Filomat 38:18 (2024), 6477–6491
https://doi.org/10.2298/FIL2418477P

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Generic $\xi^{\perp}$-Riemannian submersions from Kenmotsu manifolds onto Riemannian manifolds

Rajendra Prasad[a], Pooja Gupta[a,*]

[a]*Department of Mathematics and Astronomy, University of Lucknow, Lucknow-226007, India.*

Abstract. The goal of this article is to define and investigate the generic $\xi^{\perp}$- Riemannian submersions from Kenmotsu manifolds onto Riemannian manifolds along with the examples. We also examine the integrability as well as totally geodesicness of distributions involved in the definition of a generic $\xi^{\perp}$-Riemannian submersion. Along with it, we obtain decomposition theorems of this submersion. Furthermore, necessary and sufficient conditions for the base manifold to be a local product manifold are obtained. In addition with it, we also explore the totally umbilical nature of generic $\xi^{\perp}$-Riemannian submersion. Moreover, we obtain some curvature relations from Kenmotsu space forms between the total space, the base space and the fibers.

1. Introduction

O'Neill and Gray [9], [16] initially investigated Riemannian submersions between Riemannian manifolds. Following this, studies of these submersions between manifolds with differentiable structures were conducted. Numerous authors investigated various geometric properties of the Riemannian submersions, including anti-invariant submersion [14], [21], [22], semi-invariant submersion [3], [23], paraquaternionic 3-submersion [28], statistical submersion [27], slant submersion [20],[13], [7], [10], [19], semi-slant submersion [11], [18], conformal slant submersion, conformal semi-slant submersion [1], bi-slant submersion [25] and Quasi bi-slant submersion [17].

Riemannian submersions have uses in physics and mathematics, including Yang-Mills theory [6], Kaluza-Klein theory [12] and the theories of supergravity and superstrings [15]. A generic Riemannian submersion from an almost Hermitian manifold onto a Riemannian manifold was introduced by Ali and Fatima [5]. A Kaehler manifold's generic submanifold submersions have been examined by several writers [8]. Also Şahin researched generic Riemannian maps in [24]. Akyol introduced generic Riemannian submersions and conformal generic Riemannian submersions from almost product Riemannian submanifolds and almost Hermitian manifold respectively [2], [4].

The geometry of the new submersions on almost contact manifolds was extensively examined by Akyol [3], who also proposed and analysed semi-invariant $\xi^{\perp}$-Riemannian submersions from almost contact metric manifolds and as the result of the generalization of it, Ramazan Sari [26], worked on generic $\xi^{\perp}$-Riemannian submersions from Sasakian manifolds.

2020 *Mathematics Subject Classification*. Primary 53C15; Secondary 53C25, 53C80.
Keywords. Kenmotsu manifolds, Riemannian Submersions, Second fundamental form of a map, totally geodesic foliations.
Received: 03 October 2023; Accepted: 27 January 2024
Communicated by Mića Stanković
* Corresponding author: Pooja Gupta
Email addresses: rp.lucknow@rediffmail.com (Rajendra Prasad), poojaguptamars14@gmail.com (Pooja Gupta)

The following describes how the paper is structured: The fundamental characteristics of a Kenmotsu manifold and a Riemannian submersion are outlined in Section 2; the generic $\xi^{\perp}$-Riemannian submersions from Kenmotsu manifolds onto Riemannian manifolds with examples are designated in Section 3; Section 4 is devoted to the investigation of integrability as well as totally geodesicness of distributions involved in the definition of generic $\xi^{\perp}$- Riemannian submersion; Section 5 discusses additional conditions for generic $\xi^{\perp}$- Riemannian submersions to be totally geodesic and totally umbilical; Finally, section 6 deals with the curvature features and Einstein conditions of distributions for a generic $\xi^{\perp}$- Riemannian submersion from Kenmotsu space forms onto Riemannian manifolds.

2. Preliminaries

Let $\mathcal{N}(\phi, \xi, \eta, \mathcal{G})$ be an almost contact manifold of dimension $n = 2m + 1$ admitting ϕ as a tensor field of (1,1) type, a vector field ξ and a 1-form η satisfying the following conditions:

$$\phi^2 = -I + \eta \otimes \xi, \eta(\xi) = 1, \phi\xi = 0, \eta \circ \phi = 0, \tag{1}$$

where I is an identity map defined on $\mathcal{TN}$. Also, on an almost contact manifold there exists a Riemannian metric $\mathcal{G}$ which satisfies the condition:

$$\mathcal{G}(\phi U, \phi V) = \mathcal{G}(U, V) - \eta(U)\eta(V), \tag{2}$$

for $U, V \in \mathcal{TN}$. A manifold $\mathcal{N}$ together with the structure $(\phi, \xi, \eta, \mathcal{G})$ is called an almost contact metric manifold.

Due to the above equations (1) and (2), we obtain following consequences:

$$\mathcal{G}(U, \xi) = \eta(U) \tag{3}$$

and

$$\mathcal{G}(\phi U, V) = -\mathcal{G}(U, \phi V) \tag{4}$$

for all vector fields $U, V \in \Gamma(\mathcal{TN})$.

Now if

$$(\nabla_U \phi)(V) = \mathcal{G}(\phi U, V)\xi - \eta(V)\phi U, \tag{5}$$

and

$$\nabla_U \xi = U - \eta(U)\xi \tag{6}$$

for any U, V tangent to $\mathcal{N}$, where ∇ is the Levi-civita connection, then $(\mathcal{N}, \phi, \xi, \eta, \mathcal{G})$ is called a Kenmotsu manifold.

Let $(\mathcal{N}, \mathcal{G}_{\mathcal{N}})$ and $(\mathcal{B}, \mathcal{G}_{\mathcal{B}})$ be Riemannian manifolds with $dim(\mathcal{N}) = m$, $dim(\mathcal{B}) = n$ and $m > n$. Now, consider a Riemannian submersion $\pi : \mathcal{N} \to \mathcal{B}$ is a map of $\mathcal{N}$ onto $\mathcal{B}$ satisfying the following axioms:

1. π has maximal rank.
2. The differential π_* preserves the lengths of horizontal vectors.

For each $p \in \mathcal{B}$, $\pi^{-1}(p)$ is an $(m - n)$-dimensional submanifold of $\mathcal{N}$. The submanifolds $\pi^{-1}(p)$ are called **fibers**.

A vector field on $\mathcal{N}$ is referred to as vertical if the fibers are tangent and referred to as horizontal if the fibers are orthogonal. A vector field U on $\mathcal{N}$ is called basic vector field if U is horizontal and π-related to a vector field U_* on $\mathcal{B}$ i.e. $\pi_* U_q = U_{\pi_*(q)}$ for all $p \in \mathcal{N}$. We denote the projection morphisms on the distributions $ker\pi_*$ and $(\ker \pi_*)^\perp$ by $\mathcal{V}$ and $\mathcal{H}$ respectively. Then for any $U \in \Gamma(\mathcal{TM})$, we put

$$U = \mathcal{V}U + \mathcal{H}U. \tag{7}$$

We recall that the sections of $\mathcal{V}$ and $\mathcal{H}$ are called vertical vector fields and horizontal vector fields respectively. A Riemannian submersion $\pi : \mathcal{N} \to \mathcal{B}$ determines two $(1,2)$ tensor fields $\mathcal{T}$ and $\mathcal{A}$ on $\mathcal{N}$, by the formulas:

$$\mathcal{T}(E, F) = \mathcal{T}_E F = \mathcal{H}\nabla^{\mathcal{N}}_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla^{\mathcal{N}}_{\mathcal{V}E}\mathcal{H}F \tag{8}$$

and

$$\mathcal{A}(E, F) = \mathcal{A}_E F = \mathcal{V}\nabla^{\mathcal{N}}_{\mathcal{H}E}\mathcal{H}F + \mathcal{H}\nabla^{\mathcal{N}}_{\mathcal{H}E}\mathcal{V}F \tag{9}$$

for any $E, F \in \Gamma(\mathcal{TN})$ where $\mathcal{V}$ and $\mathcal{H}$ are the vertical and horizontal projections. It is easy to see that a Riemannian submersion $\pi : \mathcal{N} \to \mathcal{B}$ has totally geodesic fibers if and only if $\mathcal{T}$ vanishes identically. For any $E \in \Gamma(\mathcal{TN})$, $\mathcal{T}_E$ and $\mathcal{A}_E$ are skew symmetric operators on $(\Gamma(\mathcal{TN}), \mathcal{G})$ reversing the horizontal and the vertical distributions. It is also seen that $\mathcal{T}$ is vertical, $\mathcal{T}_E = \mathcal{T}_{\mathcal{V}E}$ and $\mathcal{A}_E$ is horizontal, $\mathcal{A}_E = \mathcal{A}_{\mathcal{H}E}$.

Using (8) and (9), we have

$$\nabla^{\mathcal{N}}_V W = \mathcal{T}_V W + \widehat{\nabla}_V W, \tag{10}$$

$$\nabla^{\mathcal{N}}_V X = \mathcal{T}_V X + \mathcal{H}(\nabla^{\mathcal{N}}_V X), \tag{11}$$

$$\nabla^{\mathcal{N}}_X V = \mathcal{A}_X V + \mathcal{V}(\nabla^{\mathcal{N}}_X V) \tag{12}$$

and

$$\nabla^{\mathcal{N}}_X Y = \mathcal{A}_X Y + \mathcal{H}(\nabla^{\mathcal{N}}_X Y) \tag{13}$$

for any $V, W \in \Gamma(ker\pi_*)$ and $X, Y \in \Gamma((\ker \pi_*)^\perp)$, where $\widehat{\nabla}_V W = \mathcal{V}(\nabla_V W)$. Furthermore, if X is a basic then

$$\mathcal{H}(\nabla^{\mathcal{N}}_V X) = \mathcal{A}_X V. \tag{14}$$

We note that for $V, W \in \Gamma(ker\pi_*)$, $\mathcal{T}_V W$ coincides with the second fundamental form of the immersion of the fiber submanifolds and $\mathcal{T}$ is symmetric on the vertical distribution i.e. $\mathcal{T}_V W = \mathcal{T}_W V$, for $V, W \in \Gamma(ker\pi_*)$. Furthermore, $\mathcal{A}_X Y = \frac{1}{2}\mathcal{V}[X, Y]$ which shows the complete integrability of the horizontal distribution $\mathcal{H}$, for $X, Y \in \Gamma((ker\pi_*)^\perp)$. Moreover, $\mathcal{A}$ alternates on the horizontal distribution, $\mathcal{A}_X Y = -\mathcal{A}_Y X$, for $X, Y \in \Gamma((ker\pi_*)^\perp)$.

Lemma 2.1. *Let $\pi : \mathcal{N} \to \mathcal{B}$ be a Riemannian submersion between Riemannian manifolds and X, Y be basic vector fields of $\mathcal{N}$. Then we have*

1. $\mathcal{G}(X, Y) = \mathcal{G}_{\mathcal{B}}(X_*, Y_*) \circ \pi$,
2. *the horizontal part $\mathcal{H}[X, Y]$ of $[X, Y]$ is a basic vector field and corresponds to $[X_*, Y_*]$, i.e., $\pi_*(\mathcal{H}[X, Y]) = [X_*, Y_*]$.*
3. *$[V, X]$ is vertical for any $V \in \Gamma(ker\pi_*)$.*
4. *$\mathcal{H}(\nabla_X Y)$ is the basic vector field corresponding to $\nabla^{\mathcal{B}}_{X_*} Y_*$, where ∇ and $\nabla^{\mathcal{B}}$ are the Levi-Civita connections of $\mathcal{G}$ and $\mathcal{G}_{\mathcal{B}}$ respectively.*

Let $(\mathcal{N}, \mathcal{G})$ and $(\mathcal{B}, \mathcal{G}_{\mathcal{B}})$ be Riemannian manifolds and $\pi : \mathcal{N} \to \mathcal{B}$ is a smooth map between them. Then the differential π_* of π can be viewed as a section of the bundle $Hom((\mathcal{TN}), \pi^{-1}(\mathcal{TB})) \to \mathcal{N}$, where $\pi^{-1}(\mathcal{TB})$ is the pullback bundle which has fibers $(\pi^{-1}(\mathcal{TB}))_p = \mathcal{T}_{\pi(p)}\mathcal{B}, p \in \mathcal{N}$. $Hom((\mathcal{TN}), \pi^{-1}(\mathcal{TB}))$ has a connection ∇ induced from Levi-Civita connection $\nabla^{\mathcal{N}}$ and the pullback connection. The second fundamental form of π is then given by

$$(\nabla \pi_*)(X, Y) = \nabla^{\pi}_X \pi_* Y - \pi_*(\nabla^{\mathcal{N}}_X Y), \tag{15}$$

for $X, Y \in \Gamma(\mathcal{TN})$, where we denote the Levi-Civita connections of the metrics $\mathcal{G}$ and $\mathcal{G}_{\mathcal{B}}$ conveniently by ∇. Recall that π is called a totally geodesic map if $(\nabla \pi_*)(X, Y) = 0$ for $X, Y \in \Gamma(\mathcal{TN})$. It is known that the second fundamental form is symmetric. We note that, the tensor fields $\mathcal{A}$ and$\mathcal{T}$, their covariant derivatives play a fundamantal role in expressing the Riemannian curvature $R^{\mathcal{N}}$ of $\mathcal{N}$. In 1966, [16], O'Neill are given

$$R^{\mathcal{N}}(U, V, W, S) = \widehat{R}(U, V, W, S) + \mathcal{G}(\mathcal{T}_U W, \mathcal{T}_V S) - \mathcal{G}(\mathcal{T}_V W, \mathcal{T}_U S) \tag{16}$$

where $\widehat{R}$ is Riemannian curvature tensor of any fiber $(\pi^{-1}(x), \mathcal{G})$. Moreover if $\{U, V\}$ is orthonormal basis of the vertical 2-plane, then from (16) we have

$$K^{\mathcal{N}}(U, V) = \widehat{K}(U, V) + \|\mathcal{T}_U V\|^2 - \mathcal{G}(\mathcal{T}_U U, \mathcal{T}_V V) \tag{17}$$

where $K^{\mathcal{N}}$ and $\widehat{K}$ is sectional curvature of $\mathcal{N}$ and $\pi^{-1}(x)$.

A plane section in the tangent space $T_p\mathcal{N}$ at $p \in \mathcal{N}$ is called a ϕ-section if it is spanned by a vector X orthogonal to ξ and ϕX. The sectional curvature of ϕ-section is called ϕ-sectional curvature. A Kenmotsu manifold with constant ϕ-sectional curvature c is called as Kenmotsu space form and denoted by $\mathcal{N}(c)$. The Riemannian curvature tensor of a Kenmotsu space form is given by

$$\begin{aligned} R(X, Y, Z, W) =& \frac{c-3}{4}[\mathcal{G}(Y, Z)\mathcal{G}(X, W) - \mathcal{G}(X, Z)\mathcal{G}(Y, W)] + \\ & \frac{c+1}{4}[\eta(X)\eta(Z)\mathcal{G}(Y, W) - \eta(Y)\eta(Z)\mathcal{G}(X, W) + \eta(Y)\mathcal{G}(X, Z)\eta(W) \\ & - \eta(X)\mathcal{G}(Y, Z)\eta(W) + \mathcal{G}(X, \phi Z)\mathcal{G}(\phi Y, W) - \mathcal{G}(Y, \phi Z)\mathcal{G}(\phi X, W) \\ & + 2\mathcal{G}(X, \phi Y)\mathcal{G}(\phi Z, W)] \end{aligned} \tag{18}$$

for all $X, Y, Z, W \in \Gamma(\mathcal{TN})$.

3. Generic $\xi^{\perp}$-Riemannian submersions

We define generic $\xi^{\perp}$-Riemannian submersion from a Kenmotsu manifold onto a Riemannian manifold with examples. We begin with the following definition: Let $\pi : (\mathcal{N}, \phi, \xi, \eta, \mathcal{G}) \to (\mathcal{B}, \mathcal{G}_{\mathcal{B}})$ be a Riemannian submersion such that $\mathcal{N}$ is a Kenmotsu manifold , $\mathcal{B}$ is a Riemannian manifold and ξ is normal to $ker\pi_*$. Then, the complex subspace of the vertical subspace $\mathcal{V}_p$ is defined by

$$\mathfrak{D}_p = (ker\pi_{*p} \cap \phi(ker\pi_{*p}))$$

where $p \in \mathcal{N}$.

Definition 3.1. *Let $\mathcal{N}$ be a Kenmotsu manifold with almost contact structure ϕ and metric $\mathcal{G}$ and $\mathcal{B}$ be a Riemannian manifold with Riemannian metric $\mathcal{G}_{\mathcal{B}}$. Also consider ξ is normal to $ker\pi_*$ then a Riemannian submersion $\pi : (\mathcal{N}, \phi, \xi, \eta, \mathcal{G}) \to (\mathcal{B}, \mathcal{G}_{\mathcal{B}})$ is called a generic $\xi^{\perp}$-Riemannian submersion if there is a distribution $\mathfrak{D} \subset ker\pi_*$ such that*

$$(ker\pi_*) = \mathfrak{D} \oplus \mathfrak{D}^{\perp}, \phi(\mathfrak{D}) = \mathfrak{D}$$

where, $\mathfrak{D}^{\perp}$ is the orthogonal complement of $\mathfrak{D}$ in $\Gamma(ker\pi_*)$ and is purely real distribution on the fibers of the submersion π.

Example 3.2. *Every semi-invariant $\xi^{\perp}$-Riemannian submersion from a Kenmotsu manifold onto a Riemannian manifold is a generic $\xi^{\perp}$-Riemannian submersion such that $\mathfrak{D}^{\perp}$ is total real distribution.*

Example 3.3. *Every slant $\xi^{\perp}$-Riemannian submersion from a Kenmotsu manifold onto a Riemannian manifold is a generic $\xi^{\perp}$-Riemannian submersion such that $\mathfrak{D} = \{0\}$ and $\mathfrak{D}^{\perp}$ is a slant distribution.*

Example 3.4. *Every semi-slant $\xi^{\perp}$-Riemannian submersion from a Kenmotsu manifold onto a Riemannian manifold is a generic $\xi^{\perp}$-Riemannian submersion such that $\mathfrak{D}^{\perp}$ is a slant distribution.*

Example 3.5. *Let $\mathbb{M}^7$ be a 7-dimensional manifold given by the following:*

$$\mathbb{M}^7 = \{(x_1, x_2, x_3, y_1, y_2, y_3, z) \in \mathbb{R}^7 | z > 0\}.$$

We define the Kenmotsu structure $(\phi, \xi, \eta, \mathcal{G})$ on $\mathbb{M}^7$ given by the following:

$$\xi = \frac{\partial}{\partial z}, \qquad \eta = dz,$$

$$\mathcal{G} = \begin{bmatrix} e^z & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^z & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^z & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^z & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^z & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \phi = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

A ϕ-basis for this structure can be given by $\{\varepsilon_1 = e^z \frac{\partial}{\partial y_1}, \varepsilon_2 = e^z \frac{\partial}{\partial y_2}, \varepsilon_3 = e^z \frac{\partial}{\partial y_3}, \varepsilon_4 = e^z \frac{\partial}{\partial x_1}, \varepsilon_5 = e^z \frac{\partial}{\partial x_2}, \varepsilon_6 = e^z \frac{\partial}{\partial x_3}, \varepsilon_1 = \xi\}$.

Let $\mathcal{B}=\{(u_1, u_2, u_3) \in \mathbb{R}^3 | u_3 = z \neq 0\}$. We choose the Riemannian metric $\mathcal{G}_{\mathcal{B}}$ on $\mathcal{B}$ in the following form: $\mathcal{G}_{\mathcal{B}} = \begin{bmatrix} e^{3z} & 0 & 0 \\ 0 & e^{3z} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$

Now, we define the map $\pi : \mathbb{M}^7 \to \mathcal{B}$ such that

$$\pi(x_1, x_2, x_3, y_1, y_2, y_3, z) = (\frac{x_2 + y_3}{\sqrt{2}}, \frac{x_3 + y_2}{\sqrt{2}}, z).$$

By direct calculations, we have

$$ker\pi_* = span\{X_1 = \varepsilon_1, X_2 = \frac{1}{\sqrt{2}}(\varepsilon_5 - \varepsilon_3), X_3 = \varepsilon_4, X_4 = \frac{1}{\sqrt{2}}(\varepsilon_6 - \varepsilon_2)\}$$

$$\mathfrak{D} = span\{X_1 = \varepsilon_1, X_3 = \varepsilon_4\}, \qquad \mathfrak{D}^{\perp} = span\{X_2 = \frac{1}{\sqrt{2}}(\varepsilon_5 - \varepsilon_3), X_4 = \frac{1}{\sqrt{2}}(\varepsilon_6 - \varepsilon_2)\}$$

$$(ker\pi_*)^{\perp} = span\{Y_1 = \frac{1}{\sqrt{2}}(\varepsilon_5 + \varepsilon_3), Y_2 = \frac{1}{\sqrt{2}}(\varepsilon_6 + \varepsilon_2), Y_3 = \xi\}.$$

After some computations, one can see that

$$\pi_*(Y_1) = e^{-z}\frac{\partial}{\partial u_1}, \ \pi_*(Y_2) = e^{-z}\frac{\partial}{\partial u_2}, \ \pi_*(Y_3) = \pi_*(\xi) = \frac{\partial}{\partial u_3}$$

and

$$\mathcal{G}(Y_i, Y_j) = \mathcal{G}_{\mathcal{B}}(\pi_*(Y_i), \pi_*(Y_j))$$

for all $Y_i, Y_j \in (ker\pi_*)^{\perp}, i, j = 1, 2, 3$. Thus π is semi-invariant $\xi^{\perp}$-Riemannian submersion . Moreover it is easy to verify that $\phi(X_1) = X_3, \phi(X_2) = Y_2, \phi(X_3) = X_1, \phi(X_4) = Y_1$. Thus, π is a generic $\xi^{\perp}$-Riemannian submersion .

Let $\pi : (\mathcal{N}, \phi, \xi, \eta, \mathcal{G}) \to (\mathcal{B}, \mathcal{G}_{\mathcal{B}})$ be a generic $\xi^{\perp}$-Riemannian submersion such that $\mathcal{N}$ is Kenmotsu manifold and $\mathcal{B}$ is a Riemannian manifold. Now, for any $Z \in \Gamma(\mathcal{T}\mathcal{N})$, we have the following condition

$$Z = \mathcal{V}Z + \mathcal{H}Z \tag{19}$$

where $\mathcal{V}Z \in \Gamma(ker\pi_*)$ and $\mathcal{H}Z \in \Gamma((ker\pi_*)^{\perp})$. For $U \in \Gamma(ker\pi_*)$, we write

$$\phi U = \rho U + \omega U \tag{20}$$

where ρU and ωU are vertical and horizontal components of ϕU, respectively.

Further, let μ be the complementary distribution to $\omega\mathfrak{D}^{\perp}$ in $(ker\pi_*)^{\perp}$. Then, we have

$$\phi\mathfrak{D}^{\perp} \subset \mathfrak{D}^{\perp}, \qquad (ker\pi_*)^{\perp} = \omega\mathfrak{D}^{\perp} \oplus \mu,$$

where $\phi(\mu) \subset \mu$. Hence, μ contains ξ. Similarly, for any $X \in \Gamma((ker\pi_*)^{\perp})$, we have

$$\phi X = \mathbb{B}X + \mathbb{C}X, \tag{21}$$

where, $\mathbb{B}X$ and $\mathbb{C}X$ are vertical and horizontal components of ϕX, respectively.

Now with the help of above equations (10), (11), (20), (21) and covariant derivative of ρ and ω which are defined as follows:

$$(\nabla_V^{\mathcal{N}}\omega)W = \mathcal{H}\nabla_V^{\mathcal{N}}\omega W - \omega\widehat{\nabla}_V W,$$

$$(\nabla_V^{\mathcal{N}}\rho)W = \widehat{\nabla}_V\rho W - \rho\widehat{\nabla}_V W,$$

we obtain the following relations:

$$(\nabla_V^{\mathcal{N}}\rho)W = \mathbb{B}\mathcal{T}_V W - \mathcal{T}_V\omega W \tag{22}$$

$$(\nabla_V^{\mathcal{N}}\omega)W = \mathbb{C}\mathcal{T}_V W - \mathcal{T}_V\rho W \tag{23}$$

for any $V, W \in \Gamma(ker\pi_*)$

4. Geometry of foliations

In this section, we examine the integrability as well as totally geodesicness of distributions involved in the definition of a generic $\xi^{\perp}$-Riemannian submersion. Along with it, we also obtain decomposition theorems of this submersion.

Theorem 4.1. *Let $\pi : (\mathcal{N}, \phi, \xi, \eta, \mathcal{G}) \to (\mathcal{B}, \mathcal{G}_{\mathcal{B}})$ be a generic $\xi^{\perp}$-Riemannian submersion such that $\mathcal{N}$ and $\mathcal{B}$ are Kenmotsu and Riemannian manifolds, respectively. Then, the distribution $\mathfrak{D}$ is integrable if and only if*

$$\mathcal{G}(\mathcal{T}_V\phi W - \mathcal{T}_W\phi V, \omega Z) = \mathcal{G}(\widehat{\nabla}_W\phi V - \widehat{\nabla}_V\phi W, \rho Z)$$

for any $V, W \in \Gamma(\mathfrak{D})$.

Proof. For $V, W \in \Gamma(\mathfrak{D}), Z \in \Gamma(\mathfrak{D}^{\perp}), X \in \Gamma((ker\pi_*)^{\perp})$, since $[V, W] \in \Gamma(ker\pi_*)$, we have that $\mathcal{G}([V, W], X) = 0$. Thus, $\mathfrak{D}$ is integrable if and only if $\mathcal{G}([V, W], Z) = 0$.

Initially we note that, for any $V, W \in \Gamma(\mathfrak{D})$ and $Z \in \Gamma(\mathfrak{D}^{\perp})$, from (2) and (5), we have

$$\begin{aligned}\mathcal{G}(\nabla_V W, Z) &= \mathcal{G}(\phi(\nabla_V W), \phi Z) + \eta(\nabla_V W)\eta(Z)\\ &= \mathcal{G}(\phi(\nabla_V W), \phi Z)\\ &= \mathcal{G}(\nabla_V(\phi W) - (\nabla_V\phi)W, \phi Z)\\ &= \mathcal{G}(\nabla_V\phi W - \mathcal{G}(\phi V, W)\xi + \eta(W)\phi V, \phi Z)\\ &= \mathcal{G}(\nabla_V\phi W, \phi Z).\end{aligned}$$

Now, consider

$$\begin{aligned}\mathcal{G}([V,W],Z) &= \mathcal{G}(\nabla_V W - \nabla_W V, Z)\\ &= \mathcal{G}(\nabla_V \phi W, \phi Z) - \mathcal{G}(\nabla_W \phi V, \phi Z),\end{aligned}$$

which further gives

$$\begin{aligned}\mathcal{G}([V,W],Z) &= \mathcal{G}(\mathcal{T}_V \phi W + \widehat{\nabla}_V \phi W, \phi Z) - \mathcal{G}(\mathcal{T}_W \phi V + \widehat{\nabla}_W \phi V, \phi Z)\\ &= \mathcal{G}(\mathcal{T}_V \phi W - \mathcal{T}_W \phi V, \omega Z) + \mathcal{G}(\widehat{\nabla}_V \phi W - \widehat{\nabla}_W \phi V, \rho Z)\end{aligned}$$

by using equations (10) and hence we have theorem. □

Theorem 4.2. *Let $\pi : (\mathcal{N}, \phi, \xi, \eta, \mathcal{G}) \to (\mathcal{B}, \mathcal{G}_{\mathcal{B}})$ be a generic $\xi^\perp$-Riemannian submersion such that $\mathcal{N}$ and $\mathcal{B}$ are Kenmotsu and Riemannian manifolds respectively. Then, the distribution $\mathfrak{D}^\perp$ is integrable if and only if*

$$-\widehat{\nabla}_Z \phi W - \mathcal{T}_Z \omega W + \widehat{\nabla}_W \phi Z + \mathcal{T}_W \omega Z \in \Gamma(\mathfrak{D}^\perp)$$

for any $Z, W \in \Gamma(\mathfrak{D}^\perp)$ and $K \in \Gamma(\mathfrak{D})$.

Proof. Firstly, in the account of equations (2) and (5), for any $Z, W \in \Gamma(\mathfrak{D}^\perp)$ and $K \in \Gamma(\mathfrak{D})$, we have the following

$$\begin{aligned}\mathcal{G}(\nabla_Z W, K) &= \mathcal{G}(\phi(\nabla_Z W), \phi K) + \eta(\nabla_Z W)\eta(K)\\ &= \mathcal{G}(\nabla_Z(\phi W) - \mathcal{G}(\phi Z, W)\xi + \eta(W)\phi Z, \phi K)\\ &= \mathcal{G}(\nabla_Z(\phi W), \phi K).\end{aligned}$$

Now, with help of above condition and using the equations (2), (10), (11), we obtain

$$\begin{aligned}\mathcal{G}([Z,W],K) &= \mathcal{G}(\nabla_Z W - \nabla_W Z, K)\\ &= \mathcal{G}(\mathcal{T}_Z \rho W + \widehat{\nabla}_Z \rho W + \mathcal{T}_Z \omega W + \mathcal{H}\nabla_Z \omega W, \phi K)\\ &- \mathcal{G}(\mathcal{T}_W \rho Z + \widehat{\nabla}_W \rho Z + \mathcal{T}_W \omega Z + \mathcal{H}\nabla_W \omega Z, \phi K).\end{aligned}$$

By virtue of (20), (21), we arrive

$$\begin{aligned}\mathcal{G}([Z,W],K) &= \mathcal{G}(\rho(-\widehat{\nabla}_Z \rho W - \mathcal{T}_Z \omega W + \widehat{\nabla}_W \rho Z + \mathcal{T}_W \omega Z)\\ &+ \mathbb{B}(-\mathcal{T}_Z \rho W - \mathcal{H}\nabla_Z \omega W + \mathcal{T}_W \rho Z + \mathcal{H}\nabla_W \omega Z), K)\\ &+ \mathcal{G}(\omega(-\widehat{\nabla}_Z \rho W - \mathcal{T}_Z \omega W + \widehat{\nabla}_W \rho Z + \mathcal{T}_W \omega Z)\\ &+ \mathbb{C}(-\mathcal{T}_Z \rho W - \mathcal{H}\nabla_Z \omega W + \mathcal{T}_W \rho Z + \mathcal{H}\nabla_W \omega Z), K).\end{aligned}$$

After some calculations, we get

$$\begin{aligned}\mathcal{G}([Z,W],K) =& \mathcal{G}(\rho(-\widehat{\nabla}_Z \rho W - \mathcal{T}_Z \omega W + \widehat{\nabla}_W \rho Z + \mathcal{T}_W \omega Z)\\ &+ \mathbb{B}(-\mathcal{T}_Z \rho W - \mathcal{H}\nabla_Z \omega W + \mathcal{T}_W \rho Z + \mathcal{H}\nabla_W \omega Z), K).\end{aligned}$$

Since, $\mathbb{B}(-\mathcal{T}_Z \rho W - \mathcal{H}\nabla_Z \omega W + \mathcal{T}_W \rho Z + \mathcal{H}\nabla_W \omega Z), K) \in \Gamma(\mathfrak{D}^\perp)$, therefore we can conclude that

$$\mathcal{G}([Z,W],K) = \mathcal{G}(\rho(-\widehat{\nabla}_Z \rho W - \mathcal{T}_Z \omega W + \widehat{\nabla}_W \rho Z + \mathcal{T}_W \omega Z)$$

which demonstrates the statement. □

Theorem 4.3. *Let* $\pi : (\mathcal{N}, \phi, \xi, \eta, \mathcal{G}) \to (\mathcal{B}, \mathcal{G}_{\mathcal{B}})$ *be a generic* $\xi^{\perp}$*-Riemannian submersion such that* $\mathcal{N}$ *and* $\mathcal{B}$ *are Kenmotsu and Riemannian manifolds respectively. Then, the distribution* $\mathfrak{D}$ *defines a totally geodesic foliation on* $\mathcal{N}$ *if and only if*

$$\rho(\widehat{\nabla}_K \rho L + \mathcal{T}_K \omega L) = -\mathbb{B}(\mathcal{T}_K \rho L + \mathcal{H}(\nabla_K \omega L))$$

and

$$\mathcal{G}_{\mathcal{B}}((\nabla \pi_*)(K, \phi L), \pi_* \mathbb{C}X) = \mathcal{G}(\nabla_K \phi L, \mathbb{B}X)$$

for any $K, L \in \Gamma(\mathfrak{D}), Z \in \Gamma(\mathfrak{D}^{\perp})$ *and* $X \in \Gamma(ker\pi_*)^{\perp}$.

Proof. The distribution $\mathfrak{D}$ defines a totally geodesic foliation on $\mathcal{N}$ if and only if $\mathcal{G}(\nabla_K L, Z) = 0$ and $\mathcal{G}(\nabla_K L, X) = 0$, for any $K, L \in \mathfrak{D}, Z \in \Gamma(\mathfrak{D}^{\perp})$ and $X \in \Gamma(ker\pi_*)^{\perp}$.

Now, for any $K, L \in \mathfrak{D}, Z \in \Gamma(\mathfrak{D}^{\perp})$ using (2),(5) and (20) we have

$$\begin{aligned}\mathcal{G}(\nabla_K L, Z) &= \mathcal{G}(\nabla_K \phi L, \phi Z)\\ &= \mathcal{G}(\nabla_K(\rho L + \omega L), \phi Z).\end{aligned}$$

By virtue of (4), (10) and (11), above equation implies that

$$\begin{aligned}\mathcal{G}(\nabla_K L, Z) &= -\mathcal{G}(\phi(\mathcal{T}_K \rho L + \widehat{\nabla}_K \rho L + \mathcal{T}_K \omega L + \mathcal{H}(\nabla_K \omega L)), Z)\\ &= -\mathcal{G}(\mathbb{B}\mathcal{T}_K \rho L + \rho\widehat{\nabla}_K \rho L + \rho\mathcal{T}_K \omega L + \mathbb{B}\mathcal{H}(\nabla_K \omega L), Z).\end{aligned}$$

On the other hand, for $X \in \Gamma((ker\pi_*)^{\perp})$, using (11),(15) and (21), we arrive

$$\begin{aligned}\mathcal{G}(\nabla_K L, X) &= \mathcal{G}(\nabla_K \phi L, \phi X)\\ &= \mathcal{G}(\nabla_K \phi L, \mathbb{B}X + \mathbb{C}X).\end{aligned}$$

Since π is generic $\xi^{\perp}$-Riemannian submersion, we have

$$\mathcal{G}(\nabla_K L, X) = \mathcal{G}(\nabla_K \phi L, \mathbb{B}X) + \mathcal{G}_{\mathcal{B}}(\pi_* \nabla_K \phi L, \pi_* \mathbb{C}X).$$

Then, using (15), we get

$$\begin{aligned}\mathcal{G}(\nabla_K L, X) &= \mathcal{G}(\nabla_K \phi L, \mathbb{B}X) + \mathcal{G}_{\mathcal{B}}(\nabla^{\pi}_K \pi_* \phi L - (\nabla \pi_*)(K, \phi L), \pi_* \mathbb{C}X)\\ &= \mathcal{G}(\nabla_K \phi L, \mathbb{B}X) + \mathcal{G}_{\mathcal{B}}((\nabla \pi_*)(K, \phi L), \pi_* \mathbb{C}X).\end{aligned}$$

Thus, we get the required assertion. □

Theorem 4.4. *Let* $\pi : (\mathcal{N}, \phi, \xi, \eta, \mathcal{G}) \to (\mathcal{B}, \mathcal{G}_{\mathcal{B}})$ *be a generic* $\xi^{\perp}$*-Riemannian submersion such that* $\mathcal{N}$ *and* $\mathcal{B}$ *are Kenmotsu and Riemannian manifolds, respectively. Then, the distribution* $\mathfrak{D}^{\perp}$ *defines a totally geodesic foliation on* $\mathcal{N}$ *if and only if*

$$\mathcal{G}_{\mathcal{B}}(\pi_*(\nabla_Z \mathbb{C}X), \pi_* \omega W) + \mathcal{G}(\widehat{\nabla}_Z \mathbb{B}X, \rho W) + \mathcal{G}(\mathcal{T}_Z \mathbb{B}X, \omega W) + \mathcal{G}(\mathcal{T}_Z \mathbb{C}X, \rho W) = 0$$

and

$$(\nabla \pi_*)(Z, \phi W) \in \Gamma(\mu)$$

for any $Z, W \in \Gamma(\mathfrak{D}^{\perp}), K \in \Gamma(\mathfrak{D})$ *and* $X \in \Gamma((ker\pi_*)^{\perp})$.

Proof. For any $Z, W \in \Gamma(\mathfrak{D}^{\perp}), K \in \Gamma(\mathfrak{D})$ and $X \in \Gamma((ker\pi_*)^{\perp})$, in the account of equations (2) and (20) we obtain

$$\mathcal{G}(\nabla_Z W, K) = \mathcal{G}(\nabla_Z \phi W, \phi K) = \mathcal{G}(\nabla_Z \phi W, \rho K + \omega K)$$

which turns into

$$\begin{aligned}\mathcal{G}(\nabla_Z W, K) &= \mathcal{G}(\mathcal{H}\nabla_Z \phi W, \omega K)\\ &= \mathcal{G}_{\mathcal{B}}((\nabla \pi_*)(Z, \phi W), \pi_* \omega K)\end{aligned}$$

with help of (15), (19) and following the reason that π is generic $\xi^{\perp}$-Riemannian submersion.

Hence we get one of the given assertions.

Now again, with the help of equations (2) and (20), we have

$$\begin{aligned}\mathcal{G}(\nabla_Z W, X) &= \mathcal{G}(\nabla_Z \phi W, \phi X)\\ &= -\mathcal{G}(\phi W, \nabla_Z \phi X)\\ &= \mathcal{G}(\nabla_Z \phi X, \phi W).\end{aligned}$$

Then, using (10),(11) and (21), we arrive

$$\begin{aligned}\mathcal{G}(\nabla_Z W, X) &= \mathcal{G}(\nabla_Z(\mathbb{B}X + \mathbb{C}X), \phi W)\\ &= \mathcal{G}(\widehat{\nabla}_Z \mathbb{B}X + \mathcal{T}_Z \mathbb{B}X + \mathcal{T}_Z \mathbb{C}X + \mathcal{H}(\nabla_Z \mathbb{C}X), \phi W)\end{aligned}$$

which further transforms into

$$\begin{aligned}\mathcal{G}(\nabla_Z W, X) = \mathcal{G}(\widehat{\nabla}_Z \mathbb{B}X, \rho W) + \mathcal{G}(\mathcal{T}_Z \mathbb{B}X, \omega W) + \mathcal{G}(\mathcal{T}_Z \mathbb{C}X, \rho W)\\ + \mathcal{G}_{\mathcal{B}}(\pi_*(\nabla_Z \mathbb{C}X), \pi_* \omega W)\end{aligned}$$

with the virtue of (15). Thus, we have obtained our assertion. □

Corollary 4.5. *Let $\pi : (\mathcal{N}, \phi, \xi, \eta, \mathcal{G}) \to (\mathcal{B}, \mathcal{G}_{\mathcal{B}})$ be a generic $\xi^{\perp}$-Riemannian submersion such that $\mathcal{N}$ and $\mathcal{B}$ are Kenmotsu and Riemannian manifolds, respectively. Then, the fibers of π are the locally product Riemannian manifold of leaves of $\mathfrak{D}$ and $\mathfrak{D}^{\perp}$ if and only if*

$$\rho(\widehat{\nabla}_K \rho L + \mathcal{T}_K \omega L) = -\mathbb{B}(\mathcal{T}_K \rho L + \mathcal{H}(\nabla_K \omega L)),$$

$$\mathcal{G}_{\mathcal{B}}((\nabla \pi_*)(K, \phi L), \pi_* \mathbb{C}X) = \mathcal{G}(\nabla_K \phi L, \mathbb{B}X),$$

$$\mathcal{G}_{\mathcal{B}}(\pi_*(\nabla_Z \mathbb{C}X), \pi_* \omega W) + \mathcal{G}(\widehat{\nabla}_Z \mathbb{B}X, \rho W) + \mathcal{G}(\mathcal{T}_Z \mathbb{B}X, \omega W) + \mathcal{G}(\mathcal{T}_Z \mathbb{C}X, \rho W) = 0$$

and

$$(\nabla \pi_*)(Z, \phi W) \in \Gamma(\mu)$$

for any $Z, W \in \Gamma(\mathfrak{D}^{\perp}), K, L \in \Gamma(\mathfrak{D})$ and $X \in \Gamma((ker\pi_*)^{\perp})$.

Theorem 4.6. *Let $\pi : (\mathcal{N}, \phi, \xi, \eta, \mathcal{G}) \to (\mathcal{B}, \mathcal{G}_{\mathcal{B}})$ be a generic $\xi^{\perp}$-Riemannian submersion such that $\mathcal{N}$ and $\mathcal{B}$ are Kenmotsu and Riemannian manifolds, respectively. Then $ker\pi_*$ defines a totally geodesic foliation on $\mathcal{N}$ if and only if*

$$\widehat{\nabla}_U \rho V + \mathcal{T}_U \omega V \in \Gamma(\mathfrak{D})$$

and

$$\mathcal{H}(\nabla_U \omega V) + \mathcal{T}_U \rho V \in \Gamma(\mathfrak{D}^{\perp})$$

for any $U, V \in \Gamma(ker\pi_*)$.

Proof. For all $U, V \in \Gamma(ker\pi_*)$, using (1), (5), (6) and (20), we have

$$\begin{aligned}\nabla_U V &= -\phi \nabla_U \phi V\\ &= -\phi[\nabla_U(\rho V + \omega V)]\end{aligned}$$

which further gives

$$\begin{aligned}\nabla_U V &= -\phi[\widehat{\nabla}_U \rho V + \mathcal{T}_U \rho V + \mathcal{T}_U \omega V + \mathcal{H}(\nabla_U \omega V)]\\ &= -[\rho \widehat{\nabla}_U \rho V + \omega \widehat{\nabla}_U \rho V + \mathbb{B}\mathcal{T}_U \rho V + \mathbb{C}\mathcal{T}_U \rho V + \rho \mathcal{T}_U \omega V\\ &\quad + \omega \mathcal{T}_U \omega V + \mathbb{B}\mathcal{H}(\nabla_U \omega V) + \mathbb{C}\mathcal{H}(\nabla_U \omega V)]\end{aligned}$$

with the help of equations (10), (11), (20) and (21).

As we know that, $ker\pi_*$ defines a totally geodesic foliation iff $\mathcal{G}(\nabla_U V, Z) = 0$, for any $Z \in \Gamma((ker\pi_*)^\perp)$. Thus above equation concludes that, $ker\pi_*$ defines a totally geodesic foliation if and only if

$$\mathbb{C}(\mathcal{H}(\nabla_U \omega V) + \mathcal{T}_U \rho V) + \omega(\widehat{\nabla}_U \rho V + \mathcal{T}_U \omega V) = 0,$$

which is equivalent to say that $\nabla_U V \in ker\pi_*$ iff

$$\widehat{\nabla}_U \rho V + \mathcal{T}_U \omega V \in \Gamma(\mathfrak{D})$$

and

$$\mathcal{H}(\nabla_U \omega V) + \mathcal{T}_U \rho V \in \Gamma(\mathfrak{D}^\perp)$$

for any $U, V \in \Gamma(ker\pi_*)$. Consequently, we have our claim.

□

Theorem 4.7. *Let $\pi : (\mathcal{N}, \phi, \xi, \eta, \mathcal{G}) \to (\mathcal{B}, \mathcal{G}_\mathcal{B})$ be a generic $\xi^\perp$-Riemannian submersion such that $\mathcal{N}$ and $\mathcal{B}$ are Kenmotsu and Riemannian manifolds, respectively. Then $(ker\pi_*)^\perp$ defines a totally geodesic foliation on $\mathcal{N}$ if and only if*

$$\mathcal{A}_X \mathbb{B} Y + \mathcal{H}(\nabla_X \mathbb{C} Y) \in \Gamma(\mu)$$

and

$$\mathcal{V}(\nabla_X \mathbb{B} Y) + \mathcal{A}_X \mathbb{C} Y \in \Gamma(\mathfrak{D}^\perp)$$

for any $X, Y \in \Gamma((ker\pi_*)^\perp)$.

Proof. For any $X, Y \in \Gamma((ker\pi_*)^\perp)$, on the account of equation (1), (5), (6) and (20), we have

$$\nabla_X Y = -\phi \nabla_X \phi Y.$$

Now, using equations (12), (13), (20) and (21) we obtain

$$\begin{aligned}\nabla_X Y = -[&\mathbb{B}\mathcal{A}_X \mathbb{B} Y + \mathbb{C}\mathcal{A}_X \mathbb{B} Y + \rho \mathcal{V}(\nabla_X \mathbb{B} Y) + \omega \mathcal{V}(\nabla_X \mathbb{B} Y) \\ &+ \rho \mathcal{A}_X \mathbb{C} Y + \omega \mathcal{A}_X \mathbb{C} Y + \mathbb{B}\mathcal{H}(\nabla_X \mathbb{C} Y) + \mathbb{C}\mathcal{H}(\nabla_X \mathbb{C} Y)].\end{aligned}$$

Thus, $(ker\pi_*)^\perp$ defines a totally geodesic foliation iff

$$\mathbb{B}(\mathcal{A}_X \mathbb{B} Y + \mathcal{H}(\nabla_X \mathbb{C} Y)) + \rho(\mathcal{V}(\nabla_X \mathbb{B} Y) + \mathcal{A}_X \mathbb{C} Y) = 0$$

Hence, $\nabla_X Y \in (ker\pi_*)^\perp$ if and only if

$$\mathbb{B}(\mathcal{A}_X \mathbb{B} Y + \mathcal{H}(\nabla_X \mathbb{C} Y)) = 0$$

and

$$\rho(\mathcal{V}(\nabla_X \mathbb{B} Y) + \mathcal{A}_X \mathbb{C} Y) = 0$$

and consequently we get our claim.

□

Corollary 4.8. *Let $\pi : (\mathcal{N}, \phi, \xi, \eta, \mathcal{G}) \to (\mathcal{B}, \mathcal{G}_\mathcal{B})$ be a generic $\xi^\perp$-Riemannian submersion such that $\mathcal{N}$ and $\mathcal{B}$ are Kenmotsu and Riemannian manifolds, respectively. Then, the total space $\mathcal{N}$ is a generic product manifold of the leaves of $ker\pi_*$ and $(ker\pi_*)^\perp$, i.e. $\mathcal{N} = \mathcal{N}_{ker\pi_*} \times \mathcal{N}_{(ker\pi_*)^\perp}$ if and only if*

$$\mathcal{A}_X \mathbb{B} Y + \mathcal{H}(\nabla_X \mathbb{C} Y) \in \Gamma(\mu),$$

$$\mathcal{V}(\nabla_X \mathbb{B} Y) + \mathcal{A}_X \mathbb{C} Y \in \Gamma(\mathfrak{D}^\perp),$$

$$\widehat{\nabla}_U \rho V + \mathcal{T}_U \omega V \in \Gamma(\mathfrak{D})$$

and

$$\mathcal{H}(\nabla_U \omega V) + \mathcal{T}_U \rho V \in \Gamma(\mathfrak{D}^\perp)$$

for any $U, V \in \Gamma(ker\pi_*)$, $X, Y \in \Gamma((ker\pi_*)^\perp)$.

5. Totally umbilical and totally geodesicness of π

In this part, we explore further requirements for the totally geodesic and totally umbilical nature of generic $\xi^{\perp}$-Riemannian submersion. We refer a Riemannian submersion between two Riemannian manifolds as totally geodesic iff $\nabla\pi_* = 0$. On the other hand, consider π be Riemannian submersion. Then π is called Riemannian submersion with totally umbilical fiber if

$$\mathcal{T}_U V = \mathcal{G}(U, V)\mathbb{H} \tag{24}$$

for all $U, V \in \Gamma(ker\pi_*)$ and $\mathbb{H}$ is mean curvature vector fields of fiber.

Theorem 5.1. *Let $\pi : (\mathcal{N}, \phi, \xi, \eta, \mathcal{G}) \to (\mathcal{B}, \mathcal{G}_{\mathcal{B}})$ be a generic $\xi^{\perp}$-Riemannian submersion such that $\mathcal{N}$ and $\mathcal{B}$ are Kenmotsu and Riemannian manifolds, respectively. Then π is totally geodesic if*

$$\widehat{\nabla}_V \rho W + \mathcal{T}_V \omega W \in \Gamma(\mathfrak{D}), \tag{25}$$

$$\mathcal{T}_V \rho W + \mathcal{A}_{\omega W} V \in \Gamma(\omega\mathfrak{D}^{\perp}), \tag{26}$$

$$\mathbb{C}(\mathcal{A}_X \rho V + \mathcal{H}\nabla^{\mathcal{N}}_X \omega V) + \omega(\mathcal{V}\nabla^{\mathcal{N}}_X \rho V + \mathcal{A}_X \omega V) = 0, \tag{27}$$

for any $X, Y \in \Gamma((ker\pi_*)^{\perp})$, $V \in \Gamma(ker\pi_*)$.

Proof. Since π is Riemannian submersion, we have

$$(\nabla\pi_*)(X, Y) = 0, \forall X, Y(ker\pi_*)^{\perp} \tag{28}$$

For any $V, W \in ker\pi_*$ using (2), (4), (10), (11), (15), (20) and (21) we get

$$\begin{aligned}
(\nabla\pi_*)(V, W) &= \nabla^{\pi}_V \pi_* W - \pi_*(\nabla^{\mathcal{N}}_V W) \\
&= -\pi_*(\nabla_V W) \\
&= -\pi_*(-\phi\nabla_V \phi W) \\
&= \pi_*(\phi(\nabla_V(\rho W + \omega W))) \\
&= \pi_*(\phi(\mathcal{T}_V \rho W + \widehat{\nabla}_V \rho W) + \phi(\mathcal{T}_V \omega W + \mathcal{H}(\nabla^{\mathcal{N}}_V \omega W))) \\
&= \pi_*(\phi(\mathcal{T}_V \rho W + \widehat{\nabla}_V \rho W) + \phi(\mathcal{T}_V \omega W + \mathcal{A}_{\omega W} V)) \\
&= \pi_*[(\mathbb{B}\mathcal{T}_V \rho W + \mathbb{C}\mathcal{T}_V \rho W) + (\rho\widehat{\nabla}_V \rho W + \omega\widehat{\nabla}_V \rho W) \\
&+ (\rho\mathcal{T}_V \omega W + \omega\mathcal{T}_V \omega W) + (\mathbb{B}\mathcal{A}_{\omega W} V + \mathbb{C}\mathcal{A}_{\omega W} V)]
\end{aligned}$$

Thus, $(\nabla\pi_*)(V, W) = 0$ iff

$$\omega(\widehat{\nabla}_V \rho W + \mathcal{T}_V \omega W) + \mathbb{C}(\mathcal{T}_V \rho W + \mathcal{A}_{\omega W} V) = 0. \tag{29}$$

On the other hand using (2), (4), (10), (20) and (21) for any $V \in ker\pi_*$ and $X \in (ker\pi_*)^{\perp}$ we get

$$\begin{aligned}
(\nabla\pi_*)(X, V) &= \nabla^{\pi}_X \pi_* V - \pi_*(\nabla^{\mathcal{N}}_X V) \\
&= -\pi_*(\nabla^{\mathcal{N}}_X V) \\
&= -\pi_*(-\phi\nabla^{\mathcal{N}}_X \phi V) \\
&= \pi_*(\phi(\nabla_X(\rho V + \omega V))) \\
&= \pi_*(\phi(\mathcal{V}\nabla^{\mathcal{N}}_X \rho V + \mathcal{A}_X \rho V) + \phi(\mathcal{A}_X \omega V + \mathcal{H}\nabla^{\mathcal{N}}_X \omega V)) \\
&= \phi_*(\rho\mathcal{V}\nabla^{\mathcal{N}}_X \rho V + \omega\mathcal{V}\nabla^{\mathcal{N}}_X \rho V + \mathbb{B}\mathcal{A}_X \rho V + \mathbb{C}\mathcal{A}_X \rho V \\
&+ \rho\mathcal{A}_X \omega V + \omega\mathcal{A}_X \omega V + \mathbb{B}\mathcal{H}\nabla^{\mathcal{N}}_X \omega V + \mathbb{C}\mathcal{H}\nabla^{\mathcal{N}}_X \omega V).
\end{aligned}$$

Thus, $(\nabla\pi_*)(X, V) = 0$ iff

$$\omega(\mathcal{V}\nabla^{\mathcal{N}}_X \rho V + \mathcal{A}_X \omega V) + \mathbb{C}(\mathcal{H}\nabla^{\mathcal{N}}_X \omega V + \mathcal{A}_X \rho V) = 0. \tag{30}$$

Hence the result then follows from (28), (29) and (30).
□

Theorem 5.2. *Let $\pi : (\mathcal{N}, \phi, \xi, \eta, \mathcal{G}) \to (\mathcal{B}, \mathcal{G}_{\mathcal{B}})$ be a generic $\xi^\perp$-Riemannian submersion with totally umbilical fibers such that $\mathcal{N}$ and $\mathcal{B}$ are Kenmotsu and Riemannian manifolds, respectively, then $\mathcal{G}(\mathbb{H}, X) = -\mathcal{G}(\xi, X)$.*

Proof. For any $U, V \in \Gamma(\mathfrak{D})$, using (5), (6), (20) and (21), we have

$$\mathcal{G}(\phi U, V)\xi - \eta(V)\phi U = \nabla_U \phi V - \phi\nabla_U V.$$

Taking inner product in above equation with $X \in \Gamma(\mu)$, we arrive

$$\mathcal{G}(\phi U, V)\mathcal{G}(\xi, X) - \eta(V)\mathcal{G}(\phi U, X) = \mathcal{G}(\mathcal{T}_U \phi V + \widehat{\nabla}_U \phi V - \phi(\mathcal{T}_U V) - \rho\widehat{\nabla}_U V - \omega\widehat{\nabla}_U V, X)$$

Since π is totally umbilical, using (20), we conclude that

$$\mathcal{G}(\phi U, V)\eta(X) = \mathcal{G}(U, \phi V)\mathcal{G}(\mathbb{H}, X) + \mathcal{G}(U, V)\mathcal{G}(\mathbb{H}, \phi X) \tag{31}$$

Interchanging U and V in equation (31) and subtracting these two equation, we get

$$\mathcal{G}(\phi U, V)[\eta(X) + \mathcal{G}(\mathbb{H}, X)] = 0$$

Since, $\mathcal{G}(\phi U, V) \neq 0$, hence $\eta(X) = \mathcal{G}(\xi, X) = -\mathcal{G}(\mathbb{H}, X)$, which proves the required assertion.
□

6. Generic $\xi^\perp$-Riemannian submersions with Kenmotsu space form

A plane section in the tangent space $T_p\mathcal{N}$ at $p \in \mathcal{N}$ is called a ϕ-section if there exists a unit vector X in $T_p\mathcal{N}$ orthogonal to $\xi^\perp$ such that $\{X, \phi X\}$ is an orthonormal basis of the plane section. The sectional curvature of ϕ-section is called ϕ-sectional curvature. A Kenmotsu manifold with constant ϕ-sectional curvature c is known as Kenmotsu space forms. The Riemannian curvature tensor of such a manifold is given by

$$\begin{aligned} R(X, Y)Z = &\frac{c-3}{4}[\mathcal{G}(Y, Z)X - \mathcal{G}(X, Z)Y] + \\ &\frac{c+1}{4}[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + \eta(Y)\mathcal{G}(X, Z)\xi - \eta(X)\mathcal{G}(Y, Z)\xi \\ &+ \mathcal{G}(X, \phi Z)\phi Y - \mathcal{G}(Y, \phi Z)\phi X + 2\mathcal{G}(X, \phi Y)\phi Z] \end{aligned} \tag{32}$$

for all $X, Y, Z \in \Gamma(\mathcal{T}\mathcal{N})$.

Now, we choose an orthonormal frame on $\mathcal{N}$ by

$$\{e_1, e_2, ..., e_{2r}, e_{2r+1}, ..., e_{2r+2s}, e_{2r+2s+1}\}.$$

Then, we have $\mathfrak{D} = span\{e_1, e_2, ..., e_{2r}\}$, $\mathfrak{D}^\perp = span\{e_{2r+1}, ..., e_{2r+2s}\}$ and $\xi = e_{2r+2s+1}$, where $dim\mathfrak{D} = 2r$ and $dim\mathfrak{D}^\perp = 2s$.

Theorem 6.1. *Let $\pi : (\mathcal{N}, \phi, \xi, \eta, \mathcal{G}) \to (\mathcal{B}, \mathcal{G}_{\mathcal{B}})$ be a generic $\xi^{\perp}$-Riemannian submersion such that $\mathcal{N}$ and $\mathcal{B}$ are Kenmotsu space form and Riemannian manifold, respectively. Then, we have*

$$\begin{aligned}\widehat{R}(U,V,W,S) = &\frac{c-3}{4}[\mathcal{G}(V,W)\mathcal{G}(U,S) - \mathcal{G}(U,W)\mathcal{G}(V,S)] \\ &+ \frac{c+1}{4}[\mathcal{G}(\phi V,W)\mathcal{G}(\phi U,S) - \mathcal{G}(\phi U,W)\mathcal{G}(\phi V,S) + 2\mathcal{G}(U,\phi V)\mathcal{G}(\phi W,S)] \\ &- \mathcal{G}(\mathcal{T}_U W, \mathcal{T}_V S) + \mathcal{G}(\mathcal{T}_V W, \mathcal{T}_U S)\end{aligned} \tag{33}$$

and

$$\widehat{K}(U,V) = \frac{c-3}{4}[1 - \{\mathcal{G}(U,V)^2\}] + \frac{3(c+1)}{4}\{\mathcal{G}(\phi U,V)\}^2 - \|\mathcal{T}_U V\|^2 + \mathcal{G}(\mathcal{T}_U U, \mathcal{T}_V V) \tag{34}$$

for any $U,V,W,S \in \Gamma(\mathfrak{D}^{\perp})$.

Proof. For any $U,V,W,S \in \Gamma(\mathfrak{D}^{\perp})$, through the use of (20) and (32) and using the fact that if $U \in \Gamma(\mathfrak{D}^{\perp}) \implies \eta(U) = 0$, we obtain the following:

$$\begin{aligned}R(U,V,W,S) = &\frac{c-3}{4}[\mathcal{G}(V,W)\mathcal{G}(U,S) - \mathcal{G}(U,W)\mathcal{G}(V,S)] \\ &+ \frac{c+1}{4}[\mathcal{G}(\phi V,W)\mathcal{G}(\phi U,S) - \mathcal{G}(\phi U,W)\mathcal{G}(\phi V,S) + 2\mathcal{G}(U,\phi V)\mathcal{G}(\phi W,S)].\end{aligned}$$

Thus, using above equation in (16), we get our desired claim (33).

Now, if we take $W = V$ and $S = U$ in (33), we obtain the required result of (34), which completes the proof. □

Theorem 6.2. *Let $\pi : (\mathcal{N}, \phi, \xi, \eta, \mathcal{G}) \to (\mathcal{B}, \mathcal{G}_{\mathcal{B}})$ be a generic $\xi^{\perp}$-Riemannian submersion such that $\mathcal{N}$ and $\mathcal{B}$ are Kenmotsu space form and Riemannian manifold, respectively. If $\mathfrak{D}^{\perp}$ is totally geodesic, the distribution $\mathfrak{D}^{\perp}$ is Einstein.*

Proof. For any $V,W \in \Gamma(\mathfrak{D}^{\perp})$, we recall

$$\widehat{S}_{\perp}(V,W) = \sum_{i=1}^{2s} \widehat{R}(e_i, V, W, e_i)$$

where, $\widehat{S}_{\perp}$ is Ricci tensor. Let $\mathfrak{D}^{\perp}$ is totally geodesic. Then, by using (33), we have

$$\begin{aligned}\widehat{S}_{\perp}(V,W) = \sum_{i=1}^{2s}[&\frac{c-3}{4}[\mathcal{G}(V,W)\mathcal{G}(e_i,e_i) - \mathcal{G}(e_i,W)\mathcal{G}(V,e_i)] \\ &+ \frac{c+1}{4}[\mathcal{G}(\phi V,W)\mathcal{G}(\phi e_i,e_i) - \mathcal{G}(\phi e_i,W)\mathcal{G}(\phi V,e_i) + 2\mathcal{G}(e_i,\phi V)\mathcal{G}(\phi W,e_i)].\end{aligned}$$

After, applying some elementary calculations, we obtain

$$\widehat{S}_{\perp}(V,W) = \frac{(c-3)(2s-1) + 3(c+1)}{4} g(V,W).$$

Hence, we have the required assertion. □

Corollary 6.3. *Let $\pi : (\mathcal{N}, \phi, \xi, \eta, \mathcal{G}) \to (\mathcal{B}, \mathcal{G}_{\mathcal{B}})$ be a generic $\xi^{\perp}$-Riemannian submersion such that $\mathcal{N}$ and $\mathcal{B}$ are Kenmotsu space form and Riemannian manifold, respectively. If $\mathfrak{D}^{\perp}$ is totally geodesic, then the scalar curvature $\widehat{\kappa}_{\perp}$ of $\mathfrak{D}^{\perp}$ is given as follows:*

$$\widehat{\kappa}_{\perp} = \frac{(c-3)(2s-1) + 3(c+1)}{2} s.$$

Theorem 6.4. *Let $\pi : (\mathcal{N}, \phi, \xi, \eta, \mathcal{G}) \to (\mathcal{B}, \mathcal{G}_{\mathcal{B}})$ be a generic $\xi^{\perp}$-Riemannian submersion such that $\mathcal{N}$ and $\mathcal{B}$ are Kenmotsu space form and Riemannian manifold, respectively. Then, for any $U, V, W, S \in \Gamma(\mathfrak{D})$ we have*

$$\begin{aligned}\widehat{R}(U, V, W, S) = &\frac{c-1}{2}[\mathcal{G}(V, W)\mathcal{G}(U, S) - \mathcal{G}(U, W)\mathcal{G}(V, S)] + \frac{c+1}{2}\mathcal{G}(U, V)\mathcal{G}(W, S)\\ &- \mathcal{G}(\mathcal{T}_U W, \mathcal{T}_V S) + \mathcal{G}(\mathcal{T}_V W, \mathcal{T}_U S)\end{aligned} \tag{35}$$

and

$$\widehat{K}(U, V) = \frac{c-1}{2} + \{\mathcal{G}(U, V)\}^2 - \|\mathcal{T}_U V\|^2 + \mathcal{G}(\mathcal{T}_U U, \mathcal{T}_V V) \tag{36}$$

for any $U, V, W, S \in \Gamma(\mathfrak{D}^{\perp})$.

Proof. Due the consequences of (16), (17), (32) and the fact that $\phi(\mathfrak{D}) = \mathfrak{D}$ we obtain the result (35). Now taking $W = V$ and $S = U$ in (35) we get the other result (36) of this theorem. □

Theorem 6.5. *Let $\pi : (\mathcal{N}, \phi, \xi, \eta, \mathcal{G}) \to (\mathcal{B}, \mathcal{G}_{\mathcal{B}})$ be a generic $\xi^{\perp}$-Riemannian submersion such that $\mathcal{N}$ and $\mathcal{B}$ are Kenmotsu space form and Riemannian manifold, respectively. If $\mathfrak{D}$ is totally geodesic, the distribution $\mathfrak{D}$ is Einstein.*

Proof. With the help of (35), for any $V, W \in \Gamma(\mathfrak{D})$, we obtain the following

$$\begin{aligned}\widehat{S}(V, W) &= \sum_{i=1}^{2r} \widehat{R}(e_i, V, W, e_i)\\ &= \sum_{i=1}^{2r}[\frac{c-1}{2}[\mathcal{G}(V, W)\mathcal{G}(e_i, e_i) - \mathcal{G}(e_i, W)\mathcal{G}(V, e_i)] + \frac{c+1}{2}\mathcal{G}(e_i, V)\mathcal{G}(W, e_i)]\\ &= \frac{r(c-1)+1}{2}\mathcal{G}(V, W)\end{aligned}$$

where, $\widehat{S}$ is Ricci tensor for distribution $\mathfrak{D}$ and $\mathfrak{D}$ is totally geodesic. Therefore, we have the desired claim. □

Corollary 6.6. *Let $\pi : (\mathcal{N}, \phi, \xi, \eta, \mathcal{G}) \to (\mathcal{B}, \mathcal{G}_{\mathcal{B}})$ be a generic $\xi^{\perp}$-Riemannian submersion such that $\mathcal{N}$ and $\mathcal{B}$ are Kenmotsu space form and Riemannian manifold, respectively. If the distribution $\mathfrak{D}$ is totally geodesic, then the scalar curvature, $\widehat{\kappa}$ of distribution $\mathfrak{D}$ is given as follows:*

$$\widehat{\kappa} = \{r(c-1)+1\}r.$$

References

[1] M.A. Akyol, Conformal semi-slant submersions, International Journal of Geometric Methods in Modern Physics, 14(7) (2017) 1750114.
[2] M.A. Akyol, Generic Riemannian submersions from almost product Riemannian manifolds, Gazi Univ. J. Sci. 30 (3), 89-100, 2017.
[3] M.A. Akyol, R. Sari and E. Aksoy, Semi-invariant $\xi^{\perp}$-Riemannian submersions from almost contact metric manifolds, Int. J. Geom. Methods Mod. Phys. 14 (5), 1750074, 2017.
[4] M.A. Akyol, Conformal generic submersions, Turkish J. Math. 45, 201-219, 2021.
[5] D.E. Blair, Contact manifold in Riemannain geometry, Lecture Notes in Math. 509, Springer-Verlag, Berlin-New York, (1976).
[6] J.P. Bourguignon and H.B. Lawson, Stability and isolation phenomena for Yangmills fields, Commun. Math. Phys. 79, 189-230, 1981.
[7] I. K. Erken, C. Murathan, On slant Riemannian submersions for cosymplectic manifolds, Bull. Korean Math. Soc. 51(6)(2014) 1749-1771.
[8] T. Fatima and S. Ali, Submersions of generic submanifolds of a Kaehler manifold, Arab J. Math. Sci., 20 (1), 119-131, 2014.
[9] A. Gray, Pseudo-Riemannian almost product manifolds and submersions, J. Math. Mech 16 (1967), 715-737.
[10] Y. Gündüzalp, Slant submersions from almost product Riemannian manifolds, Turkish Journal of Mathematics 37(2013) 863-873.
[11] Y. Gündüzalp, Semi-slant submersions from almost product Riemannian manifolds,Demonstratio Mathematica, 49(3)(2016) 345-356.
[12] S. Ianus and M. Visinescu, Kaluza-Klein theory with scalar fields and generalized Hopf manifolds, Class, Quantum Gravity, 4, 1317-1325, 1987.

[13] S. Ianus and M. Visinescu, Space-time compaction and Riemannian submersions, In: Rassias, G.(ed.), The Mathematical Heritage of C.F. Gauss, World Scientific, River Edge (1991) 358-371.

[14] J. W. Lee, Anti-invariant $\xi^{\perp}$-Riemannian submersions from almost contact manifolds, Hacettepe Journal of Mathematics and Statistics, 42(3)(2013), 231-241.

[15] M.T. Mustafa, Applications of harmonic morphisms to gravity, J. Math. Phys. 41 (2000) 6918-6929.

[16] B. O'Neill, The fundamental equations of a submersion, Mich. Math. J. 13(1966) 458-469.

[17] R. Prasad, S. S. Shukla and S. Kumar, On Quasi bi-slant submersions, Mediterr. J. Math. 16 (2019) 155. https://doi.org/10.1007/s00009-019-1434-7.

[18] K.S. Park, H-V-semi-slant submersions from almost quaternionic Hermitian manifolds, Bull. Korean Math. Soc., 53(2) (2016) 441-460.

[19] K.S. Park, h-slant submersions, Bull. Korean Math. Soc. 49(2) (2012) 329-338.

[20] B. Şahin, Slant submersions from almost Hermitian manifolds, Bulletin mathématique de la Société des Sciences Mathématiques de Roumanie, Nouvelle Série, Vol. 54 (102), No. 1 (2011), pp. 93-105

[21] B. Şahin, Anti-invariant Riemannian submersions from almost Hermitian manifolds, Central European J. Math. 3 (2010) 437-447.

[22] B. Şahin, Riemannian Submersions, Riemannian Maps in Hermitian Geometry and their Applications, Elsevier, Academic Press, (2017).

[23] B. Şahin, Semi-invariant Riemannian submersions from almost Hermitian manifolds, Canad. Math. Bull. 56 (2011) 173-183.

[24] B. Şahin, Generic Riemannian Maps, Miskolc Math. Notes, 18 (1), 453-467, 2017.

[25] C. Sayar, M.A. Akyol and R. Prasad, On bi-slant submersions in complex geometry, International journal of Geometric Methods in Modern Physics, 17(4), (2020), 2050055.

[26] Ramazan Sari, Generic $\xi^{\perp}$-Riemannian submersions, Hacet. J. Math. Stat., Volume 51 (2) (2022), 390 – 403. DOI : 10.15672/hujms.723602

[27] A. D. Vilcu, G.E. Vilcu, Statistical manifolds with almost quaternionic structures and quaternionic Kähler-like statistical submersions, Entropy 17(9) (2015) 6213-6228.

[28] G. E. Vilcu, Mixed paraquaternionic 3-submersions, Indag. Math, (N.S.) 24(2) (2013) 474-488.