Filomat 38:18 (2024), 6493–6508
https://doi.org/10.2298/FIL2418493S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

# Optimization methods to statistical submanifolds in statistical warped product manifolds

**Aliya Naaz Siddiqui[a,*], Fatemah Mofarreh[b], Ali Hussain Alkhaldi[c], Akram Ali[c]**

[a]*Division of Mathematics, School of Basic Sciences, Galgotias University, Greater Noida, Uttar Pradesh 203201, India*
[b]*Mathematical Science Department, Faculty of Science, Princess Nourah bint Abdulrahman University, Riyadh 11546, Saudi Arabia*
[c]*Department of Mathematics, College of Science, King Khalid University, 9004 Abha, Saudi Arabia*

**Abstract.** This research paper strives to address the utilization of an optimization technique on submanifolds. Specifically, it explores the concept of treating geometric inequalities as optimization challenges. Within this framework, the paper examines two categories of inequalities: the optimal Casorati inequalities and the Chen-Ricci inequality. These inequalities are investigated in the context of a statistical submanifold situated within an almost Kenmotsu statistical manifold. Notably, this manifold represents a statistical warped product of a trivial statistical manifold and an almost Kähler statistical manifold.

## 1. Development of statistical warped products

The productive exploration of warped product manifolds as an inherent extension of Riemannian product manifolds began in 1969 through the pioneering work of R.L. Bishop and B. O'Neil [3], primarily focused on manifolds exhibiting a curvature that is negative. These manifolds are a fascinating class of mathematical structures that emerge from the field of differential geometry, offering a versatile framework for understanding and studying various geometric phenomena. These manifolds arise by combining two distinct manifolds using a special type of product, known as the warped product. This construction introduces a notion of nontrivial warping or scaling along one of the manifold's directions, resulting in intriguing and rich geometries.

At its core, a warped product manifold is formed by taking the Cartesian product of two different manifolds and then warping one of the manifolds using a smooth, positive function known as the warping function. This warping function acts as a scaling factor, varying along the directions of the other manifold, and determines how the two manifolds are interlinked. As a consequence, the geometry of the warped product manifold can significantly differ from the simple product manifold.

Warped product manifolds have found applications in various areas of mathematics and theoretical physics. They play a crucial role in general relativity, where they are used to describe certain solutions to

2020 *Mathematics Subject Classification*. 53C25, 53C40, 53D10, 53C05
*Keywords*. Optimization techniques, Casorati curvatures; Ricci curvatures, statistical warped product manifolds; Chen-Ricci inequality
Received: 05 October 2023; Accepted: 28 March 2024
Communicated by Mića Stanković
* Corresponding author: Aliya Naaz Siddiqui
*Email addresses:* aliyanaazsiddiqui9@gmail.com (Aliya Naaz Siddiqui), fyalmofarrah@pnu.edu.sa (Fatemah Mofarreh), ahalkhaldi@kku.edu.sa (Ali Hussain Alkhaldi), akramali133@gmail.com (Akram Ali)

Einstein's field equations, including the well-known Schwarzschild and Kerr metrics, which describe the geometry around non-rotating and rotating black holes, respectively.

A statistical structure can be regarded as a fruitful extension of a Riemannian structure, encompassing a Riemannian metric along with its Levi-Civita connection (referred to as *LCC* for brevity). Building upon this notion within complex and contact geometries has led to the definition of several novel manifolds, as outlined below:

1. Serving as a elegant extension beyond the combination of a Kähler structure and its associated Levi-Civita connection (in short, *LCC*), H. Furuhata introduced the concept of a holomorphic statistical manifold in [11, 12]. This construct essentially serves as the statistical counterpart to the idea of a complex manifold. Subsequently, Siddiqui et al. [32] delved into the exploration of totally real statistical submanifolds within holomorphic statistical manifolds, shedding new insights in this context.
2. In the realm of contact geometry, Furuhata and colleagues [13] presented the statistical analog of a Sasakian manifold, introducing the notion of a Sasakian statistical manifold.
3. Vilcu et al. [40] explored the presence of almost quaternionic structures on statistical manifolds, a category termed as quaternionic Kähler-like statistical manifolds. Building upon K. Takano's findings pertaining to statistical manifolds equipped with almost complex structures [36] and almost contact structures [37], they delved into the curvature characteristics of quaternionic Kähler-like statistical submersions.
4. In [38], L. Todjihounde formulated a dualistic structure within the context of a warped product manifold.
5. Taking inspiration from his research, Furuhata et al. [14] delved into the statistical analog of a Kenmotsu manifold, which they referred to as Kenmotsu statistical manifolds. In addition, they achieved success in outlining a procedure for structuring a Kenmotsu statistical manifold in the manner of a warped product involving a holomorphic statistical manifold and a line.
6. In a recent development, taking inspiration from Todjihounde's research, Murathan et al. [21] delved into the formulation of Kenmotsu-like statistical manifolds and cosymplectic-like statistical manifolds. This construction relies on the presence of a Kähler-like statistical manifold in conjunction with a line.
7. Hulya et al. [16] showcased the Einstein statistical warped product configurations.

Within this article, we present an application situated within the realm of optimization on manifolds, encapsulated as a concise summary of the content in [25], enriched with pointers to the latest scholarly sources. Our contribution culminates in a particularly intriguing outcome concerning optimization on Riemannian manifolds, as detailed below:

**Optimizations on submanifolds:** Consider a Riemannian submanifold $(N, G)$ in a Riemannian manifold $(\overline{M}, \overline{G})$, alongside a differentiable function $f : \overline{M} \to \mathbb{R}$.

**Theorem 1.1.** *[25] When $x \in N$ represents a solution to the constrained extremum problem $\min_{x_0 \in N} f(x_0)$, then*

1. $\Big(grad(f)\Big)(x) \in T_x^{\perp}N$,
2. *the bilinear form $\mathbf{B} : T_xN \times T_xN \to \mathbb{R}$ is positive semi-definite, defined by*

$$\mathbf{B}(U_1, V_1) = Hess_f(U_1, V_1) + \overline{G}(h^*(U_1, V_1), \Big(grad(f)\Big)(x)),$$

*where $grad(f)$ indicates the gradient of $f$ and $h^*$ is the second fundamental form of $N$ in $\overline{M}$.*

To sum up, the concept of optimization on manifolds revolves around harnessing the tools of differential geometry to construct optimization strategies applicable to abstract manifolds. Subsequently, these abstract geometric methodologies are translated into practical numerical techniques suited for specific manifolds. These methods find relevance in solving problems that can be reformulated as the optimization of differentiable functions over a manifold. This research initiative has brought fresh perspectives to existing algorithms and introduced innovative approaches supported by robust convergence analyses. Optimization on manifolds has applications in a wide range of fields, including machine learning like training neural networks with constraints; computer graphics (for example shape optimization), robotics (for example motion planning) and physics such as optimizing physical systems subject to constraints. It's important to emphasize that while the optimization of real-valued functions on manifolds, as expressed in Theorem 1.1, involves locating function minima or maxima on a manifold, this isn't the sole intersection point between optimization and differential geometry. Another notable instance is the Riemannian geometry's involvement in the central path of linear programming. Therefore, within this current study, we derive optimal Casorati inequalities and the Chen-Ricci inequality for a statistical submanifold in a statistical warped product manifold. These inequalities, involving a set of paired conjugate affine connections, incorporate the intrinsic and extrinsic curvature invariants of statistical submanifolds residing in distinct ambient spaces. For example, in [2] Aydin et al. obtained Euler and Chen-Ricci inequalities on submanifolds in statistical manifolds of constant curvature. Mihai et al. [20] inspired by above inequalities and considered statistical submanifolds of Hessian manifolds of constant Hessian curvature, then obtained both inequalities with respect to a sectional curvature of the ambient Hessian manifold. Infact, Chen-Ricci inequality for CR-statistical submanifolds of holomorphic statistical manifolds of constant holomorphic curvature was also discussed in [29]. Later on, inequalities involving Casorati curvature for statistical submanifolds in statistical manifolds of constant curvature, as well as in Kenmotsu statistical manifolds of constant $\phi$-sectional curvature, were established in [19] and [9], respectively. Siddiqui et al. [33] also derived similar inequality for statistical hypersurfaces in statistical manifolds of constant curvature.

## 2. Statistical manifolds and their associated submanifolds

A statistical manifold constitutes a Riemannian manifold $(\overline{M}, \overline{G})$ accompanied by a duo of torsion-free affine connections, denoted as $\overline{\nabla}$ and $\overline{\nabla}'$, which adhere to the condition:

$$U_1\overline{G}(V_1, U_2) = \overline{G}(\overline{\nabla}_{U_1} V_1, U_2) + \overline{G}(V_1, \overline{\nabla}'_{U_1} U_2),$$

for all $U_1, V_1, U_2 \in \Gamma(T\overline{M})$. These connections are named as dual connections.

**Definition 2.1.** *A Riemannian manifold $(\overline{M}, \overline{G})$ with an affine connection $\overline{\nabla}$ is said to be a statistical manifold [11] $(\overline{M}, \overline{G}, \overline{\nabla})$ if*

1. *$\overline{\nabla}$ is a torsion free connection on $\overline{M}$,*
2. *the covariant derivative $\overline{\nabla G}$ is symmetric.*

Amari [1] initially introduced the concept of a conjugate connection within the field of statistics, and subsequently, Lauritzen [18] further advanced this idea through subsequent studies. Evidently, $\left(\overline{\nabla}'\right)' = \overline{\nabla}$. Additionally, $2\overline{\nabla}^0 = \overline{\nabla} + \overline{\nabla}'$, where $\overline{\nabla}^0$ is *LCC* on $\overline{M}$. Furthermore, if $(\overline{\nabla}, \overline{G})$ is a statistical structure on $\overline{M}$, then $(\overline{\nabla}', \overline{G})$ is also a statistical structure.

An almost Hermitian manifold $(\overline{M}, \overline{G}, \mathcal{J})$ is classified as an almost Kähler manifold if its fundamental 2-form $\Omega$ is closed, defined by

$$\Omega(U_1, V_1) = \overline{G}(U_1, \mathcal{J} V_1),$$

This concept is elaborated in reference [42].

**Definition 2.2.** *[10] Let $(\overline{M}, \overline{\nabla}, \overline{G})$ be a statistical manifold. If $(\overline{M}, \overline{G}, \mathcal{J})$ is*

1. *an almost Hermitian manifold, then $(\overline{M}, \overline{\nabla}, \overline{G}, \mathcal{J})$ is known as an almost Hermitian statistical manifold.*
2. *a (almost) Kähler manifold then $(\overline{M}, \overline{\nabla}, \overline{G}, \mathcal{J})$ is called a (almost) Kähler statistical manifold.*

**Definition 2.3.** *[12] Consider an affine connection $\overline{\nabla}$ on a Kähler manifold $(\overline{M}, \overline{\nabla}, \overline{G}, \mathcal{J})$. Then $(\overline{M}, \overline{\nabla}, \overline{G}, \mathcal{J})$ is referred to as a holomorphic statistical manifold if*

1. *$(\overline{M}, \overline{\nabla}, \overline{G})$ is a statistical manifold,*
2. *a 2−form $\Omega$ on $\overline{M}$ is $\overline{\nabla}$−parallel (means $\overline{\nabla}\Omega = 0$).*

**Definition 2.4.** *[12] A holomorphic statistical manifold $(\overline{M}, \overline{\nabla}, \overline{G}, \mathcal{J})$ is of constant holomorphic curvature $\bar{c} \in \mathbb{R}$ if and only if*

$$\begin{aligned}\overline{\mathcal{S}}(U_1, V_1)U_2 \;&=\; \frac{\bar{c}}{4}\Big\{\overline{G}(V_1, U_2)U_1 - \overline{G}(U_1, U_2)V_1 + \overline{G}(\mathcal{J}V_1, U_2)\mathcal{J}U_1 \\ &\quad -\overline{G}(\mathcal{J}U_1, U_2)\mathcal{J}V_1 + 2\overline{G}(U_1, \mathcal{J}V_1)\mathcal{J}U_2\Big\}, \end{aligned} \tag{1}$$

*where $\overline{\mathcal{S}}$ denotes the statistical curvature tensor field of $\overline{M}$. It is symbolized by $\overline{M}(c)$.*

Consider $(\overline{M}, \overline{\nabla}, \overline{G})$ as a statistical manifold and let $N$ represent a submanifold of $\overline{M}$. Consequently, $\big(N, \nabla, G\big)$ forms another statistical manifold, inheriting the statistical structure $(\nabla, G)$ from $(\overline{\nabla}, \overline{G})$. We denote $\big(N, \nabla, G\big)$ as a statistical submanifold within the context of $\big(\overline{M}, \overline{\nabla}, \overline{G}\big)$. The core equations in the Riemannian submanifold geometry (see [42]) consist of the Gauss and Weingarten formulas, along with the Gauss and Ricci equations. In the context of statistics, the Gauss and Weingarten formulas are defined as follows [41]:

$$\begin{aligned}&\overline{\nabla}_{U_1} V_1 = \nabla_{U_1} V_1 + h(U_1, V_1)V, \quad \overline{\nabla}'_{U_1} V_1 = \nabla'_{U_1} V_1 + h'(U_1, V_1)V, \\ &\overline{\nabla}_{U_1}\mathcal{N} = -\mathrm{A}_{\mathcal{N}}(U_1) + \nabla^{\perp}_{U_1}\mathcal{N}, \quad \overline{\nabla}'_{U_1}\mathcal{N} = -\mathrm{A}'_{\mathcal{N}}(U_1) + \nabla^{\perp\prime}_{U_1}\mathcal{N},\end{aligned}$$

for all $U_1, V_1 \in \Gamma(TN)$ and $\mathcal{N} \in \Gamma(T^{\perp}N)$, where

1. $\overline{\nabla}$ and $\overline{\nabla}'$ : dual connections on $\overline{M}$,
2. $\nabla$ and $\nabla'$ : dual connections on $N$,
3. $h$ and $h'$ : second fundamental forms of $N$ in regard to $\overline{\nabla}$ and $\overline{\nabla}'$.
4. A and A′ : shape operators of $N$ in regard to $\overline{\nabla}$ and $\overline{\nabla}'$

The relation between $h$ (resp. $h'$) and $\mathrm{A}_{\mathcal{N}}$ (resp. $\mathrm{A}'_{\mathcal{N}}$) is established through the following [41]:

$$\begin{aligned}&\overline{G}(h(U_1, V_1), V) = G(\mathrm{A}'_{\mathcal{N}}U_1, V_1), \\ &\overline{G}(h'(U_1, V_1), V) = G(\mathrm{A}_{\mathcal{N}}U_1, V_1).\end{aligned}$$

Consider $N$ as a $d$-dimensional submanifold in a $n$-dimensional statistical manifold $\overline{M}$. When $\{v_1, \ldots, v_d\}$ constitutes an orthonormal basis for $T_{\wp}N$, where $\wp \in N$, the mean curvature vectors of $N$ can be expressed as

$$\mathcal{H} = \frac{1}{d}\sum_{i=1}^{d} h(v_i, v_i) \quad \text{and} \quad \mathcal{H}' = \frac{1}{d}\sum_{i=1}^{d} h'(v_i, v_i).$$

The squared norms of $h$ and $h'$ are expressed by $C$ and $C'$ respectively, termed as the *Casorati curvatures* of $N$ in $\overline{M}$:

$$C = \frac{1}{d}\|h\|^2 \quad \text{and} \quad C' = \frac{1}{d}\|h'\|^2. \tag{2}$$

For a $s$−dimensional subspace $W$ of $TN$, provided $s \geq 2$, $scal(W)$ is defined as

$$scal(W) = \sum_{1 \leq i < j \leq s} \mathcal{S}(v_i, v_j, v_j, v_i),$$

where $\mathcal{S}$ denotes the statistical curvature tensor field of $N$ and the Casorati curvatures of $W$ are as follows:

$$C(W) = \frac{1}{s}\sum_{k=d+1}^{n}\sum_{i,j=1}^{s}\left(h_{ij}^{k}\right)^2 \quad \text{and} \quad C'(W) = \frac{1}{s}\sum_{k=d+1}^{n}\sum_{i,j=1}^{s}\left(h_{ij}'^{k}\right)^2.$$

where $\{v_1, \ldots, v_s\}$ is an orthonormal basis of $W$.

Also, the normalized Casorati curvatures $\delta_C(d-1)$ and $\widehat{\delta_C}(d-1)$ are defined as

$$[\delta_C(d-1)]_\wp = \frac{1}{2}C_\wp + (\frac{d+1}{2d})\inf\{C(W)|W : \text{a hyperplane of } T_\wp N\} \tag{3}$$

and

$$[\widehat{\delta_C}(d-1)]_\wp = 2C_\wp - (\frac{2d-1}{2d})\sup\{C(W)|W : \text{a hyperplane of } T_\wp N\}. \tag{4}$$

Note that dual of (3) and (4) are $\delta'_C(d-1)$ and $\widehat{\delta'_C}(d-1)$ at $\wp \in N$.

Consider the curvature tensor fields $\overline{R}$ and $R$ associated with $\overline{\nabla}$ and $\nabla$, respectively. The relevant Gauss equation is [41]

$$\begin{aligned} \overline{G}(\overline{R}(U_1,V_1)U_2,V_2) &= G(R(U_1,V_1)U_2,V_2) + \overline{G}(h(U_1,U_2),h'(V_1,V_2)) \\ &\quad -\overline{G}(h'(U_1,V_2),h(V_1,U_2)). \end{aligned} \tag{5}$$

Likewise, $\overline{R}'$ and $R'$ correspondingly denote the curvature tensor fields in regard to $\overline{\nabla}'$ and $\nabla'$. The counterpart of (5) under $\overline{\nabla}'$ and $\nabla'$ can be derived.

The statistical curvature tensor fields $\overline{\mathcal{S}}$ of $\overline{M}$ and $\mathcal{S}$ of $N$ are defined as per [23, 24] in the following manner:

$$\overline{\mathcal{S}} = \frac{1}{2}(\overline{R} + \overline{R}') \quad \text{and} \quad \mathcal{S} = \frac{1}{2}(R + R'). \tag{6}$$

Consequently, the sectional curvature $\mathbb{K}^{\nabla,\nabla'}$ on $N$ in $\overline{M}$ can be expressed as given in [23, 24]:

$$\begin{aligned} \mathbb{K}^{\nabla,\nabla'}(U_1 \wedge V_1) &= G(\mathcal{S}(U_1,V_1)V_1,U_1) \\ &= \frac{1}{2}(G(R(U_1,V_1)V_1,U_1) + G(R'(U_1,V_1)V_1,U_1)), \end{aligned} \tag{7}$$

where $U_1$ and $V_1$ are orthonormal vectors in the tangent space $T_\wp N$, and $\wp \in N$.

The concept of warped product manifolds emerged as an elegant extension of Riemannian product manifolds. As outlined by R.L. Bishop and B. O'Neil, the definition of these manifolds is as follows [3]:

**Definition 2.5.** *Let $(N_1, G_1)$ and $(N_2, G_2)$ denote two (pseudo)-Riemannian manifolds, and $\ell > 0$ represent a differentiable function on $N_1$. Consider the mappings $\rho : N_1 \times N_2 \longrightarrow N_1$ and $\varrho : N_1 \times N_2 \longrightarrow N_2$. In this context, the warped product $N = N_1 \times_\ell N_2$ represents the product manifold endowed with a Riemannian structure, such that*

$$\overline{G}(U_1,V_1) = G_1(\rho_* U_1, \rho_* V_1) + \ell^2(u)G_2(\varrho_* U_1, \varrho_* V_1), \tag{8}$$

*for all $U_1, V_1 \in \Gamma(T_{(u,v)}N)$, $u \in N_1$ and $v \in N_2$, where $*$ is the symbol for the tangent maps. The function $\ell$ is called the warping function.*

Let $\chi(N_1)$ and $\chi(N_2)$ denote the sets of all vector fields on $N_1 \times N_2$, which represent the horizontal lift of a vector field on $N_1$ and the vector lift of a vector field on $N_2$, respectively. This implies that $\rho(\chi(N_1))$ encompasses $\Gamma(TN_1)$, and $\varrho(\chi(N_2))$ encompasses $\Gamma(TN_2)$. As a result, we have $\rho_*(X) = U_1 \in \Gamma(TN_1)$, $\rho_*(Y) = V_1 \in \Gamma(TN_1)$, $\varrho_*(U) = U_2 \in \Gamma(TN_2)$, and $\varrho_*(V) = V_2 \in \Gamma(TN_2)$.

Consider a statistical warped product manifold denoted as $M = \mathbb{R} \times_\ell \overline{M}$, where the metric is $\overline{G} = G_1 + \ell^2(z)G_2$. In this context, $\mathbb{R}$ signifies a trivial statistical manifold with metric $G_1 = dz^2$, while $\overline{M}$ stands as an almost Kähler statistical manifold [10] with metric $G_2$ and dual affine connections $\nabla^{\overline{M}}$ and $\nabla^{\overline{M}\prime}$. The structure vector field on $M$ is denoted by $\xi = \partial z$. Any arbitrary vector field on $M$ can be expressed as $Z = \eta(Z) + U_1$, where $U_1$ constitutes an arbitrary vector field on $\overline{M}$ and $dz = \eta$. Moreover, a novel tensor field $\phi$ of type $(1,1)$ on $M$ can be defined by utilizing the tensor field $\mathcal{J}$, yielding $\phi Z = \mathcal{J}U_1$. This particular type of statistical warped product was termed as an almost Kenmotsu statistical manifold by R. Gorunus et al. [15]. The statistical curvature tensor of $M = \mathbb{R} \times_\ell \overline{M}(\bar{c})$ can be found in [15] as

$$\begin{aligned}\overline{\mathcal{S}}(U_1,V_1,U_2,V_2) =&\frac{1}{2}\Big(\overline{R}(U_1,V_1,U_2,V_2) + \overline{R}'(U_1,V_1,U_2,V_2)\Big)\\ =&\alpha\Big(\overline{G}(U_1,V_2)\overline{G}(V_1,U_2) - \overline{G}(U_1,U_2)\overline{G}(V_1,V_2)\Big)\\ &+ \beta\Big(\overline{G}(U_1,U_2)\overline{G}(V_1,\partial z)\overline{G}(V_2,\partial z) - \overline{G}(V_1,U_2)\overline{G}(U_1,\partial z)\overline{G}(V_2,\partial z)\\ &+ \overline{G}(V_1,V_2)\overline{G}(U_1,\partial z)\overline{G}(U_2,\partial z) - \overline{G}(U_1,V_2)\overline{G}(V_1,\partial z)\overline{G}(U_2,\partial z)\Big)\\ &+ \gamma\Big(\overline{G}(U_1,\phi U_2)\overline{G}(\phi V_1,V_2) - \overline{G}(V_1,\phi U_2)\overline{G}(\phi U_1,V_2)\\ &+ 2\overline{G}(U_1,\phi V_1)\overline{G}(\phi U_2,V_2)\Big),\end{aligned} \tag{9}$$

where $\partial z = \frac{\partial}{\partial z}$ represents the unit tangent vector field on $\mathbb{R}$ while $\alpha$, $\beta$, and $\gamma$ are

$$\alpha = \frac{\bar{c}}{4\ell^2} - \frac{(\ell')^2}{\ell^2}, \quad \beta = \frac{\bar{c}}{4\ell^2} - \frac{(\ell')^2}{\ell^2} + \frac{\ell''}{\ell} \quad \text{and} \quad \gamma = \frac{\bar{c}}{4\ell^2}.$$

## 3. Main inequality 1

In his work, F. Casorati [5] introduced an extrinsic invariant referred to as the Casorati curvature, applicable to a submanifold in a Riemannian manifold. This curvature, characterized as the normalized square length of the second fundamental form, extended the exploration of principal directions within hypersurfaces in Riemannian geometries. Eminent geometers [8, 17, 39] have rigorously examined the geometric aspects and significance of Casorati curvatures, leading to significant progress in the realm of pure Riemannian geometry. Consequently, there has been a substantial interest among geometers to establish optimal inequalities for Casorati curvatures of submanifolds across a range of ambient spaces.

**Notations:**

1. $(\mathbb{R}, dz, \overline{\nabla}^{\mathbb{R}})$ : trivial statistical manifold,
2. $(\overline{M}, \overline{G}, \overline{\nabla}, \mathcal{J})$ : holomorphic statistical manifold of constant holomorphic sectional curvature $\bar{c}$,
3. $\mathbb{R} \times_\ell \overline{M}(\bar{c})$ : statistical warped product manifold of special type.

**Theorem 3.1.** *Let $N$ be a $d$-dimensional statistical submanifold of $\mathbb{R} \times_\ell \overline{M}(\bar{c})$. Then we have the following inequality:*

$$\begin{aligned}nor(scal)^{\nabla,\nabla'} \leq& 2\delta_C^0 + \frac{1}{d-1}C^0 + \Big(\frac{\bar{c}}{4\ell^2} - \frac{(\ell')^2}{\ell^2}\Big) + \frac{3\bar{c}}{4d(d-1)\ell^2}\|\phi\|^2\\ &- \frac{2}{d}\Big(\frac{\bar{c}}{4\ell^2} - \frac{(\ell')^2}{\ell^2} + \frac{\ell''}{\ell}\Big)\|\mathrm{V}\|^2 - \frac{d}{2(d-1)}\Big(\|\mathcal{H}\|^2 + \|\mathcal{H}'\|^2\Big),\end{aligned} \tag{10}$$

*where $2C^0 = C + C'$ and $2\delta_C^0(d-1) = \delta_C(d-1) + \delta'_C(d-1)$.*

*Proof.* The statistical scalar curvature denoted as $scal^{\nabla,\nabla'}$ associated with $N$ is

$$\begin{aligned}
2scal^{\nabla,\nabla'} &= \sum_{1\le i<j\le m} \mathcal{S}(v_i, v_j, v_j, v_i) = \frac{1}{2}\sum_{1\le i<j\le m}\Big\{R(v_i, v_j, v_j, v_i) + R'(v_i, v_j, v_j, v_i)\Big\}\\
&= \sum_{1\le i<j\le m} R(v_i, v_j, v_j, v_i)\\
&= d(d-1)\Big(\frac{\bar{c}}{4\ell^2} - \frac{(\ell')^2}{\ell^2}\Big) - 2(d-1)\Big(\frac{\bar{c}}{4\ell^2} - \frac{(\ell')^2}{\ell^2} + \frac{\ell''}{\ell}\Big)\|\mathrm{V}\|^2 + \frac{3\bar{c}}{4\ell^2}\|\phi\|^2\\
&\quad + \frac{1}{2}\sum_{i,j}\Big\{\overline{G}(h'(v_i, v_i), h(v_j, v_j)) + \overline{G}(h(v_i, v_i), h'(v_j, v_j))\Big\}\\
&\quad - 2\sum_{i,j}\overline{G}(h(v_i, v_j), h'(v_i, v_j)), \qquad (11)
\end{aligned}$$

where $\mathrm{V} = \partial z - \sum_{k=1}^{p}\theta_k\xi_p$ represents a vector field that lies tangent to $N$, and $\theta_1, \dots, \theta_p$ denote smooth functions defined over $N$..

Presently, we establish a polynomial $\mathbb{Q}$ using $h^0$ in regard to $LCC$:

$$\begin{aligned}
\mathbb{Q} =& \frac{1}{2}d(d-1)C^0 + \frac{1}{2}(d-1)(d+1)C^0(W) + \frac{d}{2}(C + C') + \frac{3\bar{c}}{4\ell^2}\|\phi\|^2\\
&+ d(d-1)\Big(\frac{\bar{c}}{4\ell^2} - \frac{(\ell')^2}{\ell^2}\Big) - 2(d-1)\Big(\frac{\bar{c}}{4\ell^2} - \frac{(\ell')^2}{\ell^2} + \frac{\ell''}{\ell}\Big)\|\mathrm{V}\|^2\\
&- \frac{d^2}{2}\Big(\|\mathcal{H}\|^2 + \|\mathcal{H}'\|^2\Big) - 2scal^{\nabla,\nabla'}, \qquad (12)
\end{aligned}$$

where $W$ denotes a hyperplane located within the tangent space $T_\wp N$.

Without sacrificing generality, we assume that $W$ is spanned by $\{v_1, \dots, v_d\}$. Subsequently, utilizing (11) and (12), we obtain

$$\begin{aligned}
\mathbb{Q} &= \sum_{k=1}^{p}\Big(\sum_{i,j=1}^{d}\frac{d+3}{2}(h_{ij}^{0k})^2 + \frac{d+1}{2}\sum_{i,j=1}^{d-1}(h_{ij}^{0k})^2 - 2(\sum_{i=1} h_{ii}^{0k})^2\Big)\\
&= \sum_{k=1}^{p}\Big(2(d+2)\sum_{1\le i<j\le d-1}(h_{ij}^{0k})^2 + (d+3)\sum_{i=1}^{d-1}(h_{im}^{0k})^2\\
&\quad + d\sum_{i=1}^{d-1}(h_{ii}^{0k})^2 - 4\sum_{1\le i<j\le m}(h_{ii}^{0k}h_{jj}^{0k}) + \frac{d-1}{2}(h_{dm}^{0k})^2\Big)\\
&\ge \sum_{k=1}^{p}\Big(\sum_{i=1}^{d-1} d(h_{ii}^{0k})^2 + \frac{d-1}{2}(h_{dd}^{0k})^2 - 4\sum_{1\le i<j\le m} h_{ii}^{0k}h_{jj}^{0k}\Big).
\end{aligned}$$

For each $k \in 1, \dots, p$, we introduce a quadratic form $\mathcal{P}_k : \mathbb{R}^d \to \mathbb{R}$ defined as follows:

$$\begin{aligned}
\mathcal{P}_k(h_{11}^{0k}, h_{22}^{0k}, \dots, h_{d-1d-1}^{0k}, h_{dd}^{0k}) = &\sum_{i=1}^{d-1} d(h_{ii}^{0k})^2 + \frac{d-1}{2}(h_{dd}^{0k})^2\\
&- 4\sum_{1\le i<j\le m} h_{ii}^{0k}h_{jj}^{0k}. \qquad (13)
\end{aligned}$$

Furthermore, we delve into the constrained extremum problem of minimizing $\mathcal{P}_k$ under the constraint of

$$\overline{\mathcal{N}} : \sum_{i=1}^{d} h_{ii}^{0k} = a^k,$$

where $a^k$ is real constant. From (13), it becomes evident that the critical points

$$h^{0c} = (h_{11}^{0k}, h_{22}^{0k}, \dots, h_{d-1d-1}^{0k}, h_{dd}^{0k})$$

of $\overline{\mathcal{N}}$ are derived from the solutions of the subsequent system of linear homogeneous equations:

$$\begin{aligned} \frac{\partial \mathcal{P}_k}{\partial h_{ii}^{0k}} &= 2(d+2)(h_{ii}^{0k}) - 4\textstyle\sum_{r=1}^{d} h_{rr}^{0k} = 0 \\ \frac{\partial \mathcal{P}_k}{\partial h_{dd}^{0k}} &= (d-1)h_{dd}^{0k} - 4\textstyle\sum_{r=1}^{d-1} h_{rr}^{0k} = 0, \end{aligned} \tag{14}$$

for $i \in \{1, 2, \dots, d-1\}$ and $k \in \{1, \dots, p\}$.

Consequently, each solution denoted as $h^{0c}$ possesses:

$$h_{ii}^{0k} = \frac{1}{d+1} a^k \quad \text{and} \quad h_{dd}^{0k} = \frac{4}{d+3} a^k,$$

for $i \in \{1, 2, \dots, d-1\}$ and $k \in \{1, \dots, p\}$.

Here we fix $x \in \overline{\mathcal{N}}$, then the bilinear form $\mathbf{B} : T_x\overline{\mathcal{N}} \times T_x\overline{\mathcal{N}} \to \mathbb{R}$ is expressed as follows:

$$\mathbf{B}(U_1, V_1) = Hess_{\mathcal{P}_k}(U_1, V_1) + \langle h^*(U_1, V_1), \Big(grad(\mathcal{P}_k)\Big)(x) \rangle, \tag{15}$$

where $h^*$ signifies the second fundamental form of $\overline{\mathcal{N}}$ in $\mathbb{R}^d$, and $\langle \cdot, \cdot \rangle$ represents the standard inner product on $\mathbb{R}^d$. The Hessian matrix of $\mathcal{P}_k$ is:

$$Hess_{\mathcal{P}_k} = \begin{pmatrix} 2(d+2) & -4 & \dots & -4 & -4 \\ -4 & 2(d+2) & \dots & -4 & -4 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -4 & -4 & \dots & 2(d+2) & -4 \\ -4 & -4 & \dots & -4 & (d-1) \end{pmatrix}.$$

Consider a vector denoted as $U_1$ belonging to the tangent space $T_x\overline{\mathcal{N}}$, such that the condition $\sum_{i=1}^{d} U_{1i} = 0$ holds. Given that the hyperplane is totally geodesic in the space $\mathbb{R}^d$, it follows that

$$\begin{aligned} \mathbf{B}(U_1, U_1) = Hess_{\mathcal{P}_k}(U_1, U_1) &= 2(d+2)\sum_{i=1}^{d-1} U_{1i}^2 + (d-1)U_{1d}^2 - 8\sum_{i \neq j=1}^{d} U_{1i}U_{1j} \\ &= 2(d+2)\sum_{i=1}^{d-1} U_{1i}^2 + (d-1)U_{1d}^2 - 4\Big(\Big(\sum_{i=1}^{d} U_{1i}\Big)^2 - \sum_{i=1}^{d} U_{1i}^2\Big) \\ &= 2(d+4)\sum_{i=1}^{d-1} U_{1i}^2 + (d+3)U_{1d}^2 \geq 0. \end{aligned}$$

Nonetheless, $h^{0c}$ stands as the sole optimal solution, representing the global minimum point of the problem, and it attains a minimum value of $\mathbb{Q}(h^{0c}) = 0$ when evaluated for $h^{0c}$ that satisfies the system (14).

Consequently, it can be inferred that $\mathbb{Q} \geq 0$.

$$\begin{aligned}2scal^{\nabla,\nabla'} \leq &\frac{1}{2}d(d-1)C^0 + \frac{1}{2}(d-1)(d+1)C^0(W) + \frac{d}{2}(C+C') + \frac{3\bar{c}}{4\ell^2}\|\phi\|^2 \\ &+ d(d-1)\Big(\frac{\bar{c}}{4\ell^2} - \frac{(\ell')^2}{\ell^2}\Big) - 2(d-1)\Big(\frac{\bar{c}}{4\ell^2} - \frac{(\ell')^2}{\ell^2} + \frac{\ell''}{\ell}\Big)\|V\|^2 \\ &- \frac{d^2}{2}\Big(\|\mathcal{H}\|^2 + \|\mathcal{H}'\|^2\Big).\end{aligned}$$

The mathematical expression for the normalized statistical scalar curvature $nor(scal)^{\nabla,\nabla'}$ of $N$ is given by

$$nor(scal)^{\nabla,\nabla'} = \frac{2scal^{\nabla,\nabla'}}{d(d-1)}$$

and hence, we get

$$\begin{aligned}nor(scal)^{\nabla,\nabla'} \leq &\frac{1}{2}C^0 + \frac{d+1}{2d}C^0(W) + \frac{1}{2(d-1)}\Big(C+C'\Big) + \frac{3\bar{c}}{4d(d-1)\ell^2}\|\phi\|^2 \\ &+ \Big(\frac{\bar{c}}{4\ell^2} - \frac{(\ell')^2}{\ell^2}\Big) - \frac{2}{d}\Big(\frac{\bar{c}}{4\ell^2} - \frac{(\ell')^2}{\ell^2} + \frac{\ell''}{\ell}\Big)\|V\|^2 \\ &- \frac{d}{2(d-1)}\Big(\|\mathcal{H}\|^2 + \|\mathcal{H}'\|^2\Big).\end{aligned}$$

By considering the infimum across all tangent hyperplanes $W$, our inequality (10) is substantiated. □

Crucially, the inequality

$$\begin{aligned}nor(scal)^{\nabla,\nabla'} \leq &2\widehat{\delta^0_C} + \frac{1}{d-1}C^0 + \Big(\frac{\bar{c}}{4\ell^2} - \frac{(\ell')^2}{\ell^2}\Big) + \frac{3\bar{c}}{4d(d-1)\ell^2}\|\phi\|^2 \\ &- \frac{2}{d}\Big(\frac{\bar{c}}{4\ell^2} - \frac{(\ell')^2}{\ell^2} + \frac{\ell''}{\ell}\Big)\|V\|^2 - \frac{d}{2(d-1)}\Big(\|\mathcal{H}\|^2 + \|\mathcal{H}'\|^2\Big).\end{aligned} \tag{16}$$

in relation to $\widehat{\delta^0_C}$ can be deduced by analyzing the subsequent polynomial:

$$\begin{aligned}Q = &2d(d-1)C^0 - \frac{1}{2}(d-1)(d+1)C^0(W) + \frac{d}{2}(C+C') + \frac{3\bar{c}}{4\ell^2}\|\phi\|^2 \\ &+ d(d-1)\Big(\frac{\bar{c}}{4\ell^2} - \frac{(\ell')^2}{\ell^2}\Big) - 2(d-1)\Big(\frac{\bar{c}}{4\ell^2} - \frac{(\ell')^2}{\ell^2} + \frac{\ell''}{\ell}\Big)\|V\|^2 \\ &- \frac{d^2}{2}\Big(\|\mathcal{H}\|^2 + \|\mathcal{H}'\|^2\Big) - 2scal^{\nabla,\nabla'}\end{aligned} \tag{17}$$

The remainder of the process follows a similar pattern to the proof of Theorem 3.1.

**Theorem 3.2.** *Let $N$ be a $d$-dimensional statistical submanifold of $\mathbb{R} \times_\ell \overline{M}(\bar{c})$. Subsequently, the inequalities (10) and (16) exhibit equality if and only if the condition $h = -h'$ is satisfied.*

*Proof.* The equations (10) and (16) exhibit equality if and only if $h^0 = 0$, leading to the conclusion that $h = -h'$. This implies a partial assertion that $h$ and $h'$ are linearly dependent. □

When $\phi(TN) = T^\perp N$, the submanifold $N$ is referred to as a Legendrian submanifold. In a specific scenario, when $n = d$, an immediate application of this concept leads us to the following outcome, relying on Theorem 3.1:

**Corollary 3.3.** *Let $N$ be a $d$-dimensional Legendrian submanifold of a statistical warped product manifold of the form $\mathbb{R} \times_\ell \overline{M}^{2d}(\bar{c})$. Then we have*

$$nor(scal)^{\nabla,\nabla'} \leq 2\delta_C^0 + \frac{1}{d-1}C^0 + \Big(\frac{\bar{c}}{4\ell^2} - \frac{(\ell')^2}{\ell^2}\Big) - \frac{d}{2(d-1)}\Big(\|\mathcal{H}\|^2 + \|\mathcal{H}'\|^2\Big), \tag{18}$$

*Moreover, the equality holds in (18) if and only if $h = -h'$ holds.*

We notice that the submanifolds wherein the equality condition of the Casorati curvature inequalities hold true at each point are referred to as *Casorati ideal submanifolds* [8]. Hence, we are able to formulate the subsequent statement:

**Theorem 3.4.** *Let $N$ be a $d$-dimensional Legendrian Casorati ideal submanifold of a statistical warped product manifold of the form $M^{2d+1} = \mathbb{R} \times_\ell \overline{M}^{2d}(\bar{c})$ for (18). Then it is a totally geodesic (in regard to LCC) Legendrian submanifold.*

## 4. Main inequality 2

In the original work by Chen [6], he established relationships that involved the sectional curvature, scalar curvature, and squared norm of the mean curvature for a submanifold embedded in a real space form. Moreover, he derived inequalities that connected the $k$-Ricci curvature, squared mean curvature, and shape operator for submanifolds existing in real space forms with different codimensions. Since that time, numerous geometers have investigated similar inequalities applicable to diverse submanifolds and ambient spaces.

By employing optimization techniques within the framework of Riemannian geometry, T. Oprea [26] derived the Chen-Ricci inequality for a submanifold within a real space form. More recently, Siddiqui et al. [34] delved into the Chen-Ricci inequality for a submanifold within a Kenmotsu statistical manifold characterized by a constant $\phi$-sectional curvature, employing optimization techniques. As a result, a significant application of Theorem 1.1 manifests in the following context:

**Theorem 4.1.** *Let $N$ be a $d$-dimensional statistical submanifold of $\mathbb{R} \times_\ell \overline{M}(\bar{c})$. Then, for each unit vector $U_1 \in T_\wp N$, $\wp \in N$, we have*

$$\begin{aligned} Ric^{\nabla,\nabla'}(U_1) \geq & 2Ric^0(U_1) - \frac{d^2}{8}[\|\mathcal{H}\|^2 + \|\mathcal{H}'\|^2] - \Big[\Big(\frac{\bar{c}}{4\ell^2} - \frac{(\ell')^2}{\ell^2}\Big)(d-1) \\ & + \frac{3\bar{c}}{4\ell^2}\|\phi U_1\|^2 + \Big(\frac{\bar{c}}{4\ell^2} - \frac{(\ell')^2}{\ell^2} + \frac{\ell''}{\ell}\Big)\Big((2-d)G^2(U_1, \mathrm{V}) - \|\mathrm{V}\|^2\Big)\Big], \end{aligned} \tag{19}$$

*where $\nabla, \nabla'$ and $Ric^0$ stand for, respectively, the statistical Ricci curvature and Ricci curvature in regard to LCC. Furthermore, equality is valid in (19) if and only if*

$$h(U_1, U_1) = \frac{d}{2}\mathcal{H}(\wp) \quad and \quad h(U_1, V_1) = 0, \tag{20}$$

$$h'(U_1, U_1) = \frac{d}{2}\mathcal{H}'(\wp) \quad and \quad h'(U_1, V_1) = 0, \tag{21}$$

*for all $V_1 \in T_\wp N$ orthogonal to $U_1$.*

*Proof.* We select $\{v_1, \ldots, v_d\}$ as the orthonormal frame for $T_\wp N$, with the condition that $v_1 = U_1$ and $\|U_1\| = 1$. Additionally, $\{\tilde{e}1, \ldots, \tilde{e}p\}$ is chosen as the orthonormal frame for $T_\wp N$. Utilizing equations (9) and (6), we arrive at

$$\begin{aligned}\sum_{i=2}^{d}\overline{\mathcal{S}}(v_1,v_i,v_1,v_i) =&\Big(\frac{\bar{c}}{4\ell^2}-\frac{(\ell')^2}{\ell^2}\Big)(d-1)-\Big(\frac{\bar{c}}{4\ell^2}-\frac{(\ell')^2}{\ell^2}\frac{\ell''}{\ell}\Big)\\ &+\Big(\sum_{i=2}^{d}g(v_i,v_i)G^2(U_1,\mathrm{V})+\sum_{i=1}^{d}G^2(\mathrm{V},v_i)-G^2(U_1,\mathrm{V})\Big)\\ &+\frac{3\bar{c}}{4\ell^2}\|\phi U_1\|^2\\ &=\Big(\frac{\bar{c}}{4\ell^2}-\frac{(\ell')^2}{\ell^2}\Big)(d-1)+\frac{3\bar{c}}{4\ell^2}\|\phi U_1\|^2\\ &+\Big(\frac{\bar{c}}{4\ell^2}-\frac{(\ell')^2}{\ell^2}+\frac{\ell''}{\ell}\Big)\Big((2-d)G^2(U_1,\mathrm{V})-\|\mathrm{V}\|^2\Big).\end{aligned}\tag{22}$$

By utilizing (5), the dual of (5), and (6), we derive

$$\begin{aligned}2\overline{\mathcal{S}}(v_1,v_i,v_1,v_i) =&2\mathcal{S}(v_1,v_i,v_1,v_i)-g(h(v_1,v_1),h'(v_i,v_i))-g(h'(v_1,v_1),h(v_i,v_i))\\ &+2g(h(v_1,v_i),h'(v_1,v_i))\\ &=2\mathcal{S}(v_1,v_i,v_1,v_i)-\Big\{4g(h^0(v_1,v_1),h^0(v_i,v_i))-g(h(v_1,v_1),h(v_i,v_i))\\ &-g(h'(v_1,v_1),h'(v_i,v_i))-4g(h^0(v_1,v_i),h^0(v_1,v_i))\\ &+g(h(v_1,v_i),h(v_1,v_i))+g(h'(v_1,v_i),h'(v_1,v_i))\Big\}\\ &=2\mathcal{S}(v_1,v_i,v_1,v_i)-4\sum_{k=1}^{p}(h^{0k}_{11}h^{0k}_{ii}-(h^{0k}_{1i})^2)\\ &+\sum_{k=1}^{p}(h^{k}_{11}h^{k}_{ii}-(h^{k}_{1i})^2)+\sum_{k=1}^{p}(h'^{k}_{11}h'^{k}_{ii}-(h'^{k}_{1i})^2).\end{aligned}$$

By summing over $2\le i\le d$ and employing (22), we obtain

$$\begin{aligned}2\Big[\Big(\frac{\bar{c}}{4\ell^2}-\frac{(\ell')^2}{\ell^2}\Big)(d-1)+\frac{3\bar{c}}{4\ell^2}\|\phi U_1\|^2+\Big(\frac{\bar{c}}{4\ell^2}-\frac{(\ell')^2}{\ell^2}+\frac{\ell''}{\ell}\Big)((2-d)G^2(U_1,\mathrm{V})-\|\mathrm{V}\|^2)\Big]&\\ =2Ric^{\nabla,\nabla'}(U_1)-4\sum_{k=1}^{p}\sum_{i=2}^{d}(h^{0k}_{11}h^{0k}_{ii}-(h^{0k}_{1i})^2)&\\ +\sum_{k=1}^{p}\sum_{i=2}^{d}(h^{k}_{11}h^{k}_{ii}-(h^{k}_{1i})^2)+\sum_{k=1}^{p}\sum_{i=2}^{d}(h'^{k}_{11}h'^{k}_{ii}-(h'^{k}_{1i})^2),&\end{aligned}$$

where $Ric^{\nabla,\nabla'}(U_1)$ indicates the statistical Ricci curvature of $N$ in regard to $\nabla$ and $\nabla'$ at $\wp$.

Moreover, we deduce

$$
\begin{aligned}
2Ric^{\nabla,\nabla'}(U_1) - 2\Big[&\Big(\frac{\bar{c}}{4\ell^2} - \frac{(\ell')^2}{\ell^2}\Big)(d-1) + \frac{3\bar{c}}{4\ell^2}\|\phi U_1\|^2 \\
&+ \Big(\frac{\bar{c}}{4\ell^2} - \frac{(\ell')^2}{\ell^2} + \frac{\ell''}{\ell}\Big)((2-d)G^2(U_1, \mathrm{V}) - \|\mathrm{V}\|^2)\Big] \\
&= 4\sum_{k=1}^{p}\sum_{i=2}^{d}(h_{11}^{0k}h_{ii}^{0k} - (h_{1i}^{0k})^2) - \sum_{k=1}^{p}\sum_{i=2}^{d}(h_{11}^{k}h_{ii}^{k} - (h_{1i}^{k})^2) \\
&- \sum_{k=1}^{p}\sum_{i=2}^{d}(h_{11}'^{k}h_{ii}'^{k} - (h_{1i}'^{k})^2).
\end{aligned} \tag{23}
$$

According to the Gauss equation in regard to $LCC$, it can be inferred that

$$
\begin{aligned}
Ric^{0}(U_1) - \Big[&\Big(\frac{\bar{c}}{4\ell^2} - \frac{(\ell')^2}{\ell^2}\Big)(d-1) + \frac{3\bar{c}}{4\ell^2}\|\phi U_1\|^2 \\
&+ \Big(\frac{\bar{c}}{4\ell^2} - \frac{(\ell')^2}{\ell^2} + \frac{\ell''}{\ell}\Big)((2-d)G^2(U_1, \mathrm{V}) - \|\mathrm{V}\|^2)\Big] \\
&= \sum_{k=1}^{p}\sum_{i=2}^{d}(h_{11}^{0k}h_{ii}^{0k} - (h_{1i}^{0k})^2).
\end{aligned}
$$

Now, upon substituting this into (23), we reach

$$
\begin{aligned}
-2Ric^{\nabla,\nabla'}(U_1) - 2\Big[&\Big(\frac{\bar{c}}{4\ell^2} - \frac{(\ell')^2}{\ell^2}\Big)(d-1) + \frac{3\bar{c}}{4\ell^2}\|\phi U_1\|^2 \\
&+ \Big(\frac{\bar{c}}{4\ell^2} - \frac{(\ell')^2}{\ell^2} + \frac{\ell''}{\ell}\Big)((2-d)G^2(U_1, \mathrm{V}) - \|\mathrm{V}\|^2)\Big] \\
&+ 4Ric^{0}(U_1) \\
&= \sum_{k=1}^{p}\sum_{i=2}^{d}(h_{11}^{k}h_{ii}^{k} - (h_{1i}^{k})^2) + \sum_{k=1}^{p}\sum_{i=2}^{d}(h_{11}'^{k}h_{ii}'^{k} - (h_{1i}'^{k})^2)
\end{aligned} \tag{24}
$$

$$
\le \sum_{k=1}^{p}\sum_{i=2}^{d}h_{11}^{k}h_{ii}^{k} + \sum_{k=1}^{p}\sum_{i=2}^{d}h_{11}'^{k}h_{ii}'^{k}. \tag{25}
$$

In this context, we introduce two quadratic forms denoted as $\mathcal{P}_k$ and $\mathcal{P}'_k$, which map from $\mathbb{R}^d$ to $\mathbb{R}$, defined as follows:

$$
\mathcal{P}_k(h_{11}^k, h_{22}^k, \ldots, h_{dd}^k) = \sum_{k=1}^{p}\sum_{i=2}^{d}h_{11}^{k}h_{ii}^{k},
$$

and

$$
\mathcal{P}'_k(h_{11}'^k, h_{22}'^k, \ldots, h_{dd}'^k) = \sum_{k=1}^{p}\sum_{i=2}^{d}h_{11}'^{k}h_{ii}'^{k},
$$

Again we follow the same steps followed in proving Theorem 3.1, but here we consider the constrained extremum problem as $\max\mathcal{P}_k$ subject to

$$
\overline{\mathcal{N}} : \sum_{i=1}^{d}h_{ii}^{k} = a^k.
$$

Considering an optimal solution $h^{0c} = (h^k_{11}, h^k_{22}, \dots h^k_{dd})$ for the given problem, the vector $grad(\mathcal{P}k) = (\sum_{i=2}^{d} h^k_{ii}, h^k_{11}, h^k_{11}, \dots, h^k_{11})$ is orthogonal to $\overline{\mathcal{N}}$ at $h^{0c}$. This implies that

$$h^k_{11} = \sum_{i=2}^{d} h^k_{ii} = \frac{a^k}{2}.$$

The bilinear form $\mathbf{B} : T_x\overline{\mathcal{N}} \times T_x\overline{\mathcal{N}} \to \mathbb{R}$ has the expression (15). Then, here we have

$$Hess_{\mathcal{P}_k} = \begin{pmatrix} 0 & 1 & \dots & 1 \\ 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \end{pmatrix}.$$

By considering a vector $U_1 \in T_x\overline{\mathcal{N}}$, we perform the following calculations, analogous to those conducted in Theorem 3.1:

$$\begin{aligned} \mathbf{B}(U_1, U_1) =& 2\sum_{i=2}^{d} U_{11}U_{1i} = (U_{11} + \sum_{i=2}^{d} U_{1i})^2 - (U_{11})^2 - (\sum_{i=2}^{d} U_{1i})^2 \\ =& -2(U_{11})^2 \leq 0. \end{aligned}$$

Consequently, we discover

$$\mathcal{P}_k \leq \frac{1}{4}(\sum_{i=1}^{d} h^k_{ii})^2 = \frac{d^2}{4}(\mathcal{H}^k)^2. \tag{26}$$

Likewise, for the constrained extremum problem of maximizing $\mathcal{P}'_k$ subject to the condition

$$\overline{\mathcal{N}}' : \sum_{i=1}^{d} h'^k_{ii} = a'^k,$$

using analogous reasoning as mentioned earlier, we reach

$$\mathcal{P}'_k \leq \frac{1}{4}(\sum_{i=1}^{d} h'^k_{ii})^2 = \frac{d^2}{4}(\mathcal{H}'^k)^2. \tag{27}$$

By merging equations (24), (26), and (27), we arrive at equation (19). Furthermore, $U_1$ adheres to the equality case if and only if

$$h^k_{1i} = 0, \quad h'^k_{1i} = 0, \quad i \in \{2, \dots, d\} \quad \text{and} \quad h^k_{11} = \frac{d}{2}\mathcal{H}, \quad h'^k_{11} = \frac{d}{2}\mathcal{H}'.$$

□

**Corollary 4.2.** *Let $N$ be a $d$-dimensional anti-invariant submanifold of $\mathbb{R} \times_\ell \overline{M}(\bar{c})$. Then, for each unit vector $U_1 \in T_\wp N$, $\wp \in N$, we have:*

$$\begin{aligned} Ric^{\nabla,\nabla'}(U_1) \geq& 2Ric^0(U_1) - \frac{d^2}{8}[\|\mathcal{H}\|^2 + \|\mathcal{H}'\|^2] - \Big[\Big(\frac{\bar{c}}{4\ell^2} - \frac{(\ell')^2}{\ell^2}\Big)(d-1) \\ &+ \Big(\frac{\bar{c}}{4\ell^2} - \frac{(\ell')^2}{\ell^2} + \frac{\ell''}{\ell}\Big)\Big((2-d)G^2(U_1, \mathrm{V}) - \|\mathrm{V}\|^2\Big)\Big]. \end{aligned}$$

**Corollary 4.3.** *Let $N$ be a $d$-dimensional invariant submanifold of $\mathbb{R}\times_\ell \overline{M}(\bar{c})$. Then, for each unit vector $U_1 \in T_\wp N$, $\wp \in N$, we have*

$$Ric^{\nabla,\nabla'}(U_1) \geq 2Ric^0(U_1) - \frac{d^2}{8}[\|\mathcal{H}\|^2 + \|\mathcal{H}'\|^2] - \Big[\Big(\frac{\bar{c}}{4\ell^2} - \frac{(\ell')^2}{\ell^2}\Big)(d-1) + \frac{3\bar{c}}{4\ell^2} + \Big(\frac{\bar{c}}{4\ell^2} - \frac{(\ell')^2}{\ell^2} + \frac{\ell''}{\ell}\Big)\Big((2-d)G^2(U_1, \mathrm{V}) - \|\mathrm{V}\|^2\Big)\Big].$$

## 5. Some non-trivial examples

We present two non-trivial instances of statistical immersions into statistical warped product manifolds, which serve to illustrate the key outcomes outlined earlier.

**Example 5.1.** *Following [31], we give a non-trivial example satisfying Theorem 3.2. We consider a statistical manifold $(\mathbb{H}^3, G_{\mathbb{H}^3}, \mathcal{D}^{(-1)}, \mathcal{D}^{(+1)})$ of constant statistical sectional curvature $\bar{c} = r^2 - 1$, $r \in \mathbb{R}$ and the translation surface*

$$\Big(\overline{M}^2, G_{\overline{M}^2} = \frac{1}{y^2}((p^2+1)dx^2 + dy^2), D^{(-1)}, D^{(+1)}\Big),\ p \in \mathbb{R},\ (x,y) \in \mathbb{R}^2,\ y > 0$$

*which is a statistical submanifold of*

$$\Big(\mathbb{H}^3 = \{(u,v,t) \in \mathbb{R}^3 | t > 0\}, G_{\mathbb{H}^3} = \frac{1}{t^2}(du^2 + dv^2 + dt^2), \mathcal{D}^{(-1)}, \mathcal{D}^{(+1)}\Big).$$

*Then it becomes straightforward to prove that under the following isometric immersion, $h$ and $h'$ vanish*

$$\begin{aligned} f : \Big(N = \mathbb{R}\times_{\cosh(z)} \overline{M}^2, dz^2 + \cosh^2(z)G_{\overline{M}^2}\Big) &\to \Big(\mathbb{R}\times_{\cosh(z)} \mathbb{H}^3, G = dz^2 + \cosh^2(z)G_{\mathbb{H}^3}\Big), \\ f(z,x,y) &= (z, x, ax+b, y),\ a, b \in \mathbb{R}. \end{aligned}$$

*This means that $N$ is a totally geodesic submanifold of $\mathbb{R}\times_{\cosh(z)} \mathbb{H}^3$ in regard to the induced LCC.*

**Example 5.2.** *In the work presented in [27], certain non-trivial examples of statistical warped product manifolds are elaborated upon. In this reason, we leverage Example 4.9 from [27] as a non-trivial instance that fulfills the equality condition as described in Theorem 4.1. First we consider the standard statistical warped product manifold*

$$M = \mathbb{R}\times_{\cosh(\lambda t)} \mathbb{H}^{d+p-1}(\bar{c}) = \{t, y^1, \ldots, y^{d+p-1} \in \mathbb{R}^{d+p} | y^{d+p-1} > 0\},$$

*where*

$$\bar{c} = \kappa(r^2-1),\ \lambda, r \in \mathbb{R},\ \kappa > 0,\ \ m \geq 2,\ \ p \geq 1.$$

*For constants $(a^1, \ldots, a^p) \in \mathbb{R}^p$, we have the submanifold $N$ of $M$ as*

$$N = \{(a^1, \ldots, a^p, x^1, \ldots, x^d) \in M | (x^1, \ldots, x^d) \in \mathbb{R}^{d-1}\times\mathbb{R}^+\}.$$

*The second fundamental forms in regard to dual connections are given by*

$$h(v_i, v_j) = h'(v_i, v_j) = -\delta_{ij}\lambda\tanh(\lambda a^1)\tilde{e}_1,$$

*where*

$$\{v_i = \sqrt{\kappa}x^d\cosh^{-1}(\lambda a^1)\frac{\partial}{\partial y^{i+p-1}} | i = 1, \ldots, d\}$$

*and*

$$\{\tilde{e}_1 = \frac{\partial}{\partial t}, \tilde{e}_k = \sqrt{\kappa}x^d\cosh^{-1}(\lambda a^1)\frac{\partial}{\partial y^{k-1}} | k = 2, \ldots, p\}$$

*are the adopted orthonormal frames. Therefore, the equality condition of Theorem 4.1 is fulfilled at each point of $N$ if and only if either $d \geq 3$ and $\lambda a^1 = 0$, or if $d = 2$.*

## 6. Closing thoughts and observations

Curvature is a measure of how a curve or surface deviates from being straight or flat and we know that the curvature invariants are widely used in the field of physics and in differential geometry also. The mean curvature vector is a mathematical concept used in the field of differential geometry to describe the curvature of a surface at a given point. It is important in various areas of mathematics and physics, including computer graphics, and materials science, where it is used to analyze the behavior of surfaces and their interactions with light, fluids, and other forces. On the other hand, the Ricci curvature plays a significant role in the study of general relativity, where the curvature of spacetime is described by the Einstein field equations. It also has applications in various fields, such as geometry, topology, and mathematical physics. Extensively explored in the field of differential geometry, this concept offers a means to quantify the extent to which the geometry defined by a given Riemannian metric may deviate from the standard Euclidean $n$-space. In a more intuitive sense, the Ricci curvature at a point on a manifold provides information about the concentration or divergence of geodesics (the analogs of straight lines) starting from that point. Positive Ricci curvature indicates that nearby geodesics tend to converge, implying a concentration of volume. Negative Ricci curvature indicates that geodesics tend to diverge, leading to a spread-out behavior of nearby points. Ricci curvature equal to zero suggests that volume preservation is maintained along geodesics. Given the importance of regulating extrinsic quantities in relation to intrinsic ones, establishing elementary correlations between extrinsic and intrinsic curvature invariants is a fundamental pursuit in contemporary Riemannian geometry. For instance, deriving lower bounds on the Ricci tensor within a Riemannian manifold provides a means to extract global geometric and topological insights by contrasting with the geometry of a constant curvature space form. Within this paper, we have demonstrated a range of optimal inequalities that involve fundamental curvature invariants for statistical submanifolds within a statistical warped product manifold of the type $\mathbb{R} \times_{\ell} \overline{M}(\bar{c})$. Furthermore, we have explored the instances where these inequalities attain equality. Through a meticulous analysis of the terms within the primary inequalities established above, it becomes evident that the simplest intrinsic curvature invariant is bounded from above by means of certain basic extrinsic curvature invariants. Concluding with examples (see [27, 31]) that illustrate the cases of equality in the main inequalities, we have demonstrated the feasibility of achieving these states. Looking ahead to further research, a fascinating avenue would involve generating novel or akin types of inequalities, or exploring various outcomes for diverse categories of statistical submersions (for example see [28, 30, 35]).

## Acknowledgments

The second author, Fatemah Mofarreh, expresses her gratitude to Princess Nourah bint Abdulrahman University Researchers Supporting Project No. PNURSP2024R27, Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia. The other authors would like to express their gratitude to the Deanship of Scientific Research at King Khalid University, Saudi Arabia for providing funding for this work through the General Research Project under grant number GRP/239/44.

## References

[1] Amari, S., *Differential-Geometrical Methods in Statistics. Lecture Notes in Statistics*, Springer-Verlag: New York, USA, **28** (1985).

[2] Aydin, M.E., Mihai, A., Mihai, I., *Some Inequalities on Submanifolds in Statistical Manifolds of Constant Curvature*. Filomat **29(3)**, 465-477 (2015).

[3] Bishop, R.L., O'Neill, B., *Manifolds of negative curvature*. Trans. Amer. Math. Soc. 145, 1–49 (1969).

[4] Bahadir, O.,Tripathi, M.M., *Geometry of lightlike hypersurfaces of a statistical manifold*, arXiv:1901.09251[math.DG] (2019).

[5] Casorati, F., Mesure de la courbure des surfaces suivant l'idée commune, Acta Math. **14**, 95–110 (1890).

[6] Chen, B.-Y., *Relationship between Ricci curvature and shape operator for submanifolds with arbitrary codimensions*. Glasg. Math. J. **41**, 33-41 (1999).

[7] Chen, B.-Y., *Differential geometry of warped product manifolds and submanifolds*, Worlds Scientific, Hackensack, New Jersey, (2017).

[8] Decu, S., Haesen, S., Verstraelen, L., *Optimal inequalities involving Casorati curvatures*. Bull. Transylv. Univ. Brasov, Ser B. **49** 85-93 (2007).

[9] Decu, S., Haesen, S., Verstraelen, L., Vilcu, G.E., *Curvature Invariants of Statistical Submanifolds in Kenmotsu Statistical Manifolds of Constant $\phi-$Sectional Curvature*. Entropy **20**, 529; DOI:10.3390/e20070529 (2018).

[10] Erken, I.K., Murathan, C., Yazla, A. *Almost cosymplectic statistical manifolds*. Quaestiones Mathematicae, **43(2)**, 265-282 (2020).

[11] Furuhata, H., *Hypersurfaces in statistical manifolds*. Diff. Geom. Appl. **27**, 420-429 (2009).

[12] Furuhata, H., Hasegawa, I., *Submanifold theory in holomorphic statistical manifolds*. In: Dragomir, S., Shahid, M.H., Al-Solamy, F.R. (eds.) Geometry of Cauchy-Riemann Submanifolds, pp. 179–215, Springer, Singapore (2016).

[13] Furuhata, H., Hasegawa, I., Okuyama, Y., Sato, K., Shahid, M. H., *Sasakian statistical manifolds*, J. Geom. Phys., **117**, 179-186 (2017).

[14] Furuhata, H., Hasegawa, I., Okuyama, Y., Sato, K., Kenmotsu statistical manifolds and warped product. J. Geom. DOI 10.1007/s00022-017-0403-1 (2017).

[15] Gorunus, R., Erken, I.K., Yazla, A., Murathan, C., *Generalized Wintgen inequality for Legendrian submanifolds in almost Kenmotsu manifolds*. Inter. Elec. J. Geom., **12(1)**, 43-56 (2019).

[16] Hulya, A., Cihan, O., *Einstein statistical warped product manifolds*. Filomat **32 (11)**, 3891-3897 (2018).

[17] Kowalczyk, D., Casorati curvatures. Bull. Transilvania Univ. Brasov Ser. III. **50(1)**, 2009-2013 (2008).

[18] Lauritzen, S.: In statistical manifolds. In: Amari, S., Barndorff-Nielsen, O., Kass, R., Lauritzen, S., Rao, C.R. (eds.) Differential Geometry in Statistical Inference, **10**, pp. 163–216. IMS Lecture Notes Institute of Mathematical Statistics, Hayward (1987).

[19] Lee, C. W., Yoon, D. W., Lee, J. W., *A pinching theorem for statistical manifolds with Casorati curvatures*. J. Non. Sci. Appl., **10**, 4908–4914 (2017).

[20] Mihai, A., Mihai, I., *Curvature Invariants for Statistical Submanifolds of Hessian Manifolds of Constant Hessian Curvature*. Mathematics **44(6)** (2018).

[21] Murathan, C., Sahin, B., *A study of Wintgen like inequality for submanifolds in statistical warped product manifolds*. J. Geom. **109(30)** (2018).

[22] Nguiffo Boyom, M., Siddiqui, A.N., Mior Othman, W.A., Shahid, M.H., *Classification of totally umbilical CR-statistical submanifolds in holomorphic statistical manifolds with constant holomorphic curvature*. In: Nielsen F., Barbaresco F. (eds) Geometric Science of Information. GSI 2017. Lecture Notes in Computer Science, **10589**. Springer, Cham

[23] Opozda, B., *Bochner's technique for statistical structures*. Ann. Glob. Anal. Geom. **48(4)**, 357-395 (2015).

[24] Opozda, B., *A sectional curvature for statistical structures*. Linear Algebra Appl. **497**, 134-161 (2016).

[25] Oprea, T., *Chen's inequality in the Lagrangian case*. Colloq. Math. **108** 163-169 (2007).

[26] Oprea, T., *On a geometric inequality*. arXiv:math/0511088v1 [math.DG] 3 Nov 2005.

[27] Satoh, N., Furuhata, H., Hasegawa, I., Nakane, T., Okuyama, Y., Sato, K., Shahid, M.H., Siddiqui, A.N., *Statistical submanifolds from a viewpoint of the Euler inequality*, Info. Geo. **4**, 189-213 (2021).

[28] Siddiqui, A.N., Alkhaldi, A.H., Siddiqi, M.D., Ali, A., *Lower Bounds on Statistical Submersions with Vertical Casorati Curvatures*. Intern. J. Geom. Meth. Mod. Phys. **19(3)** (2022).

[29] Siddiqui, A.N., Al-Solamy, F.R., Shahid, M.H., Mihai, I., *On CR-Statistical Submanifolds of Holomorphic Statistical Manifolds*. Filomat **35(11)**, 3571–3584 (2021).

[30] Siddiqui, A.N., Chen, B.-Y., Siddiqi, M.D., *Chen inequalities for statistical submersions between statistical manifolds*, Intern. J. Geom. Meth. Mod. Phys. **18(40)**, pp. 17 (2021).

[31] Siddiqui, A.N., Murathan, C., Siddiqi, M.D., *The Chen's first inequality for submanifolds of statistical warped product manifolds*. J. Geom. Phys. **169**, 1-13 (2021).

[32] Siddiqui, A.N., Shahid, M.H., *On totally real statistical submanifolds*. Filomat **32(13)**, 4473-4483 (2018).

[33] Siddiqui, A.N., Shahid, M.H., *Optimizations on statistical hypersurfaces with Casorati curvatures*. Kragujevac J. Math. **45(3)**, 449–463 (2021).

[34] Siddiqui, A.N., Suh, Y.J., Bahadir, O., *Extremities for statistical submanifolds in Kenmotsu statistical manifolds*. Filomat **35(2)**, 591-603 (2021).

[35] Siddiqui, A.N., Uddin, S., Shahid, M.H., *B.-Y. Chen's inequalities for Kähler-like statistical submersions*. Intern. Elec. J. Geom. **15(2)**, 278-287 (2022).

[36] Takano, K., *Statistical manifolds with almost complex structures and its statistical submersions*. Tensor N.S. **65**, 128-142 (2004).

[37] Takano, K., *Statistical manifolds with almost contact structures and its statistical submersions*. J. Geom. **85**, 171-187 (2006).

[38] Todjihounde, L., *Dualistic structures on warped product manifolds*. Diff. Geom. Dyn. Syst. **8**, 278-284 (2006).

[39] Verstraelen, L., *Geometry of submanifolds I, The first Casorati curvature indicatrices*. Kragujevac J. Math. **37** 5-23 (2013).

[40] Vilcu, A.D., Vilcu, G.E., Statistical manifolds with almost quaternionic structures and quaternionic Kähler-like statistical submersions. Entropy **17**, 6213-6228 (2015).

[41] Vos, P. W., *Fundamental equations for statistical submanifolds with applications to the Bartlett correction*. Ann. Inst. Stat. Math. **41(3)** 429-450 (1989).

[42] Yano, K., Kon, M., *Structures on manifolds*. Worlds Scientific, Singapore (1984).