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Properties of harmonically-h-convex functions related to the Hermite-Hadamard-Fejér type inequalities

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Abstract. The Hermite-Hadamard-Fejér inequalities for a harmonically-h-convex function are explored in this study, and the findings for specific classes of functions are highlighted. Several generalizations of the Hermite-Hadamard inequalities are also discussed. Some properties of functions $\mathcal{H}$ and $\mathcal{F}$, which are naturally defined for Hermite-Hadamard-Fejér type inequalities for harmonically-h-convex functions, have also been studied. Finally, we find applications of the results concerning the p-logarithmic mean and the order p mean.

1. Introduction

Mathematical inequalities have emerged as a new area of study mathematicians in recent decades. This topic has far-reaching implications for mathematical analysis, functional analysis, numerical analysis, applied mathematics, physics, and other applied sciences. This subject is so rich in mathematical reasoning due to which many new proofs of classical results have arisen in mathematical literature. This topic is largely reliant on convex sets and convex functions, as well as its innovative generalizations. The following is the classical definition of convex functions:

Definition 1.1. *Let $\varnothing \neq I \subseteq \mathbb{R}$. The function $\chi : I \to \mathbb{R}$ is said to be convex on I if for all $\mu, \varsigma \in I$ and $\alpha \in [0,1]$, one has the inequality:*

$$\chi(\alpha\mu + (1-\alpha)\varsigma) \leq \alpha\chi(\mu) + (1-\alpha)\chi(\varsigma).$$

Let $\chi : \varnothing \neq I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function. Then the following double inequality:

$$\chi\left(\frac{\rho+\sigma}{2}\right) \leq \frac{1}{\sigma-\rho}\int_{\rho}^{\sigma}\chi(\mu)d\mu \leq \frac{\chi(\rho)+\chi(\sigma)}{2} \tag{1}$$

holds for $\rho, \sigma \in I$ with $\rho < \sigma$. This is known as the Hermite-Hadamard inequality for convex mapping. The inequalities in (1) hold in reversed direction if χ is a concave function on I.

In [22], Hudzik and Maligranda considered some properties of two classes of s-convex real valued functions already exist in the literature.

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Definition 1.2. *[22] A function $\chi:[0,\infty)\to\mathbb{R}$ is said to be s-convex in the first sense if*

$$\chi(\alpha\mu+(1-\alpha)\varsigma)\le\alpha^s\chi(\mu)+(1-\alpha^s)\chi(\varsigma)$$

for all $\mu,\varsigma\in[0,\infty)$ and all $\alpha\in[0,1]$. The class K_s^1 contains all the s-convex in the first sense.

Definition 1.3. *[22] A function $\chi:[0,\infty)\to\mathbb{R}$ is said to be s-convex in the second sense if*

$$\chi(\alpha\mu+(1-\alpha)\varsigma)\le\alpha^s\chi(\mu)+(1-\alpha)^s\chi(\varsigma)$$

for all $\mu,\varsigma\in[0,\infty)$ and all $\alpha\in[0,1]$. The class K_s^2 contains all the s-convex in the second sense.

Remark 1.4. *It is clear that s-convexity in the first sense and in the second sense mean just the convexity when $s=1$.*

Dragomir and Fitzpatrick demonstrated in [22] the following form of Hermite-Hadamard's inequality for s-convex functions in the second sense:

Theorem 1.5. *[22] Suppose that $\chi:[0,\infty)\to[0,\infty)$ is an s-convex function in the second sense, where $s\in(0,1)$ and let $\rho,\sigma\in[0,\infty)$, $\rho<\sigma$. If $\chi\in L_1([\rho,\sigma])$ then the following inequalities hold:*

$$2^{s-1}\chi\left(\frac{\rho+\sigma}{2}\right)\le\frac{1}{\sigma-\rho}\int_\rho^\sigma\chi(\mu)d\mu\le\frac{\chi(\rho)+\chi(\sigma)}{s+1} \tag{2}$$

The constant $\frac{1}{s+1}$ is the best possible in the second inequality in (2).

In the paper Varošanec [35], considered a larger class of non-negative functions, which is known as h-convex functions. This class contains several well-known classes of functions such as non-negative convex functions, s-convex in the second sense, Godunova-Levin functions and P-functions.

Definition 1.6. *[35] Let $h:J\subseteq\mathbb{R}\to\mathbb{R}$, where $(0,1)\subseteq J$, be a positive function. A function $\chi:I\subseteq\mathbb{R}\to\mathbb{R}$ is said to be h-convex or that χ is said to belong to the class $SX(h,I)$, if χ is non-negative and for all $\mu,\varsigma\in I$ and $\alpha\in(0,1)$, we have*

$$\chi(\alpha\mu+(1-\alpha)\varsigma)\le h(\alpha)\chi(\mu)+h(1-\alpha)\chi(\varsigma). \tag{3}$$

If the inequality (3) is reversed then χ is said to be h-concave and we say that χ belongs to the class $SV(h,I)$.

Fejér [15], established the following double inequality as a weighted generalization of (1):

$$\chi\left(\frac{\rho+\sigma}{2}\right)\int_\rho^\sigma\vartheta(\mu)d\mu\le\int_\rho^\sigma\chi(\mu)\vartheta(\mu)d\mu\le\frac{\chi(\rho)+\chi(\sigma)}{2}\int_\rho^\sigma\vartheta(\mu)d\mu, \tag{4}$$

where $\chi:I\longrightarrow\mathbb{R}$, $\emptyset\ne I\subseteq\mathbb{R}$, $\rho,\sigma\in I$ with $\rho<\sigma$ is any convex function and $\vartheta:[\rho,\sigma]\to\mathbb{R}$ is non-negative integrable and symmetric about $\mu=\frac{\rho+\sigma}{2}$.

When the class of convex functions is extended to the class of h-convex functions, the features associated with the integral mean of the function χ do not change, according to Bombardelli and Varošanec [1]. The authors additionally illustrated the Hermite-Hadamard-Fejér inequality for an h-convex function, as well as specific examples for other classes of function such as convex functions and s-convex functions. It was also discovered in this study that the left-hand side inequality of their result is stronger than the right-hand side inequality. This research also includes various features of the following functions:

$$H(t)=\frac{1}{\sigma-\rho}\int_\rho^\sigma\chi\left(t\mu+(1-t)\frac{\rho+\sigma}{2}\right)d\mu$$

and

$$F(t)=\frac{1}{(\sigma-\rho)^2}\int_\rho^\sigma\int_\rho^\sigma \chi(t\mu+(1-t)\varsigma)\,d\mu d\varsigma$$

that arise when the function χ is an h-convex function.

We recall that harmonically convex functions, also known as HA-convex functions, are prominent generalizations of convex functions.

Definition 1.7. *[23] A function $\chi : I \subseteq (0,\infty) \to \mathbb{R}$ is considered to be harmonically-convex, if*

$$\chi\left(\frac{\mu\varsigma}{(1-\alpha)\mu+\alpha\varsigma}\right)\leq \alpha\chi(\mu)+(1-\alpha)\chi(\varsigma) \tag{5}$$

for all $\mu,\varsigma \in I$ and $\alpha \in (0,1)$. The function $\chi : I \to \mathbb{R}$ is HA-concave if the inequality in (5) hold in reversed.

The results below include some significant information underlying HA-convex and convex functions.

Theorem 1.8. *[9, 10] If $[\rho,\sigma] \subset I \subset (0,\infty)$ and if we consider the function $g : \left[\frac{1}{\sigma},\frac{1}{\rho}\right] \to \mathbb{R}$ defined by $g(t) = (\chi\circ k)(t)$, where $k(t)=\frac{1}{t}$, then χ is harmonically convex on $[\rho,\sigma]$ if and only if g is convex in the usual sense on $\left[\frac{1}{\sigma},\frac{1}{\rho}\right]$.*

Theorem 1.8 can easily be extended to harmonically h-convex functions as follows:

Theorem 1.9. *If $[\rho,\sigma] \subset I \subset (0,\infty)$ and if we consider the function $g : \left[\frac{1}{\sigma},\frac{1}{\rho}\right] \to \mathbb{R}$ defined by $g(t) = (\chi\circ k)(t)$, where $k(t)=\frac{1}{t}$, then χ is harmonically h-convex on $[\rho,\sigma]$ if and only if g is h-convex in the usual sense on $\left[\frac{1}{\sigma},\frac{1}{\rho}\right]$.*

Theorem 1.10. *[9, 10] If $I \subset (0,\infty)$ and χ is convex and nondecreasing function then χ is HA-convex and if χ is HA-convex and nonincreasing function then χ is convex.*

The following inequality of Hermite-Hadamard type for HA-convex (harmonically-convex) functions holds (see for instance [33] its extension for for an HA-h-convex functions):

Theorem 1.11. *[23] Let $\chi : I \subseteq (0,\infty) \to \mathbb{R}$ be a HA-convex function and $\rho,\sigma \in I$ with $\rho<\sigma$. If $\chi \in L([\rho,\sigma])$ then the following inequalities hold:*

$$\chi\left(\frac{2\rho\sigma}{\rho+\sigma}\right)\leq\frac{\rho\sigma}{\sigma-\rho}\int_\rho^\sigma\frac{\chi(\mu)}{\mu^2}d\mu\leq\frac{\chi(\rho)+\chi(\sigma)}{2}. \tag{6}$$

The notion of harmonically symmetric functions was introduced in [31].

Definition 1.12. *[31] A function $\vartheta : [\rho,\sigma] \subseteq (0,\infty) \to \mathbb{R}$ is harmonically symmetric with respect to $(0,\infty)$ if*

$$\vartheta(\mu)=\vartheta\left(\frac{1}{\frac{1}{\rho}+\frac{1}{\sigma}-\frac{1}{\mu}}\right)$$

holds for all $\mu \in [\rho,\sigma]$.

Fejér type inequalities using HA-convex functions using harmonically symmetric functions were presented in Latif et al. [31].

Theorem 1.13. *[31] Let $\chi : I \subseteq (0,\infty) \to \mathbb{R}$ be a HA-convex function and $\rho, \sigma \in I$ with $\rho < \sigma$. If $\chi \in L([\rho,\sigma])$ and $\vartheta : [\rho,\sigma] \subseteq (0,\infty) \to \mathbb{R}$ is nonnegative integrable harmonically symmetric with respect to $\frac{2\rho\sigma}{\rho+\sigma}$, then*

$$\chi\left(\frac{2\rho\sigma}{\rho+\sigma}\right)\int_\rho^\sigma \frac{\vartheta(\mu)}{\mu^2}d\mu \le \int_\rho^\sigma \frac{\chi(\mu)\vartheta(\mu)}{\mu^2}d\mu \le \frac{\chi(\rho)+\chi(\sigma)}{2}\int_\rho^\sigma \frac{\vartheta(\mu)}{\mu^2}d\mu. \tag{7}$$

Noor et all. [33] considered a larger class of *HA*-convex functions, know as the class of *HA-h*-convex functions. This class contains many classes of functions such as non-negative *HA*-convex functions, *HA-s*-convex in the second sense, *HA*-Godunova-Levin functions and *HA-P*-functions. This class is defined as:

Definition 1.14. *[33] Let $h : [0,1] \to [0,\infty)$ be a non-negative function. A function $\chi : I \subseteq (0,\infty) \to \mathbb{R}$ is said to be HA-h-convex, if*

$$\chi\left(\frac{\mu\varsigma}{(1-\alpha)\mu+\alpha\varsigma}\right) \le h(\alpha)\chi(\mu) + h(1-\alpha)\chi(\varsigma) \tag{8}$$

for all $\mu, \varsigma \in I$ and $\alpha \in (0,1)$. The function $\chi : I \to \mathbb{R}$ is HA-h-concave if the inequality in (8) is reversed. Note that harmonically-h-convex function becomes harmonically-s-convex function for $h(\alpha) = s$ and harmonically-h-convex function is reduced to harmonically-P-function for $h(\alpha) = 1$.

The interested readers are referred to [33] for integral inequalities for the class of *HA-h*-convex functions.

The main motivation of this research is the study given in Bombardelli and Varošanec [1]. In the next section, we will prove that there will be no change in the properties of $\frac{\rho\sigma}{\sigma-\rho}\int_\rho^\sigma \frac{\chi(\mu)}{\mu^2}d\mu$ if the class of *HA*-convex functions is extended to the class of *HA-h*-convex functions. We will also prove Hermite-Hadamard-Fejér type inequalities for an *HA-h*-convex function and we will consider some special cases for other classes of functions such as *HA*-convex functions and *HA-s*-convex functions.We will discuss in this research that in our obtained result, the left-hand side of the inequality is stronger than the right-hand side of the inequality. Lastly, some properties will be observed as well for the mappings $\mathcal{H}, \mathcal{F} : [0,1] \to \mathbb{R}$ be defined by

$$\mathcal{H}(t) = \frac{\rho\sigma}{\sigma-\rho}\int_\rho^\sigma \frac{1}{\mu^2}\chi\left(\frac{2\rho\sigma\mu}{2\rho\sigma t + (1-t)\mu(\rho+\sigma)}\right)d\mu,$$

$$\mathcal{F}(t) = \left(\frac{\rho\sigma}{\sigma-\rho}\right)^2\int_\rho^\sigma\int_\rho^\sigma \frac{1}{\mu^2\varsigma^2}\chi\left(\frac{\mu\varsigma}{(1-t)\mu+t\varsigma}\right)d\mu d\varsigma,$$

where $\chi : I \subseteq (0,\infty) \to \mathbb{R}$ is *HA*-convex on I and $\rho, \sigma \in I$.

2. The Hermite Hadamard Fejér inequalities for a *HA-h*-convex function

We begin this section with the following Hermite-Hadamard-Fejér Inequality for an *HA-h*-convex function.

Theorem 2.1. *Let $\chi : [\rho,\sigma] \subseteq (0,\infty) \to \mathbb{R}$ be an HA-h-convex function and $\vartheta : [\rho,\sigma] \to \mathbb{R}$ be non-negative, integrable and symmetric with respect to $\frac{2\rho\sigma}{\rho+\sigma}$. Then*

$$\frac{\rho\sigma}{\sigma-\rho}\int_\rho^\sigma \frac{\chi(\mu)\vartheta(\mu)}{\mu^2}d\mu \le [\chi(\rho)+\chi(\sigma)]\int_0^1 h(t)\vartheta\left(\frac{\rho\sigma}{(1-t)\rho+t\sigma}\right)dt. \tag{9}$$

If $\chi : [\rho,\sigma] \subseteq (0,\infty) \to \mathbb{R}$ is an HA-h-concave function, then the inequality in (9) holds in reversed direction.

Proof. Let $\mu \in (\rho,\sigma)$ there exists $\alpha \in (0,1)$ such that $\mu = \frac{\rho\sigma}{\bar{\alpha}\rho+\alpha\sigma}$, where $\bar{\alpha} = 1-\alpha$. Since χ is a *HA-h*-convex function, we have

$$\chi\left(\frac{\rho\sigma}{\bar{\alpha}\rho+\alpha\sigma}\right)\vartheta\left(\frac{\rho\sigma}{\bar{\alpha}\rho+\alpha\sigma}\right) \le \left[h(\alpha)\chi(\rho)+h(\bar{\alpha})\chi(\sigma)\right]\vartheta\left(\frac{\rho\sigma}{\bar{\alpha}\rho+\alpha\sigma}\right) \tag{10}$$

and

$$\chi\left(\frac{\rho\sigma}{\alpha\rho+\bar{\alpha}\sigma}\right)\vartheta\left(\frac{\rho\sigma}{\alpha\rho+\bar{\alpha}\sigma}\right) \le \left[h(\bar{\alpha})\chi(\rho)+h(\alpha)\chi(\sigma)\right]\vartheta\left(\frac{\rho\sigma}{\alpha\rho+\bar{\alpha}\sigma}\right). \tag{11}$$

Adding (10) and (11) and integrating with respect to α over the interval $[0,1]$, we obtain

$$\begin{aligned}
&\int_0^1 \chi\left(\frac{\rho\sigma}{\bar{\alpha}\rho+\alpha\sigma}\right)\vartheta\left(\frac{\rho\sigma}{\bar{\alpha}\rho+\alpha\sigma}\right)d\alpha + \int_0^1 \chi\left(\frac{\rho\sigma}{\alpha\rho+\bar{\alpha}\sigma}\right)\vartheta\left(\frac{\rho\sigma}{\alpha\rho+\bar{\alpha}\sigma}\right)d\alpha \\
&\le \chi(\rho)\int_0^1 h(\alpha)\vartheta\left(\frac{\rho\sigma}{\bar{\alpha}\rho+\alpha\sigma}\right)d\alpha + \chi(\rho)\int_0^1 h(\bar{\alpha})\vartheta\left(\frac{\rho\sigma}{\alpha\rho+\bar{\alpha}\sigma}\right)d\alpha \\
&\qquad + \chi(\sigma)\int_0^1 h(\bar{\alpha})\vartheta\left(\frac{\rho\sigma}{\bar{\alpha}\rho+\alpha\sigma}\right)d\alpha + \chi(\sigma)\int_0^1 h(\alpha)\vartheta\left(\frac{\rho\sigma}{\alpha\rho+\bar{\alpha}\sigma}\right)d\alpha.
\end{aligned} \tag{12}$$

By making use of the substitution $\bar{\alpha} = t$ in (12) and using the assuption that ϑ is symmetric with respect to $\frac{2\rho\sigma}{\rho+\sigma}$, we have

$$\begin{aligned}
&\int_0^1 \chi\left(\frac{\rho\sigma}{t\rho+(1-t)\sigma}\right)\vartheta\left(\frac{\rho\sigma}{t\rho+(1-t)\sigma}\right)dt + \int_0^1 \chi\left(\frac{\rho\sigma}{(1-t)\rho+t\sigma}\right)\vartheta\left(\frac{\rho\sigma}{(1-t)\rho+t\sigma}\right)dt \\
&\le 2\chi(\rho)\int_0^1 h(t)\vartheta\left(\frac{\rho\sigma}{(1-t)\rho+t\sigma}\right)dt + 2\chi(\sigma)\int_0^1 h(t)\vartheta\left(\frac{\rho\sigma}{t\rho+(1-t)\sigma}\right)dt \\
&= 2\left[\chi(\rho)+\chi(\sigma)\right]\int_0^1 h(t)\vartheta\left(\frac{\rho\sigma}{t\rho+(1-t)\sigma}\right)dt.
\end{aligned} \tag{13}$$

By using the change of variable techniques, we observe that each integral on right hand side of (13) is equal to $\frac{\rho\sigma}{\sigma-\rho}\int_\rho^\sigma \frac{\chi(\mu)\vartheta(\mu)}{\mu^2}d\mu$. Hence the theorem is established. □

Remark 2.2. *If in Theorem 2.1*

(i) The function χ is convex, that is, if $h(t) = t$, then

$$\int_\rho^\sigma \frac{\chi(\mu)\vartheta(\mu)}{\mu^2}d\mu \le \left(\frac{\rho\sigma}{\sigma-\rho}\right)\left[\chi(\rho)+\chi(\sigma)\right]\int_\rho^\sigma \left(\frac{1}{\mu}-\frac{1}{\sigma}\right)\frac{\vartheta(\mu)}{\mu^2}d\mu. \tag{14}$$

(ii) The function χ is an s-convex, that is, if $h(t) = t^s$, $s \in (0,1)$, then

$$\int_\rho^\sigma \frac{\chi(\mu)\vartheta(\mu)}{\mu^2}d\mu \le \left(\frac{\rho\sigma}{\sigma-\rho}\right)\left[\chi(\rho)+\chi(\sigma)\right]\int_\rho^\sigma \left(\frac{1}{\mu}-\frac{1}{\sigma}\right)^s\frac{\vartheta(\mu)}{\mu^2}d\mu. \tag{15}$$

Theorem 2.3. *Let h be defined over the interval $\left[0,\max\left\{0,\frac{\sigma-\rho}{\rho\sigma}\right\}\right]$ and $\chi : [\rho,\sigma] \subseteq (0,\infty) \to \mathbb{R}$ be a HA-h-convex function and $\vartheta : [\rho,\sigma] \to \mathbb{R}$ be non-negative, integrable and symmetric with respect to $\frac{2\rho\sigma}{\rho+\sigma}$ with $\int_\rho^\sigma \frac{\vartheta(t)}{t^2}dt > 0$. Then*

$$\chi\left(\frac{2\rho\sigma}{\rho+\sigma}\right) \le C\int_\rho^\sigma \frac{\chi(\mu)\vartheta(\mu)}{\mu^2}d\mu, \tag{16}$$

where $C = \frac{2h\left(\frac{1}{2}\right)}{\int_{\rho}^{\sigma}\frac{\vartheta(\mu)}{\mu^2}d\mu}$.

Furthermore, if $\int_{\frac{2\rho\sigma}{\rho+\sigma}}^{\sigma}\int_{\rho}^{\frac{2\rho\sigma}{\rho+\sigma}} h\left(\frac{1}{\varsigma}-\frac{1}{\mu}\right)d\varsigma d\mu \neq 0$, $h(\mu)\neq 0$ *for* $\mu>0$ *and*

(i) If h is multiplicative or

(ii) If h is supermultiplicative and χ *is non-negative*
and if χ *is an HA-h-convex function, then inequality (16) holds for*

$$C=\min\left\{\frac{2h\left(\frac{1}{2}\right)}{\int_{\rho}^{\sigma}\frac{\vartheta(\mu)}{\mu^2}d\mu},\frac{\int_0^{\frac{\sigma-\rho}{2\rho\sigma}}h(\mu)\,\vartheta\left(\frac{2\rho\sigma}{\rho+\sigma+2\rho\sigma\mu}\right)d\mu}{\int_{\frac{2\rho\sigma}{\rho+\sigma}}^{\sigma}\int_{\rho}^{\frac{2\rho\sigma}{\rho+\sigma}}\frac{h\left(\frac{1}{\varsigma}-\frac{1}{\mu}\right)\vartheta(\mu)\vartheta(\varsigma)}{\mu^2\varsigma^2}d\varsigma d\mu}\right\}.$$

Proof. Since χ be a *HA-h*-convex function, then for $\alpha=\frac{1}{2}$, $\mu=\frac{\rho\sigma}{(1-t)\rho+t\sigma}$ and $\varsigma=\frac{\rho\sigma}{t\rho+(1-t)\sigma}$, from the definition of a *HA-h*-convex function, we have the following inequality

$$\chi\left(\frac{2\rho\sigma}{\rho+\sigma}\right)\leq h\left(\frac{1}{2}\right)\left[\chi\left(\frac{\rho\sigma}{(1-t)\rho+t\sigma}\right)+\chi\left(\frac{\rho\sigma}{t\rho+(1-t)\sigma}\right)\right].$$

Now we multiply it with $\vartheta\left(\frac{\rho\sigma}{t\rho+(1-t)\sigma}\right)=\vartheta\left(\frac{\rho\sigma}{(1-t)\rho+t\sigma}\right)$ and integrate by t over $[0,1]$ to obtain

$$\begin{aligned}&\chi\left(\frac{2\rho\sigma}{\rho+\sigma}\right)\int_0^1\vartheta\left(\frac{\rho\sigma}{(1-t)\rho+t\sigma}\right)dt\\&\leq h\left(\frac{1}{2}\right)\left[\int_0^1\chi\left(\frac{\rho\sigma}{(1-t)\rho+t\sigma}\right)\vartheta\left(\frac{\rho\sigma}{(1-t)\rho+t\sigma}\right)dt+\int_0^1\chi\left(\frac{\rho\sigma}{t\rho+(1-t)\sigma}\right)\vartheta\left(\frac{\rho\sigma}{t\rho+(1-t)\sigma}\right)dt\right].\end{aligned}\tag{17}$$

Making suitable substitution, we get that

$$\chi\left(\frac{2\rho\sigma}{\rho+\sigma}\right)\leq\frac{2h\left(\frac{1}{2}\right)}{\int_{\rho}^{\sigma}\frac{\vartheta(\mu)}{\mu^2}d\mu}\int_{\rho}^{\sigma}\frac{\chi(\mu)\,\vartheta(\mu)}{\mu^2}d\mu.$$

The equality (16) is thus established.
We oberve due to the h-convexity g on $\left[\frac{1}{\sigma},\frac{1}{\rho}\right]$ that

$$\begin{aligned}&\chi\left(\frac{2\rho\sigma}{\rho+\sigma}\right)=(\chi\circ k)\left(\frac{\rho+\sigma}{2\rho\sigma}\right)\\&=g\left(\frac{\rho+\sigma}{2\rho\sigma}\right)=g\left(\left(\frac{\frac{1}{\varsigma}-\frac{\rho+\sigma}{2\rho\sigma}}{\frac{1}{\varsigma}-\frac{1}{\mu}}\right)\frac{1}{\mu}+\left(\frac{\frac{\rho+\sigma}{2\rho\sigma}-\frac{1}{\mu}}{\frac{1}{\varsigma}-\frac{1}{\mu}}\right)\frac{1}{\varsigma}\right)\\&\leq h\left(\frac{\frac{1}{\varsigma}-\frac{\rho+\sigma}{2\rho\sigma}}{\frac{1}{\varsigma}-\frac{1}{\mu}}\right)g\left(\frac{1}{\mu}\right)+h\left(\frac{\frac{\rho+\sigma}{2\rho\sigma}-\frac{1}{\mu}}{\frac{1}{\varsigma}-\frac{1}{\mu}}\right)g\left(\frac{1}{\varsigma}\right)\\&\quad=h\left(\frac{\frac{1}{\varsigma}-\frac{\rho+\sigma}{2\rho\sigma}}{\frac{1}{\varsigma}-\frac{1}{\mu}}\right)(\chi\circ k)\left(\frac{1}{\mu}\right)+h\left(\frac{\frac{\rho+\sigma}{2\rho\sigma}-\frac{1}{\mu}}{\frac{1}{\varsigma}-\frac{1}{\mu}}\right)(\chi\circ k)\left(\frac{1}{\varsigma}\right)=h\left(\frac{\frac{1}{\varsigma}-\frac{\rho+\sigma}{2\rho\sigma}}{\frac{1}{\varsigma}-\frac{1}{\mu}}\right)\chi(\mu)+h\left(\frac{\frac{\rho+\sigma}{2\rho\sigma}-\frac{1}{\mu}}{\frac{1}{\varsigma}-\frac{1}{\mu}}\right)\chi(\varsigma).\end{aligned}\tag{18}$$

Let $\alpha=\frac{\frac{1}{\varsigma}-\frac{\rho+\sigma}{2\rho\sigma}}{\frac{1}{\varsigma}-\frac{1}{\mu}}$ and $\bar{\alpha}=1-\alpha=\frac{\frac{\rho+\sigma}{2\rho\sigma}-\frac{1}{\mu}}{\frac{1}{\varsigma}-\frac{1}{\mu}}$, hence (18) takes the form

$$\chi\left(\frac{2\rho\sigma}{\rho+\sigma}\right)\leq h(\alpha)\,\chi(\mu)+h(\bar{\alpha})\,\chi(\varsigma).\tag{19}$$

Since h is supermultiplicative, we have

$$h(\alpha)=h\left(\frac{\frac{1}{\varsigma}-\frac{\rho+\sigma}{2\rho\sigma}}{\frac{1}{\varsigma}-\frac{1}{\mu}}\right)\le\frac{h\left(\frac{1}{\varsigma}-\frac{\rho+\sigma}{2\rho\sigma}\right)}{h\left(\frac{1}{\varsigma}-\frac{1}{\mu}\right)}$$

and

$$h(\bar{\alpha})=h\left(\frac{\frac{\rho+\sigma}{2\rho\sigma}-\frac{1}{\mu}}{\frac{1}{\varsigma}-\frac{1}{\mu}}\right)\le\frac{h\left(\frac{\rho+\sigma}{2\rho\sigma}-\frac{1}{\mu}\right)}{h\left(\frac{1}{\varsigma}-\frac{1}{\mu}\right)}.$$

Thus (19) becomes

$$h\left(\frac{1}{\varsigma}-\frac{1}{\mu}\right)\chi\left(\frac{2\rho\sigma}{\rho+\sigma}\right)\le h\left(\frac{1}{\varsigma}-\frac{\rho+\sigma}{2\rho\sigma}\right)\chi(\mu)+h\left(\frac{\rho+\sigma}{2\rho\sigma}-\frac{1}{\mu}\right)\chi(\varsigma). \tag{20}$$

Multiplying (20) with $\frac{\vartheta(\mu)}{\mu^2}$ and integrating over interval $\left[\frac{2\rho\sigma}{\rho+\sigma},\sigma\right]$ with respect to μ and then multiplying with $\frac{\vartheta(\mu)}{\mu^2}$ and integrating over interval $\left[\rho,\frac{2\rho\sigma}{\rho+\sigma}\right]$ with respect to ς we get

$$\begin{aligned}
&\chi\left(\frac{2\rho\sigma}{\rho+\sigma}\right)\int_{\frac{2\rho\sigma}{\rho+\sigma}}^{\sigma}\int_{\rho}^{\frac{2\rho\sigma}{\rho+\sigma}}\frac{h\left(\frac{1}{\varsigma}-\frac{1}{\mu}\right)\vartheta(\mu)\vartheta(\varsigma)}{\mu^2\varsigma^2}d\varsigma d\mu\\
&\le\int_{\frac{2\rho\sigma}{\rho+\sigma}}^{\sigma}\frac{\chi(\mu)\vartheta(\mu)}{\mu^2}d\mu\int_{\rho}^{\frac{2\rho\sigma}{\rho+\sigma}}h\left(\frac{1}{\varsigma}-\frac{\rho+\sigma}{2\rho\sigma}\right)\frac{\vartheta(\varsigma)}{\varsigma^2}d\varsigma+\int_{\frac{2\rho\sigma}{\rho+\sigma}}^{\sigma}h\left(\frac{\rho+\sigma}{2\rho\sigma}-\frac{1}{\mu}\right)\frac{\vartheta(\mu)}{\mu^2}d\mu\int_{\rho}^{\frac{2\rho\sigma}{\rho+\sigma}}\frac{\chi(\varsigma)\vartheta(\varsigma)}{\varsigma^2}d\varsigma\\
&=\int_{\frac{2\rho\sigma}{\rho+\sigma}}^{\sigma}\frac{\chi(\mu)\vartheta(\mu)}{\mu^2}d\mu\int_{\rho}^{\frac{2\rho\sigma}{\rho+\sigma}}h\left(\frac{1}{\varsigma}-\frac{\rho+\sigma}{2\rho\sigma}\right)\frac{\vartheta(\varsigma)}{\varsigma^2}d\varsigma.
\end{aligned}\tag{21}$$

Substitution $t=\frac{1}{\varsigma}-\frac{\rho+\sigma}{2\rho\sigma}$ in the first integral and substitution $t=\frac{\rho+\sigma}{2\rho\sigma}-\frac{1}{\mu}$ in the second integral on the rifht hand side of (21), we get

$$\begin{aligned}
&\chi\left(\frac{2\rho\sigma}{\rho+\sigma}\right)\int_{\frac{2\rho\sigma}{\rho+\sigma}}^{\sigma}\int_{\rho}^{\frac{2\rho\sigma}{\rho+\sigma}}\frac{h\left(\frac{1}{\varsigma}-\frac{1}{\mu}\right)\vartheta(\mu)\vartheta(\varsigma)}{\mu^2\varsigma^2}d\varsigma d\mu\\
&\le\int_{\frac{2\rho\sigma}{\rho+\sigma}}^{\sigma}\frac{\chi(\mu)\vartheta(\mu)}{\mu^2}d\mu\int_{\rho}^{\frac{2\rho\sigma}{\rho+\sigma}}h\left(\frac{1}{\varsigma}-\frac{\rho+\sigma}{2\rho\sigma}\right)\frac{\vartheta(\varsigma)}{\varsigma^2}d\varsigma+\int_{\frac{2\rho\sigma}{\rho+\sigma}}^{\sigma}h\left(\frac{\rho+\sigma}{2\rho\sigma}-\frac{1}{\mu}\right)\frac{\vartheta(\mu)}{\mu^2}d\mu\int_{\rho}^{\frac{2\rho\sigma}{\rho+\sigma}}\frac{\chi(\varsigma)\vartheta(\varsigma)}{\varsigma^2}d\varsigma\\
&=\int_{\frac{2\rho\sigma}{\rho+\sigma}}^{\sigma}\frac{\chi(\mu)\vartheta(\mu)}{\mu^2}d\mu\int_{0}^{\frac{\sigma-\rho}{2\rho\sigma}}h(t)\vartheta\left(\frac{\rho+\sigma}{\rho+\sigma+2\rho\sigma t}\right)dt\\
&\quad+\int_{0}^{\frac{\sigma-\rho}{2\rho\sigma}}h(t)\vartheta\left(\frac{\rho+\sigma}{\rho+\sigma-2\rho\sigma t}\right)dt\int_{\frac{2\rho\sigma}{\rho+\sigma}}^{\sigma}\frac{\chi(\mu)\vartheta(\mu)}{\mu^2}d\mu.
\end{aligned}\tag{22}$$

Since the mapping ϑ is geometrically symmetric with respect to $\frac{2\rho\sigma}{\rho+\sigma}$, hence $\vartheta\left(\frac{\rho+\sigma}{\rho+\sigma+2\rho\sigma t}\right)=\vartheta\left(\frac{\rho+\sigma}{\rho+\sigma-2\rho\sigma t}\right)$ for all $t\in\left[0,\frac{\sigma-\rho}{2\rho\sigma}\right]$. Thus from (22), we get

$$\chi\left(\frac{2\rho\sigma}{\rho+\sigma}\right)\int_{\frac{2\rho\sigma}{\rho+\sigma}}^{\sigma}\int_{\rho}^{\frac{2\rho\sigma}{\rho+\sigma}}\frac{h\left(\frac{1}{\varsigma}-\frac{1}{\mu}\right)\vartheta(\mu)\vartheta(\varsigma)}{\mu^2\varsigma^2}d\varsigma d\mu\le\int_{0}^{\frac{\sigma-\rho}{2\rho\sigma}}h(t)\vartheta\left(\frac{2\rho\sigma}{\rho+\sigma+2\rho\sigma t}\right)dt\int_{\rho}^{\sigma}\frac{\chi(\mu)\vartheta(\mu)}{\mu^2}d\mu.$$

Hence (16) is established. □

Remark 2.4. *Suppose that the conditions of Theorem 2.3 are satisfied and*

(i) *If χ is an HA-h-concave function, then the inequality in (16) is reversed.*

(ii) *If h submultiplicative with $\int_{\frac{2\rho\sigma}{\rho+\sigma}}^{\sigma}\int_{\rho}^{\frac{2\rho\sigma}{\rho+\sigma}}\frac{h\left(\frac{1}{\varsigma}-\frac{1}{\mu}\right)\vartheta(\mu)\vartheta(\varsigma)}{\mu^2\varsigma^2}d\varsigma d\mu\neq 0$, $h>0$, and if χ is an HA-h-concave function then the inequality in (16) is reversed with constant C as given in Theorem 2.3 by changing* min *to* max.

Remark 2.5. *In Theorem 2.3*

(a) *If χ is HA-convex, i.e. $h(t)=t$, then inequality (16) holds for $C=\frac{1}{\int_\rho^\sigma\frac{\vartheta(t)}{t^2}dt}$.*
Furthermore, if $\int_{\frac{2\rho\sigma}{\rho+\sigma}}^{\sigma}\int_{\rho}^{\frac{2\rho\sigma}{\rho+\sigma}}\frac{\left(\frac{1}{\varsigma}-\frac{1}{\mu}\right)\vartheta(\mu)\vartheta(\varsigma)}{\mu^2\varsigma^2}d\varsigma d\mu\neq 0$, $h(\mu)\neq 0$ for $\mu>0$ and
If h is multiplicative or if h is supermultiplicative and χ is non-negative, then inequality (16) holds for

$$C=\min\left\{\frac{1}{\int_\rho^\sigma\frac{\vartheta(\mu)}{\mu^2}d\mu},\frac{\int_0^{\frac{\sigma-\rho}{2\rho\sigma}}\mu\vartheta\left(\frac{2\rho\sigma}{\rho+\sigma+2\rho\sigma\mu}\right)d\mu}{\int_{\frac{2\rho\sigma}{\rho+\sigma}}^{\sigma}\int_{\rho}^{\frac{2\rho\sigma}{\rho+\sigma}}\frac{\left(\frac{1}{\varsigma}-\frac{1}{\mu}\right)\vartheta(\mu)\vartheta(\varsigma)}{\mu^2\varsigma^2}d\varsigma d\mu}\right\}.$$

(b) *If χ is HA-s-convex, i.e. $h(t)=t^s$, then inequality (16) holds for $C=\frac{2^{1-s}}{\int_\rho^\sigma\frac{\vartheta(\mu)}{\mu^2}d\mu}$.*
Furthermore, if $\int_{\frac{2\rho\sigma}{\rho+\sigma}}^{\sigma}\int_{\rho}^{\frac{2\rho\sigma}{\rho+\sigma}}\frac{\left(\frac{1}{\varsigma}-\frac{1}{\mu}\right)^s\vartheta(\mu)\vartheta(\mu)}{\mu^2\varsigma^2}d\varsigma d\mu\neq 0$, $h(\mu)\neq 0$ for $\mu>0$ and
If h is multiplicative or if h is supermultiplicative and χ is non-negative, then inequality (16) holds for

$$C=\min\left\{\frac{2^{1-s}}{\int_\rho^\sigma\frac{\vartheta(\mu)}{\mu^2}d\mu},\frac{\int_0^{\frac{\sigma-\rho}{2\rho\sigma}}\mu^s\vartheta\left(\frac{2\rho\sigma}{\rho+\sigma+2\rho\sigma\mu}\right)d\mu}{\int_{\frac{2\rho\sigma}{\rho+\sigma}}^{\sigma}\int_{\rho}^{\frac{2\rho\sigma}{\rho+\sigma}}\frac{\left(\frac{1}{\varsigma}-\frac{1}{\mu}\right)^s\vartheta(\mu)\vartheta(\varsigma)}{\mu^2\varsigma^2}d\varsigma d\mu}\right\}.$$

Remark 2.6. *In Theorem 2.3*

(a) *If χ is HA-convex, i.e. $h(t)=t$ and $\vartheta(\mu)=\frac{\rho\sigma}{\sigma-\rho}$, $\mu\in[\rho,\sigma]$, then inequality (16) becomes the first inequality in (6).*

(b) *If χ is HA-s-convex, i.e. $h(t)=t^s$and $\vartheta(\mu)=\frac{\rho\sigma}{\sigma-\rho}$, $\mu\in[\rho,\sigma]$., then inequality (16) becomes the first inequality proved in [33, Corollary 3.3, page 5.].*

Let us now consider non-weighted Hermite Hadamard inequalities for *HA-h*-convex function from [33]:

$$\frac{1}{2h\left(\frac{1}{2}\right)}\chi\left(\frac{2\rho\sigma}{\rho+\sigma}\right)\leq\frac{\rho\sigma}{\sigma-\rho}\int_\rho^\sigma\frac{\chi(\mu)}{\mu^2}d\mu\leq[\chi(\rho)+\chi(\sigma)]\int_0^1 h(t)\,dt, \tag{23}$$

where $h\left(\frac{1}{2}\right)>0$.
Now we define $L:[\rho,\sigma]\to\mathbb{R}$ and $P:[\rho,\sigma]\to\mathbb{R}$ by

$$L(\varsigma)=[\chi(\rho)+\chi(\varsigma)]\left(\frac{1}{\rho}-\frac{1}{\varsigma}\right)\int_0^1 h(t)\,dt-\int_\rho^\mu\frac{\chi(\mu)}{\mu^2}d\mu$$

and

$$P(\varsigma)=\int_\rho^\varsigma\frac{\chi(\varsigma)}{\varsigma^2}d\varsigma-\chi\left(\frac{2\rho\varsigma}{\rho+\varsigma}\right)\frac{(\varsigma-\rho)}{2\rho\varsigma h\left(\frac{1}{2}\right)}$$

respectively.

Theorem 2.7. *If the function χ is HA-h-convex, $\chi \geq 0$, $h\left(\frac{1}{2}\right) > 0$ and $\frac{1}{4h\left(\frac{1}{2}\right)} \geq \int_0^1 h(t)\,dt$, then*

$$L(\varsigma) \geq P(\varsigma) \geq 0, \varsigma \in [\rho, \sigma]. \tag{24}$$

Proof. Applying the second Hermite-Hadamard type inequalities over the intervals $\left[\rho, \frac{2\rho\varsigma}{\rho+\varsigma}\right]$ and $\left[\frac{2\rho\varsigma}{\rho+\varsigma}, \mu\right]$, we obtain

$$\int_\rho^{\frac{2\rho\mu}{\rho+\mu}} \frac{\chi(\mu)}{\mu^2}d\mu \leq \left[\frac{\chi(\rho) + \chi\left(\frac{2\rho\varsigma}{\rho+\varsigma}\right)}{2}\right]\left(\frac{\varsigma - \rho}{\rho\varsigma}\right)\int_0^1 h(t)\,dt \tag{25}$$

and

$$\int_{\frac{2\rho\varsigma}{\rho+\varsigma}}^{\varsigma} \frac{\chi(\mu)}{\mu^2}d\mu \leq \left[\frac{\chi\left(\frac{2\rho\varsigma}{\rho+\varsigma}\right) + \chi(\varsigma)}{2}\right]\left(\frac{\varsigma - \rho}{\rho\varsigma}\right)\int_0^1 h(t)\,dt. \tag{26}$$

Adding (25) and (26), we obtain

$$\int_\rho^{\varsigma} \frac{\chi(\mu)}{\mu^2}d\mu \leq \left(\frac{\varsigma - \rho}{\rho\varsigma}\right)\left[\chi\left(\frac{2\rho\varsigma}{\rho+\varsigma}\right) + \frac{\chi(\rho) + \chi(\mu)}{2}\right]\int_0^1 h(t)\,dt. \tag{27}$$

Multiplying both sides of (27), we get

$$\int_\rho^{\varsigma} \frac{\chi(\mu)}{\mu^2}d\mu - \left(\frac{\varsigma - \rho}{\rho\varsigma}\right)[\chi(\rho) + \chi(\varsigma)]\int_0^1 h(t)\,dt \leq 2\left(\frac{\varsigma - \rho}{\rho\varsigma}\right)\chi\left(\frac{2\rho\varsigma}{\rho+\varsigma}\right)\int_0^1 h(t)\,dt - \int_\rho^{\mu} \frac{\chi(\mu)}{\mu^2}d\mu. \tag{28}$$

We can observe now that

$$P(\varsigma) = \int_\rho^{\varsigma} \frac{\chi(\mu)}{\mu^2}d\mu - \chi\left(\frac{2\rho\varsigma}{\rho+\varsigma}\right)\frac{(\varsigma - \rho)}{2\rho\varsigma h\left(\frac{1}{2}\right)} \leq \int_\rho^{\varsigma} \frac{\chi(\mu)}{\mu^2}d\mu - 2\left(\frac{\varsigma - \rho}{\rho\varsigma}\right)\chi\left(\frac{2\rho\varsigma}{\rho+\varsigma}\right)\int_0^1 h(t)\,dt$$
$$\leq \left(\frac{\varsigma - \rho}{\rho\varsigma}\right)[\chi(\rho) + \chi(\varsigma)]\int_0^1 h(t)\,dt - \int_\rho^{\varsigma} \frac{\chi(\mu)}{\mu^2}d\mu = L(\varsigma).$$

Hence the first inequality in (24) is proved. The second inequality in (24) follows from the first inequality in (23). The proof is thus accomplished. □

3. Mappings connected with the Hermite-Hadamard type inequalities for *HA*-convex functions

Consider the mappings $\mathcal{H}, \mathcal{F} : [0, 1] \to \mathbb{R}$ be defined by

$$\mathcal{H}(t) = \frac{\rho\sigma}{\sigma - \rho}\int_\rho^{\sigma} \frac{1}{\mu^2}\chi\left(\frac{2\rho\sigma\mu}{2\rho\sigma t + (1 - t)\mu(\rho + \sigma)}\right)d\mu$$

and

$$\mathcal{F}(t) = \left(\frac{\rho\sigma}{\sigma - \rho}\right)^2\int_\rho^{\sigma}\int_\rho^{\sigma} \frac{1}{\mu^2\varsigma^2}\chi\left(\frac{\mu\varsigma}{(1 - t)\mu + t\varsigma}\right)d\mu d\varsigma,$$

where $\chi : I \subseteq (0, \infty) \to \mathbb{R}$ is *HA*-convex on I and $\rho, \sigma \in I$.

The author has proved that $\mathcal{H}(0) = \chi\left(\frac{2\rho\sigma}{\rho+\sigma}\right)$ and $\mathcal{H}(1) = \frac{\rho\sigma}{\sigma-\rho}\int_\rho^{\sigma} \frac{\chi(\mu)}{\mu^2}d\mu$. The author has discussed some properties for *HA*-convex functions and now we investigate which of those properties of the mappings $\mathcal{H}$ and $\mathcal{F}$ for *HA*-*h*-convex mappings.

Theorem 3.1. *Let χ be HA-h-convex on $[\rho,\sigma] \subseteq (0,\infty)$ and $h : J \to \mathbb{R}$, $[0,1] \subseteq J$. Then the mapping $\mathcal{H}$ is HA-h-convex on $(0,1]$ for $t \in (0,1]$*

$$\mathcal{H}(0) \leq tC_1\mathcal{H}(t),$$

where

$$tC_1 = \begin{cases} 2h\left(\frac{1}{2}\right), & \text{in general case,} \\ \min\left\{2h\left(\frac{1}{2}\right), \frac{\int_0^1 h\left(\frac{\sigma-\rho}{2\rho\sigma}t\mu\right)d\mu}{\int_0^1\int_0^1 \frac{h\left(\frac{\sigma-\rho}{2\rho\sigma}t\left(\frac{1}{\varsigma}+\frac{1}{\mu}\right)\right)}{\mu^2\varsigma^2}d\varsigma d\mu}\right\}, \end{cases}$$

h satisfies (i) or (ii) of Theorem 2.3.

Proof. We kknow that if $\chi : [\rho,\sigma] \to \mathbb{R}$ is HA-h-convex on $[\rho,\sigma] \subset (0,\infty)$, then $\chi \circ k$ is h-convex on $[\frac{1}{\sigma}, \frac{1}{\rho}]$. In order to show that $\mathcal{H}$ is HA-h-convex on $(0,1]$, it suffices to prove that the mapping $\bar{\mathcal{H}} : [0,1] \to \mathbb{R}$ defined by

$$\bar{\mathcal{H}}(t) = \frac{\rho\sigma}{\sigma-\rho}\int_{\frac{1}{\sigma}}^{\frac{1}{\rho}} (\chi \circ k)\left(t\frac{1}{\mu} + (1-t)\left(\frac{\rho+\sigma}{2\rho\sigma}\right)\right)d\mu$$

is h-convex on $(0,1]$. Let $\alpha, \beta \in [0,1]$ with $\alpha+\beta = 1$ and $t_1, t_2 \in (0,1]$, then

$$\begin{aligned}
&\bar{\mathcal{H}}(\alpha t_1 + t_2\beta) \\
&= \frac{\rho\sigma}{\sigma-\rho}\int_{\frac{1}{\sigma}}^{\frac{1}{\rho}} (\chi \circ k)\left((\alpha t_1 + t_2\beta)\frac{1}{\mu} + (1-(\alpha t_1+t_2\beta))\left(\frac{\rho+\sigma}{2\rho\sigma}\right)\right)d\mu \\
&= \frac{\rho\sigma}{\sigma-\rho}\int_{\frac{1}{\sigma}}^{\frac{1}{\rho}} (\chi \circ k)\left((\alpha t_1 + t_2\beta)\frac{1}{\mu} + (\alpha+\beta-(\alpha t_1+t_2\beta))\left(\frac{\rho+\sigma}{2\rho\sigma}\right)\right)d\mu \\
&= \frac{\rho\sigma}{\sigma-\rho}\int_{\frac{1}{\sigma}}^{\frac{1}{\rho}} (\chi \circ k)\left(\alpha\left[t_1\frac{1}{\mu} + (1-t_1)\left(\frac{\rho+\sigma}{2\rho\sigma}\right)\right] + \beta\left[t_2\frac{1}{\mu} + (1-t_2)\left(\frac{\rho+\sigma}{2\rho\sigma}\right)\right]\right)d\mu.
\end{aligned}$$

Since $\chi \circ k$ is h-convex, we get

$$\begin{aligned}
\bar{\mathcal{H}}(\alpha t_1 + t_2\beta) &\leq \alpha\left[\frac{\rho\sigma}{\sigma-\rho}\int_{\frac{1}{\sigma}}^{\frac{1}{\rho}} (\chi \circ k)\left(t_1\frac{1}{\mu} + (1-t_1)\left(\frac{\rho+\sigma}{2\rho\sigma}\right)\right)d\mu\right] \\
&\quad + \beta\left[\frac{\rho\sigma}{\sigma-\rho}\int_{\frac{1}{\sigma}}^{\frac{1}{\rho}} (\chi \circ k)\left(t_2\frac{1}{\mu} + (1-t_2)\left(\frac{\rho+\sigma}{2\rho\sigma}\right)\right)d\mu\right] = \alpha\bar{\mathcal{H}}(t_1) + \beta\bar{\mathcal{H}}(t_2).
\end{aligned}$$

By making the substitution $\frac{1}{u} = t\frac{1}{\mu} + (1-t)\frac{\rho+\sigma}{2\rho\sigma}$, we get

$$\begin{aligned}
\mathcal{H}(t) &= \frac{\rho\sigma}{t(\sigma-\rho)}\int_{\frac{2\rho\sigma}{2\sigma t+(1-t)(\rho+\sigma)}}^{\frac{2\rho\sigma}{2\rho t+(1-t)(\rho+\sigma)}} \frac{\chi(u)}{u^2}du \\
&= \frac{\left(\frac{2\rho\sigma}{2\sigma t+(1-t)(\rho+\sigma)}\right)\left(\frac{2\rho\sigma}{2\rho t+(1-t)(\rho+\sigma)}\right)}{\left(\frac{2\rho\sigma}{2\rho t+(1-t)(\rho+\sigma)}\right) - \left(\frac{2\rho\sigma}{2\sigma t+(1-t)(\rho+\sigma)}\right)}\int_{\frac{2\rho\sigma}{2\sigma t+(1-t)(\rho+\sigma)}}^{\frac{2\rho\sigma}{2\rho t+(1-t)(\rho+\sigma)}} \frac{\chi(u)}{u^2}du = \frac{u_U u_L}{u_U - u_L}\int_{u_L}^{u_U} \frac{\chi(u)}{u^2}du. \quad (29)
\end{aligned}$$

Multiplying both sides of (29) by C_1, we obtain

$$C_1\mathcal{H}(t)=\frac{C_1u_Lu_U}{u_U-u_L}\int_{u_L}^{u_U}\frac{\chi(u)}{u^2}du\geq\chi\left(\frac{2u_Lu_U}{u_L+u_U}\right)=\chi\left(\frac{2\rho\sigma}{\rho+\sigma}\right), \tag{30}$$

where C_1 is a constant defined in Theorem 2.3 over the interval $[u_L,u_U]$, where $u_L=\frac{2\rho\sigma}{2\sigma t+(1-t)(\rho+\sigma)}$ and $u_U=\frac{2\rho\sigma}{2\rho t+(1-t)(\rho+\sigma)}$ and $\vartheta(u)=\frac{u_Lu_U}{u_U-u_L}$, $u\in[u_L,u_U]$. □

Remark 3.2. *If χ is a HA-convex function, then we get*

$$\mathcal{H}(0)\leq\mathcal{H}(t), \tag{31}$$

for all $t\in[0,1]$. It is a known result for a HA-convex function proved in [32]. If χ is an HA-s-convex function in the second sense, then $tC_1=2^{s+2}(s+2)$ so we have

$$\mathcal{H}(0)\leq 2^{s+2}(s+2)\,\mathcal{H}(t). \tag{32}$$

Theorem 3.3. *Let χ be HA-h-convex on $[\rho,\sigma]\subseteq(0,\infty)$ and $h:J\to\mathbb{R}$, $[0,1]\subseteq J$. Then the mapping $\mathcal{F}$ is is symmetric with respect to $\frac{1}{2}$ and HA-h-convex on $(0,1]$. Furthermore, the following inequalities hold*

$$2h\left(\frac{1}{2}\right)\mathcal{F}(t)\geq\mathcal{F}\left(\frac{1}{2}\right)\text{ and }tC_1\mathcal{F}(t)\geq\mathcal{H}(1-t) \tag{33}$$

for $t\in(0,1]$, where C_1 is defined as in the Theorem 3.1.

Proof. We observe that the following equality holds for all $\mu,\varsigma\in[\rho,\sigma]$ and $t\in(0,1]$:

$$\frac{2\mu\varsigma}{\mu+\varsigma}=\frac{\left(\frac{\mu\varsigma}{(1-t)\mu+t\varsigma}\right)\left(\frac{\mu\varsigma}{t\mu+(1-t)\varsigma}\right)}{\frac{1}{2}\left(\frac{\mu\varsigma}{(1-t)\mu+t\varsigma}\right)+\frac{1}{2}\left(\frac{\mu\varsigma}{t\mu+(1-t)\varsigma}\right)}.$$

Since χ is a *HA-h*-convex on $[\rho,\sigma]$, we have

$$\chi\left(\frac{2\mu\varsigma}{\mu+\varsigma}\right)\leq h\left(\frac{1}{2}\right)\chi\left(\frac{\mu\varsigma}{(1-t)\mu+t\varsigma}\right)+h\left(\frac{1}{2}\right)\chi\left(\frac{\mu\varsigma}{t\mu+(1-t)\varsigma}\right). \tag{34}$$

Multiplying the inequality (34) by $\frac{1}{\mu^2\varsigma^2}$, integrating with respect to μ over $[\rho,\sigma]$, with respect to ς over $[\rho,\sigma]$ and using the fact that

$$\int_\rho^\sigma\int_\rho^\sigma\frac{1}{\mu^2\varsigma^2}\chi\left(\frac{\mu\varsigma}{(1-t)\mu+t\varsigma}\right)d\varsigma d\mu=\int_\rho^\sigma\int_\rho^\sigma\frac{1}{\mu^2\varsigma^2}\chi\left(\frac{\mu\varsigma}{t\mu+(1-t)\varsigma}\right)d\varsigma d\mu$$

we obtain the inequality

$$\begin{aligned}\int_\rho^\sigma\int_\rho^\sigma\frac{1}{\mu^2\varsigma^2}\chi\left(\frac{2\mu\varsigma}{\mu+\varsigma}\right)d\varsigma d\mu&\leq 2h\left(\frac{1}{2}\right)\int_\rho^\sigma\int_\rho^\sigma\frac{1}{\mu^2\varsigma^2}\chi\left(\frac{\mu\varsigma}{(1-t)\mu+t\varsigma}\right)d\varsigma d\mu\\&=2h\left(\frac{1}{2}\right)\left(\frac{1}{\rho}-\frac{1}{\sigma}\right)^2\cdot\frac{1}{\left(\frac{1}{\rho}-\frac{1}{\sigma}\right)^2}\int_\rho^\sigma\int_\rho^\sigma\frac{1}{\mu^2\varsigma^2}\chi\left(\frac{\mu\varsigma}{(1-t)\mu+t\varsigma}\right)d\varsigma d\mu=2\left(\frac{\sigma-\rho}{\rho\sigma}\right)^2h\left(\frac{1}{2}\right)\mathcal{F}(t).\end{aligned} \tag{35}$$

Thus the first inequality in (33) is proved.
Let us consider the mapping

$$\mathcal{H}_\varsigma(t)=\frac{\rho\sigma}{\sigma-\rho}\int_\rho^\sigma\frac{1}{\mu^2}\chi\left(\frac{\mu\varsigma}{(1-t)\mu+t\varsigma}\right)d\mu$$

for a fixed ς.
By making use of the substitution $u = \frac{\mu\varsigma}{(1-t)\mu+t\varsigma}$, we get

$$\mathcal{H}_{\varsigma}(t) = \frac{\rho\sigma}{t(\sigma-\rho)} \int_{\frac{\rho\varsigma}{(1-t)\rho+t\varsigma}}^{\frac{\sigma\varsigma}{(1-t)\sigma+t\varsigma}} \frac{\chi(u)}{u^2} du = \frac{u_U u_L}{u_U - u_L} \int_{\rho_L}^{\rho_U} \frac{\chi(u)}{u^2} du,$$

where $u_L = \frac{\rho\varsigma}{(1-t)\rho+t\varsigma}$ and $u_U = \frac{\sigma\varsigma}{(1-t)\sigma+t\varsigma}$.
Using the result from Theorem 2.3, we get

$$C_1\mathcal{H}_{\varsigma}(t) = C_1 \int_{u_L}^{u_U} \frac{\chi(u)}{u} \frac{u_U u_L}{u_U - u_L} du \geq \chi\left(\frac{\left(\frac{2\rho\sigma}{\rho+\sigma}\right)\varsigma}{(1-t)\left(\frac{2\rho\sigma}{\rho+\sigma}\right) + t\varsigma} \right) \tag{36}$$

Multiplying both sides of the inequality (36) by $\frac{1}{\varsigma^2}$, integrating with respect to μ over $[\rho, \sigma]$ and multiplying both sides by $\frac{u_U u_L}{u_U - u_L}$, we get

$$tC_1\mathcal{F}(t) \geq \mathcal{H}(1-t).$$

Hence the second inequality in (33) is also established, where C_1 is given in Theorem 3.1. □

Remark 3.4. *If $tC_1 > 0$, then we have*

$$\mathcal{F}(t) \geq \frac{1}{tC_1}\mathcal{H}(1-t) \tag{37}$$

for all $t \in (0, 1]$. Replacing t with $1-t$ in (37), we have

$$\mathcal{F}(1-t) \geq \frac{1}{(1-t)C_1}\mathcal{H}(t) \tag{38}$$

for all $t \in (0, 1]$.
Since $\mathcal{F}$ is symmetric with respect to $\frac{1}{2}$, we have

$$\mathcal{F}(t) \geq \max\left\{ \frac{1}{tC_1}\mathcal{H}(1-t), \frac{1}{(1-t)C_1}\mathcal{H}(t) \right\}. \tag{39}$$

If χ is a HA-convex function, then we get the following result:

$$\mathcal{F}(t) \geq \max\{\mathcal{H}(1-t), \mathcal{H}(t)\}. \tag{40}$$

If h is a multiplicative function, then $tC_1 = (1-t)C_1$. Thus, we obtain

$$\mathcal{F}(t) \geq \frac{1}{tC_1}\max\{\mathcal{H}(1-t), \mathcal{H}(t)\}. \tag{41}$$

If χ is a HA-s-convex function, then $h(t) = t^s$ and we get the following result

$$\mathcal{F}(t) \geq \frac{1}{2^{s+2}(s+2)}\max\{\mathcal{H}(1-t), \mathcal{H}(t)\}. \tag{42}$$

4. Applications to Special Means

Suppose that χ is *HA*-concave and *HA*-*h*-convex simultaneously, or vice versa, when χ is *HA*-convex and *HA*-*h*-concave. If χ is a *HA*-concave and *HA*-*h*-convex function with $\int_0^1 h(t)\,dt > 0$, then the Hermite-Hadamard type inequalities of Theorem 1.11, Theorems 2.1 and 2.3 lead us to the following inequalities

$$\frac{\rho\sigma}{\sigma-\rho} \int_{\rho}^{\sigma} \frac{\chi(\mu)}{\mu^2} d\mu \leq \chi\left(\frac{2\rho\sigma}{\rho+\sigma}\right) \leq C \int_{\rho}^{\sigma} \frac{\chi(\mu)}{\mu^2} d\mu \tag{43}$$

and

$$\frac{\rho\sigma}{(\sigma-\rho)\int_{\rho}^{\sigma}h(t)\,dt}\int_{\rho}^{\sigma}\frac{\chi(\mu)}{\mu^{2}}d\mu\leq\chi(\rho)+\chi(\sigma)\leq\frac{2\rho\sigma}{\sigma-\rho}\int_{\rho}^{\sigma}\frac{\chi(\mu)}{\mu^{2}}d\mu. \tag{44}$$

If χ is a *HA*-convex and *HA*-*h*-concave function simultaneously, then the inequalities (43) and (44) hold in reversed directions.

Let ρ and σ be two non-negative real numbers, then the p-logarithmic mean L_p and geometric-mean of the order p are defined as follows:

$$L_p(\rho,\sigma)=\left[\frac{\sigma^{p+1}-\rho^{p+1}}{(p+1)(\sigma-\rho)}\right]^{\frac{1}{p}},\ p\in\mathbb{R}\setminus\{0,-1\}$$

and

$$M_p(\rho,\sigma)=\left(\frac{\rho^{p}+\sigma^{p}}{2}\right)^{\frac{1}{p}}.$$

It has been shown in [35] that for the functions χ and h_k defined as $h_k(\mu)=\mu^k$, $g(\mu)=\mu^p$, $\mu>0$, $k,p\in\mathbb{R}$, we have the following facts:

(i) The function χ is h_k-convex if
(a) $p\in(-\infty,0]\cup[1,\infty)$ and $k\leq 1$;
(b) $p\in(0,1)$ and $k\leq p$.
(ii) The function χ is h_k-concave if
(a) $p\in(0,1)$ and $k\geq 1$;
(b) $p>1$ and $k\geq p$.

According to Theorem 1.9 for the functions $h_k(\mu)=\mu^k$, $g(\mu)=\mu^p$, $\mu>0$, $k,p\in\mathbb{R}$, we have that

(i) The function $\chi(\mu)=g\left(\frac{1}{\mu}\right)=\mu^{-p}$ is *HA*-h_k-convex if
(a) $p\in(-\infty,0]\cup[1,\infty)$ and $k\leq 1$;
(b) $p\in(0,1)$ and $k\leq p$.
(ii) The function $\chi(\mu)=g\left(\frac{1}{\mu}\right)=\mu^{-p}$ is *HA*-h_k-concave if
(a) $p\in(0,1)$ and $k\geq 1$;
(b) $p>1$ and $k\geq p$.

Let $p\in(0,1)$ and $0\leq k\leq p$, then we have the following inequalities:

$$L_{-p-2}^{-p-2}(\rho,\sigma)\leq M_{-1}^{-2p}\left(\rho^{-1},\sigma^{-1}\right)\leq\left(\frac{k+2}{2^{k+1}-1}\right)L_{-p-2}^{-p-2}(\rho,\sigma) \tag{45}$$

and

$$\frac{L_{-p-2}^{-p-2}(\rho,\sigma)}{2(\sigma-\rho)L_{k}^{k}(\rho,\sigma)}\leq M_1\left(\rho^{k},\sigma^{k}\right)\leq L_{-p-2}^{-p-2}(\rho,\sigma). \tag{46}$$

5. Conclusion

Since the last four decades, the field of mathematical inequalities has emerged as an emerging topic, and a lot of research has been generated by a number of mathematicians with innovative results. There are numerous applications for this concept in applied mathematics, pure mathematics, and other applied disciplines. Convexity and its generalizations have yielded a variety of unique results with applications in numerical analysis, fixed point theory, differential equations, and optimization theory. In this study, we have used the harmonic convexity as a generalization of convexity to get new Fejér type inequalities with the help of some mappings defined over the interval $[0,1]$. We have discussed some of very interesting properties of those mappings and as a consequence we get refinements of number of number of results previously obtained in this topic.

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