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On weaker versions of R-star-Lindelöf and M-star-Lindelöf properties

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Abstract. In this paper, we introduce and analyze the selection principles weakly (almost) R-star-Lindelöf and weakly (almost) M-star-Lindelöf, which turn out to be weaker versions of the R-star-Lindelöf and M-star-Lindelöf properties, respectively, and we provide some relationships with other known properties in literature. Also, we introduce the selection principles $\mathbf{SS}^{\star}_{\Pi_\Delta(\Lambda),1}(\mathbb{C}_\Delta(\Lambda),\mathscr{B})$ and $\mathbf{SS}^{\star}_{\Pi_\Delta(\Lambda),\mathsf{fin}}(\mathbb{C}_\Delta(\Lambda),\mathscr{B})$ to characterize the properties weakly (almost) R-star-Lindelöf and weakly (almost) M-star-Lindelöf in the hyperspace (Λ,τ^+_Δ), respectively.

1. Introduction and preliminaries

The research on selection principles (with many different terminologies) started in [9, 21, 31, 37–39]. Some authors have studied selection principles concerning weaker versions of Rothberger and Menger properties and star type selection principles [3, 6–8, 24, 25, 27, 34, 41]. Now, we recall two known selection principles defined in 1996 by M. Scheepers [38]. Given an infinite set X, let $\mathscr{A}$ and $\mathscr{B}$ be collections of families of subsets of X.

- $\mathbf{S}_1(\mathscr{A},\mathscr{B})$ denotes the principle: For any sequence $(\mathcal{A}_n : n \in \mathbb{N})$ of elements of $\mathscr{A}$, there is a sequence $(B_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $B_n \in \mathcal{A}_n$ and $\{B_n : n \in \mathbb{N}\}$ is an element of $\mathscr{B}$.
- $\mathbf{S}_{\mathsf{fin}}(\mathscr{A},\mathscr{B})$ denotes the principle: for each sequence $(\mathcal{A}_n : n \in \mathbb{N})$ of elements of $\mathscr{A}$ there is a sequence $(\mathcal{B}_n : n \in \mathbb{N})$ such that $\mathcal{B}_n$ is a finite subset of $\mathcal{A}_n$ for each $n \in \mathbb{N}$ and $\bigcup_{n\in\mathbb{N}} \mathcal{B}_n \in \mathscr{B}$.

For a topological space (X,τ) we denote by:

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$\mathscr{O}_X = \{\mathcal{A} \subseteq \tau : \bigcup \mathcal{A} = X\}$, $\mathscr{D}_X = \{D \subseteq X : \mathrm{cl}(D) = X\}$,

$\mathscr{D}'_X = \{\mathcal{A} \subseteq \tau : \bigcup \mathcal{A} \in \mathscr{D}_X\}$, $\mathscr{D}''_X = \{\mathcal{A} \subseteq \tau : \bigcup\{\mathrm{cl}(A) : A \in \mathcal{A}\} = X\}$.

We write simply $\mathscr{O}$, $\mathscr{D}$, $\mathscr{D}'$ and $\mathscr{D}''$ whenever it is clear what the space X is. When we take $\mathbf{S}_1(\mathscr{O},\mathscr{O})$ and $\mathbf{S}_{\mathsf{fin}}(\mathscr{O},\mathscr{O})$, we get the well known *Rothberger property* [37] and the *Menger property* [21, 31], respectively. Moreover, $\mathbf{S}_1(\mathscr{O},\mathscr{D}')$ and $\mathbf{S}_{\mathsf{fin}}(\mathscr{O},\mathscr{D}')$ are known as *weakly Rothberger property* and *weakly Menger property* [15], respectively.

Furthermore, recall that given $A \subseteq X$ and a collection $\mathcal{U}$ of subsets of X, the *star* of A with respect to $\mathcal{U}$ is defined by $\mathrm{St}(A,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}$. As usual, we write $\mathrm{St}(x,\mathcal{U})$ instead of $\mathrm{St}(\{x\},\mathcal{U})$, for every $x \in X$. In [26] Kočinac introduced the star type selection principles $\mathbf{S}^*_1(\mathscr{A},\mathscr{B})$, $\mathbf{S}^*_{\mathsf{fin}}(\mathscr{A},\mathscr{B})$, $\mathbf{SS}^*_1(\mathscr{A},\mathscr{B})$ and $\mathbf{SS}^*_{\mathsf{fin}}(\mathscr{A},\mathscr{B})$. The particular cases $\mathbf{SS}^*_1(\mathscr{O},\mathscr{D}')$ and $\mathbf{SS}^*_{\mathsf{fin}}(\mathscr{O},\mathscr{D}')$ are known as *weakly strong star-Rothberger property* and *weakly strong star-Menger property*, respectively (see [27]). Moreover, $\mathbf{S}^*_1(\mathscr{O},\mathscr{D}')$ and $\mathbf{S}^*_{\mathsf{fin}}(\mathscr{O},\mathscr{D}')$ are known as *weakly star-Rothberger property* and *weakly star-Menger property*, respectively (see [34]).

Recently, in [4] Bal, Bhowmik and Gaulden introduced the following selection principles, for collections of families $\mathscr{C}$ and $\mathscr{B}$ of subsets of X and for a family $\mathcal{A}$ of subsets of X. As is common, $[A]^{<\kappa}$ (respectively, $[A]^{\leq\kappa}$) denotes the collection of all subsets of any set A with less than (respectively, at most) κ elements, where κ is any cardinal.

- $\mathbf{SS}^*_{\mathscr{C},1}(\mathcal{A},\mathscr{B})$, if for every sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ and every $C \in \mathscr{C}$, there is $x_n \in A_n$ for any $n \in \mathbb{N}$, such that $\{\mathrm{St}(x_n, C) : n \in \mathbb{N}\}$ is an element of $\mathscr{B}$.

- $\mathbf{SS}^*_{\mathscr{C},\mathsf{fin}}(\mathcal{A},\mathscr{B})$, if for every sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ and every $C \in \mathscr{C}$, there is $F_n \in [A_n]^{<\omega}$ for any $n \in \mathbb{N}$, such that $\{\mathrm{St}(F_n, C) : n \in \mathbb{N}\}$ is an element of $\mathscr{B}$.

Moreover, in [2, Definitions 3.1, 2.5], [4, Definition 1.5] and, with another terminology, in [5], the authors introduced the next principles for a topological space X.

- $\mathbf{SS}^*_{\mathscr{O},1}(\mathscr{D},\mathscr{O})$: X is said to be *R-star-Lindelöf if* for every sequence $(D_n : n \in \mathbb{N})$ of dense subsets of X and for every open cover $\mathcal{U}$ of X, there exist $x_n \in D_n$, for any $n \in \mathbb{N}$, such that $\bigcup_{n\in\mathbb{N}} \mathrm{St}(x_n,\mathcal{U}) = X$.

- $\mathbf{SS}^*_{\mathscr{O},\mathsf{fin}}(\mathscr{D},\mathscr{O})$: X is said to be *M-star-Lindelöf* if for every sequence $(D_n : n \in \mathbb{N})$ of dense subsets of X and for every open cover $\mathcal{U}$ of X there exists a family $\{F_n : n \in \mathbb{N}\}$ of finite subsets of X such that $F_n \subseteq D_n$ for each $n \in \mathbb{N}$ and $\bigcup_{n\in\mathbb{N}} \mathrm{St}(F_n,\mathcal{U}) = X$.

In this paper, in Section 2, we introduce the notions weakly (almost) R-star-Lindelöf and weakly (almost) M-star-Lindelöf, which turn out to be weaker versions of the R-star-Lindelöf and M-star-Lindelöf properties, respectively (see Definition 2.1). Also, we provide some relationships (see Figure 1) and results of them.

On the other hand, the study of hyperspace theory started in [19, 32, 35, 44]. We denote by $\mathrm{CL}(X)$ the family of all nonempty closed subsets of a topological space X. The family $\mathrm{CL}(X)$, endowed with some topology, is known as hyperspace of X. In [16] the authors used π-networks to characterize topological spaces whose hyperspace, endowed with the upper Fell topology, satisfies the Rothberger property. Then, in [29] are defined the concepts of π_F-network, π_V-network, k_F-cover and c_V-cover and they are used to study the $\mathbf{S}_1(\mathscr{A},\mathscr{B})$ and $\mathbf{S}_{\mathsf{fin}}(\mathscr{A},\mathscr{B})$ principles in $\mathrm{CL}(X)$ endowed with the Fell and Vietoris topologies, for different families $\mathscr{A}$ and $\mathscr{B}$. Later, in [10, 17] the authors introduce the generic notions of $\pi_\Delta(\Lambda)$-networks (and $c_\Delta(\Lambda)$-covers), which are a generalization of π_F-networks and π_V-networks (and of k_F-cover and c_V-cover, respectively). These concepts are used to characterize Menger-type star selection principles [10], star and strong star-type versions of Rothberger and Menger principles [11], Hurewicz like properties [12], weaker forms of Rothberger and Menger properties and groupability [13] and weakly star Rothberger and Menger properties [14] in hyperspaces endowed with the hit-and-miss topology. Continuing with this research line, in this paper, in Section 3, we will introduce the selection principles $\mathbf{SS}^\star_{\Pi_\Delta(\Lambda),1}(\mathbb{C}_\Delta(\Lambda),\mathscr{B})$ and $\mathbf{SS}^\star_{\Pi_\Delta(\Lambda),\mathsf{fin}}(\mathbb{C}_\Delta(\Lambda),\mathscr{B})$ to obtain characterizations of weakly (almost) R-star-Lindelöf and weakly (almost) M-star-Lindelöf properties in hyperspaces.

Now, we present some basic concepts about the theory of hyperspaces. All spaces are assumed to be Hausdorff noncompact and, even, nonparacompact. For a space (X, τ), we denote by $\mathrm{CL}(X)$, $\mathbb{K}(X)$, $\mathbb{F}(X)$ and $\mathbb{CS}(X)$ the family of all nonempty closed subsets, the family of all nonempty compact subsets, the family of all nonempty finite subsets of X and the family of all convergent sequences of X, respectively.

For any subset $U \subseteq X$ and every family $\mathcal{U}$ of subsets of X, we denote:

$$\begin{aligned} U^- &= \{A \in \mathrm{CL}(X) : A \cap U \neq \emptyset\}; \\ U^+ &= \{A \in \mathrm{CL}(X) : A \subseteq U\}; \\ U^c &= X \setminus U; \\ \mathcal{U}^c &= \{U^c : U \in \mathcal{U}\}. \end{aligned}$$

Let Δ be a subfamily of $\mathrm{CL}(X)$ closed under finite unions and containing all singletons. Then, the *hit-and-miss topology on* $\mathrm{CL}(X)$ *respect to* Δ, denoted by τ^+_Δ, is generated by the base

$$\left\{\left(\bigcap_{i=1}^m V_i^-\right) \cap (B^c)^+ : B \in \Delta \text{ and } V_i \in \tau \text{ for } i \in \{1, \dots, m\}\right\}.$$

We use $(V_1, \dots, V_m)^+_B$ to denote the basic element $(\bigcap_{i=1}^m V_i^-) \cap (B^c)^+$ (see [48]).

Two important known particular cases of the hit-and-miss topology are the *Vietoris topology*, τ_V, when $\Delta = \mathrm{CL}(X)$ (see [32, 44]), and the *Fell topology*, τ_F, when $\Delta = \mathbb{K}(X)$ (see [18]). Along this paper, unless we say the opposite, we will consider a subspace Λ of $(\mathrm{CL}(X), \tau^+_\Delta)$, which contains the singletons and is closed under finite unions.

In another context, inspired by Li [29], we introduced in [10] the notions of $\pi_\Delta(\Lambda)$-network and $c_\Delta(\Lambda)$-cover of a space X. We remember them and a couple of lemmas which will be used along this work.

Given a family $\Delta \subseteq \mathrm{CL}(X)$, we denote

$$\zeta_\Delta = \{(B; V_1, \dots, V_n) : B \in \Delta \text{ and } V_1, \dots, V_n \text{ are open subsets of } X \text{ with } V_i \cap B^c \neq \emptyset \ (1 \leq i \leq n),\ \ n \in \mathbb{N}\}.$$

Definition 1.1. Let (X, τ) be a topological space. A family $\mathcal{J} \subseteq \zeta_\Delta$ is called a $\pi_\Delta(\Lambda)$*-network of* X, if for each $U \in \Lambda^c$, there exist $(B; V_1, \dots, V_n) \in \mathcal{J}$ with $B \subseteq U$ and $F \in [X]^{<\omega}$ such that $F \cap U = \emptyset$ and for each $i \in \{1, \dots, n\}$, $F \cap V_i \neq \emptyset$. The family of all $\pi_\Delta(\Lambda)$-networks is denoted by $\Pi_\Delta(\Lambda)$ (see [10, Definition 2.1]).

Lemma 1.2. *Let* (X, τ) *be a topological space and* $\Delta, \Lambda \subseteq \mathrm{CL}(X)$*. Consider the family* $\mathcal{J} = \{(B_s; V_{1,s}, \dots, V_{m_s,s}) : s \in S\}$ *and define the collection* $\mathscr{U} = \left\{(V_{1,s}, \dots, V_{m_s,s})^+_{B_s} : (B_s; V_{1,s}, \dots, V_{m_s,s}) \in \mathcal{J}\right\}$*. Then,* $\mathcal{J}$ *is a* $\pi_\Delta(\Lambda)$*-network of* X *if and only if* $\mathscr{U}$ *is an open cover of* (Λ, τ^+_Δ) *(see [10, Lemma 2.4]).*

Definition 1.3. Let (X, τ) be a topological space. A family $\mathcal{U} \subseteq \Lambda^c$ is called a $c_\Delta(\Lambda)$*-cover of* X, if for any $B \in \Delta$ and open subsets $V_1, \dots, V_m$ of X, with $B^c \cap V_i \neq \emptyset$ for any $i \in \{1, \dots, m\}$, there exist $U \in \mathcal{U}$ and $F \in [X]^{<\omega}$ such that $B \subseteq U$, $F \cap U = \emptyset$ and for each $i \in \{1, \dots, m\}$, $F \cap V_i \neq \emptyset$. We denote by $\mathbb{C}_\Delta(\Lambda)$ the family of all $c_\Delta(\Lambda)$-covers of X (see [10, Definition 2.20]).

Lemma 1.4. *Let* (X, τ) *be a topological space and* $\Delta, \Lambda \subseteq \mathrm{CL}(X)$*. A family* $\mathcal{U} \subseteq \Lambda^c$ *is a* $c_\Delta(\Lambda)$*-cover of* X *if and only if the family* $\mathcal{U}^c$ *is a dense subset of* (Λ, τ^+_Δ) *(see [10, Lemma 2.22]).*

2. Weaker versions of R and M star-Lindelöf properties

In this section, we introduce some weaker versions of R-star-Lindelöf and M-star-Lindelöf properties.

Definition 2.1. We say that a topological space (X, τ) is:

- *almost R-star-Lindelöf* if for every sequence $(D_n : n \in \mathbb{N})$ of dense subsets of X and for every open cover $\mathcal{U}$ of X, there exist $x_n \in D_n$, for any $n \in \mathbb{N}$, such that $\bigcup_{n\in\mathbb{N}} \mathrm{cl}\,(\mathrm{St}\,(x_n, \mathcal{U})) = X$.
- *weakly R-star-Lindelöf* if for every sequence $(D_n : n \in \mathbb{N})$ of dense subsets of X and for every open cover $\mathcal{U}$ of X, there exist $x_n \in D_n$, for any $n \in \mathbb{N}$, such that $\mathrm{cl}\,(\bigcup_{n\in\mathbb{N}} \mathrm{St}\,(x_n, \mathcal{U})) = X$.

- *almost M-star-Lindelöf* if for every sequence $(D_n : n \in \mathbb{N})$ of dense subsets of X and for every open cover $\mathcal{U}$ of X there exists a family $\{F_n : n \in \mathbb{N}\}$ of finite subsets of X such that $F_n \subseteq D_n$ for each $n \in \mathbb{N}$ and $\bigcup_{n\in\mathbb{N}} \mathrm{cl}\,(\mathrm{St}\,(F_n, \mathcal{U})) = X$.
- *weakly M-star-Lindelöf* if for every sequence $(D_n : n \in \mathbb{N})$ of dense subsets of X and for every open cover $\mathcal{U}$ of X, there exists $F_n \in [D_n]^{<\omega}$, for each $n \in \mathbb{N}$, such that $\mathrm{cl}\,(\bigcup_{n\in\mathbb{N}} (\mathrm{St}\,(F_n, \mathcal{U}))) = X$.

Remark 2.2. By using the families $\mathscr{D}'$ and $\mathscr{D}''$, the principles almost R-star-Lindelöf, weakly R-star-Lindelöf, almost M-star-Lindelöf and weakly M-star-Lindelöf correspond to $\mathbf{SS}^*_{\mathscr{O},1}(\mathscr{D}, \mathscr{D}'')$, $\mathbf{SS}^*_{\mathscr{O},1}(\mathscr{D}, \mathscr{D}')$, $\mathbf{SS}^*_{\mathscr{O},\mathrm{fin}}(\mathscr{D}, \mathscr{D}'')$ and $\mathbf{SS}^*_{\mathscr{O},\mathrm{fin}}(\mathscr{D}, \mathscr{D}')$, respectively.

Remark 2.3. In Definition 2.1, we obtain the same if we take the open cover $\mathcal{U}$ consisting of basic open sets.

The Figure 1 shows the relationships between the notions introduced which follow immediately from the definitions. The Examples 2.4, 2.7 and 2.11, show which arrows are not reversible. However, the authors do not know if the other arrows are or are not reversible.

$$\begin{array}{ccc}
\text{weakly R-star-Lindelöf} & \longrightarrow & \text{weakly M-star-Lindelöf} \\
\not\downarrow\uparrow & & \not\downarrow\uparrow \\
\text{almost R-star-Lindelöf} & \longrightarrow & \text{almost M-star-Lindelöf} \\
\not\downarrow\uparrow & & \not\downarrow\uparrow \\
\text{R-star-Lindelöf} & \longrightarrow & \text{M-star-Lindelöf}
\end{array}$$

Figure 1: Relationships between the selection principles defined.

Remember that a space X is called <u>*almost star countable*</u> if for each open cover $\mathcal{U}$ of X, there exists a countable subset $\{x_n : n \in \mathbb{N}\}$ of X such that $\bigcup_{n\in\mathbb{N}} \overline{\mathrm{St}(x_n, \mathcal{U})} = X$. Note that every almost R-star-Lindelöf space is almost star countable (see [42]).

Example 2.4. There exists a Tychonoff weakly M-star-Lindelöf space which is not almost M-star-Lindelöf.

Let D be a discrete space of cardinality ω_1 and consider the subspace

$$X = (\beta D \times (\omega + 1)) \setminus ((\beta D \setminus D) \times \{\omega\})$$

of the product of βD and $\omega + 1$ endowed with the order topology. This space was presented in [42, Example 2.5] where the author proved that X is not almost star countable. Hence, it is not almost M-star Lindelöf. On the other hand, note that if Y is a dense in X, then $Y \cap (\beta D \times \omega)$ is dense in $\beta D \times \omega$. Let $(Y_n : n \in \mathbb{N})$ a sequence of dense subsets in X and $\mathcal{U}$ be an open cover of X. For each $n \in \mathbb{N}$, let $Y'_n = Y_n \cap (\beta D \times \omega)$ and $\mathcal{U}' = \{U \cap (\beta D \times \omega) : U \in \mathcal{U}\}$. So, $\mathcal{U}'$ is an open cover of $\beta D \times \omega$. Given that for each $n \in \omega$, $\beta D \times \{n\}$ is a compact subset of $\beta D \times \omega$, we have that there is $\mathcal{U}'_n = \{U^n_j : 1 \le j \le m_n\} \subseteq \mathcal{U}'$ such that $\beta D \times \{n\} \subseteq \bigcup \mathcal{U}'_n$. As Y'_n is dense in $\beta D \times \omega$, we fix an element y^n_j in $U^n_j \cap Y'_n$, for $1 \le j \le m_n$ and we put $F_n = \left\{y^n_1, \ldots, y^n_{m_n}\right\}$. We see that $\mathrm{cl}\,[\bigcup\{\mathrm{St}(F_n, \mathcal{U}) : n \in \mathbb{N}\}] = X$. Indeed, it follows from the facts $\beta D \times \omega \subseteq \bigcup\{\mathrm{St}(F_n, \mathcal{U}) : n \in \mathbb{N}\}$ and $\beta D \times \omega$ is dense in X. Therefore, X is weakly M-star-Lindelöf.

Recall that a space X is *H-closed* (see [1, 28]) if for every open cover $\mathcal{U}$ of X there exists $\mathcal{V} \in [\mathcal{U}]^{<\omega}$ such that $\bigcup \left\{\overline{U} : U \in \mathcal{V}\right\} = X$. Also recall that a space X is called *almost Lindelöf* if for every open cover $\mathcal{U}$ there exists $\mathcal{V} \in [\mathcal{U}]^{\le\omega}$ such that $\bigcup \left\{\overline{U} : U \in \mathcal{V}\right\} = X$ (see [46]). Of course every H-closed space is almost Lindelöf. It follows from the definitions the next result.

Proposition 2.5. *Every almost Lindelöf space is almost R-star-Lindelöf. In particular, if X is H-closed, then X is almost R-star-Lindelöf.*

Example 2.6. There exists an almost R-star-Lindelöf space which is not almost Lindelöf.

Consider the Mrówka-Isbell space $\Psi(\mathcal{A}) = \omega \cup \mathcal{A}$ (see [33]), where $\mathcal{A}$ is a MAD family with $|\mathcal{A}| = 2^{\omega}$. As ω is dense in $\Psi(\mathcal{A})$ and every dense set in $\Psi(\mathcal{A})$ contains ω, it follows that $\Psi(\mathcal{A})$ is R-star-Lindelöf. So, by Figure 1, $\Psi(\mathcal{A})$ is almost R-star-Lindelöf. On the other hand, as $\Psi(\mathcal{A})$ is regular and not Lindelöf, by [46, Theorem 1.1-(ii)], we conclude that X is not almost Lindelöf.

Example 2.7. There exists an almost R-star-Lindelöf space which is not R-star-Lindelöf (nor M-star-Lindelöf).

Let ω_1 be endowed with discrete topology and $X = \omega_1 \cup T(\omega_1)$ its Katětov extension, where $T(\omega_1) = \{p : p \text{ is a free ultrafilter of } \omega_1\}$. It is known that X is an H-closed space ([23]). So, by Proposition 2.5, X is almost R-star-Lindelöf. On the other hand, we see that X is not M-star-Lindelöf. Indeed, let $\mathcal{A} = \{A_\alpha : \alpha \in \omega_1\}$ a pairwise disjoint collection of nonempty subsets of ω_1 and $Y = \{p_\alpha : \alpha \in \omega_1\}$ a collection of free ultrafilters on ω_1 such that $A_\alpha \in p_\alpha$ (see [20, Example p. 2202]). Note that Y is a closed and discrete subset of X. Now, let $\mathcal{U} = \{\{p_\alpha\} \cup A_\alpha : \alpha \in \omega_1\} \cup \{X \setminus Y\}$, clearly $\mathcal{U}$ is an open cover of X. For every $n \in \mathbb{N}$, we take $D_n = \omega_1$, which is dense in X. If $F_n \in [D_n]^{<\omega}$, there exists $\alpha_0 \in \omega_1$ such that $A_{\alpha_0} \cap (\bigcup_{n\in\mathbb{N}} F_n) = \emptyset$. Hence, as $\{p_{\alpha_0}\} \cup A_{\alpha_0}$ is the unique element in $\mathcal{U}$ which contains p_{α_0}, it follows that $p_{\alpha_0} \notin \bigcup\{\mathrm{St}(F_n, \mathcal{U}) : n \in \mathbb{N}\}$. We conclude that X is not M-star-Lindelöf.

Remember that a topological space X is called *weakly Lindelöf* if for every open cover $\mathcal{U}$ of X there exist $\mathcal{V} \in [\mathcal{U}]^{\leq\omega}$ such that $\mathrm{cl}\,(\bigcup \mathcal{V}) = X$.

Proposition 2.8. *Let X be a topological space. If X is weakly Lindelöf, then X is weakly R-star-Lindelöf.*

Example 2.9. There exists a weakly R-star-Lindelöf space which is not weakly Lindelöf.

Consider ω_1 endowed with the order topology. It is known that this space is R-star-Lindelöf (see [4, Example 2.19]. Hence, by Figure 1, ω_1 is weakly R-star-Lindelöf. On the other hand, it can be shown that ω_1 is not weakly Lindelöf.

Recall that a space X is *absolutely countably compact* if for every open cover $\mathcal{U}$ of X and every dense subset D of X, there exists a finite subset F of D such that $\mathrm{St}(F, \mathcal{U}) = X$ (see [30]). Now, consider ω_1 endowed with the order topology and ist Alexandroff duplicate $A(\omega_1)$. From [43, Proposition 3.8], we have that $A(\omega_1)$ is not weakly Lindelöf. However, from [40, Lemma 2.1] (result established by Vaughan in [45]), $A(\omega_1)$ is absolutely countably compact. Hence, $A(\omega_1)$ is M-star-Lindelöf. So, by Figure 1, $A(\omega_1)$ is weakly M-star-Lindelöf.

Remark 2.10. The authors do not know if $A(\omega_1)$ is R-star-Lindelöf. If the answer is positive, we would conclude that $A(\omega_1)$ is a weakly R-star-Lindelöf space which is not weakly Lindelöf. Otherwise, if the answer is negative, $A(\omega_1)$ would be an example of an M-star-Lindelöf space which is not R-star-Lindelöf (see [2, Problem 3.16]).

The next example, due to Junnila [22], shows that definitions of weakly R-star-Lindelöf and almost R-star-Lindelöf are not equivalent.

Example 2.11. There exists a weakly R-star-Lindelöf space which is not almost R-star-Lindelöf.

Let A_0 be an uncountable set, $A_{n+1} = [\mathcal{P}(A_n)]^{<\omega}$ and $X = \bigcup_{n\in\mathbb{N}} A_n$. For $x \in X$, let $n(x)$ be the smallest $n \in \mathbb{N}$ such that $x \in A_n$. For $x \in X$ and $z \in A_{n(x)+1}$, let $z_x = \{a \in z : x \in a\}$ and $P_z(x) = \{u \in A_{n(x)+1} : u \cap z = z_x\}$. Now define $U \subseteq X$ to be open if and only if for each $x \in U$, $P_z(x) \subseteq U$ for some $z \in A_{n(x)+1}$. In [47, Example 3.7], was shown that X is not almost star countable and observed that X is weakly Lindelöf. Thus, X is not almost R-star-Lindelöf and by Proposition 2.8, X is weakly R-star-Lindelöf.

Remark 2.12. The properties weakly (almost) R-star-Lindelöf and weakly (almost) M-star-Lindelöf are not preserved to closed subsets (compare with Proposition 2.13). Indeed, we see that:

- In Example 2.4, the set $D \times \{\omega\}$ is a closed subset of X, however it is not weakly M-star-Lindelöf.

- The subset Y in Example 2.7, is a closed subset of X which is not almost R-star-Lindelöf (nor M-star Lindelöf).
- The subset A_0 is a closed subset of the space X in the Example 2.11 and it is not weakly R-star-Lindelöf.

Next, we show some results for weakly (almost) R-star-Lindelöf and weakly (almost) M-star-Lindelöf properties. Their proofs follow some ideas from [2, 4].

Proposition 2.13. *The properties weakly (almost) R-star-Lindelöf and weakly (almost) M-star-Lindelöf are preserved to clopen subsets.*

Proof. We will show just the case in which X is a weakly M-star-Lindelöf space. Suppose that Y is a clopen subset of X. Let $\mathcal{U}$ be an open cover of Y and D_n a dense subset of Y, for every $n \in \mathbb{N}$. By hypothesis, we can choose finite sets $F_n \subseteq D_n \cup (X \setminus Y)$ such that $\mathrm{cl}\,(\bigcup_{n\in\mathbb{N}} \mathrm{St}(F_n, \mathcal{U} \cup \{X \setminus Y\})) = X$. It follows immediately that $\mathrm{cl}\,(\bigcup_{n\in\mathbb{N}} \mathrm{St}(F_n \cap D_n, \mathcal{U})) = Y$ and the result holds. □

Lemma 2.14. *Let X and Y topological spaces and $f : X \to Y$ be a continuous surjection. Let $\mathcal{V}$ be a family of subsets of Y, $\mathcal{U} = \{f^{-1}(V) : V \in \mathcal{V}\}$ and for each $n \in \mathbb{N}$, $A_n \subseteq X$.*

(a) If $\mathrm{cl}\,(\bigcup_{n\in\mathbb{N}} \mathrm{St}(A_n, \mathcal{U})) = X$, then $\mathrm{cl}\,(\bigcup_{n\in\mathbb{N}} \mathrm{St}(f(A_n), \mathcal{V})) = Y$;
(b) If $\bigcup_{n\in\mathbb{N}} \mathrm{cl}\,(\mathrm{St}(A_n, \mathcal{U})) = X$, then $\bigcup_{n\in\mathbb{N}} \mathrm{cl}\,(\mathrm{St}(f(A_n), \mathcal{V})) = Y$.

Proof. (a) Let W be a nonempty open set of Y. Then $f^{-1}(W)$ intersects $\bigcup_{n\in\mathbb{N}} \mathrm{St}(A_n, \mathcal{U})$. So, there exist $N \in \mathbb{N}$, $V \in \mathcal{V}$, $a_N \in A_N$ and $z \in X$ such that $a_N \in f^{-1}(V)$ and $z \in f^{-1}(V) \cap f^{-1}(W)$. Hence, $f(a_N) \in V$ and $f(z) \in V \cap W$, that is, $f(z) \in W \cap \mathrm{St}(f(A_N), \mathcal{V})$ and the result follows.

(b) Let $y \in Y$ and $x \in X$ with $f(x) = y$. So, there exist $N \in \mathbb{N}$ such that $x \in \mathrm{cl}\,(\mathrm{St}(A_N, \mathcal{U}))$. We claim that $y \in \mathrm{cl}\,(\mathrm{St}(f(A_N), \mathcal{V}))$. Indeed, let W an open neighborhood of y. Hence, $f^{-1}(W)$ intersects $\mathrm{St}(A_N, \mathcal{U})$. Thus, there exist $V \in \mathcal{V}$, $a_N \in A_N$ and $z \in X$ such that $a_N \in f^{-1}(V)$ and $z \in f^{-1}(V) \cap f^{-1}(W)$. So, $f(a_N) \in V$ and $f(z) \in V \cap W$, that is, $f(z) \in W \cap \mathrm{St}(f(A_N), \mathcal{V})$ and the result follows. □

Proposition 2.15. *The properties weakly (almost) R-star-Lindelöf and weakly (almost) M-star-Lindelöf are preserved under open continuous functions.*

Proof. Let X and Y topological spaces and $f : X \to Y$ be an open continuous surjection. We will show just the case in which X is a weakly M-star-Lindelöf space. Let $\mathcal{V}$ be an open cover of Y and for each $n \in \mathbb{N}$, E_n a dense subset of Y. Note that $\mathcal{U} = \{f^{-1}(V) : V \in \mathcal{V}\}$ is an open cover of X and for each $n \in \mathbb{N}$, $D_n = f^{-1}(E_n)$ is a dense subset of X. As X is weakly M-star-Lindelöf, there exists $F_n \in [D_n]^{<\omega}$ such that $\mathrm{cl}\,(\bigcup_{n\in\mathbb{N}} \mathrm{St}(F_n, \mathcal{U})) = X$. It follows immediately that $f(F_n) \in [E_n]^{<\omega}$. From Lemma 2.14-(a), we have $\mathrm{cl}\,(\bigcup_{n\in\mathbb{N}} \mathrm{St}(f(F_n), \mathcal{V})) = Y$ and the result follows. □

Proposition 2.16. *Let (X, τ) be a topological space and suppose that $X = \bigcup_{n\in\mathbb{N}} Y_n$, where Y_n is an open set. If, for each $n \in \mathbb{N}$, Y_n satisfies any of the following properties: weakly R-star-Lindelöf, almost R-star-Lindelöf, weakly M-star-Lindelöf or almost M-star-Lindelöf, then X does it too.*

Proof. We will prove the theorem just for the weakly M-star-Lindelöf property. Suppose that $X = \bigcup_{n\in\mathbb{N}} Y_n$, where Y_n is an open weakly M-star-Lindelöf subset of X, for any $n \in \mathbb{N}$. Let $\mathcal{U}$ be an open cover of X and D_n a dense subset of X, for every $n \in \mathbb{N}$. Let $\{L_n : n \in \mathbb{N}\}$ a partition of $\mathbb{N}$ into infinite countable sets. By hypothesis, for any $k \in L_n$ we can choose finite sets $F_k \subseteq D_k \cap Y_n$ such that $Y_n \subseteq \mathrm{cl}\left(\bigcup_{k\in L_n} \mathrm{St}(F_k, \mathcal{U})\right)$. It follows immediately that $X = \bigcup_{n\in\mathbb{N}} \mathrm{cl}\left(\bigcup_{k\in L_n} \mathrm{St}(F_k, \mathcal{U})\right) \subseteq \mathrm{cl}\left(\bigcup_{n\in\mathbb{N}} \bigcup_{k\in L_n} \mathrm{St}(F_k, \mathcal{U})\right)$ and the result follows. □

3. Weaker versions of R-star-Lindelöf and M-star-Lindelöf properties in hyperspaces

The R-star-Lindelöf and M-star-Lindelöf properties in the hyperspace $(\Lambda, \tau_{\Delta}^{+})$, have been characterized in [36, Theorems 4.5, 4.2, respectively]. Now, we present the corresponding characterizations for weakly (almost) R-star-Lindelöf and weakly (almost) M-star-Lindelöf in the same hyperspace. In order to characterize the weakly R-star-Lindelöf property in hyperspaces, we introduce the following.

Notation 3.1. Let (X, τ) be a topological space and consider $\Delta, \Lambda \subseteq \mathrm{CL}(X)$. Let $U \in \Lambda^c$ and $\mathcal{J} \in \Pi_{\Delta}(\Lambda)$. We denote

$$\mathrm{St}^{\star}(U, \mathcal{J}) = \{V \in \Lambda^c : \text{there is } (B; V_1, \ldots, V_n) \in \mathcal{J} \text{ such that } B \subseteq V \cap U, V_i \cap V^c \neq \emptyset \neq V_i \cap U^c\}.$$

Lemma 3.2. *Let (X, τ) be a topological space and consider the hyperspace $(\Lambda, \tau_{\Delta}^{+})$. Let $\mathcal{J} \in \Pi_{\Delta}(\Lambda)$ and $\mathscr{U}$ as Lemma 1.2. For any $U \in \Lambda^c$, $\mathrm{St}^{\star}(U, \mathcal{J}) = (\mathrm{St}(U^c, \mathscr{U}))^c$.*

Proof. Let $\mathscr{U} = \left\{(V_1, \ldots, V_m)_B^+ : (B; V_1, \ldots, V_m) \in \mathcal{J}\right\}$. It is enough to note that $V \in \mathrm{St}^{\star}(U, \mathcal{J})$ is the same as $V^c, U^c \in (V_1, \ldots, V_n)_B^+$, for some $(V_1, \ldots, V_n)_B^+ \in \mathscr{U}$. □

Definition 3.3. A topological space (X, τ) satisfies $\mathbf{SS}^{\star}_{\Pi_{\Delta}(\Lambda),1}(\mathbb{C}_{\Delta}(\Lambda), \mathscr{B})$, where $\mathscr{B} \subseteq \tau$, if for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathbb{C}_{\Delta}(\Lambda)$ and every $\mathcal{J} \in \Pi_{\Delta}(\Lambda)$, there is $U_n \in \mathcal{U}_n$ for any $n \in \mathbb{N}$, such that $\{\mathrm{St}^{\star}(U_n, \mathcal{J}) : n \in \mathbb{N}\} \in \mathscr{B}$.

Notation 3.4. Let (X, τ) be a topological space and consider $\Delta, \Lambda \subseteq \mathrm{CL}(X)$. We denote by $\mathbb{C}'_{\Delta}(\Lambda) = \{\mathcal{A} \subseteq \mathcal{P}(\tau) : \bigcup \mathcal{A} \in \mathbb{C}_{\Delta}(\Lambda)\}$.

Theorem 3.5. *Let (X, τ) be a topological space. The following conditions are equivalent:*

(1) *$(\Lambda, \tau_{\Delta}^{+})$ is weakly R-star-Lindelöf or equivalently Λ satisfies $\mathbf{SS}^{*}_{\mathscr{O},1}\left(\mathscr{D}_{\Lambda}, \mathscr{D}'_{\Lambda}\right)$;*

(2) *(X, τ) satisfies $\mathbf{SS}^{\star}_{\Pi_{\Delta}(\Lambda),1}\left(\mathbb{C}_{\Delta}(\Lambda), \mathbb{C}'_{\Delta}(\Lambda)\right)$.*

Proof. (1) ⇒ (2) Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of $c_{\Delta}(\Lambda)$-covers of X and $\mathcal{J}$ be a $\pi_{\Delta}(\Lambda)$-network. For every $n \in \mathbb{N}$, consider $\mathcal{D}_n = \mathcal{U}_n^c$ and $\mathscr{U} = \left\{(V_{1,s}, \ldots, V_{m_s,s})_{B_s}^+ : (B_s; V_{1,s}, \ldots, V_{m_s,s}) \in \mathcal{J}\right\}$. By Lemmas 1.4, and 1.2, we obtain that, for any $n \in \mathbb{N}$, $\mathcal{D}_n$ is a dense subset of $(\Lambda, \tau_{\Delta}^{+})$ and $\mathscr{U}$ is an open cover of Λ, respectively. Hence, applying (1) to the sequence $(\mathcal{D}_n : n \in \mathbb{N})$ and $\mathscr{U}$, we obtain, for each $n \in \mathbb{N}$, $F_n \in \mathcal{D}_n$ such that $\bigcup_{n\in\mathbb{N}} \mathrm{St}(F_n, \mathscr{U})$ is dense in Λ. Put $U_n = F_n^c$, so $U_n \in \mathcal{U}_n$. We claim that $\bigcup_{n\in\mathbb{N}}(\mathrm{St}^{\star}(U_n, \mathcal{J}))$ is an element of $\mathbb{C}_{\Delta}(\Lambda)$. Indeed, let $B \in \Delta$ and $V_1, \ldots, V_m$ open subsets of X, with $B^c \cap V_i \neq \emptyset$, for any $i \in \{1, \ldots, m\}$. Let $G \in (V_1, \ldots, V_m)_B^+ \cap \bigcup_{n\in\mathbb{N}} \mathrm{St}(F_n, \mathscr{U})$ and put $U = G^c$. Clearly, $B \subseteq U$, furthermore, we can take $x_i \in V_i \setminus U$ and $F = \{x_i : 1 \leq i \leq m\}$. As $U^c \in \mathrm{St}(F_N, \mathscr{U})$, for some $N \in \mathbb{N}$, from Lemma 3.2, we conclude that $U \in \mathrm{St}^{\star}(U_N, \mathcal{J})$ and the claim holds. This means that $\{\mathrm{St}^{\star}(U_n, \mathcal{J}) : n \in \mathbb{N}\} \in \mathbb{C}'_{\Delta}(\Lambda)$ and (2) holds.

(2) ⇒ (1) Let $(\mathcal{D}_n : n \in \mathbb{N})$ be a sequence of dense subsets of $(\Lambda, \tau_{\Delta}^{+})$ and $\mathscr{U}$ be an open cover of Λ. By Lemma 1.4, we have that, for each $n \in \mathbb{N}$, $\mathcal{U}_n = \mathcal{D}_n^c$ is a $c_{\Delta}(\Lambda)$-cover of X and by Lemma 1.2, $\mathcal{J} = \{(B; V_1, \ldots, V_m) : (V_1, \ldots, V_m)_B^+ \in \mathscr{U}\}$ is a $\pi_{\Delta}(\Lambda)$-network of X. Hence, applying (2), there is $U_n \in \mathcal{U}_n$ for any $n \in \mathbb{N}$, such that $\bigcup_{n\in\mathbb{N}}(\mathrm{St}^{\star}(U_n, \mathcal{J}))$ is an element of $\mathbb{C}_{\Delta}(\Lambda)$. Thus, by Lemmas 3.2 and 1.4, $\bigcup_{n\in\mathbb{N}}(\mathrm{St}(U_n^c, \mathscr{U})) = \left(\bigcup_{n\in\mathbb{N}}(\mathrm{St}^{\star}(U_n, \mathcal{J}))\right)^c$ is dense in Λ. Since $U_n^c \in \mathcal{D}_n$, the result follows. □

As an immediate consequence of Theorem 3.5, we obtain the following corollaries.

Corollary 3.6. *Let (X, τ) be a topological space. If Λ is any of the hyperspaces $\mathrm{CL}(X)$, $\mathbb{K}(X)$, $\mathbb{F}(X)$ or $\mathbb{CS}(X)$, then $(\Lambda, \tau_{\Delta}^{+})$ is weakly R-star-Lindelöf if and only if X satisfies the principle $\mathbf{SS}^{\star}_{\Pi_{\Delta}(\Lambda),1}(\mathbb{C}_{\Delta}(\Lambda), \mathbb{C}'_{\Delta}(\Lambda))$.*

Corollary 3.7. *Let (X, τ) be a topological space. Let Λ be some of the spaces $\mathbb{K}(X)$, $\mathbb{F}(X)$, $\mathbb{CS}(X)$, we have:*

(a) Suppose that $\Delta = \mathbb{K}(X)$. Then (Λ, τ_F) is weakly R-star-Lindelöf if and only if X satisfies the principle $\mathbf{SS}^{\star}_{\Pi_{\mathbb{K}(X)}(\Lambda),1}\left(\mathbb{C}_{\mathbb{K}(X)}(\Lambda), \mathbb{C}'_{\mathbb{K}(X)}(\Lambda)\right)$.

(b) *Suppose that* $\Delta = \mathrm{CL}(X)$. *Then* (Λ, τ_V) *is weakly R-star-Lindelöf if and only if X satisfies the principle* $\mathbf{SS}^{\star}_{\Pi_{\mathrm{CL}(X)}(\Lambda),1}\left(\mathbb{C}_{\mathrm{CL}(X)}(\Lambda), \mathbb{C}'_{\mathrm{CL}(X)}(\Lambda)\right)$.

Remember that $\Pi_{\mathbb{K}(X)}(\mathrm{CL}(X)) = \Pi_F$ and $\Pi_{\mathrm{CL}(X)}(\mathrm{CL}(X)) = \Pi_V$, (see [10, Remark 2.2] and [29]). Furthermore, we have $\mathbb{C}_{\mathbb{K}(X)}(\mathrm{CL}(X)) = \mathbb{K}_F$ and $\mathbb{C}_{\mathrm{CL}(X)}(\mathrm{CL}(X)) = \mathbb{C}_V$ (see [10, Remark 2.21] and [29]). Now we denote by $\mathbb{K}'_F = \{\mathcal{A} : \bigcup \mathcal{A} \in \mathbb{K}_F\}$ and $\mathbb{C}'_V = \{\mathcal{A} : \bigcup \mathcal{A} \in \mathbb{C}_V\}$.

Corollary 3.8. *Let* (X, τ) *be a topological space, we have:*

(a) $(\mathrm{CL}(X), \tau_F)$ *is weakly R-star-Lindelöf if and only if X satisfies the principle* $\mathbf{SS}^{\star}_{\Pi_F,1}(\mathbb{K}_F, \mathbb{K}'_F)$.

(b) $(\mathrm{CL}(X), \tau_V)$ *is weakly R-star-Lindelöf if and only if X satisfies the principle* $\mathbf{SS}^{\star}_{\Pi_V,1}\left(\mathbb{C}_V, \mathbb{C}'_V\right)$.

To characterize the weakly M-star-Lindelöf property in hyperspaces, we introduce the next notion. The proof of Theorem 3.10 follows the same ideas as Theorem 3.5 and we omitted it.

Definition 3.9. A topological space (X, τ) satisfies $\mathbf{SS}^{\star}_{\Pi_\Delta(\Lambda),\mathrm{fin}}(\mathbb{C}_\Delta(\Lambda), \mathscr{B})$, where $\mathscr{B} \subseteq \tau$, if for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathbb{C}_\Delta(\Lambda)$ and every $\mathcal{J} \in \Pi_\Delta(\Lambda)$, there is $\mathcal{V}_n \in [\mathcal{U}_n]^{<\omega}$ for any $n \in \mathbb{N}$, such that $\bigcup_{n\in\mathbb{N}}\{\mathrm{St}^\star(U, \mathcal{J}) : U \in \mathcal{V}_n\} \in \mathscr{B}$.

Theorem 3.10. *Let* (X, τ) *be a topological space. The following conditions are equivalent:*

(1) (Λ, τ^+_Δ) *is weakly M-star-Lindelöf or equivalently* Λ *satisfies* $\mathbf{SS}^{*}_{\mathcal{O},\mathrm{fin}}\left(\mathscr{D}_\Lambda, \mathscr{D}'_\Lambda\right)$;

(2) (X, τ) *satisfies* $\mathbf{SS}^{\star}_{\Pi_\Delta(\Lambda),\mathrm{fin}}\left(\mathbb{C}_\Delta(\Lambda), \mathbb{C}'_\Delta(\Lambda)\right)$.

From Theorem 3.10, we obtain the following corollaries.

Corollary 3.11. *Let* (X, τ) *be a topological space. If* Λ *is any of the hyperspaces* $\mathrm{CL}(X)$, $\mathbb{K}(X)$, $\mathbb{F}(X)$ *or* $\mathbb{CS}(X)$, *then* (Λ, τ^+_Δ) *is weakly M-star-Lindelöf if and only if X satisfies the principle* $\mathbf{SS}^{\star}_{\Pi_\Delta(\Lambda),\mathrm{fin}}(\mathbb{C}_\Delta(\Lambda), \mathbb{C}'_\Delta(\Lambda))$.

Corollary 3.12. *Let* (X, τ) *be a topological space. Let* Λ *be some of the spaces* $\mathbb{K}(X)$, $\mathbb{F}(X)$, $\mathbb{CS}(X)$, *we have:*

(a) *Suppose that* $\Delta = \mathbb{K}(X)$. *Then* (Λ, τ_F) *is weakly M-star-Lindelöf if and only if X satisfies the principle* $\mathbf{SS}^{\star}_{\Pi_{\mathbb{K}(X)}(\Lambda),\mathrm{fin}}\left(\mathbb{C}_{\mathbb{K}(X)}(\Lambda), \mathbb{C}'_{\mathbb{K}(X)}(\Lambda)\right)$.

(b) *Suppose that* $\Delta = \mathrm{CL}(X)$. *Then* (Λ, τ_V) *is weakly M-star-Lindelöf if and only if X satisfies the principle* $\mathbf{SS}^{\star}_{\Pi_{\mathrm{CL}(X)}(\Lambda),\mathrm{fin}}\left(\mathbb{C}_{\mathrm{CL}(X)}(\Lambda), \mathbb{C}'_{\mathrm{CL}(X)}(\Lambda)\right)$.

Corollary 3.13. *Let* (X, τ) *be a topological space, we have:*

(a) $(\mathrm{CL}(X), \tau_F)$ *is weakly M-star-Lindelöf if and only if X satisfies the principle* $\mathbf{SS}^{\star}_{\Pi_F,\mathrm{fin}}(\mathbb{K}_F, \mathbb{K}'_F)$.

(b) $(\mathrm{CL}(X), \tau_V)$ *is weakly M-star-Lindelöf if and only if X satisfies the principle* $\mathbf{SS}^{\star}_{\Pi_V,\mathrm{fin}}(\mathbb{C}_V, \mathbb{C}'_V)$.

Next, we characterize the almost R-star-Lindelöf and almost M-star-Lindelöf properties in the same hyperspaces.

Notation 3.14. Let (X, τ) be a topological space and consider the hyperspace (Λ, τ^+_Δ). Suppose that $\mathcal{A} \subseteq \Lambda^c$, then we denote by $\mathcal{A}^{cl} = (\mathrm{cl}_\Lambda(\mathcal{A}^c))^c$.

Remark 3.15. Let (X, τ) be a topological space and consider $\Delta, \Lambda \subseteq \mathrm{CL}(X)$. Let $\mathcal{A} \subseteq \Lambda^c$. Then $\mathcal{A}^{cl}$ is the family of every $W \in \Lambda^c$ such that for any $K \in \Delta$ and open sets $W_1, \ldots, W_m$, with $K \subseteq W$ and $W_i \cap W^c \neq \emptyset$, there is $V \in \mathcal{A}$ such that $K \subseteq V$ and $W_i \cap V^c \neq \emptyset$.

The proof of the next lemma follows immediately from the definitions.

Lemma 3.16. *Let* (X, τ) *be a topological space and consider the hyperspace* (Λ, τ^+_Δ). *Let* $\mathcal{J} \in \Pi_\Delta(\Lambda)$ *and* $\mathscr{U}$ *as Lemma 1.2. Then for any* $U \in \Lambda^c$, $\left(\mathrm{St}^\star(U, \mathcal{J})\right)^{\mathrm{cl}} = [cl_\Lambda(\mathrm{St}(U^c, \mathscr{U}))]^c$.

Notation 3.17. Let (X, τ) be a topological space and let $\Delta, \Lambda \subseteq \mathrm{CL}(X)$. We denote by $\Pi''_{\Delta}(\Lambda) = \{\mathscr{B} \subseteq \mathcal{P}(\Lambda^c) : \bigcup\{\mathcal{B}^{\mathrm{cl}} : \mathcal{B} \in \mathscr{B}\} = \Lambda^c\}$.

Theorem 3.18. *Let (X, τ) be a topological space. The following conditions are equivalent:*

(1) $(\Lambda, \tau^+_{\Delta})$ *is almost R-star-Lindelöf or equivalently Λ satisfies* $\mathbf{SS}^*_{\mathcal{O},1}\left(\mathscr{D}_{\Lambda}, \mathscr{D}''_{\Lambda}\right)$;

(2) (X, τ) *satisfies* $\mathbf{SS}^{\star}_{\Pi_{\Delta}(\Lambda),1}\left(\mathbb{C}_{\Delta}(\Lambda), \Pi''_{\Delta}(\Lambda)\right)$.

Proof. (1) ⇒ (2) Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of $c_{\Delta}(\Lambda)$-covers of X and $\mathcal{J}$ be a $\pi_{\Delta}(\Lambda)$-network. For each $n \in \mathbb{N}$, consider $\mathcal{D}_n = \mathcal{U}^c_n$ and $\mathscr{U} = \left\{(V_{1,s}, \ldots, V_{m_s,s})^+_{B_s} : (B_s; V_{1,s}, \ldots, V_{m_s,s}) \in \mathcal{J}\right\}$. By Lemmas 1.4, and 1.2, we obtain that, for any $n \in \mathbb{N}$, $\mathcal{D}_n$ is a dense subset of $(\Lambda, \tau^+_{\Delta})$ and $\mathscr{U}$ is an open cover of Λ, respectively. Hence, applying (1) to the sequence $(\mathcal{D}_n : n \in \mathbb{N})$ and $\mathscr{U}$, we obtain, for each $n \in \mathbb{N}$, $F_n \in \mathcal{D}_n$ such that $\bigcup_{n\in\mathbb{N}} \mathrm{cl}_{\Lambda}(\mathrm{St}(F_n, \mathscr{U})) = \Lambda$. Put $U_n = F^c_n$, so $U_n \in \mathcal{U}_n$. We claim that $\bigcup_{n\in\mathbb{N}}(\mathrm{St}^{\star}(U_n, \mathcal{J}))^{\mathrm{cl}} = \Lambda^c$. Indeed, let $W \in \Lambda^c$, so there is $N \in \mathbb{N}$ such that $W^c \in \mathrm{cl}_{\Lambda}(\mathrm{St}(F_N, \mathscr{U}))$. Thus, by Lemma 3.16, $W \in (\mathrm{St}^{\star}(U_N, \mathcal{J}))^{\mathrm{cl}}$ and the claim holds. Hence, $\{\mathrm{St}^{\star}(U_n, \mathcal{J}) : n \in \mathbb{N}\} \in \Pi''_{\Delta}(\Lambda)$ and (2) holds.

(2) ⇒ (1) Let $(\mathcal{D}_n : n \in \mathbb{N})$ be a sequence of dense subsets of $(\Lambda, \tau^+_{\Delta})$ and $\mathscr{U}$ be an open cover of Λ. By Lemma 1.4, we have that, for each $n \in \mathbb{N}$, $\mathcal{U}_n = \mathcal{D}^c_n$ is a $c_{\Delta}(\Lambda)$-cover of X and by Lemma 1.2, $\mathcal{J} = \{(B; V_1, \ldots, V_m) : (V_1, \ldots, V_m)^+_B \in \mathscr{U}\}$ is a $\pi_{\Delta}(\Lambda)$-network of X. Hence, applying (2), there is $U_n \in \mathcal{U}_n$ for any $n \in \mathbb{N}$, such that $\bigcup_{n\in\mathbb{N}}(\mathrm{St}^{\star}(U_n, \mathcal{J}))^{\mathrm{cl}} = \Lambda^c$. Thus, from Lemma 3.16, $\bigcup_{n\in\mathbb{N}} \mathrm{cl}_{\Lambda}(\mathrm{St}(U^c_n, \mathscr{U})) = \Lambda$ and the result holds. □

As a consequence of Theorem 3.18, we obtain the following corollaries.

Corollary 3.19. *Let (X, τ) be a topological space. If Λ is any of the hyperspaces* $\mathrm{CL}(X)$, $\mathbb{K}(X)$, $\mathbb{F}(X)$ *or* $\mathbb{CS}(X)$, *then* $(\Lambda, \tau^+_{\Delta})$ *is almost R-star-Lindelöf if and only if X satisfies the principle* $\mathbf{SS}^{\star}_{\Pi_{\Delta}(\Lambda),1}(\mathbb{C}_{\Delta}(\Lambda), \Pi''_{\Delta}(\Lambda))$.

Corollary 3.20. *Let (X, τ) be a topological space. Let Λ be some of the spaces* $\mathbb{K}(X)$, $\mathbb{F}(X)$, $\mathbb{CS}(X)$, *we have:*

(a) Suppose that $\Delta = \mathbb{K}(X)$. *Then* (Λ, τ_F) *is almost R-star-Lindelöf if and only if X satisfies the principle* $\mathbf{SS}^{\star}_{\Pi_{\mathbb{K}(X)}(\Lambda),1}\left(\mathbb{C}_{\mathbb{K}(X)}(\Lambda), \Pi''_{\mathbb{K}(X)}(\Lambda)\right)$.

(b) Suppose that $\Delta = \mathrm{CL}(X)$. *Then* (Λ, τ_V) *is almost R-star-Lindelöf if and only if X satisfies the principle* $\mathbf{SS}^{\star}_{\Pi_{\mathrm{CL}(X)}(\Lambda),1}\left(\mathbb{C}_{\mathrm{CL}(X)}(\Lambda), \Pi''_{\mathrm{CL}(X)}(\Lambda)\right)$.

Now we denote by $\Pi''_F = \{\mathscr{B} \subseteq \mathcal{P}(\mathrm{CL}(X)^c) : \bigcup\{\mathcal{B}^{\mathrm{cl}} : \mathcal{B} \in \mathscr{B}\} = \mathrm{CL}(X)^c\}$, where $\mathcal{B}^{\mathrm{cl}}$ is obtained considering $\Delta = \mathbb{K}(X)$. Similarly we denote by $\Pi''_V = \{\mathscr{B} \subseteq \mathcal{P}(\mathrm{CL}(X)^c) : \bigcup\{\mathcal{B}^{\mathrm{cl}} : \mathcal{B} \in \mathscr{B}\} = \mathrm{CL}(X)^c\}$, where $\mathcal{B}^{\mathrm{cl}}$ is obtained considering $\Delta = \mathrm{CL}(X)$.

Corollary 3.21. *Let (X, τ) be a topological space, we have:*

(a) $(\mathrm{CL}(X), \tau_F)$ *is almost R-star-Lindelöf if and only if X satisfies the principle* $\mathbf{SS}^{\star}_{\Pi_F,1}(\mathbb{K}_F, \Pi''_F)$.

(b) $(\mathrm{CL}(X), \tau_V)$ *is almost R-star-Lindelöf if and only if X satisfies the principle* $\mathbf{SS}^{\star}_{\Pi_V,1}(\mathbb{C}_V, \Pi''_V)$.

The Theorem 3.22 characterizes the almost M-star-Lindelöf property and its proof follows the same ideas as Theorem 3.18 and we omitted it.

Theorem 3.22. *Let (X, τ) be a topological space. The following conditions are equivalent:*

(1) $(\Lambda, \tau^+_{\Delta})$ *is almost M-star-Lindelöf or equivalently* $\mathbf{SS}^*_{\mathcal{O},fin}\left(\mathscr{D}_{\Lambda}, \mathscr{D}''_{\Lambda}\right)$;

(2) (X, τ) *satisfies* $\mathbf{SS}^{\star}_{\Pi_{\Delta}(\Lambda),fin}\left(\mathbb{C}_{\Delta}(\Lambda), \Pi''_{\Delta}(\Lambda)\right)$.

As an immediate consequence of Theorem 3.22, we obtain the following corollaries.

Corollary 3.23. *Let (X, τ) be a topological space. If Λ is any of the hyperspaces* $\mathrm{CL}(X)$, $\mathbb{K}(X)$, $\mathbb{F}(X)$ *or* $\mathbb{CS}(X)$, *then* $(\Lambda, \tau^+_{\Delta})$ *is almost M-star-Lindelöf if and only if X satisfies the principle* $\mathbf{SS}^{\star}_{\Pi_{\Delta}(\Lambda),fin}(\mathbb{C}_{\Delta}(\Lambda), \Pi''_{\Delta}(\Lambda))$.

Corollary 3.24. *Let (X, τ) be a topological space. Let Λ be some of the spaces $\mathbb{K}(X)$, $\mathbb{F}(X)$, $\mathbb{CS}(X)$, we have:*

(a) Suppose that $\Delta = \mathbb{K}(X)$. Then (Λ, τ_F) is almost M-star-Lindelöf if and only if X satisfies the principle $\mathbf{SS}^{\star}_{\Pi_{\mathbb{K}(X)}(\Lambda),fin}\left(\mathbb{C}_{\mathbb{K}(X)}(\Lambda), \Pi''_{\mathbb{K}(X)}(\Lambda)\right)$.

(b) Suppose that $\Delta = \mathrm{CL}(X)$. Then (Λ, τ_V) is almost M-star-Lindelöf if and only if X satisfies the principle $\mathbf{SS}^{\star}_{\Pi_{\mathrm{CL}(X)}(\Lambda),fin}\left(\mathbb{C}_{\mathrm{CL}(X)}(\Lambda), \Pi''_{\mathrm{CL}(X)}(\Lambda)\right)$.

Corollary 3.25. *Let (X, τ) be a topological space, we have:*

(a) $(\mathrm{CL}(X), \tau_F)$ is almost M-star-Lindelöf if and only if X satisfies the principle $\mathbf{SS}^{\star}_{\Pi_F,fin}(\mathbb{K}_F, \Pi''_F)$.

(b) $(\mathrm{CL}(X), \tau_V)$ is almost M-star-Lindelöf if and only if X satisfies the principle $\mathbf{SS}^{\star}_{\Pi_V,fin}(\mathbb{C}_V, \Pi''_V)$.

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