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Deferred statistical convergence of double sequences in the context of gradual normed linear spaces

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Abstract. In this paper, we introduce the concepts of "deferred statistical convergence" and "strong deferred convergence" for double sequences in the setting of gradual normed linear spaces. Our study uncovers the differences between these two concepts. Additionally, we perform a thorough analysis of the mathematical properties associated with these notions and establish several implication relationships.

1. Introduction and literature review

The concept of fuzzy sets was originally introduced by Zadeh [28] in 1965 as an extension of traditional set theory. Nowadays, it enjoys wide-ranging applications in numerous scientific and engineering disciplines. The term "fuzzy number" holds a central position in fuzzy set theory but deviates from conventional numbers, lacking adherence to specific algebraic properties. This variance has sparked debates among various authors regarding its characteristics. To clarify this ambiguity, some authors opt for the term "fuzzy intervals" rather than "fuzzy numbers".

To alleviate researcher's confusion, Fortin et al. [11] introduced the concept of gradual real numbers within fuzzy intervals in 2008. Gradual real numbers are primarily characterized by their assignment function defined over the interval $(0, 1]$, and it is possible to regard every real number as a gradual real number with a constant assignment function. In contrast to fuzzy numbers, gradual real numbers adhere to all the algebraic properties associated with classical real numbers, making them applicable in various computational and optimization contexts.

In 2011, Sadeqi and Azari [20] were the pioneers in introducing the concept of gradual normed linear spaces. They conducted an in-depth analysis of various properties, considering both algebraic and topological aspects, and demonstrated that a gradual normed linear space can be classified as a locally convex space. This categorization carries significant implications, as it indicates that the four fundamental theorems of locally convex spaces - namely, the Hahn-Banach theorem, the uniform boundedness theorem, the open mapping theorem, and the closed graph theorem are applicable within the framework of gradual normed spaces.

Considering that the scope of gradual normed linear spaces extends beyond that of classical spaces, researchers have recognized the importance of delving deeper into this direction. In recent years, significant advancements in this field have been driven by the work of Ettefagh et al. [8, 9], Choudhury and Debnath

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[5], and many others. For a comprehensive study on gradual real numbers, one may refer to [2, 7, 16] where many more references can be found.

In 1951, the concept of statistical convergence emerged independently through the work of Fast [10] and Steinhaus [24], aiming to offer deeper insights into summability theory. Subsequently, this concept gained significant attention in the domain of sequence spaces, with notable contributions from researchers like Fridy [12, 13], Šalát [21], Di Maio and Kočinac [6], and others [4]. In 2003, Mursaleen and Edely [18] extended this concept to double sequences, focusing on its relationship with statistical Cauchy double sequences and strong Cesàro summable double sequences. Furthermore, in 2003, Tripathy [25] conducted a study exploring various properties of sequence spaces formed by statistically convergent double sequences and established a decomposition theorem. It's worth noting that statistical convergence has applications across a wide range of mathematical disciplines, including number theory, mathematical analysis, probability theory, and various other fields.

In 1932, Agnew [1] introduced a generalization of the Cesàro mean, known as the deferred Cesàro mean, which offered enhanced features and utility. Using the deferred Cesàro mean as a foundation, Küçükaslan and Yilmaztürk [15] in 2016 introduced the concept of deferred statistical convergence. Their work involved proving fundamental properties and establishing several implication relationships between deferred statistical convergence, strong deferred Cesàro mean, and statistical convergence. For more comprehensive information on deferred statistical convergence and its various generalizations, [3, 14, 17, 22, 23, 26] can be addressed where many more references can be found.

2. Definitions and preliminaries

Throughout the paper, $\mathbb{N}$ and $\mathbb{R}$ denote the set of all positive integers and the set of all real numbers respectively and by the convergence of a double sequence, we mean the convergence in Pringsheim's [19] sense.

Definition 2.1. ([11]) A gradual real number $\tilde{r}$ is defined by an assignment function $\mathcal{A}_{\tilde{r}} : (0, 1] \to \mathbb{R}$. The set of all gradual real numbers is denoted by $G(\mathbb{R})$. A gradual real number $\tilde{r}$ is said to be non-negative if for every $\kappa \in (0, 1]$, $\mathcal{A}_{\tilde{r}}(\kappa) \geq 0$. The set of all non-negative gradual real numbers is denoted by $G^*(\mathbb{R})$.

In [11], the gradual operations between the elements of $G(\mathbb{R})$ was defined as follows:

Definition 2.2. Let "$*$" be any operation in $\mathbb{R}$ and suppose $\tilde{r}_1, \tilde{r}_2 \in G(\mathbb{R})$ with assignment functions $\mathcal{A}_{\tilde{r}_1}$ and $\mathcal{A}_{\tilde{r}_2}$ respectively. Then, $\tilde{r}_1 * \tilde{r}_2 \in G(\mathbb{R})$ is defined with the assignment function $\mathcal{A}_{\tilde{r}_1 * \tilde{r}_2}$ given by

$$\mathcal{A}_{\tilde{r}_1 * \tilde{r}_2}(\kappa) = \mathcal{A}_{\tilde{r}_1}(\kappa) * \mathcal{A}_{\tilde{r}_2}(\kappa),$$

for all $\kappa \in (0, 1]$. Then, the gradual addition $\tilde{r}_1 + \tilde{r}_2$ and the gradual scalar multiplication $c\tilde{r}(c \in \mathbb{R})$ are defined by

$$\mathcal{A}_{\tilde{r}_1 + \tilde{r}_2}(\kappa) = \mathcal{A}_{\tilde{r}_1}(\kappa) + \mathcal{A}_{\tilde{r}_2}(\kappa) \text{ and } \mathcal{A}_{c\tilde{r}}(\kappa) = c\mathcal{A}_{\tilde{r}}(\kappa),$$

for all $\kappa \in (0, 1]$.

Definition 2.3. ([20]) Let X be a real vector space. The function $\|\cdot\|_{\mathcal{G}} : X \to G^*(\mathbb{R})$ is said to be a gradual norm on X, if for every $\kappa \in (0, 1]$, the following conditions are true for any $x, y \in X$:

(G_1) $\mathcal{A}_{\|x\|_{\mathcal{G}}}(\kappa) = \mathcal{A}_{\tilde{0}}(\kappa)$ if and only if $x = 0$;

(G_2) $\mathcal{A}_{\|\lambda x\|_{\mathcal{G}}}(\kappa) = |\lambda| \mathcal{A}_{\|x\|_{\mathcal{G}}}(\kappa)$ for any $\lambda \in \mathbb{R}$;

(G_3) $\mathcal{A}_{\|x+y\|_{\mathcal{G}}}(\kappa) \leq \mathcal{A}_{\|x\|_{\mathcal{G}}}(\kappa) + \mathcal{A}_{\|y\|_{\mathcal{G}}}(\kappa)$.

The pair $(X, \|\cdot\|_{\mathcal{G}})$ is called a gradual normed linear space (GNLS).

Example 2.4. ([20]) Let $X = \mathbb{R}^w$ and $x = (x_1, x_2, ..., x_w) \in \mathbb{R}^w$. Define

$$\|\cdot\|_{\mathcal{G}} : \mathbb{R}^w \to G^*(\mathbb{R})$$

for $\kappa \in (0, 1]$, as follows

$$\mathcal{A}_{\|x\|_{\mathcal{G}}}(\kappa) = e^{\kappa} \sum_{i=1}^{w} |x_i|.$$

Then, $\|\cdot\|_{\mathcal{G}}$ is a gradual norm on $\mathbb{R}^w$ and $(\mathbb{R}^w, \|\cdot\|_{\mathcal{G}})$ is a GNLS.

Definition 2.5. ([20]) Let $x = (x_k)$ be a sequence in the GNLS $(X, \|\cdot\|_{\mathcal{G}})$. Then, x is said to be gradual convergent to $x_0 \in X$, if for every $\kappa \in (0, 1]$ and $\varepsilon > 0$, there exists $N(= N_{\varepsilon}(\kappa)) \in \mathbb{N}$ such that

$$\mathcal{A}_{\|x_k - x_0\|_{\mathcal{G}}}(\kappa) < \varepsilon,$$

satisfies for all $k \geq N$. Symbolically, $x_k \to x_0(G)$.

Let $p = (p_n)$ and $q = (q_n)$ be the sequence of non-negative integers satisfying

$$p_n < q_n \text{ and } \lim_{n\to\infty} q_n = \infty.$$

Definition 2.6. ([1]) A real-valued sequence $x = (x_k)$ is said to be strong deferred Cesàro convergent to $x_0 \in \mathbb{R}$ if

$$\lim_{n\to\infty} \frac{1}{q_n - p_n} \sum_{k=p_n+1}^{q_n} |x_k - x_0| = 0.$$

Symbolically, it is denoted by $x_k \to x_0(D[p, q])$.

Definition 2.7. ([27]) Let $K \subset \mathbb{N}$ and $K_{p,q}(n)$ denote the set $\{p_n + 1 \leq k \leq q_n : k \in K\}$. Then, the deferred density of K is denoted and defined as

$$\delta_{p,q}(K) = \lim_{n\to\infty} \tfrac{1}{q_n - p_n} |K_{p,q}(n)|$$

provided that the limit exists. Here, $|\cdot|$ indicates the cardinality of the inside set.

Definition 2.8. ([15]) A real-valued sequence $x = (x_k)$ is said to be deferred statistical convergent to $x_0 \in \mathbb{R}$ if for every $\varepsilon > 0$,

$$\delta_{p,q}(B(\varepsilon)) = 0,$$

where $B(\varepsilon) = \{k \in \mathbb{N} : |x_k - x_0| \geq \varepsilon\}$. Symbolically, it is represented as $x_k \to x_0(DS[p, q])$.

In particular, if we take $p(n) = 0$ and $q(n) = n$, then Definition 2.6, Definition 2.7, and Definition 2.8 reduces to the definition of strong Cesàro summability, natural density, and statistical convergence respectively.

Definition 2.9. ([19]) A real valued double sequence $x = (x_{ij})$ is said to be convergent to a real number x_0, if for any $\varepsilon > 0$, there exists a positive integer $k_0 = k_0(\varepsilon)$ such that for all $i, j \geq k_0$,

$$\left| x_{ij} - x_0 \right| < \varepsilon.$$

Definition 2.10. ([18]) Let $K \subseteq \mathbb{N} \times \mathbb{N}$ and $K_{l,m}$ denote the set

$$\{(i, j) \in K : i \leq l, j \leq m\}.$$

The double natural density of K is denoted and defined by

$$\delta^2(K) = \lim_{l,m\to\infty} \frac{|K_{l,m}|}{lm},$$

provided that the limit exists.

Definition 2.11. Let $(X, \|\cdot\|_{\mathcal{G}})$ be a GNLS. Then, a double sequence (x_{ij}) in X is said to be gradual bounded if there exists $M > 0$ such that

$$\mathcal{A}_{\|x_{ij}\|_{\mathcal{G}}}(\kappa) < M$$

holds for all $\kappa \in (0, 1]$ and $(i, j) \in \mathbb{N} \times \mathbb{N}$.

Definition 2.12. A double sequence $x = (x_{ij})$ in the GNLS $(X, \|\cdot\|_{\mathcal{G}})$ is said to be statistical convergent to $x_0 \in X$ if for every $\varepsilon > 0$ and $\kappa \in (0, 1]$,

$$\delta^2\left(\left\{(i, j) \in \mathbb{N} \times \mathbb{N} : \mathcal{A}_{\|x_{ij}-x_0\|_{\mathcal{G}}}(\kappa) \geq \varepsilon\right\}\right) = 0.$$

In this case, we write $x_{ij} \to x_0(GS)$.

3. Main results

In this section, we will introduce the definitions of strong gradual deferred convergence and gradual deferred statistical convergence for double sequences. Throughout the paper, $\mathbf{0} \in \mathbb{R}^w$ denotes the $w-$tuple $(0, 0, ..., 0)$ and $\varpi_l = b_l - a_l, \varpi'_l = b'_l - a'_l, \varrho_m = q_m - p_m, \varrho'_m = q'_m - p'_m$, where $a = (a_l), a' = (a'_l), b = (b_l), b' = (b'_l), p = (p_m), p' = (p'_m), q = (q_m), q' = (q'_m)$ be the sequences of nonnegative integers satisfying

$$a_l < b_l, a'_l < b'_l, p_m < q_m, p'_m < q'_m$$

and

$$\lim_{l\to\infty} b_l = \infty, \lim_{m\to\infty} q_m = \infty \text{ and } \lim_{l\to\infty} b'_l = \infty, \lim_{m\to\infty} q'_m = \infty. \tag{1}$$

Definition 3.1. Let $x = (x_{ij})$ be a double sequence in the GNLS $(X, \|\cdot\|_{\mathcal{G}})$. Then, x is said to be strong gradual deferred convergent to $x_0 \in X$ if for every $\kappa \in (0, 1]$,

$$\lim_{l,m\to\infty} \frac{1}{\varpi_l \varrho_m} \sum_{i=a_l+1}^{b_l} \sum_{j=p_m+1}^{q_m} \mathcal{A}_{\|x_{ij}-x_0\|_{\mathcal{G}}}(\kappa) = 0.$$

In this case, we write $x_{ij} \to x_0(GD_{ap}^{bq})$.

Definition 3.2. Let $K \subseteq \mathbb{N} \times \mathbb{N}$ and $K_{ap}^{bq}(l, m)$ denote the set

$$\{(i, j) \in K : a_l < i \leq b_l, p_m < j \leq q_m\}.$$

The double deferred density of K is denoted and defined by

$$\delta_{ap}^{2,bq}(K) = \lim_{l,m\to\infty} \frac{\left|K_{ap}^{bq}(l,m)\right|}{\varpi_l \varrho_m},$$

provided that the limit exists.

In particular if $b_l = l, a_l = 0, q_m = m, p_m = 0$ then the above definition reduces to the Definition 2.10.

Definition 3.3. Let $x = (x_{ij})$ be a double sequence in the GNLS $(X, \|\cdot\|_{\mathcal{G}})$. Then, x is said to be gradual deferred statistical convergent to $x_0 \in X$ if for every $\varepsilon > 0$ and $\kappa \in (0, 1]$,

$$\delta_{ap}^{2,bq}\left(\left\{(i,j)\in\mathbb{N}\times\mathbb{N}:\mathcal{A}_{\|x_{ij}-x_0\|_{\mathcal{G}}}(\kappa)\geq\varepsilon\right\}\right)=0,$$

$$\text{i.e., }\lim_{l,m\to\infty}\frac{1}{\varpi_l\varrho_m}\left|\left\{a_l<i\leq b_l,p_m<j\leq q_m:\mathcal{A}_{\|x_{ij}-x_0\|_{\mathcal{G}}}(\kappa)\geq\varepsilon\right\}\right|=0.$$

In this case, we write $x_{ij}\to x_0(GDS_{ap}^{bq})$.

Theorem 3.4. *Let (x_{ij}) be any double sequence in the GNLS $(X,\|\cdot\|_{\mathcal{G}})$ such that $x_{ij}\to x_0(\mathcal{G})$ in X. Then, $x_{ij}\to x_0(GDS_{ap}^{bq})$ for any a,p,b,q.*

Proof. Since $x_{ij}\to x_0(\mathcal{G})$, then the set

$$\left\{(i,j)\in\mathbb{N}\times\mathbb{N}:A_{\|x_{ij}-x_0\|_{G}}(\kappa)\geq\varepsilon\right\}$$

contains a finite number of elements and consequently has deferred double density zero. □

The converse of the above theorem is not true, in general.

Example 3.5. Let $X=\mathbb{R}^w$ and $\|\cdot\|_{\mathcal{G}}$ be the norm given in Example 2.4. Suppose, b_l,q_m are strictly increasing sequences and a_l,p_m satisfies the conditions $0<a_l<[|\sqrt{b_l}|]-1, 0<p_m<[|\sqrt{q_m}|]-1$ for all $(l,m)\in\mathbb{N}\times\mathbb{N}$, where $[|\cdot|]$ denotes the greatest integer function. Consider the double sequence (x_{ij}) in $\mathbb{R}^w$ as follows:

$$x_{ij}=\begin{cases}(0,0,...,0,(ij)^2), & if \quad [|\sqrt{b_l}|]-1<i\leq[|\sqrt{b_l}|],\\ & \quad\quad [|\sqrt{q_m}|]-1<j\leq[|\sqrt{q_m}|],\\ & \quad\quad l,m=1,2,3,\dots\\ \mathbf{0}, & otherwise.\end{cases}$$

Then, for any $\varepsilon>0$ and $\kappa\in(0,1]$,

$$\frac{\left|\left\{a_l<i\leq b_l,p_m<j\leq q_m:\mathcal{A}_{\|x_{ij}-x_0\|_{\mathcal{G}}}(\kappa)\geq\varepsilon\right\}\right|}{\varpi_l\varrho_m}=\frac{1}{\varpi_l\varrho_m}. \tag{2}$$

Letting $l,m\to\infty$ on both sides of (2) we obtain $x_{ij}\to\mathbf{0}(GDS_{ap}^{bq})$. But it is clear that $x_{ij}\nrightarrow\mathbf{0}(\mathcal{G})$.

Theorem 3.6. *Let (x_{ij}) be any double sequence in the GNLS $(X,\|\cdot\|_{\mathcal{G}})$ such that $x_{ij}\to x_0(GDS_{ap}^{bq})$ in X. Then, x_0 is uniquely determined.*

Proof. If possible suppose $x_{ij}\to x_0(GDS_{ap}^{bq})$ and $x_{ij}\to y_0(GDS_{ap}^{bq})$ for some $x_0\neq y_0$ in X. Let $\varepsilon>0$ be arbitrary. Then, by Definition 3.3 we have, for any $\varepsilon>0$ and $\kappa\in(0,1]$,

$$\delta_{ap}^{2,bq}(B_1(\kappa,\varepsilon))=\delta_{ap}^{2,bq}(B_2(\kappa,\varepsilon))=1,$$

where

$$B_1(\kappa,\varepsilon)=\left(\left\{(i,j)\in\mathbb{N}\times\mathbb{N}:\mathcal{A}_{\|x_{ij}-x_0\|_{\mathcal{G}}}(\kappa)<\varepsilon\right\}\right)$$

and

$$B_2(\kappa,\varepsilon)=\left(\left\{(i,j)\in\mathbb{N}\times\mathbb{N}:\mathcal{A}_{\|x_{ij}-y_0\|_{\mathcal{G}}}(\kappa)<\varepsilon\right\}\right).$$

Choose $(i_0,j_0)\in B_1(\kappa,\varepsilon)\cap B_2(\kappa,\varepsilon)$, then $\mathcal{A}_{\|x_{i_0j_0}-x_0\|_{\mathcal{G}}}(\kappa)<\varepsilon$ and $\mathcal{A}_{\|x_{i_0j_0}-y_0\|_{\mathcal{G}}}(\kappa)<\varepsilon$.

Hence,

$$\mathcal{A}_{||x_0-y_0||_{\mathcal{G}}}(\kappa) \le \mathcal{A}_{||x_{i_0j_0}-x_0||_{\mathcal{G}}}(\kappa) + \mathcal{A}_{||x_{i_0j_0}-y_0||_{\mathcal{G}}}(\kappa) < \varepsilon + \varepsilon = 2\varepsilon.$$

Since ε is arbitrary, so $\mathcal{A}_{||x_0-y_0||_{\mathcal{G}}}(\kappa) = \mathcal{A}_{\tilde{0}}(\kappa)$ and so we must have $x_0 = y_0$. □

Theorem 3.7. *Let (x_{ij}) and (y_{ij}) be two double sequences in the GNLS $(X, ||\cdot||_{\mathcal{G}})$ such that $x_{ij} \to x_0(GDS_{ap}^{bq})$ and $y_{ij} \to y_0(GDS_{ap}^{bq})$. Then:*
(i) $x_{ij} + y_{ij} \to x_0 + y_0(GDS_{ap}^{bq})$ and
(ii) $cx_{ij} \to cx_0(GDS_{ap}^{bq})$, $c \in \mathbb{R}$.

Proof. (i) Suppose $x_{ij} \to x_0(GDS_{ap}^{bq})$ and $y_{ij} \to y_0(GDS_{ap}^{bq})$. Then, by Definition 3.3, for given $\varepsilon > 0$,

$$\delta_{ap}^{2,bq}(C_1) = \delta_{ap}^{2,bq}(C_2) = 0,$$

where

$$C_1 = \left\{(i,j) \in \mathbb{N} \times \mathbb{N} : \mathcal{A}_{||x_{ij}-x_0||_G}(\kappa) \ge \tfrac{\varepsilon}{2}\right\}$$

and

$$C_2 = \left\{(i,j) \in \mathbb{N} \times \mathbb{N} : \mathcal{A}_{||y_{ij}-y_0||_G}(\kappa) \ge \tfrac{\varepsilon}{2}\right\}.$$

Now as the inclusion

$$((\mathbb{N} \times \mathbb{N}) \setminus C_1) \cap ((\mathbb{N} \times \mathbb{N}) \setminus C_2) \subseteq \left\{(i,j) \in \mathbb{N} \times \mathbb{N} : \mathcal{A}_{||x_{ij}+y_{ij}-x_0-y_0||_G}(\kappa) < \varepsilon\right\}$$

holds, so we must have

$$\delta_{ap}^{2,bq}\left(\left\{(i,j) \in \mathbb{N} \times \mathbb{N} : \mathcal{A}_{||x_{ij}+y_{ij}-x_0-y_0||_G}(\kappa) \ge \varepsilon\right\}\right) \le \delta_{ap}^{2,bq}(C_1 \cup C_2) = 0;$$

and consequently,

$$x_{ij} + y_{ij} \to x_0 + y_0(GDS_{ap}^{bq}).$$

(ii) If $c = 0$, then there is nothing to prove. So let us assume $c \neq 0$. Then, since $x_{ij} \to x_0(GDS_{ap}^{bq})$, we have for given $\varepsilon > 0$, $\delta_{ap}^{2,bq}(C_1) = 0$, where

$$C_1 = \left\{(i,j) \in \mathbb{N} \times \mathbb{N} : \mathcal{A}_{||x_{ij}-x_0||_G}(\kappa) \ge \tfrac{\varepsilon}{|c|}\right\}.$$

Now since

$$\mathcal{A}_{||cx_{ij}-cx_0||_G}(\kappa) = |c|\mathcal{A}_{||x_{ij}-x_0||_G}(\kappa)$$

holds for any $c \in \mathbb{R}$, we must have $C_2 \subseteq C_1$, where

$$C_2 = \left\{(i,j) \in \mathbb{N} \times \mathbb{N} : \mathcal{A}_{||cx_{ij}-cx_0||_G}(\kappa) \ge \varepsilon\right\},$$

which as a consequence implies $\delta_{ap}^{2,bq}(C_2) = 0$. This completes the proof. □

Theorem 3.8. *Let (x_{ij}) be any double sequence in the GNLS $(X, ||\cdot||_{\mathcal{G}})$. Then, $x_{ij} \to x_0(GD_{ap}^{bq})$ implies $x_{ij} \to x_0(GDS_{ap}^{bq})$.*

Proof. Let $x_{ij} \to x_0(GD_{ap}^{bq})$ and $\varepsilon > 0$ be arbitrary. Then, for any $\kappa \in (0, 1]$,

$$\lim_{l,m\to\infty} \frac{1}{\varpi_l \varrho_m} \sum_{i=a_l+1}^{b_l} \sum_{j=p_m+1}^{q_m} \mathcal{A}_{\|x_{ij}-x_0\|_{\mathcal{G}}}(\kappa) = 0, \tag{3}$$

holds. Now,

$$\begin{aligned}
&\frac{1}{\varpi_l \varrho_m} \sum_{i=a_l+1}^{b_l} \sum_{j=p_m+1}^{q_m} \mathcal{A}_{\|x_{ij}-x_0\|_{\mathcal{G}}}(\kappa) \\
&\qquad \geq \frac{1}{\varpi_l \varrho_m} \underbrace{\sum_{i=a_l+1}^{b_l} \sum_{j=p_m+1}^{q_m}}_{\mathcal{A}_{\|x_{ij}-x_0\|_{\mathcal{G}}}(\kappa)\geq\varepsilon} \mathcal{A}_{\|x_{ij}-x_0\|_{\mathcal{G}}}(\kappa) \\
&\qquad \geq \frac{1}{\varpi_l \varrho_m} \underbrace{\sum_{i=a_l+1}^{b_l} \sum_{j=p_m+1}^{q_m}}_{\mathcal{A}_{\|x_{ij}-x_0\|_{\mathcal{G}}}(\kappa)\geq\varepsilon} \varepsilon \\
&\qquad \geq \frac{\varepsilon}{\varpi_l \varrho_m} \left| \left\{ a_l < i \leq b_l, p_m < j \leq q_m : \mathcal{A}_{\|x_{ij}-x_0\|_{\mathcal{G}}}(\kappa) \geq \varepsilon \right\} \right|.
\end{aligned}$$

Taking $l, m \to \infty$ on both sides of the above inequation and using (3), we obtain

$$\lim_{l,m\to\infty} \frac{\left| \left\{ a_l < i \leq b_l, p_m < j \leq q_m : \mathcal{A}_{\|x_{ij}-x_0\|_{\mathcal{G}}}(\kappa) \geq \varepsilon \right\} \right|}{\varpi_l \varrho_m} = 0.$$

Hence, $x_{ij} \to x_0(GDS_{ap}^{bq})$. □

It should be noted that the converse of Theorem 3.8 is not necessarily true. Consider Example 3.5. It was shown that $x_{ij} \to \mathbf{0}(GDS_{ap}^{bq})$. But since the right-hand side of the following inequation

$$\frac{1}{\varpi_l \varrho_m} \sum_{i=a_l+1}^{b_l} \sum_{j=p_m+1}^{q_m} \mathcal{A}_{\|x_{ij}-\mathbf{0}\|_{\mathcal{G}}}(\kappa) \geq \frac{([|\sqrt{b_l}|] - 1)^2([|\sqrt{q_m}|] - 1)^2}{\varpi_l \varrho_m}$$

tends to 1 as $l, m \to \infty$, so the left-hand side never approaches zero. In other words, $x_{ij} \nrightarrow \mathbf{0}(GD_{ap}^{bq})$.

From the above remark, naturally, a question arises under which condition the converse of Theorem 3.8 holds. The next theorem answers.

Theorem 3.9. *Let (x_{ij}) be a gradual bounded double sequence in the GNLS $(X, \|\cdot\|_{\mathcal{G}})$. Then, $x_{ij} \to x_0(GDS_{ap}^{bq})$ implies $x_{ij} \to x_0(GD_{ap}^{bq})$.*

Proof. Let $x_{ij} \to x_0(GDS_{ap}^{bq})$. Since (x_{ij}) is gradual bounded, so there exists a $M > 0$ such that for all $\kappa \in (0, 1]$ and $i, j \in \mathbb{N}$,

$$\mathcal{A}_{\|x_{ij}-x_0\|_{\mathcal{G}}}(\kappa) \leq M.$$

Now

$$\begin{aligned}
&\frac{1}{\varpi_l \varrho_m}\sum_{i=a_l+1}^{b_l}\sum_{j=p_m+1}^{q_m}\mathcal{A}_{\|x_{ij}-x_0\|_{\mathcal{G}}}(\kappa)\\
&=\frac{1}{\varpi_l \varrho_m}\left(\underbrace{\sum_{i=a_l+1}^{b_l}\sum_{j=p_m+1}^{q_m}}_{\mathcal{A}_{\|x_{ij}-x_0\|_{\mathcal{G}}}(\kappa)\geq\varepsilon}\mathcal{A}_{\|x_{ij}-x_0\|_{\mathcal{G}}}(\kappa)+\underbrace{\sum_{i=a_l+1}^{b_l}\sum_{j=p_m+1}^{q_m}}_{\mathcal{A}_{\|x_{ij}-x_0\|_{\mathcal{G}}}(\kappa)<\varepsilon}\mathcal{A}_{\|x_{ij}-x_0\|_{\mathcal{G}}}(\kappa)\right)\\
&\leq\frac{1}{\varpi_l \varrho_m}\left(\underbrace{\sum_{i=a_l+1}^{b_l}\sum_{j=p_m+1}^{q_m}}_{\mathcal{A}_{\|x_{ij}-x_0\|_{\mathcal{G}}}(\kappa)\geq\varepsilon}M+\underbrace{\sum_{i=a_l+1}^{b_l}\sum_{j=p_m+1}^{q_m}}_{\mathcal{A}_{\|x_{ij}-x_0\|_{\mathcal{G}}}(\kappa)<\varepsilon}\varepsilon\right)\\
&\leq\frac{M}{\varpi_l \varrho_m}\left|\left\{a_l<i\leq b_l, p_m<j\leq q_m:\mathcal{A}_{\|x_{ij}-x_0\|_{\mathcal{G}}}(\kappa)\geq\varepsilon\right\}\right|\\
&\qquad+\frac{\varepsilon}{\varpi_l \varrho_m}\left|\left\{a_l<i\leq b_l, p_m<j\leq q_m:\mathcal{A}_{\|x_{ij}-x_0\|_{\mathcal{G}}}(\kappa)<\varepsilon\right\}\right|.
\end{aligned}$$

From the assumption and the above inequation, we conclude that for any $\kappa\in(0,1]$,

$$\lim_{l,m\to\infty}\frac{1}{\varpi_l \varrho_m}\sum_{i=a_l+1}^{b_l}\sum_{j=p_m+1}^{q_m}\mathcal{A}_{\|x_{ij}-x_0\|_{\mathcal{G}}}(\kappa)=0.$$

Hence, $x_{ij}\to x_0(GD_{ap}^{bq})$. □

Theorem 3.10. *Let (x_{ij}) be a double sequence in the GNLS $(X,\|\cdot\|_{\mathcal{G}})$ such that $x_{ij}\to x_0(GS)$. Then, $x_{ij}\to x_0(GD_{ap}^{bq})$ provided that the sequences $\left(\frac{a_l}{\varpi_l}\right)$ and $\left(\frac{p_m}{\varrho_m}\right)$ are bounded.*

Proof. Since, $x_{ij}\to x_0(GS)$, then for any $\varepsilon>0$ and $\kappa\in(0,1]$,

$$\lim_{l,m\to\infty}\frac{1}{lm}\left|\left\{i\leq l, j\leq m:\mathcal{A}_{\|x_{ij}-x_0\|_{\mathcal{G}}}(\kappa)\geq\varepsilon\right\}\right|=0.$$

Again since the sequences $b=(b_l)$ and $q=(q_m)$ satisfy (1), from the above limit, we must have

$$\lim_{l,m\to\infty}\frac{1}{b_l q_m}\left|\left\{i\leq b_l, j\leq q_m:\mathcal{A}_{\|x_{ij}-x_0\|_{\mathcal{G}}}(\kappa)\geq\varepsilon\right\}\right|=0. \tag{4}$$

Clearly the inclusion

$$\left\{a_l<i\leq b_l, p_m<j\leq q_m:\mathcal{A}_{\|x_{ij}-x_0\|_{\mathcal{G}}}(\kappa)\geq\varepsilon\right\}\subseteq\left\{i\leq b_l, j\leq q_m:\mathcal{A}_{\|x_{ij}-x_0\|_{\mathcal{G}}}(\kappa)\geq\varepsilon\right\}$$

yields the inequation

$$\left|\left\{a_l<i\leq b_l, p_m<j\leq q_m:\mathcal{A}_{\|x_{ij}-x_0\|_{\mathcal{G}}}(\kappa)\geq\varepsilon\right\}\right|\leq\left|\left\{i\leq b_l, j\leq q_m:\mathcal{A}_{\|x_{ij}-x_0\|_{\mathcal{G}}}(\kappa)\geq\varepsilon\right\}\right|$$

which subsequently gives the following inequation

$$\frac{1}{\varpi_l \varrho_m}\left|\left\{a_l < i \leq b_l, p_m < j \leq q_m : \mathcal{A}_{\|x_{ij}-x_0\|_{\mathcal{G}}}(\kappa) \geq \varepsilon\right\}\right| \leq \left(1+\frac{a_l}{\varpi_l}\right)\left(1+\frac{p_m}{\varrho_m}\right)\frac{1}{b_l q_m}\left|\left\{i \leq b_l, j \leq q_m : \mathcal{A}_{\|x_{ij}-x_0\|_{\mathcal{G}}}(\kappa) \geq \varepsilon\right\}\right|. \quad (5)$$

Since the sequences $\left(\frac{a_l}{\varpi_l}\right)$ and $\left(\frac{p_m}{\varrho_m}\right)$ are bounded, so letting $l, m \to \infty$ on both sides of (5) and using (4) we obtain,

$$\lim_{l,m\to\infty} \frac{\left|\left\{a_l < i \leq b_l, p_m < j \leq q_m : \mathcal{A}_{\|x_{ij}-x_0\|_{\mathcal{G}}}(\kappa) \geq \varepsilon\right\}\right|}{\varpi_l \varrho_m} = 0.$$

Hence, $x_{ij} \to x_0(GD^{bq}_{ap})$ holds and the proof is complete. □

Theorem 3.11. *Let $a' = (a'_l), b' = (b'_l), p' = (p'_m)$, and $q' = (q'_m)$ be sequences of positive integers such that*

$$a_l \leq a'_l < b'_l \leq b_l, \text{ and } p_m \leq p'_m < q'_m \leq q_m$$

holds for all $l, m \in \mathbb{N}$. Then"

(i) $x_{ij} \to x_0(GDS^{b'q'}_{a'p'})$ implies $x_{ij} \to x_0(GDS^{bq}_{ap})$ provided that the sets

$$\left\{(i,j) \in \mathbb{N}\times\mathbb{N} : a_l < i \leq a'_l, p_m < j \leq p'_m\right\}$$

and

$$\left\{(i,j) \in \mathbb{N}\times\mathbb{N} : b'_l < i \leq b_l, q'_m < j \leq q_m\right\}$$

are finite sets for all $(l,m) \in \mathbb{N}\times\mathbb{N}$.

(ii) $x_{ij} \to x_0(GDS^{bq}_{ap})$ implies $x_{ij} \to x_0(GDS^{b'q'}_{a'p'})$ provided that

$$\lim_{l,m\to\infty} \frac{\varpi_l \varrho_m}{\varpi'_l \varrho'_m} = d > 0.$$

(iii) If $x_{ij} \to x_0(GDS^{a'p'}_{ap})$ and $x_{ij} \to x_0(GDS^{bq}_{b'q'})$ holds simultaneously, then, $x_{ij} \to x_0(GDS^{bq}_{ap})$ provided that the set

$$\left\{(i,j) \in \mathbb{N}\times\mathbb{N} : a'_l < i \leq b'_l, p'_m < j \leq q'_m\right\}$$

is finite for all $(l,m) \in \mathbb{N}\times\mathbb{N}$.

Proof. (i) For any $\varepsilon > 0$ and $\kappa \in (0,1]$, the equality

$$\left\{a_l < i \leq b_l, p_m < j \leq q_m : \mathcal{A}_{\|x_{ij}-x_0\|_{\mathcal{G}}}(\kappa) \geq \varepsilon\right\} \subseteq \left\{a_l < i \leq a'_l, p_m < j \leq p'_m : \mathcal{A}_{\|x_{ij}-x_0\|_{\mathcal{G}}}(\kappa) \geq \varepsilon\right\} \cup \left\{a'_l < i \leq b'_l, p'_m < j \leq q'_m : \mathcal{A}_{\|x_{ij}-x_0\|_{\mathcal{G}}}(\kappa) \geq \varepsilon\right\} \cup \left\{b'_l < i \leq b_l, q'_m < j \leq q_m : \mathcal{A}_{\|x_{ij}-x_0\|_{\mathcal{G}}}(\kappa) \geq \varepsilon\right\}$$

and

$$\frac{1}{\varpi_l\varrho_m}\left|\left\{a_l < i \le b_l, p_m < j \le q_m : \mathcal{A}_{\|x_{ij}-x_0\|_{\mathcal{G}}}(\kappa) \ge \varepsilon\right\}\right| \le \frac{1}{\varpi'_l\varrho'_m}\left|\left\{a_l < i \le a'_l, p_m < j \le p'_m : \mathcal{A}_{\|x_{ij}-x_0\|_{\mathcal{G}}}(\kappa) \ge \varepsilon\right\}\right| + \frac{1}{\varpi'_l\varrho'_m}\left|\left\{a'_l < i \le b'_l, p'_m < j \le q'_m : \mathcal{A}_{\|x_{ij}-x_0\|_{\mathcal{G}}}(\kappa) \ge \varepsilon\right\}\right| + \frac{1}{\varpi'_l\varrho'_m}\left|\left\{b'_l < i \le b_l, q'_m < j \le q_m : \mathcal{A}_{\|x_{ij}-x_0\|_{\mathcal{G}}}(\kappa) \ge \varepsilon\right\}\right|$$

holds.

On taking $l, m \to \infty$ we obtain

$$\lim_{l,m\to\infty} \frac{\left|\left\{a_l < i \le b_l, p_m < j \le q_m : \mathcal{A}_{\|x_{ij}-x_0\|_{\mathcal{G}}}(\kappa) \ge \varepsilon\right\}\right|}{\varpi_l\varrho_m} = 0.$$

Hence, $x_{ij} \to x_0(GDS^{bq}_{ap})$.

(ii) For any $\varepsilon > 0$ and $\kappa \in (0, 1]$, the inclusion

$$\left\{a'_l < i \le b'_l, p'_m < j \le q'_m : \mathcal{A}_{\|x_{ij}-x_0\|_{\mathcal{G}}}(\kappa) \ge \varepsilon\right\} \subseteq \left\{a_l < i \le b_l, p_m < j \le q_m : \mathcal{A}_{\|x_{ij}-x_0\|_{\mathcal{G}}}(\kappa) \ge \varepsilon\right\}$$

and the inequality

$$\left|\left\{a'_l < i \le b'_l, p'_m < j \le q'_m : \mathcal{A}_{\|x_{ij}-x_0\|_{\mathcal{G}}}(\kappa) \ge \varepsilon\right\}\right| \le \left|\left\{a_l < i \le b_l, p_m < j \le q_m : \mathcal{A}_{\|x_{ij}-x_0\|_{\mathcal{G}}}(\kappa) \ge \varepsilon\right\}\right|$$

holds good. So we have,

$$\frac{1}{\varpi'_l\varrho'_m}\left|\left\{a'_l < i \le b'_l, p'_m < j \le q'_m : \mathcal{A}_{\|x_{ij}-x_0\|_{\mathcal{G}}}(\kappa) \ge \varepsilon\right\}\right| \le \frac{\varpi_l\varrho_m}{\varpi'_l\varrho'_m}\frac{1}{\varpi_l\varrho_m}\left|\left\{a_l < i \le b_l, p_m < j \le q_m : \mathcal{A}_{\|x_{ij}-x_0\|_{\mathcal{G}}}(\kappa) \ge \varepsilon\right\}\right|$$

and by taking $l, m \to \infty$ the desired result is obtained.

(iii) The proof is easy, so omitted. □

4. Concluding remarks

In this paper, we have explored several fundamental characteristics of deferred statistical convergence of double sequences in gradual normed linear spaces. Theorem 3.8 and Theorem 3.9 have uncovered the relationship between strong deferred convergence and gradual deferred statistical convergence of double sequences. The domains of summability theory and sequence convergence have broad applications across diverse mathematical disciplines, especially in the field of mathematical analysis. Investigating this research avenue within gradual normed linear spaces is relatively unexplored and is at an early stage of development. The findings from this study may hold significance for future researchers as they delve further into the different facets of convergence within gradual normed linear spaces.

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