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On classes of Fredholm type operators

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Abstract. Given an idempotent p in a Banach algebra and following the study in [6] of p-invertibility, we consider here left p-invertibility, right p-invertibility and p-invertibility in the Calkin Algebra $C(X)$, where X is a Banach space. Then we define and study left and right generalized Drazin invertibility and we characterize left and right Drazin invertible elements in the Calkin algebra. Globally, this leads to define and characterize the classes of P-Fredholm, pseudo B-Fredholm and weak B-Fredholm operators.

1. Introduction

Let X be a Banach space, let $L(X)$ be the Banach algebra of bounded linear operators acting on the Banach space X and let $T \in L(X)$. We will denote by $N(T)$ the null space of T, by $\alpha(T)$ the nullity of T, by $R(T)$ the range of T and by $\beta(T)$ its defect. If the range $R(T)$ of T is closed and $\alpha(T) < \infty$ (resp. $\beta(T) < \infty$), then T is called an upper semi-Fredholm (resp. a lower semi-Fredholm) operator. A semi-Fredholm operator is an upper or a lower semi-Fredholm operator. If both of $\alpha(T)$ and $\beta(T)$ are finite then T is called a Fredholm operator and the index of T is defined by $ind(T) = \alpha(T) - \beta(T)$. The notations $\Phi_+(X), \Phi_-(X)$ and $\Phi(X)$ will design respectively the set of upper semi-Fredholm, lower semi-Fredholm and Fredholm operators.

For $T \in L(X)$, the Fredholm spectrum of T is defined by:

$$\sigma_F(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a Fredholm operator}\}.$$

Define also the sets:

$\Phi_l(X) = \{T \in \Phi_+(X) \mid \text{there exists a bounded projection of } X \text{ onto } R(T)\}$,

and

$\Phi_r(X) = \{T \in \Phi_-(X) \mid \text{there exists a bounded projection of } X \text{ onto } N(T)\}$.

Recall that the Calkin algebra over X is the quotient algebra $C(X) = L(X)/K(X)$, where $K(X)$ is the closed ideal of compact operators on X. Let G_r and G_l be the right and left, respectively, invertible elements of $C(X)$. From [11, Theorem 4.3.2] and [11, Theorem 4.3.3], it follows that $\Phi_l(X) = \Pi^{-1}(G_l)$ and $\Phi_r(X) = \Pi^{-1}(G_r)$, where $\Pi : L(X) \to C(X)$ is the natural projection. We observe that $\Phi(X) = \Phi_l(X) \cap \Phi_r(X)$.

Definition 1.1. *The elements of $\Phi_l(X)$ and $\Phi_r(X)$ will be called respectively left semi-Fredholm operators and right semi-Fredholm operators .*

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In 1958, in his paper [14], the author extended the concept of invertibility in rings and semigroups and introduced a new kind of inverse, known now as the Drazin inverse.

Definition 1.2. *An element a of a semigroup $\mathcal{S}$ is called Drazin invertible if there exists an element $b \in \mathcal{S}$ written $b = a^d$ and called the Drazin inverse of a, satisfying the following equations:*

$$ab = ba, b = ab^2, a^k = a^{k+1}b,$$

for some nonnegative integer k. The least nonnegative integer k for which these equations holds is the Drazin index $i(a)$ of a.

It follows from [14] that a Drazin invertible element in a semigroup has a unique Drazin inverse.

In 1996, in [15, Definition 2.3], the author extended the notion of Drazin invertibility.

Definition 1.3. *An element a of a Banach algebra A will be said to be generalized Drazin invertible if there exists $b \in A$ such that $bab = b, ab = ba$ and $aba - a$ is a quasinilpotent element in A.*

In [15], the author proved that $a \in A$ is generalized Drazin invertible if and only if there exists $\epsilon > 0$, such that for all λ such that $0 <| \lambda |< \epsilon$, the element $a - \lambda e$ is invertible and he proved in [15, Theorem 4.2], that a generalized Drazin invertible element has a unique generalized Drazin inverse. He also proved that an element $a \in A$ is generalized Drazin invertible if and only if there exists an idempotent $p \in A$ commuting with a, such that $a + p$ is invertible in A and ap is quasinilpotent.

Let A be a ring with a unit and let p be an idempotent in A. Recall that the commutant C_p of p is the subring of A defined by $C_p = \{x \in A \mid xp = px\}$. In [6, Definition 2.2], the concepts of left $p-$invertibility, right $p-$invertibility and $p-$invertibility where defined as follows.

Definition 1.4. *Let $a \in A$. We will say that:*
1- a is left p-invertible if $ap = pa$ and $a + p$ is left invertible in C_p.
2- a is right p-invertible if $ap = pa$ and $a + p$ is right invertible in C_p.
3- a is p-invertible if $ap = pa$ and $a + p$ is invertible.

Moreover in [6, Definition 2.11], left and right Drazin invertibility where defined as follows:

Definition 1.5. *We will say that an element $a \in A$ is left Drazin invertible (respectively right Drazin invertible) if there exists an idempotent $p \in A$ such that a is left p-invertible (respectively right p-invertible) and ap is nilpotent.*

For $T \in L(X)$, we will say that a subspace M of X is *T-invariant* if $T(M) \subset M$. We define $T_{|M} : M \to M$ as $T_{|M}(x) = T(x)$, $x \in M$. If M and N are two closed T-invariant subspaces of X such that $X = M \oplus N$, we say that T is *completely reduced* by the pair (M, N) and it is denoted by $(M, N) \in Red(T)$. In this case we write $T = T_{|M} \oplus T_{|N}$ and we say that T is a *direct sum* of $T_{|M}$ and $T_{|N}$.

It is said that $T \in L(X)$ admits a generalized Kato decomposition, abbreviated as GKD, if there exists $(M, N) \in Red(T)$ such that $T_{|M}$ is Kato and $T_{|N}$ is quasinilpotent. Recall that an operator $T \in L(X)$ is *Kato* if $R(T)$ is closed and $Ker(T) \subset R(T^n)$ for every $n \in \mathbb{N}$.

Remark 1.6. For $T \in L(X)$, to say that a pair (M, N) of closed subspaces of X is in $Red(T)$, means simply that there exists an idempotent $P \in L(X)$ commuting with T. Indeed if $(M, N) \in Red(T)$, let P be the projection on M in parallel to N. Then since $(M, N) \in Red(T)$, we see that P commutes with T. Conversely, given an idempotent $P \in L(X)$ commuting with T, if we set $M = P(X)$ and $N = (I - P)X$, it is clear that $(M, N) \in Red(T)$.

For $T \in L(X)$ and a nonnegative integer n, define $T_{[n]}$ to be the restriction of T to $R(T^n)$ viewed as a map from $R(T^n)$ into $R(T^n)$ (in particular $T_{[0]} = T$). If for some integer n the range space $R(T^n)$ is closed and $T_{[n]}$ is an upper (resp. a lower) semi-Fredholm operator, then T is called an upper (resp. a lower) semi-B-Fredholm operator. Moreover, if $T_{[n]}$ is a Fredholm operator, then T is called a B-Fredholm operator, see [7]. From [7, Theorem 2.7], we know that $T \in L(X)$ is a B-Fredholm operator if there exists $(M, N) \in Red(T)$ such that $T_{|M}$ is a nilpotent operator and $T_{|N}$ is a Fredholm operator. The notations $\Phi_B(X)$, $\Phi_{B^+}(X)$ and $\Phi_{B^-}(X)$ will designate respectively the set of upper semi-B-Fredholm, lower semi-B-Fredholm and B-Fredholm operators. We have $\Phi_B(X) = \Phi_{B^+}(X) \cap \Phi_{B^-}(X)$.

The results of this paper can be summarized as follows. First let us recall the following important result, which we will use throughout this paper. If p is an idempotent element of $C(X)$, then from [3, Lemma 1] there exists an idempotent $P \in L(X)$ such that $\Pi(P) = p$. So letting $p = \Pi(P)$ being an idempotent in the Calkin algebra and using the results of [6], we give in the Example 1.7 algebraic characterization of p-invertible (respectively right p-invertible and left p-invertible) elements in the Calkin algebra $C(X)$. This analysis leads to the introduction of the class $\Phi_P(X)$ of P-Fredholm (respectively the class $\Phi_{lP}(X)$ of left semi-P-Fredholm and the class $\Phi_{rP}(X)$ of right semi-P-Fredholm) operators in a similar way as the corresponding classes of Fredholm, left semi-Fredholm and right semi-Fredholm operators. We will see that $\Phi_P(X) = \Phi_{lP}(X) \cap \Phi_{rP}(X)$.

In the second section, we introduce the notions of left and right generalized invertibility in a Banach algebra and we characterize left and right generalized Drazin invertible elements in the Calkin algebra. In particular we prove that an element of $C(X)$ is generalized Drazin invertible if and only if it is left and right generalized Drazin invertible. This leads us to introduce the classes $\Phi_{l\mathcal{P}B}(X)$ and $\Phi_{r\mathcal{P}B}(X)$ of left and right pseudo semi-B-Fredholm operators, completing in this way the study of the class $\Phi_{\mathcal{P}\mathcal{B}}(X)$ of pseudo B-Fredholm operators inaugurated in [9] and [5]. We will prove that $\Phi_{\mathcal{P}B}(X) = \Phi_{l\mathcal{P}B}(X) \cap \Phi_{r\mathcal{P}B}(X)$.

Similarly in the third section, after characterizing left and right Drazin invertible elements in the Calkin algebra, we prove that an element of $C(X)$ is Drazin invertible if and only if it is left and right Drazin invertible. From [8, Theorem 3.4], we know that $T \in L(X)$ is a B-Fredholm operator if and only if $\pi(T)$ is Drazin invertible in the algebra $L(X)/F_0(X)$, where $F_0(X)$ is the ideal of finite rank operators in $L(X)$, and $\pi : L(X) \to L(X)/F_0(X)$ is the natural projection. Thus to differentiate the class of operators studied in this section from B-Fredholm operators, we name them "weak B-Fredholm operators". We will prove that the class of weak B-Fredholm operators contains strictly the class of B-Fredholm operators.

This leads us to introduce the classes $\Phi_{l\mathcal{W}B}(X)$, $\Phi_{r\mathcal{W}B}(X)$ of left and right weak semi-B-Fredholm operators, completing in this way the study of the class $\Phi_{\mathcal{W}B}(X)$ of weak B-Fredholm operators inaugurated in [9] and [5]. We will prove that $\Phi_{\mathcal{W}B}(X) = \Phi_{l\mathcal{W}B}(X) \cap \Phi_{r\mathcal{W}B}(X)$.

Though, the weak B-Fredholm operators and pseudo B-Fredholm operators, where not explicitly defined in [9], they where characterized by Drazin (generalized Drazin) invertible elements in the Calkin algebra in [9, Theorem 6.1,ii] and [9, Theorem 6.1, i] respectively.

Example 1.7. P-Fredholm Operators

Definition 1.8. *Let $T \in L(X)$ and let $p = \Pi(P)$ be an idempotent element of $C(X)$. We will say that:*
1- T is a left semi-P-Fredholm operator if $\Pi(T)$ is left p-invertible in $C(X)$.
2- T is a right semi-P-Fredholm operator if $\Pi(T)$ is right p-invertible in $C(X)$.
3- T is a P-Fredholm operator if $\Pi(T)$ is p-invertible in $C(X)$.

Let p be an idempotent element of $C(X)$, we define the sets G_{rp} and G_{lp} to be the right and left, respectively, p-invertible elements of $C(X)$. Then $\Phi_{lP}(X) = \Pi^{-1}(G_{lp})$ and $\Phi_{rP}(X) = \Pi^{-1}(G_{rp})$. If the idempotent p=0, then $\Phi_{l0}(X) = \Phi_l(X), \Phi_{r0}(X) = \Phi_r(X), G_{l0} = G_l$ and $G_{r0} = G_r$. So we retrieve the previous relations $\Phi_l(X) = \Pi^{-1}(G_l)$ and $\Phi_r(X) = \Pi^{-1}(G_r)$.

The following results characterizes algebraically left semi-P-Fredholm, right semi-P-Fredholm and P-Fredholm operators. Since these characterizations are obtained straightforwardly by using the corresponding results obtained in [6], we give them without proofs, referring the reader to [6].

Using [6, Theorem 2.5], we obtain the following characterization of left semi-p-Fredholm operators.

Theorem 1.9. *Let $T \in L(X)$. Then there exist an idempotent $p = \Pi(P) \in C(X)$ such that T is a left semi-P-Fredholm operator if and only if there exists an element $S \in L(X)$ such that $STS - S, S^2T - S, TST - ST^2$ are compact, there exists $V \in L(X)$ such that the commutator $[V, ST]$ and $V(I + T - ST) - I$ are compact. In this case $p = \Pi(I - ST)$.*

Using [6, Theorem 2.6], we obtain the following characterization of right semi-P-Fredholm operators..

Theorem 1.10. *Let $T \in L(X)$. Then there exist an idempotent $p = \Pi(P) \in C(X)$ such that T is a right semi-P-Fredholm operator if and only if there exists an element $S \in L(X)$ such that $STS - S, TS^2 - S, TST - T^2S$ are compact, there exists $V \in L(X)$ such that the commutator $[V, TS]$ and $(I + T - TS)V - I$ are compact. In this case $p = \Pi(I - TS)$.*

Using [6, Theorem 2.7], we obtain the following characterization of P-Fredholm operators..

Theorem 1.11. *Let $T \in L(X)$. Then there exist an idempotent $p = \Pi(P) \in C(X)$ such that T is a P-Fredholm operator if and only if there exists an element $S \in L(X)$ such that the commutator $[T, S]$ and $STS - S$ are compact and that $I + T - ST$ is a Fredholm operator. In this case $p = \Pi(I - ST)$.*

Remark 1.12. Let $p = \Pi(P) \in C(X)$ be an idempotent. It is clear that $T \in L(X)$ is P-Fredholm if and only if $T \in L(X)$ is left and right semi-P-Fredholm. So we have $\Phi_P(X) = \Phi_{lP}(X) \cap \Phi_{rP}(X)$.

2. On Pseudo B-Fredholm Operators

Definition 2.1.
1. *$T \in L(X)$ is called a Riesz- left semi-Fredholm operator if there exists $(M, N) \in Red(T)$ such that $T_{|M}$ is a Riesz operator and $T_{|N} \in \Phi_l(N)$.*
2. *$T \in L(X)$ is called a Riesz- right semi-Fredholm operator if there exists $(M, N) \in Red(T)$ such that $T_{|M}$ is a Riesz operator and $T_{|N} \in \Phi_r(N)$.*
3. *Following [9, Section 6, p.133], $T \in L(X)$ is called a Riesz- Fredholm operator if there exists $(M, N) \in Red(T)$ such that $T_{|M}$ is a Riesz operator and $T_{|N}$ is a Fredholm operator.*

Definition 2.2. *Let A be a Banach algebra. We will say that an element $a \in A$ is:*

1. *left generalized Drazin invertible if there exists an idempotent $p \in A$ such that a is left p-invertible in A and ap is quasinilpotent.*
2. *right generalized Drazin invertible if there exists an idempotent $p \in A$ such that a is right p-invertible in A and ap is quasinilpotent.*
3. *generalized Drazin invertible, as defined in [15], if there exists an idempotent $p \in A$ such that a is p-invertible in A and ap is quasinilpotent.*

Definition 2.3. *Let $T \in L(X)$. We will say that:*

1. *T is a left pseudo semi-B-Fredholm operator if $\Pi(T)$ is left generalized Drazin invertible in $C(X)$.*
2. *T is a right pseudo semi-B-Fredholm operator if $\Pi(T)$ is right generalized Drazin invertible in $C(X)$.*
3. *T is a pseudo semi-B-Fredholm operator if it is right or left pseudo semi-B-Fredholm.*
4. *T is a pseudo B-Fredholm operator if $\Pi(T)$ is generalized Drazin invertible in $C(X)$.*

Remark 2.4.
1. *The class of pseudo B-Fredholm was defined in [5, Definition 2.4]. It involves the class of B-Fredholm operators defined in [7].*
2. *In [9], in 2015, the author studied generalized Drazin invertible elements under Banach algebra homomorphisms. In the case of the Calkin Algebra, he proved in [9, Theorem 6.1, i] that an operator whose image is generalized Drazin invertible in the Calkin algebra is a compact perturbation of Riesz-Fredholm operator. Later in 2019, in [5, Theorem 2.10], a second characterization of this result was given in the same terms.*
Here in this paper, the concept of p-invertibility enable us to give left and right versions of [9, Theorem 6.1].

Theorem 2.5. *Let $T \in L(X)$. Then the following properties are equivalent:*

1. *T is a left pseudo semi-B-Fredholm operator.*
2. *T is a compact perturbation of a Riesz- left semi-Fredholm operator.*

Proof. 1) $\Rightarrow$ 2) Assume that T is a left pseudo semi-B-Fredholm operator, then $\Pi(T)$ is left generalized Drazin invertible in $C(X)$. So there exist an idempotent $p = \Pi(P) \in C(X)$ such that:

- $p\Pi(T) = \Pi(T)p$
- $p\Pi(T)$ is quasinilpotent in $C(X)$
- There exists $S \in L(X)$ such that $p\Pi(S) = \Pi(S)p$ and $\Pi(S)(\Pi(T) + p)) = \Pi(I)$

Let $X = X_1 \oplus X_2$ be the decomposition of X associated to P, that's $X_1 = R(P)$ and $X_2 = N(P)$. Since $\Pi(T)$ and $\Pi(P)$ commutes, we have that $\Pi(PTP) = \Pi(TP)$ and $\Pi((I-P)T(I-P)) = \Pi(T(I-P))$. It follows that PTP is a Riesz operator, $TP = PTP + K_1$, $T(I-P) = (I-P)T(I-P) + K_2$, where $K_1, K_2 \in K(X)$, and so:

$$T = TP + T(I-P) = PTP + (I-P)T(I-P) + K,$$

where $K = K_1 + K_2 \in K(X)$. We have that $(X_1, X_2) \in Red(PTP)$ and $(X_1, X_2) \in Red((I-P)T(I-P))$, hence:

$$PTP = (PTP)_{|X_1} \oplus (PTP)_{|X_2} = (PTP)_{|X_1} \oplus 0,$$

and

$$(I-P)T(I-P) = ((I-P)T(I-P))_{|X_1} \oplus ((I-P)T(I-P))_{|X_2} = 0 \oplus ((I-P)T(I-P))_{|X_2},$$

Therefore,

$$T = (PTP)_{|X_1} \oplus ((I-P)T(I-P))_{|X_2} + K. \tag{1}$$

It's easily seen that $(PTP)_{|R(P)}$ is Riesz operator. Moreover we show that $((I-P)T(I-P))_{|R(I-P)}$ is a left semi-Fredholm operator. Since $\Pi(S)(\Pi(T) + p) = \Pi(I)$, we have:

$$(I-P)S(I-P)(I-P)(T+P)(I-P) = I - P + F_1,$$

where F_1 is a compact operator such that $(X_1, X_2) \in Red(F_1)$. As $I - P$ is the identity on $L(X_2)$, it follows that $((I-P)T(I-P))_{|X_2} = ((I-P)(T+P)(I-P))_{|X_2}$ is a left semi-Fredholm operator. According to (1), we see that T is a compact perturbation of a Riesz-left semi-Fredholm operator.

2) $\Rightarrow$ 1) Conversely assume that $X = X_1 \oplus X_2$ is the direct sum of the two closed subspaces X_1, X_2 of X such that $T = T_1 \oplus T_2 + K$ where $T_1 \in L(X_1)$ is a Riesz operator and $T_2 \in L(X_2)$ is a left semi-Fredholm operator and $K \in K(X)$. Let P be the projection of X on X_1 in parallel to X_2. Then $PT = (T_1 \oplus 0) + PK$ and $TP = (T_1 \oplus 0) + KP$. Thus $PT - TP = PK - KP$ and $PK - KP$ is compact. Let $p = \Pi(P)$, then p commutes with $\Pi(T)$. Let also $I_1 = P_{|X_1}$ and $I_2 = (I-P)_{|X_2}$. As T_2 is a left semi-Fredholm operator in $L(X_2)$, there exists $S_2 \in L(X_2)$ such that $S_2T_2 - I_2$ is compact and $I_1 \oplus S_2$ commutes with P because $(I_1 \oplus S_2)P = P(I_1 \oplus S_2) = I_1 \oplus 0$.

We have $T_1 \oplus T_2 + P = (T_1 \oplus 0) + P + (0 \oplus T_2) = P[(T_1 \oplus 0) + I)]P + (I-P)(I_1 \oplus T_2)(I-P)$. As T_1 is a Riesz operator, then $T_1 \oplus 0$ is also a Riesz operator and $\Pi((T_1 \oplus 0) + I)) = \Pi((T_1 \oplus 0)) + \Pi(I)$ is invertible in $C(X)$. Let $\Pi(S_1)$ be its inverse, where $S_1 \in L(X)$. We observe that $\Pi(I_1 \oplus T_2)$ is left invertible in $C(X)$ having $\Pi(I_1 \oplus S_2)$ as a left inverse.Then:

$$\Pi((PS_1P + (I-P)(I_1 \oplus S_2)(I-P)))\Pi(T+P) = \Pi((PS_1P + (I-P)(I_1 \oplus S_2)(I-P)))\Pi(T_1 \oplus T_2 + P)$$

$$= \Pi([PS_1P + (I-P)(I_1 \oplus S_2)(I-P)])\Pi(P[(T_1 \oplus 0) + I)]P + (I-P)(I_1 \oplus T_2)(I-P)) = \Pi(I).$$

It is easily seen that $\Pi((PS_1P + (I-P)(I_1 \oplus S_2)(I-P)))$ commutes with $\Pi(P)$.
Moreover we have $\Pi(PT) = \Pi(T_1 \oplus 0)$ is quasinilpotent in $C(X)$ because $T_1 \oplus 0$ is a Riesz operator. Thus T is a left pseudo semi-B-Fredholm operator □

Following the same method as in Theorem 2.5, we obtain the following characterization of right pseudo semi-B-Fredholm operators. Due to the similarity of its proof with that of Theorem 2.5, we give it without proof.

Theorem 2.6. *Let $T \in L(X)$. Then the following properties are equivalent:*

1. *T is a right pseudo semi-B-Fredholm operator.*
2. *T is a compact perturbation of a Riesz-right semi-Fredholm operator.*

Combining Theorem 2.5 and Theorem 2.6, we obtain the following natural characterization of pseudo B-Fredholm operators.

Theorem 2.7. *Let $T \in L(X)$. Then the following properties are equivalent:*

1. *T is a pseudo B-Fredholm operator.*
2. *T is left and right pseudo semi-B-Fredholm operator.*

Proof. 1) $\Rightarrow$ 2) Obviously if we assume that T is a pseudo B-Fredholm operator, then T is left and right pseudo semi-B-Fredholm operator.

2) $\Rightarrow$ 1) Assume now that T is left and right pseudo semi-B-Fredholm operator. So there exists an idempotent p in $C(X)$ such that $\Pi(T)$ is left p-invertible and $\Pi(T)p$ is quasinilpotent. We have $\Pi(T) = p\Pi(T)p + (e-p)\Pi(T)(e-p) = x_1 + x_2$, where $x_1 = p\Pi(T)p$, $x_2 = (e-p)\Pi(T)(e-p)$ and $e = \Pi(I)$ is the identity element of $C(X)$. We know that $pC(X)p$ is a closed subalgebra of $C(X)$ having p as a unit element and $(e-p)C(X)(e-p)$ is a closed subalgebra of $C(X)$ having $e-p$ as a unit element. Now since x_1 is quasinilpotent in $pC(X)p$ and x_2 is left invertible in $(e-p)C(X)(e-p)$, then if $\lambda \in \mathbb{C} \setminus \{0\}$ and $|\lambda|$ is small enough, we have $\Pi(T) - \lambda e = (x_1 - \lambda p) + (x_2 - \lambda(e-p))$ is left invertible in $C(X)$.

Similarly as $\Pi(T)$ is right generalized Drazin invertible in $C(X)$, we can prove that if $\lambda \in \mathbb{C} \setminus \{0\}$ and $|\lambda|$ is small enough, then $\Pi(T) - \lambda e$ is right invertible in $C(X)$.

Consequently if $\lambda \in \mathbb{C} \setminus \{0\}$ and $|\lambda|$ is small enough, then $\Pi(T) - \lambda e$ is invertible. So if $0 \in \sigma(\Pi(T))$, then it is an isolated point. From [15, Theorem 3.1], it follows that $\Pi(T)$ is generalized Drazin invertible in $C(X)$ and so T is a pseudo B-Fredholm operator. □

Remark 2.8.

1. Theorem 2.7 gives a left and right versions of [9, Theorem 6.1, i)], where pseudo B-Fredholm operators where characterized.
2. In the recent works [2], [12] and [17], the authors studied operators similar to pseudo semi-B-Fredholm operators considered here. They defined them as the direct sum of a semi-Fredholm operator and a quasi-nilpotent one. However there exists operators which are pseudo semi-B-Fredholm operators in the sense of Definition 2.3, but do not have a decomposition as the direct sum of a semi-Fredholm operator and a quasi-nilpotent operator as seen by the Example 2.9.
3. The set of essentially left (right) generalized Drazin invertible operators was introduced and investigated in [13], An operator is essentially left (right)generalized Drazin invertible operator if and only if it is a direct sum a quasinilpotent operator and a left (right) semi-Fredholm operator. Example 2.9 shows that the set of essentially left (right) generalized Drazin invertible operators, is contained strictly in the set of left (right) pseudo semi-B-Fredholm operators.

Example 2.9. *Let T be a compact operator having infinite spectrum. Since $\Pi(T) = 0$, then T is a pseudo B-Fredholm operator in the sense of Definition 2.3. We prove that T cannot be written as the direct sum of a semi-Fredholm operator and a quasi-nilpotent one.*

Assume that there exists a pair $(M, N) \in Red(T)$ such that $T = T_1 \oplus T_2$ where $T_1 = T_{|M}$ is a quasi-nilpotent operator and $T_2 = T_{|N}$ is a semi-Fredholm operator. As T is compact, T_2 is also compact. Then necessarily N is finite dimensional and then $\sigma(T_2)$ is finite. Since $\sigma(T) = \sigma(T_1) \cup \sigma(T_2)$, it follows that $\sigma(T)$ is finite and this is a contradiction.

3. On Weak B-Fredholm Operators

Definition 3.1.

1. *$T \in L(X)$ is called a power compact-left semi-Fredholm operator if there exists $(M, N) \in Red(T)$ such that $T_{|M}$ is a power compact operator and $T_{|N} \in \Phi_l(N)$.*

2. *$T \in L(X)$ is called a power compact-right semi-Fredholm operator if there exists $(M, N) \in Red(T)$ such that $T_{|M}$ is a power compact operator and $T_{|N} \in \Phi_r(N)$.*
3. *Following [9, Section 6, p. 133], $T \in L(X)$ is called a power compact-Fredholm operator if there exists $(M, N) \in Red(T)$ such that $T_{|M}$ is a power compact operator and $T_{|N}$ is a Fredholm operator.*

Definition 3.2. *[6, Definition 2.11] Let A be a Banach algebra. We will say that an element $a \in A$ is left Drazin invertible (respectively right generalized Drazin invertible) if there exists an idempotent $p \in A$ such that a is left p-invertible (respectively right p-invertible) and ap is nilpotent.*

Definition 3.3. *Let $T \in L(X)$. We will say that:*

1. *T is a left weak semi-B-Fredholm operator if $\Pi(T)$ is left Drazin invertible in $C(X)$.*
2. *T is a right weak semi-B-Fredholm operator if $\Pi(T)$ is right Drazin invertible in $C(X)$.*
3. *T is a weak semi-B-Fredholm operator if it is right or left weak semi-B-Fredholm operator in $C(X)$.*
4. *T is a weak B-Fredholm operator if $\Pi(T)$ is Drazin invertible in $C(X)$.*

Remark 3.4. As each Drazin invertible element in the algebra $L(X)/F_0(X)$ is a Drazin invertible element in the Calkin algebra, the class of B-Fredholm operators is contained in the class of weak B-Fredholm operators.

Let $(\lambda_n)_n$ be a sequence in $\mathbb{C}$ such that $\lambda_n \neq 0$ for all n and $\lambda_n \longrightarrow 0$ as $n \to \infty$ and let us consider the operator T defined on the Hilbert space $l^2(\mathbb{N})$ by:

$$T(\xi_1, \xi_2, \xi_3, ...) = (\lambda_1\xi_1, \lambda_2\xi_2, \lambda_3\xi_3, ...).$$

Then

$$T^n(\xi_1, \xi_2, \xi_3, ...) = ((\lambda_1)^n\xi_1, (\lambda_2)^n\xi_2, (\lambda_3)^n\xi_3, ...).$$

Since $(\lambda_m)^n \neq 0$ for all $m \geq 0$ and $(\lambda_m)^n \longrightarrow 0$ as $m \longrightarrow \infty$ for all $n \geq 0$ then $K^n \in K(X)$ and K^n is not a finite rank operator for all $n \geq 1$. Hence $R(K^n)$ is not closed for all $n \geq 1$. Thus K is not a B-Fredholm operator. However as $\Pi(K) = 0$ is Drazin invertible in $C(X)$, K is a weak B-Fredholm operator.

Thus the class of weak B-Fredholm operators $\Phi_{\mathcal{WB}}(X)$ contains strictly the class of B-Fredholm operators $\Phi_B(X)$, which itself contains strictly the class of Fredholm operators $\Phi(X)$ as seen in [7] and Obviously the class of pseudo B-Fredholm operators $\Phi_{\mathcal{PB}}(X)$ contains the class of weak B-Fredholm operators $\Phi_{\mathcal{WB}}(X)$.

Let T be a Riesz operator which is not power compact, [10, Theorem 4.2] gives an example of such operator. Then $T \in \Phi_{\mathcal{PB}}(X)$ but $T \notin \Phi_{\mathcal{WB}}(X)$ and so $\Phi_{\mathcal{WB}}(X) \subsetneq \Phi_{\mathcal{PB}}(X)$.

Moreover let $\Phi_{\mathbb{P}}(X) = \{T \in L(X) \mid \exists P,\ P^2 = P \text{ and } T \text{ is P-Fredholm}\}$ be the class of all P-Fredholm operators, when P varies in the set of all idempotents of $L(X)$. Then:

$$\Phi_{\mathbb{P}}(X) = \bigcup_{\{P \in L(X) | P^2 = P\}} \Phi_P(X)$$

and $\Phi_{\mathbb{P}}(X)$ contains strictly the class $\Phi_{\mathcal{PB}}(X)$ as shown by the following example.

Example 3.5. *Let $T : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be the forward shift operator defined by:*

$$T(x_1, x_2, x_3, ...) = (0, x_1, x_2, x_3, ...).$$

Then from [1, Problem 7.5.3], the essential spectrum of T coincides with the unit circle, that is, $\sigma_e(T) = \{z \in \mathbb{C} \mid |z| = 1\}$.

Let $S = T - I$, then $0 \in \sigma_e(S)$ and 0 is not isolated in $\sigma_e(S)$. Thus S is not a pseudo B-Fredholm operator but $S + I$ is a Fredholm operator and I commutes with S. Hence $S \in \Phi_I(X)$ and so $S \in \Phi_{\mathbb{P}}(X)$ but $S \notin \Phi_{\mathcal{PB}}(X)$.

Thus we have the following strict inclusions between these different classes :

$$\Phi(X) \subsetneq \Phi_B(X) \subsetneq \Phi_{\mathcal{WB}}(X) \subsetneq \Phi_{\mathcal{PB}}(X) \subsetneq \Phi_{\mathbb{P}}(X).$$

Similar inclusion relations can be obtained by considering the left and right versions of the previous classes.

Theorem 3.6. *Let $T \in L(X)$. Then the following properties are equivalent:*

1. *T is a left weak semi-B-Fredholm operator.*
2. *T is a compact perturbation of a power compact-left semi-Fredholm operator.*

Proof. Since a power compact operator is a Riesz operator, the proof of this result can be obtained by adapting the proof Theorem 2.5 to this case. □

Theorem 3.7. *Let $T \in L(X)$. Then the following properties are equivalent:*

1. *T is a right weak semi-B-Fredholm operator.*
2. *T is a compact perturbation of a power compact-right semi-Fredholm operator.*

Proof. Since a power compact operator is a Riesz operator, the proof of this result can be obtained by adapting the proof Theorem 2.5 to this case. □

Combining Theorem 3.6 and Theorem 3.7, we obtain the following natural characterization of pseudo B-Fredholm operators.

Theorem 3.8. *Let $T \in L(X)$. Then the following properties are equivalent:*

1. *T is a weak B-Fredholm operator.*
2. *T is a left and right weak semi-B-Fredholm operator.*

Proof. 1) ⇒ 2) Assume that T is a weak B-Fredholm operator. So $\Pi(T)$ is Drazin invertible in $\mathcal{C}(X)$ and from [16, Proposition 4.9], there exists an idempotent $p \in \mathcal{C}(X)$ such that $\Pi(T)$ is p-invertible and $\Pi(T)p$ is nilpotent. So $\Pi(T)$ is left Drazin invertible and right Drazin invertible in $\mathcal{C}(X)$ and T is a left and right weak semi-B-Fredholm operator.

2) ⇒ 1) Conversely assume now that $\Pi(T)$ is left and right Drazin invertible in $\mathcal{C}(X)$, so $\Pi(T)$ is left and right generalized Drazin invertible in $\mathcal{C}(X)$. From Theorem 2.7, it follows that $\Pi(T)$ is generalized Drazin invertible in $\mathcal{C}(X)$. Then from [15, Theorem 3.1] there exists an idempotent $p \in \mathcal{C}(X)$, commuting with $\Pi(T)$ such that $\Pi(T)p$ is quasi-nilpotent and $\Pi(T) + p$ is invertible. From [15, Theorem 3.1], we know that p lies in the closed subalgebra of $\mathcal{C}(X)$ generated by $\Pi(T)$.

Now as $\Pi(T)$ is left Drazin invertible in $\mathcal{C}(X)$, there exists an idempotent $p_0 \in \mathcal{C}(X)$, commuting with $\Pi(T)$ such that $\Pi(T)p_0$ is nilpotent and $\Pi(T) + p_0$ is let invertible. Assume that $(\Pi(T)p_0)^n = (\Pi(T))^n p_0 = 0$, for an integer $n > 0$. Since p_0 commutes with $\Pi(T)$, it commutes also with p. We have:

$$(\pi(T))^n p = [(\pi(T))^n p_0 + (\pi(T))^n (e - p_0)]p = [(\pi(T))^n (e - p_0)]p,$$

where $e = \Pi(I)$ is the identity element of $\mathcal{C}(X)$. From [6, Theorem 2.15], it follows that $(\pi(T))^n (e - p_0)$ is left invertible in the closed subalgebra $(e - p_0)\mathcal{C}(X)(e - p_0)$ of $\mathcal{C}(X)$. So there exists $b \in (e - p_0)\mathcal{C}(X)(e - p_0)$ such that $b(\pi(T))^n (e - p_0)) = e - p_0$. Hence:

$$b^m((\pi(T))^n)^m (e - p_0)p = (e - p_0)p \text{ and } b^m((\pi(T))^n p)^m (e - p_0) = p(e - p_0),$$

for all integer $m > 0$. As p commutes with p_0, then $p(e - p_0)$ is an idempotent. If $p(e - p_0) \neq 0$, then $\|(e - p_0)p\| \geq 1$. So $1 \leq \|b\| \|(\pi(T)^n p)^m\|^{\frac{1}{m}} \|e - p_0\|^{\frac{1}{m}}$ for all integer $m > 0$. However this is impossible since $(\pi(T))^n p$ is quasinilpotent. So necessarily $(e - p_0)p = 0$ and $((\pi(T))^n p) = 0$. Thus $\pi(T)p$ is nilpotent and $\Pi(T)$ is Drazin invertible in $\mathcal{C}(X)$. Therefore T is a weak B-Fredholm operator. □

Remark 3.9.
1. Theorem 3.8 gives a left and right versions of [9, Theorem 6.1, ii)], where weak B-Fredholm operators where characterized.
2. The set of essentially left (right) Drazin invertible operators was introduced and investigated in [18]. An operator is essentially left (right) Drazin invertible operator if and only if it is a direct sum of a nilpotent operator and a left (right) semi-Fredholm operator. Example 3.9 shows that the class of essentially left (right) Drazin invertible operators, is contained strictly in the class of left (right) weak semi-B-Fredholm operators.

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