in \mathbb{C}$*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{2(\beta-\alpha)}{(1+q)\pi}\sin\frac{\pi(1-\alpha)}{\beta-\alpha}\max\left\{1, \left|\frac{B_2}{B_1} + \frac{(1-(1+q)\mu)B_1}{q}\right|\right\},$$

where

$$B_1 = \frac{\beta-\alpha}{\pi}qi\left(1 - e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}\right) \quad \text{and} \quad B_2 = \frac{\beta-\alpha}{2\pi}q^2 i\left(1 - e^{4\pi i\frac{1-\alpha}{\beta-\alpha}}\right). \tag{16}$$

The result is sharp.

Proof. If $f \in \mathcal{S}_q^*(\alpha, \beta)$, then we have

$$\varphi(z) \prec f_{q,\alpha,\beta}(z) \qquad (z \in \mathbb{U}),$$

where

$$\varphi(z) = \frac{z\partial_q f(z)}{f(z)} = 1 + c_1 z + c_2 z^2 + \cdots \qquad (z \in \mathbb{U}) \tag{17}$$

and

$$f_{q,\alpha,\beta}(z) = 1 + \sum_{n=1}^{\infty} B_n z^n = 1 + \sum_{n=1}^{\infty}\frac{\beta-\alpha}{n\pi}q^n i\left(1 - e^{2n\pi i\frac{1-\alpha}{\beta-\alpha}}\right)z^n \qquad (z \in \mathbb{U}).$$

As explained in the proof of Theorem 2.1, from (15) we get

$$c_1 = \left([2]_q - 1\right)a_2, \qquad c_2 = \left([3]_q - 1\right)a_3 - \left([2]_q - 1\right)a_2^2$$

or equivalently

$$c_1 = qa_2, \qquad c_2 = q\left[(1+q)a_3 - a_2^2\right]. \tag{18}$$

Since $f_{q,\alpha,\beta}(z)$ is univalent and $\varphi(z) \prec f_{q,\alpha,\beta}(z)$, the function

$$h(z) = \frac{1 + f_{q,\alpha,\beta}^{-1}(\varphi(z))}{1 - f_{q,\alpha,\beta}^{-1}(\varphi(z))} = 1 + h_1 z + h_2 z^2 + \cdots \qquad (z \in \mathbb{U})$$

is analytic and has a positive real part in $\mathbb{U}$. Also we have

$$\varphi(z) = f_{q,\alpha,\beta}\left(\frac{h(z)-1}{h(z)+1}\right) = 1 + \frac{B_1 h_1}{2}z + \left[\frac{B_1}{2}\left(h_2 - \frac{h_1^2}{2}\right) + \frac{B_2}{4}h_1^2\right]z^2 + \cdots. \tag{19}$$

Thus by (17)-(19) we get

$$a_2 = \frac{B_1}{2q}h_1, \tag{20}$$

$$a_3 = \frac{B_1}{2q(1+q)}\left[h_2 - \frac{1}{2}\left(1 - \frac{B_2}{B_1} - \frac{B_1}{q}\right)h_1^2\right]. \tag{21}$$

Taking into account (20) and (21), we obtain

$$a_3 - \mu a_2^2 = \frac{B_1}{2q(1+q)}\left(h_2 - \lambda h_1^2\right), \tag{22}$$

where

$$\lambda = \frac{1}{2}\left[1 - \frac{B_2}{B_1} - \frac{(1-(1+q)\mu)B_1}{q}\right]. \tag{23}$$

Our result now follows by an application of Lemma 1.8. The result is sharp for the functions

$$\frac{z\partial_q f(z)}{f(z)} = f_{q,\alpha,\beta}\left(z^2\right) \quad \text{and} \quad \frac{z\partial_q f(z)}{f(z)} = f_{q,\alpha,\beta}(z).$$

□

3. Coefficient inequalities for the class $\mathcal{S}_{q,g}(\alpha,\beta)$

Theorem 3.1. *Let α, β and δ be real numbers such that $0 \le \alpha, \delta < 1 < \beta$ and let the function $f \in \mathcal{A}$ be defined by (2). If $f \in \mathcal{S}_{q,g}(\alpha,\beta)$, then*

$$|a_2| \le \frac{2(\beta-\delta)}{[2]_q \pi}\sin\frac{\pi(1-\delta)}{\beta-\delta} + \frac{2q(\beta-\alpha)}{[2]_q \pi}\sin\frac{\pi(1-\alpha)}{\beta-\alpha}$$

and for $n = 3, 4, \ldots$

$$|a_n| \le \frac{\prod_{k=2}^{n}\left([k-2]_q + \frac{2(\beta-\delta)}{\pi}\sin\frac{\pi(1-\delta)}{\beta-\delta}\right)}{[n]_q!} + \frac{2q(\beta-\alpha)}{[n-2]_q!\,[n]_q\,\pi}\sin\frac{\pi(1-\alpha)}{\beta-\alpha}\prod_{k=1}^{n-2}\left([k]_q + \frac{2(\beta-\delta)}{\pi}\sin\frac{\pi(1-\delta)}{\beta-\delta}\right),$$

where $g \in \mathcal{S}_q^*(\delta,\beta)$.

Proof. Let the function $f \in \mathcal{S}_{q,g}(\alpha,\beta)$ be of the form (2). Therefore, there exists a function

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{S}_q^*(\delta,\beta) \tag{24}$$

so that

$$\alpha < \Re\left(\frac{z\partial_q f(z)}{g(z)}\right) < \beta. \tag{25}$$

Note that by Theorem 2.1, we have

$$|b_n| \leq \frac{\prod_{k=2}^{n}\left([k-2]_q + \frac{2(\beta-\delta)}{\pi}\sin\frac{\pi(1-\delta)}{\beta-\delta}\right)}{[n-1]_q!} \qquad (n = 2, 3, \ldots). \tag{26}$$

Let us define the function $\hat{p}(z)$ by

$$\hat{p}(z) = \frac{z\partial_q f(z)}{g(z)} \qquad (z \in \mathbb{U}). \tag{27}$$

Then according to the assertion of Lemma 1.6, we get

$$\hat{p}(z) \prec f_{q,\alpha,\beta}(z) \qquad (z \in \mathbb{U}), \tag{28}$$

where $f_{q,\alpha,\beta}(z)$ is defined by (4). Hence, using Lemma 1.7, we obtain

$$\left|\frac{\hat{p}^{(m)}(0)}{m!}\right| = |d_m| \leq |B_1| \qquad (m = 1, 2, \ldots), \tag{29}$$

where

$$\hat{p}(z) = 1 + d_1 z + d_2 z^2 + \cdots \qquad (z \in \mathbb{U}). \tag{30}$$

Also from (27), we find

$$z\partial_q f(z) = \hat{p}(z)g(z). \tag{31}$$

Since $a_1 = b_1 = 1$, in view of (31), we obtain

$$[n]_q a_n - b_n = d_{n-1} + d_{n-2}b_2 + \cdots + d_1 b_{n-1} = \sum_{j=1}^{n-1} d_j b_{n-j} \qquad (n = 2, 3, \ldots). \tag{32}$$

Now we get from (29) and (32),

$$|a_n| \leq \frac{1}{[n]_q}|b_n| + \frac{|B_1|}{[n]_q}\sum_{j=1}^{n-1}\left|b_{n-j}\right|.$$

Using the fact that

$$\sum_{j=1}^{n-1}\left|b_{n-j}\right| = 1 + |b_2| + \cdots + |b_{n-1}| \leq \frac{\prod_{k=1}^{n-2}\left([k]_q + \frac{2(\beta-\delta)}{\pi}\sin\frac{\pi(1-\delta)}{\beta-\delta}\right)}{[n-2]_q!},$$

the proof of the Theorem 3.1 is completed. □

Letting $\beta \to \infty$ in Theorem 3.1, we have the following result.

Corollary 3.2. *Let α and δ be real numbers such that $0 \leq \alpha, \delta < 1$ and let the function $f \in \mathcal{A}$ be defined by* (2). *If $f \in C_q(\delta, \alpha)$, then*

$$|a_2| \leq \frac{2(1-\delta)}{[2]_q} + \frac{2q(1-\alpha)}{[2]_q}$$

and for $n = 3, 4, \ldots$

$$|a_n| \leq \frac{\prod_{k=2}^{n} \left([k-2]_q + 2(1-\delta) \right)}{[n]_q!} + \frac{2q(1-\alpha)}{[n-2]_q! \, [n]_q} \prod_{k=1}^{n-2} \left([k]_q + 2(1-\delta) \right),$$

where $g \in \mathcal{S}_q^* (\delta)$.

Letting $\beta \to \infty$ and $q \to 1^-$ in Theorem 3.1, we have the coefficient bounds for close-to-convex functions of order α and type δ.

Corollary 3.3. *[14] Let* α *and* δ *be real numbers such that* $0 \leq \alpha, \delta < 1$ *and let the function* $f \in \mathcal{A}$ *be defined by* (2). *If* $f \in C(\alpha, \delta)$, *then*

$$|a_n| \leq \frac{2(3-2\delta)(4-2\delta)\cdots(n-2\delta)}{n!} \left[n(1-\alpha) + (\alpha - \delta) \right] \qquad (n = 2, 3, \ldots).$$

4. Coefficient inequalities for the class $\mathcal{B}_{q,g}(\alpha, \beta; u)$

Theorem 4.1. *Let* α, β *and* δ *be real numbers such that* $0 \leq \alpha, \delta < 1 < \beta$ *and let the function* $f \in \mathcal{A}$ *be defined by* (2). *If* $f \in \mathcal{B}_{q,g}(\alpha, \beta; u)$, *then*

$$|a_2| \leq \frac{(1+u)(2+u)}{(n+u)(n+1+u)} \left[\frac{2(\beta-\delta)}{[2]_q \pi} \sin \frac{\pi(1-\delta)}{\beta-\delta} + \frac{2q(\beta-\alpha)}{[2]_q \pi} \sin \frac{\pi(1-\alpha)}{\beta-\alpha} \right] \tag{33}$$

and for $n = 3, 4, \ldots$

$$\begin{aligned} |a_n| \leq \; & \frac{(1+u)(2+u)}{(n+u)(n+1+u)} \left[\frac{\prod_{k=2}^{n} \left([k-2]_q + \frac{2(\beta-\delta)}{\pi} \sin \frac{\pi(1-\delta)}{\beta-\delta} \right)}{[n]_q!} \right. \\ & \left. + \frac{2q(\beta-\alpha)}{[n-2]_q! \, [n]_q \, \pi} \sin \frac{\pi(1-\alpha)}{\beta-\alpha} \prod_{k=1}^{n-2} \left([k]_q + \frac{2(\beta-\delta)}{\pi} \sin \frac{\pi(1-\delta)}{\beta-\delta} \right) \right], \end{aligned} \tag{34}$$

where $u \in \mathbb{R} \setminus (-\infty, -1]$ *and* $g \in \mathcal{S}_q^*(\delta, \beta)$.

Proof. Let the function $f \in \mathcal{A}$ be given by (2). Also let

$$\varphi(z) = z + \sum_{n=2}^{\infty} \varphi_n z^n \in \mathcal{S}_{q,g}(\alpha, \beta),$$

so that

$$a_n = \frac{(1+u)(2+u)}{(n+u)(n+1+u)} \varphi_n \qquad (n = 2, 3, \ldots; \; u \in \mathbb{R} \setminus (-\infty, -1]).$$

Thus, by using Theorem 3.1 in conjunction with the above equality, we obtain desired inequalities (33) and (34). □

5. Conclusion and Future Work

In this study, for functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad \text{and} \qquad \varphi(z) = z + \sum_{n=2}^{\infty} \varphi_n z^n \quad (z \in \mathbb{U}),$$

we consider following subclasses:

$$\mathcal{S}_q^*(\alpha, \beta) = \left\{ f \in \mathcal{A} : \quad \alpha < \Re\left(\frac{z\partial_q f(z)}{f(z)} \right) < \beta \quad (0 \leq \alpha < 1 < \beta) \right\},$$

where $z \in \mathbb{U}$ and $\partial_q f$ is Jackson's q-derivative of f;

$$\mathcal{S}_{q,g}(\alpha, \beta) = \left\{ f \in \mathcal{A} : \quad \alpha < \Re\left(\frac{z\partial_q f(z)}{g(z)} \right) < \beta \quad (0 \leq \alpha < 1 < \beta) \right\},$$

where $z \in \mathbb{U}$ and $g \in \mathcal{S}_q^*(\delta, \beta)$ with $0 \leq \delta < 1 < \beta$;

$$\mathcal{B}_{q,g}(\alpha, \beta; u) = \left\{ f \in \mathcal{A} : z^2 \partial_q^{(2)} f(z) + 2(1+u)\, z\partial_q^{(1)} f(z) + u(1+u) f(z) = (1+u)(2+u)\varphi(z) \right\},$$

where $z \in \mathbb{U}$, $\varphi \in \mathcal{S}_{q,g}(\alpha, \beta)$ and $u \in \mathbb{R} \backslash (-\infty, -1]$.

For functions f belong to the classes $\mathcal{S}_q^*(\alpha, \beta)$, $\mathcal{S}_{q,g}(\alpha, \beta)$ and $\mathcal{B}_{q,g}(\alpha, \beta; u)$, we investigate upper bounds for the general coefficient $|a_n|$, respectively. Also for functions $f \in \mathcal{S}_q^*(\alpha, \beta)$, we obtain sharp bounds for the Fekete-Szegö functional $\phi_\mu(f) = a_3 - \mu a_2^2$ when $\mu \in \mathbb{C}$.

This study could inspire light on further researches such as analytic (or meromorphic) and multivalent q-starlike and q-convex functions associated with vertical strip domain

$$\Omega_{\alpha,\beta} = \{ w \in \mathbb{C} : \alpha < \Re(w) < \beta \}.$$

On the other hand Srivastava, in his recent survey-cum-expository review articles [19, p. 340] and [20, pp. 1511-1512], pointed out the fact that current trend of translating known q-results into the corresponding (p, q)-results $(0 < q < p \leq 1)$ is trivial and in-consequential, simply because the additional forced-in parameter p is obviously redundant.

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