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G-convergently separation axioms

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Abstract. A convergence sequence in a Hausdorff space X has a unique limit. Hence this idea gives us a function which is defined on convergence sequences and has the values in X. Replacing this limit function with any function G whose domain is a certain subset of the sequences extends the notion of limit and such a function G is called G-method. Then sequential definitions of continuity, compactness and connectedness have been extended to G-method setting.

In the paper we intent to study some separation axioms such that T_i $(i = 0, 1, 2, 3, 4)$ for G-methods in sets or topological spaces; and characterise them in terms of G-open and G-closed subsets. Then we give some different counterexamples of G-methods and evaluate them if these separations axioms are satisfied.

1. Introduction

The convergence of the sequences is useful to define sequential versions of some definitions in topology such as continuity, compactness, connectedness and many others. Therefore many authors have been interested in to give the sequential definitions of them.

Initiated with a work in [8], in a number of references [25], [18], [2], [3], [26] for a regular summability matrix A, A-continuity have been studied. In [27], [28], [29], [7] for almost convergence and related methods A-continuity have been considered. The paper [6] is referred for summability matrices and [14] for summability in topological groups. In [17] statistical convergence in topological spaces with some applications have been considered.

The authors in [15] have an important investigation of extending the sequential definition of a continuous function to any G-method defined on a subspace of the sequences. Then the sequential definitions are extended to G-compactness for topological groups in [13] (for topological groups with operations in [21]), to G-continuity in [11] (see [16] and [12] for different types of continuities) and to G-connectedness for topological groups in [10] (see also [9]). [23] defines G-open set, G-neighbourhood, and gives more properties of G-continuities for given a method G on X.

The reference Lin and Liu in [19] extends the G-methods and different convergence methods not only to topological spaces but also to arbitrary sets; and G-hulls, G-closures, G-kernels and G-interiors are taken into account. Mucuk and Çakallı in [22] extends sequential connectedness for the topological groups with

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operations which combine groups, rings and some others to G-methods. In the paper [1] the authors consider the notion of neutrosophic topological space and obtain some results toward this direction. In this paper we have intension to consider some separation notions T_i ($i = 0, 1, 2, 3, 4$) for G-methods not only in topological groups or topological space but also for sets. We give some counter examples of G-methods and then evaluate if these separation axioms are satisfied for these methods.

We acknowledge that the results and examples of this paper will be taken into account in PhD thesis [4] of second author, which is under preparation.

2. Preliminaries

In the paper, X denotes a set if otherwise is not stated and the letters $\mathbf{x}$ and $\mathbf{y}$ represent the sequences $\mathbf{x} = (x_n)$ and $\mathbf{y} = (y_n)$ in X. For the set of all sequences and for the set of convergent sequences in X, we write $s(X)$ and $c(X)$ respectively.

A *G-method* is defined to be a map $G\colon c_G(X) \to X$, where the domain $c_G(X)$ is a subset of $s(X)$. The sequence $\mathbf{x} = (x_n)$ is called *G-convergent* to ℓ whenever $\mathbf{x} \in c_G(X)$ and $G(\mathbf{x}) = \ell$. In particularly if X is a Hausdorff space, then limit function defined on the set of convergent sequences $c(X)$ gives a G-method. The method G is said to be *regular* whenever for any convergent sequence $\mathbf{x} = (x_n)$ one has $G(\mathbf{x}) = \lim \mathbf{x}$ and G *preserves the G-convergence of subsequences*, whenever for any sequence $\mathbf{x}$ with $G(\mathbf{x}) = \ell$, then any subsequence of $\mathbf{x}$ is also G-convergent to the same point ℓ.

Let X be a set and G a method on X. A function $f\colon X \to X$ is said to be *G-continuous* if whenever $\mathbf{x} \in c_G(X)$, then $f(\mathbf{x}) \in c_G(X)$ and $G(f(\mathbf{x})) = f(G(\mathbf{x}))$. In the case when X is a Hausdorff space and G is the limit function, the G-continuity agrees with the continuity of the function.

Let $A \subseteq X$ be a subset and $\ell \in X$. Then $\ell \in X$ is a point of the *G-hull* of A if there is a sequence $\mathbf{x} = (x_n)$ in A with $G(\mathbf{x}) = \ell$; and write $[A]^G$ for G-hull of A [19]. A is said to be *G-closed* if $[A]^G \subseteq A$. For a regular method G and $a \in A$, the constant sequence $\mathbf{x} = (a, a, \dots)$, G-convergent to a and therefore $A \subseteq [A]^G$. Hence in the case where G is regular, A is G-closed whenever $[A]^G = A$. $[A]^G$ is not necessarily closed since $[[A]^G]^G$ is not necessarily equivalent to $[A]^G$. The union of any two G-closed subsets of X is not necessarily a G-closed subset of X. Hence even in the case where G is regular, $[A]^G$ is not necessarily G-closed (see Example 4.1). For the subsets $A, B \subseteq X$, $A \subseteq B$ implies $[A]^G \subseteq [B]^G$.

If the complement $X \setminus A$ is G-closed, then we call A as *G-open*. Unlike the G-closed subsets, the union of G-open subsets of X is G-open.

In a similar way for a subset $A \subseteq X$, we can rephrase G-closed and G-open subsets of A. In other words $F \subseteq A$ is a *G-closed* subset of A if for a G-closed subset K of X we have $F = K \cap A$ and $U \subseteq A$ is *G-open* in A if for a G-open subset V of X we have $U = A \cap V$.

A subset $V \subseteq X$ is a *G-neighbourhood* of a when there is a G-open subset U of X such that $a \in U \subseteq V$. G-closure $\overline{A}^G$ of A is defined as the smallest G-closed subset including A. We remind that the union of G-closed subsets are not necessarily G-closed however the intersection of G-closed subsets are G-closed. Hence G-closure $\overline{A}^G$ of A is a G-closed subset and $A \subseteq \overline{A}^G$. One need to note that for a subset $A \subseteq X$, G-closure $\overline{A}^G$ is a G-closed but G-hull $[A]^G$ of A is not necessarily G-closed subset. A is G-closed if and only if $A = \overline{A}^G$.

Theorem 2.1. ([19, Corollary 2.7]) *If X is a set endowed with a G-method, then $[A]^G \subseteq \overline{A}^G$ for $A \subseteq X$.*

Proof. If K is a G-closed subset containing A, then we have

$$[A]^G \subseteq [K]^G \subseteq K$$

which proves that $[A]^G \subseteq \overline{A}^G$. □

Theorem 2.2. *For a G-method on X and $A \subseteq X$, $x \in \overline{A}^G$ if and only if any G-open subset U including x has some elements of A.*

Proof. If $x \in \overline{A}^G$, U is a G-open subset with $x \in U$ and $U \cap A$ is empty, then we have $A \subseteq X \setminus U$, where $X \setminus U$ is a G-closed subset and $x \notin X \setminus U$. That gives us $x \notin \overline{A}^G$ which is contradiction with the assumption $x \in \overline{A}^G$. Hence $U \cap A$ is non empty.

If the intersection of every G-open neighbourhood of x with A is non-empty, and $x \notin \overline{A}^G$, then there exists a G-closed subset K with $A \subseteq K$ and $x \notin K$. Hence $X \setminus K$ is a G-open neighbourhood of x and $A \cap (X \setminus K) = \emptyset$. This is a contradiction and therefore $x \in \overline{A}^G$. □

3. Some G-convergently separation axioms

We can define G-convergently versions of the separation axioms T_i $(i = 0, 1, 2, 3, 4)$ as follows.

Definition 3.1. Let X be a set endowed with a G-method. We call X:

i) *G-convergently* T_0 if for any pair of different points $a, b \in X$, at least one has a G-open neighbourhood not including the other.

ii) G-*convergently* T_1 if for given any pair of distinct points $a, b \in X$, each one has a G-open neighbourhood not containing another one.

iii) G-*convergently* T_2 or *G-convergently Hausdorff* if any distinct points $a, b \in X$, have disjoint G-open neighbourhoods.

iv) G-convergently T_3 if it is G-convergently T_1 and whenever F is G-closed and $x \notin F$, there are disjoint G-open subsets U and V with $F \subseteq U$ and $x \in V$.

v) G-convergently T_4 if it is G-convergently T_1 and whenever F and K are G-closed, then there are disjoint G-open subsets U and V such that $F \subseteq U$ and $K \subseteq V$.

Theorem 2.2 is useful for the proof of the following propositions.

Proposition 3.2. *If G is a method on a set X, then the following hold.*

(i) X is G-convergently T_0 *if and only if for every distinct points* $a, b \in X$ *either* $a \notin \overline{\{b\}}^G$ *or* $b \notin \overline{\{a\}}^G$.

(ii) X is G-convergently T_1 *if and only for every distinct points* $a, b \in X$ *we have* $a \notin \overline{\{b\}}^G$ *and* $b \notin \overline{\{a\}}^G$.

Proof. (i) Let X be a G-convergently T_0 set and $a, b \in A$ different points. Then there exists a G-open neighbourhoof U_a of a with $b \notin U_a$ or there is a G-open neighbourhood U_b of b with $a \notin U_b$. Hence by Theorem 2.2 either $a \notin \overline{\{b\}}^G$ or $b \notin \overline{\{a\}}^G$.

Let $a \notin \overline{\{b\}}^G$ or $b \notin \overline{\{a\}}^G$. Then there exists a G-open neighborhood U_a of a with $b \notin U_a$ or there exists a G-open neighborhood U_b of b with $a \notin U_b$. Hence X is G-convergently T_0.

(ii) The proof is similar to (i) and therefore it is omitted. □

Equivalently to Proposition 3.2, we have the following.

Proposition 3.3. *If G is a method on a set X, then the following are true.*

(i) X is not G-convergently T_0 *if and only if there is a pair of distinct points* $a, b \in X$ *such that* $a \in \overline{\{b\}}^G$ *and* $b \in \overline{\{a\}}^G$.

(ii) X is not G-convergently T_1 *if and only if there is a pair of distinct points* $a, b \in X$ *such that* $a \in \overline{\{b\}}^G$ *or* $b \in \overline{\{a\}}^G$.

G-convergently T_0 set can be characterised as follows.

Theorem 3.4. *A set X is a G-convergently* T_0 *if and only if for all different points* $a, b \in X$ *we have that* $\overline{\{a\}}^G \neq \overline{\{b\}}^G$.

Proof. Equivalently we prove that X is not G-convergently T_0 if and only if there is a pair of different points $a, b \in X$ such that $\overline{\{a\}}^G = \overline{\{b\}}^G$.

Let X be not G-convergently T_0. Then by Proposition 3.3 (i) there is a pair of different points $a, b \in X$ such that $a \in \overline{\{b\}}^G$ and $b \in \overline{\{a\}}^G$.

If $x \in \overline{\{a\}}^G$ and U is a G-open neighbourhood of x, then U is a G-open neighbourhood of a and therefore a G-open neighbourhood of b. Hence $x \in \overline{\{b\}}^G$ and therefore $\overline{\{a\}}^G \subseteq \overline{\{b\}}^G$. Similarly $\overline{\{b\}}^G \subseteq \overline{\{a\}}^G$ and therefore $\overline{\{a\}}^G = \overline{\{b\}}^G$.

Let $\overline{\{a\}}^G = \overline{\{b\}}^G$ for a pair of different points $a, b \in X$. Then any G-open subset including one also contains the other and therefore X is not G-convergently T_0. □

Proposition 3.5. *If G is a regular method on a set X, then all single point subsets are G-closed.*

Proof. Let G be regular method on a set X and $A = \{a\}$. If $u \in [A]^c$, then the constant sequence $\mathbf{a} = (a, a, \dots)$ is G-convergent to u. Since G is a regular method we have $u = G(\mathbf{a}) = \lim \mathbf{a} = a$ and therefore $u = a \in A$. Hence $[A]^G \subseteq A$ and $A = \{a\}$ is G-closed. □

Here note that if each one point set is G-closed, then G is not necessarily a regular method. For example for the G-method in Example 4.2 all subsets and therefore all one point sets are G-closed but G is not a regular method.

Theorem 3.6. *A set X is G-convergently T_1 if and only if one point subsets are G-closed.*

Proof. Let X be G-convergently T_1 and $a \in X$. If $x \in \{a\}^c$, then x and a are distinct points and therefore there are G-open subsets U_x and U_a such that $x \in U_x$, $a \notin U_x$ and $a \in U_a$, $x \notin U_a$. Hence $U_x \cap \{a\} = \emptyset$ and therefore $U_x \subseteq \{a\}^c$. That is $\{a\}^c$ is G-open and therefore $\{a\}$ is G-closed.

Let each one point set be G-closed and $x, y \in X$ be distinct points. By assumption $\{x\}$ and $\{y\}$ are G-closed subsets and therefore $X \backslash \{y\}$ and $X \backslash \{x\}$ are respectively G-open neighbourhoods of x and y not containing the other. Therefore X is G-convergently T_1 set. □

Corollary 3.7. *If G is regular method on a set X, then X is G-convergently T_1.*

Proof. This is a direct result of Propositions 3.5 and Theorem 3.6 . □

Proposition 3.8. *Let X be a set with a G-method. Then the following implications hold.*
G-convergently $\mathsf{T}_4 \Rightarrow$ G-convergently $\mathsf{T}_3 \Rightarrow$ G-convergently $\mathsf{T}_2 \Rightarrow$ G-convergently $\mathsf{T}_1 \Rightarrow$ G-convergently T_0

Proof. G-convergently $\mathsf{T}_4 \Rightarrow$ G-convergently T_3 : Let X be G-convergently T_4, $F \subseteq X$ a G-closed subset and $x \notin F$. Since X is G-convergently T_1 by Theorem 3.6 we have that $\{x\}$ is G-closed. Since X is G-convergently T_4, we have G-open subsets $U, V \subseteq X$ such that $F \subseteq U$ and $\{x\} \subseteq V$; and therefore X is G-convergently T_3.

G-convergently $\mathsf{T}_3 \Rightarrow$ G-convergently T_2: Let X be G-convergently T_3 and $x, y \in X$ be distinct point. Since the one point subset $\{x\}$ is G-closed and $y \notin \{x\}$, there are G-open disjoint subsets $U, V \subseteq X$ such that $\{x\} \subseteq U$ and $y \in V$; and therefore X is G-convergently T_2 .

The other parts of the proof is straightforward and omitted. □

We need the following theorem in some proofs of the paper.

Theorem 3.9. ([22, Theorem 13]) *For a method G on a set X preserving the G-convergences of subsequences, and the first projection map $\pi_1 \colon A \times B \to A, (a, b) \mapsto a$, the inverse image ${\pi_1}^{-1}(U)$ of a G-open subset $U \subseteq A$ is a G-open subset of $A \times B$.*

Theorem 3.10. *If A and B are G-convergently T_1, then so also is $A \times B$.*

Proof. Let $a, b \in A \times B$ be distinct points. Let $\pi_1(a)$ and $\pi_1(b)$ be distinct points of A. Since A is G-convergently T_1 the points $\pi_1(a)$ and $\pi_1(b)$ have respectively G-open neighbourhoods U and V in A which contain exactly one of these points. Then by Theorem 3.9, the subsets ${\pi_1}^{-1}(U)$ and ${\pi_1}^{-1}(V)$ are G-open neighbourhoods of a and b respectively in $A \times B$ not including the other. If $\pi_2(a)$ and $\pi_2(b)$ are distinct points of B, the proof is similar. Therefore $A \times B$ is G-convergently T_1. □

In particularly if X is G-convergently T_1, then so also is $X \times X$.

Theorem 3.11. *If A, B are G-convergently T_0, then so also is $A \times B$.*

Proof. The proof is similar to Theorem 3.10 and therefore omitted □

It is well known that X is a Hausdorf space if and only it the diagonal function $\Delta\colon X \to X \times X, x \mapsto (x, x)$ is closed. For G-methods we prove a similar result as follows.

Theorem 3.12. *Let X be a set and G be a method on X. Then the diagonal function $\Delta\colon X \to X \times X, x \mapsto (x, x)$ is closed.*

Proof. Let A be a G-closed subset of X. We prove that $\Delta A = \{(a, a)\colon a \in A\}$ is G-closed. Let (x_n, x_n) be a sequence in ΔA such that $G(x_n, x_n) = (G(x_n), G(x_n)) = (x, x)$. Then (x_n) is a sequence in A and $G(x_n) = x$. Since A is G-closed we have that $x \in A$ and therefore $(x, x) \in \Delta A$. This proves that ΔA is G-closed.

As a result of Theorem 3.12, $\Delta X = \{(x, x) \mid x \in X\}$ is a G-closed subset of $X \times X$. □

Theorem 3.13. *The product of two G-convergently T_2 subsets of X is G-convergently Hausdorff.*

Proof. Let A and B be two G-convergently T_2 subsets of X and $a, b \in A \times B$ be two distinct points. Assume that $\pi_1(a)$ and $\pi_1(b)$ are distinct points of A. Since A is G-convergently T_2 the points $\pi_1(a)$ and $\pi_1(b)$ have disjoint G-open neighbourhoods U and V in A. Then $U \times B$ and $V \times B$ become disjoint G-open neighbourhoods of a and b respectively, which completes the proof. □

Theorem 3.14. *Let $f\colon X \to X$ be a function and $A = \{(x, x')\colon f(x) = f(x')\}$. If f is G-continuous, then A is G-closed.*

Proof. Let (x_n, y_n) be a sequence in A with $G(x_n, y_n) = (G(x_n), G(y_n)) = (x, y)$. Then for all $n \in \mathbb{N}$ we have that $f(x_n) = f(y_n)$ which implies that $G(f(x_n)) = G(f(y_n))$. Since f is G-continuous we have that $f(x_n) \in c_G(X)$ and $f(G(x_n)) = G(f(x_n))$. Similarly $f(y_n) \in c_G(X)$ and $f(G(y_n)) = G(f(y_n))$. Hence $f(G(x_n)) = f(G(y_n))$ and therefore $f(x) = f(y)$. Hence $(x, y) \in A$ which proves that A is G-closed. □

If X is a topological space, Y is a Hausdorff space and $f\colon X \to Y$ is continuous, then the graph set $G_f = \{(x, f(x))\colon x \in X\}$ is closed in $X \times Y$. As we can see in the following theorem, for G-methods we do not need Hausdorff condition

Theorem 3.15. *Let X be a set and G a method on it. If $f\colon X \to X$ is a G-continuous map, then the graph set $G_f = \{(x, f(x))\colon x \in X\}$ is G-closed in $X \times X$.*

Proof. Let $f\colon X \to X$ be a G-continuous map and let $(x_n, f(x_n))$ be a sequence in the graph set G_f such that $G(x_n, f(x_n)) = (x, y)$. Then we have

$$G(x_n, f(x_n)) = (G(x_n), G(f(x_n)) = (x, y)$$

Since f is G-continuous $G(f(x_n)) = f(G(x_n))$. Hence we have

$$G(x_n, f(x_n)) = (G(x_n), f(G(x_n)) = (x, f(x))$$

That means $(x, f(x)) = (x, y) \in G_f$ and therefore G_f is G-closed. □

Theorem 3.16. *If G is a regular method on a set X, then the projection maps $\pi_1\colon : X \times X \to X$ and $\pi_2\colon X \times X \to X$ are G-closed.*

Proof. We prove for the first projection map. Let $A \subseteq X \times X$ be G-closed subset and $\mathbf{x} = (x_n)$ be a sequence in $\pi_1(A)$ with $G(\mathbf{x}) = x$. Chose a point a in $\pi_2(A)$. Then $(\mathbf{x}, \mathbf{a})$ is a sequence in A, where $\mathbf{a}$ is the constant sequence at a and $G(\mathbf{x}, \mathbf{a}) = (G(\mathbf{x}), G(\mathbf{a})) = (x, a)$. Here note that since G is regular we have that $G(\mathbf{a}) = a$. Since A is G-closed we have $(x, a) \in A$ and therefore $x \in \pi_1(A)$. Hence $\pi_1(A)$ is G-closed and therefore $\pi_1\colon : X \times X \to X$ is G-closed. The proof for second projection map is similar. □

Theorem 3.17. *Let G be a regular method on a subset X. If X is G-convergently T_3, then so also is $X \times X$.*

Proof. Let X be G-convergently T_3. By Theorem 3.10, $X \times X$ is G-convergently T_1. Let $F \subseteq X \times X$ be G-closed and $(x, y) \notin F$. Then $x \notin \pi_1(F)$ or $y \notin \pi_2(F)$. Assume that $x \notin \pi_1(F)$. By Theorem 3.16 $\pi_1(F)$ is G-closed in X. Since X is G-convergently T_3, there are disjoint G-open subsets U and V such that $\pi_1(F) \subseteq U$ and $x \in V$. Then ${\pi_1}^{-1}(U)$ and ${\pi_1}^{-1}(V)$ are disjoint G-open subsets of $X \times X$ such that $F \subseteq {\pi_1}^{-1}(U)$ and $x \in {\pi_1}^{-1}(V)$. As a result $X \times X$ is G-convergently T_3. □

Theorem 3.18. *Let G be a regular method on a subset X. If X is G-convergently T_4, then so also is $X \times X$.*

Proof. The proof is similar to that of Theorem 3.17 and therefore omitted. □

4. Some counterexamples and G-convergently separation axioms

We now give a few examples of G-methods for some separation axioms.

According to the following G-method, $\mathbb{R}$ is G-convergently T_1 but not G-convergently T_2.

Example 4.1. Define a G-method on $\mathbb{R}$ with $G(\mathbf{x}) = \lim \dfrac{x_n + x_{n+1}}{2}$. Hence the method G is defined for the convergent sequences in $\mathbb{R}$. For any convergence sequence $\mathbf{x} = (x_n)$ we have $G(\mathbf{x}) = \lim \mathbf{x}$; and therefore the method G is regular. Hence by Corollary 3.7, $\mathbb{R}$ and therefore all subsets are G-convergently T_1. The non-empty G-closed subsets are $\mathbb{R}$ and single point subsets $\{x\}$'s for $x \in \mathbb{R}$ and G-open subsets are $\mathbb{R} \setminus \{x\}$'s. Hence different points have no disjoint G-open neighbourhoods and therefore $\mathbb{R}$ is not G-convergently T_2. The method also neither G-convergently T_3 nor T_4.

(i) Let $A = \{0, 1\}$. Then $[A]^G = \{0, \frac{1}{2}, 1\}$ and $[[A]^G]]^G = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$. Hence $[[A]^G]]^G \neq [A]^G$ and therefore the hull $[A]^G$ is not G-closed. $\overline{A}^G = \mathbb{R}$, since the only G-closed subset including A is $\mathbb{R}$.

(ii) Let $a, b \in \mathbb{R}$. Then $A = \{a\}$ and $B = \{b\}$ are G-closed but $[A \cup B]^G = \{a, \frac{a+b}{2}, b\}$ and therefore $A \cup B$ is not G-closed.

For the G-method below written on a set X , all subsets are G-convergently T_4.

Example 4.2. Let X be a set. Define a G-method on X such that $G(\mathbf{x}) = x_1$ for all sequences $\mathbf{x} = (x_n)$ in X. The method G is not regular. For instance for different points $a_1, a \in X$ and the sequence $\mathbf{x} = (a_1, a, a \dots)$ we have $G(\mathbf{x}) \neq \lim \mathbf{x}$ since $G(\mathbf{x}) = a_1$ but $\lim \mathbf{x} = a$. Let $A \subseteq X$ be a subset. If $\mathbf{x} = (x_n)$ is a sequence in A, then $G(\mathbf{x}) = x_1 \in A$ and therefore A is G-closed. Hence all subsets are G-closed and therefore G-open. Hence X is G-convergently T_i $(i = 0, 1, 2, 3, 4)$.

In the following example X is G-convergently T_0 but not G-convergently T_1.

Example 4.3. Let X be a set with a constant element $c \in X$. The G-method defined by $G(\mathbf{x}) = c$ for all sequences $\mathbf{x} = (x_n)$ in X is not regular. For example if $\mathbf{x} = (x_n)$ is a convergence sequence with $\lim \mathbf{x} = \ell$ such that $\ell \neq c$, then $G(\mathbf{x}) \neq \lim \mathbf{x}$.

Let A be a proper subset of X. Then $[A]^G = \{c\}$ and therefore A is G-closed if $c \in A$, and G-open if $c \notin A$. Hence the point c has no G-open neighbourhood apart from X.

Let $a, b \in A$ be different points. If $a \neq c \neq b$, then $U_a = X \setminus \{b, c\}$ and $U_b = X \setminus \{a, c\}$ are respectively G-open neighbourhoods of a and b. If $a = c$, then $X \setminus \{c\}$ is a G-open neighbourhood of b not including a. Similarly if $b = c$, then $X \setminus \{c\}$ is a G-open neighbourhood of a not including b. Therefore X is G-convergently T_0. Only G-open neighbourhood of c is X and therefore for $a \neq c$, the point c has no G-open neighbourhood not including a. We can also say that for $a \neq c$, one point set $\{a\}$ is not G-closed. Hence X is not G-convergently T_i $(i = 1, 2, 3, 4)$.

i) If $A = \{a, b\}$, then $[A]^G = \{c\}$ and $\overline{A}^G = \{a, b, c\}$.

ii) For any non-empty subset A, we have that $[A]^G = \{c\}$ and $\overline{A}^c = A \cup \{c\}\}$.

The following is also an example of G-method in which $\mathbb{R}$ is G-convergently T_1 but not G-convergently T_2.

Example 4.4. Define a G-method on $\mathbb{R}$ by $G(\mathbf{x}) = \sum_{n=1}^{\infty} x_n$ for the sequences $\mathbf{x} = (x_n)$ whenever the series is convergent. The method G is not regular. For example for the sequence $\mathbf{x} = (\frac{1}{2^n})$ we have $\lim \mathbf{x} = 0$ but $G(\mathbf{x}) = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1$.

If $A = \{0\}$, then $[A]^G = \{0\}$ and therefore A is G-closed.

Let $a \neq 0$ and $A = \{a\}$. If $x \in [A]^G$, then the sequence $(a_n) = (a, a, \dots)$ is G-convergent to x but this is not possible and therefore $[A]^G = \emptyset$. Hence $A = \{a\}$ is G-closed.

Hence all single point sets are G-closed and therefore . by Theorem 3.6, $\mathbb{R}$ is G-convergently T_1. $\mathbb{R}$ is not G-convergently T_i $(i = 2, 3, 4)$ since different points have no disjoint G-open neighbourhoods. .

In the following example $\mathbb{R}$ is not G-convergently T_0.

Example 4.5. A G-method on $\mathbb{R}$ is defined by $G(\mathbf{x}) = \lim x_n x_{n+1}$ for some sequence $\mathbf{x} = (x_n)$ in $\mathbb{R}$.

For example the limit of the sequence $\mathbf{x} = (2, 2, \cdots)$ is 2 but $G(\mathbf{x}) = 4$. Hence the method G is not regular. The subsets $\{0\}$, $\{1\}$ and $\{0, 1\}$ are G-closed; and therefore the only G-open proper subsets are $\mathbb{R} \setminus \{0\}$, $\mathbb{R} \setminus \{1\}$ and $\mathbb{R} \setminus \{0, 1\}$. Hence $\mathbb{R}$ with this method G is not T_i $(i = 0, 1, 2, 3, 4)$.

For the following G-method, $\mathbb{R}$ is not G-convergently T_0.

Example 4.6. Let G be a method on $\mathbb{R}$ defined by $G(\mathbf{x}) = \lim(x_{n+1} - x_n)$ for some sequences $\mathbf{x} = (x_n)$ in $\mathbb{R}$.

For example for a non-zero element $a \in \mathbb{R}$, the limit of the constant sequence $\mathbf{x} = (a, a \cdots)$ is a but $G(\mathbf{x}) = 0$. Hence the method G is not regular.

If $A = \{0\}$, then $[A]^G = \{0\} = \overline{A}^G$ and therefore $A = \{0\}$ is G-closed.

If $a \neq 0$ and $A = \{a\}$, then $[A]^G = \{0\}$ and therefore $A = \{a\}$ is not G-closed.

If $A = \{a, 0\}$, then $[A]^G = \{a, 0, -a\}$; and therefore $A = \{a, 0\}$ is not G-closed. Hence G-closed subsets are closed subgroups of $\mathbb{R}$. For example the subgroup $\mathbb{Z}$ and the subgroups generated by $a \in \mathbb{R}$ are closed and therefore they are G-closed subsets.

Let a and b be non-integer and different points. Then G-open subsets includes both of these points. Hence $\mathbb{R}$ is not G-convergently T_0 and therefore not G-convergently T_i $(i = 0, 1, 2, 3, 4)$.

We now give an example of G-method defined on a topological space X such that X is G-convergently T_2 but not G-convergently T_3.

We remark that if X is a Hausdorff space, then limit function lim from the set $c(X)$ of convergent sequences in X to X itself is a G-method with $G = \lim$. If X is a first countable space, then open and closed subsets are defined in terms of convergent sequences. Hence if X is a first countable Hausdorff space and $G = \lim$, then G-open and hence G-closed subsets coincide with open and closed subsets respectively; and therefore G-convergently separation axioms agree with usual separation axioms.

In the following example X is a first countable Hausdorff space which is not T_3 and therefore it provides an example that X is a G-convergently T_2 with $G = \lim$ but not G-convergently T_3.

Example 4.7. Let X be the half of the plane $X = \{(x, y) \in \mathbb{R}^2 \colon y \geq 0\}$. Define a topology on X whose basis open subsets are open disks in X and half open disks

$$U(a, r) = B((a, 0), r) \cap X \setminus \{(x, 0) \in \mathbb{R}^2 \colon x \in (a - r, a + r) \setminus \{a\}\}$$

Hence X is a first countable Hausdorff space. The subset $F = \{(x, 0) \in \mathbb{R}^2 \colon -1 < x < 1\}$ is closed in X and $x = (1, 0) \notin F$. Any open neighbourhood of $x = (1, 0)$ intersects the open subsets containing F and therefore X is not T_3. It is clear that X is a first countable Hausdorff space.

By the similar procedure one can produce examples of sets which are G-convergently T_3 but not G-convergently T_4. For example $\mathbb{R}^2$ with lower limit topology is a first countable space which is T_3 but not T_4 [24, Example 3, p.198]. Hence we can deduce that it is G-convergently T_3 but not G-convergently T_4 with $G = \lim$.

5. Conclusion

In the paper, using G-open and G-closed subsets on a set equipped with a G-method we define and characterise G-convergent separation axioms; and give some counter examples of the results obtained.

In a topological space X, a sequence $\mathbf{x} = (x_n)$ convergences to a point $a \in X$ if almost all terms of the sequence are in the every open neighbourhood of a. In a first countable space, open and closed subsets are characterised in terms of convergence sequences. Hence by extending these to the G-method setting, we say that a sequence $\mathbf{x} = (x_n)$ is *G-sequentially converges* to a if every G-open neighbourhood of a contains almost all terms. That gives us a variety of G-convergence. Then we say a subset A to be *G-sequentially open* if any sequence converging to a point of A, is almost in A and call *G-sequentially closed* when the complement is G-sequentially open (see [5] and [20] for more discussion of convergences for G-methods). Taking these into account one can develop the new variant of G-convergently separation axioms under the different name G-sequentially separation axioms.

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