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General improved Chen's inequality for Warped product bi-slant submanifolds in Kenmotsu manifolds

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Abstract. Recently, B.Y. Chen established a relationship for the squared norm of the second fundamental form (an extrinsic invariant) of warped product bi-slant submanifolds of Kenmotsu manifolds in terms of the warping function (an intrinsic invariant). In this paper, we study warped product bi-slant submanfolds of Kenmotsu manifolds. We obtain a sufficient necessary condition that a bi-slant submanifold is locally a warped product and prove a characterisation theorem for such submanifolds. Finally, we establish a general improved Chen type of inequality. The equality case is also investigated.

1. Introduction

In [6], Bishop and O'Neill initiated the concept of the warped product manifolds in their study of manifolds of negative curvature. Warped product submanifolds have been studied rapidly since Chen introduced the notion of CR-warped product of Kaehler manifolds in his series papers [10, 11]. Moreover, Chen [12] established a general sharp geometric inequality for the squared norm of the second fundamental form for different warped product submanifolds of different ambient manifolds, which embodies the relationship between the main extrinsic invariant (the second fundamental form) and an intrinsic invariant (the warping function). Motivated by his work, many distinguished geometers extended and improved the Chen's type inequality [1, 16, 17, 19, 21, 24].

On the other hand, J.L. Cabrerizro et al. introduced the notion of bi-slant submanifolds of almost contact metric manifolds as a generalization of contact CR-submanifolds, semi-slant submanifolds and pseudo-slant submanifolds in [8]. S. Uddin and B.Y. Chen investigated warped product bi-slant submanifolds in Kaehler manifolds in [26], They proved the non-existence of warped product bi-slant submanifolds of Kaehler manifolds, and the non-existence of such submanifolds for cosymplectic manifolds was proved in [2]. Also, many researchers extended the above special type of submanifolds in some different structure manifolds (see [18], [20], [22], [25], [27], [4], [5], [3]), they have given many non-trivial examples and proved several interesting results including characterisation theorems and inequalities. Recently, the idea has been extended for CR-slant warped products and bi-warped products (see [12],[14], [21], [26], [23]).

The paper is organised as follows: In section 2, we recall some basic formulas and definitions, which are useful to the next section. Section 3 is devoted to the study of bi-slant submanifolds of almost contact

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metric manifolds and provide some basic results which are useful to the next section. In section 4, we study warped product bi-slant submanfolds of Kenmotsu manifolds. In the beginning of this section, we give an example of such submanifolds to prove its existence and then obtain a sufficient necessary condition for that a bi-slant submanifold to be a warped product. Finally, we prove a characterisation theorem for such submanifolds. The last section 5 is devoted to establish a general improved Chen's inequality for warped product bi-slant submanifolds in Kenmotsu manifolds. Also, the equality is also discussed in details.

2. Preliminaries

Let $\widetilde{M}$ be a $(2n+1)$ dimensional Riemannian manifold, then $\widetilde{M}$ is said to be an almost contact metric manifold if it admits an almost contact metric structure (φ, ξ, η, g), where the endomorphism φ of its tangent bundle is a tensor field of the form $(1,1)$, ξ is a structure vector field, η is a 1-form and g is a Riemannian metric on $\widetilde{M}$ satisfying the following conditions [7]:

$$\begin{aligned} &\varphi^2 = -I + \eta \otimes \xi, \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0, \quad \eta(\xi) = 1, \\ &g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi), \end{aligned} \tag{2.1}$$

for any vector field X, Y tangent to $\widetilde{M}$, and from (2.1), we also have

$$g(\varphi X, Y) = -g(X, \varphi Y). \tag{2.2}$$

An almost contact metric manifold $\widetilde{M}$ is said to be a Kenmotsu manifold [15], if the relation

$$\left(\widetilde{\nabla}_X \varphi\right) Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X \tag{2.3}$$

holds, where $\widetilde{\nabla}$ is the Levi-Civita connection of g. From (2.1), (2.2), it can be easily derived that

$$\widetilde{\nabla}_X \xi = X - \eta(X)\xi. \tag{2.4}$$

In addition, the covariant derivative of the tensor field φ is defined by

$$\left(\widetilde{\nabla}_X \varphi\right) Y = \widetilde{\nabla}_X(\varphi Y) - \varphi\left(\widetilde{\nabla}_X Y\right), \tag{2.5}$$

for any vector field X, Y tangent to $\widetilde{M}$.

Let $\psi : (M, g) \to \left(\widetilde{M}, g\right)$ be an isometric immersion of an almost contact metric manifold. We denote by ∇ and $\nabla^\perp$ the Levi-Civita connections on the tangent bundle TM and the normal bundle $T^\perp M$, respectively, and $\widetilde{\nabla}$ the extrinsic connection of M on $\widetilde{M}$. Then the Gauss and Weingarten formulas are repectively given by

$$\begin{aligned} &\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \\ &\widetilde{\nabla}_X N = -A_N X + \nabla^\perp_X N, \end{aligned} \tag{2.6}$$

for any $X, Y \in \Gamma(TM)$ and $N \in \Gamma(T^\perp M)$. Where $\sigma : TM \times TM \to T^\perp M$ is the second fundamental form of M in $\widetilde{M}$, and A is the shape operator of the submanifold M. Moreover, σ and A are related as follows

$$g(A_N X, Y) = g(\sigma(X, Y), N). \tag{2.7}$$

We assume that $\dim(M) = m$. Let $\{e_1, \cdots e_m\}$ be a local orthonormal frame of the tangent bunble TM and $\{e_{m+1}, \cdots e_{2n+1}\}$ be a local orthonormal frame of the normal bunble $T^\perp M$. If we set $\sigma^r_{ij} = g\left(\sigma\left(e_i, e_j\right), e_r\right), i, j = 1, 2, \cdots, m, r = 1, \cdots, 2n+1$, then the squared norm of the second fundamental form is defined by

$$\|\sigma\|^2 = \sum_{i,j=1}^{m} g\left(\sigma\left(e_i, e_j\right), \sigma\left(e_i, e_j\right)\right) = \sum_{r=m+1}^{2n+1} \sum_{i,j=1}^{m} \left(g\left(\sigma\left(e_i, e_j\right), e_r\right)\right)^2. \tag{2.8}$$

Definition 2.1. *[29] A submanifold M of $\widetilde{M}$ is said to be totally geodesic if its second fundamental form σ is identically zero, while it is called totally umbilical if its second fundamental form σ satisfies*

$$\sigma(X,Y) = g(X,Y)H, \tag{2.9}$$

for each $X, Y \in \Gamma(TM)$, where H is the mean curvature vector of $H = \frac{1}{m}\sum_{i=1}^{m}\sigma(e_i, e_i)$. In addition, M is called minimal if $H = 0$.

For any $X \in \Gamma(TM)$ and $N \in \Gamma(TM^{\perp})$, an orthogonal decomposition of φX and φN yields

$$\begin{aligned} \varphi X &= TX + FX, \\ \varphi N &= tN + fN, \end{aligned} \tag{2.10}$$

where TX (resp. tN) and FX (resp. fN) denotes the tangential and normal components of φX (resp. φN), respectively. Thus, T is an endomorphism on the tangent bundle TM and F is a normal bundle valued 1-form of TM. Moreover, M is said to be invariant if $F \equiv 0$, that is, $\varphi X \in \Gamma(TM)$, and anti-invariant if $T \equiv 0$, that is, $\varphi X \in \Gamma(T^{\perp}M)$. Furthermore, from (2.2) and (2.8), we have

$$g(TX,Y) = -g(X,TY). \tag{2.11}$$

For a differentiable function f on a submanifold M of $\widetilde{M}$, the gradient ∇f is defined as

$$g(\nabla f, X) = X(f). \tag{2.12}$$

Now, an important class of submanifold under the action of φ of $\widetilde{M}$ is given.

Definition 2.2. *[9] A submanifold M of an almost contact metric manifold $\widetilde{M}$ is said to be slant, if for each non-zero vector X tangent to M at p such that X is not proportional to $\langle \xi_p \rangle$, the slant angle $\theta(X) \in \left[0, \frac{\pi}{2}\right]$ between φX and T_pM is constant, i.e., it is independent of the choice of $p \in M$ and $X \in T_pM - \langle \xi_p \rangle$.*

Obviously, if $\theta = 0$, then M is an invariant submanifold and if $\theta = \frac{\pi}{2}$, then M is an anti-invariant submanifold. A slant submanifold is said to be proper if it is neither $\theta = 0$ nor $\theta = \frac{\pi}{2}$.

Similarly, we define the slant distribution, if the differentiable distribution $\mathfrak{D}$ is a θ-slant distribution on a submanifold M of $\widetilde{M}$, then for any non-zero $X \in D$, $p \in M$, the slant angle $\theta(X_p) \in \left[0, \frac{\pi}{2}\right]$ between φX_p and $\mathfrak{D}_p$ is constant, i.e., it is independent of the choice of $p \in M$ and $X \in \mathfrak{D}$. Also, it is an invariant distribution if $\theta = 0$, and it is called anti-invariant if $\theta = \frac{\pi}{2}$. A slant distribution is said to be proper if it is neither invariant nor anti-invariant. Moreover, it was proved in [9] that a submanifold M of $\widetilde{M}$ such that ξ tangent to M, and there exists a distribution $\mathfrak{D}$ satisfies $TM = \mathfrak{D} \oplus \langle \xi \rangle$, then M is slant if and only if $\mathfrak{D}$ is a slant distribution with the same slant angle.

For the slant submanifold, the following characterizations are known.

Lemma 2.3. *Let M be a submanifold tangent to ξ of an almost contact metric manifold $\widetilde{M}$, Then M is θ-slant submanifold if and only if [9]*

$$T^2 = cos^2\theta(-I + \eta \otimes \xi). \tag{2.13}$$

The following relations are the natural consequences of (2.11) and (2.13) as

$$\begin{aligned} g(TX,TY) &= cos^2\theta\,(g(X,Y) - \eta(X)\eta(Y)), \\ g(FX,FY) &= sin^2\theta\,(g(X,Y) - \eta(X)\eta(Y)), \end{aligned} \tag{2.14}$$

for any $X, Y \in \Gamma(TM)$. Another relation for a θ-slant submanifold of $\widetilde{M}$ is obtained by using (2.1), (2.10) and (2.13) as [20]

$$\begin{aligned} tFX &= sin^2\theta(-X + \eta(X)\xi), \\ fFX &= -FTX, \end{aligned} \tag{2.15}$$

for any $X \in \Gamma(TM)$.

3. Bi-slant submanifolds

The bi-slant immersion on a almost cantact metric manifold was first introduced in [8], Cabrerizo et al. defined bi-slant submanifolds as follows:

Definition 3.1. *A submanifold M of an almost cantact mertic manifold $\left(\widetilde{M}, \varphi, \xi, \eta, g\right)$ is said to be a bi-slant submanifold if there exists a pair of orthogonal distributions $\mathfrak{D}^1$ and $\mathfrak{D}^2$ such that*

(i) The tangent bundle TM admits the orthogonal direct decomposition: $TM = \mathfrak{D}^1 \oplus \mathfrak{D}^2 \oplus \langle\xi\rangle$,
(ii) $\varphi\mathfrak{D}^1 \perp \mathfrak{D}^2$ and $\varphi\mathfrak{D}^2 \perp \mathfrak{D}^1$,
(iii) Each $\mathfrak{D}^i$ $(i = 1, 2)$ is a slant distribution with a slant angle θ_i, and the set $\{\theta_1, \theta_2\}$ is called bi-slant angle.

In particular, if the distributions $\mathfrak{D}^1$ and $\mathfrak{D}^2$ are invariant and anti-invariant with respect to φ, respectively, then we call this type of bi-slant submanifolds as CR submanifolds. If neither $\theta_i = 0$ nor $\theta_i = \frac{\pi}{2}$ $(i = 1, 2)$, then M is called proper. In addition, M is known as semi-slant if $\mathfrak{D}^1$ and $\mathfrak{D}^2$ are invariant and proper, respectively, and it is called pseudo-slant if $\mathfrak{D}^1$ and $\mathfrak{D}^2$ are anti-invariant and proper, respectively.

Let M be a bi-slant submanifold with slant angle $\{\theta_1, \theta_2\}$ of a Kenmotsu manifold $\widetilde{M}$, for any $X \in \Gamma(TM)$, we put

$$\varphi X = T_1X + T_2X + FX, \tag{3.1}$$

where T_i denotes the orthogonal projection of T on $\mathfrak{D}^i$, for any $i = 1, 2$. Then if $X \in \mathfrak{D}^i$, we obtain $\varphi X = T_iX + FX$.

Now, we give the following useful results for bi-slant submanifolds.

Lemma 3.2. *Let M be a proper bi-slant submanifold of a Kenmotsu manifold $\left(\widetilde{M}, \varphi, \xi, \eta, g\right)$ with bi-slant angle $\{\theta_1, \theta_2\}$. Then, the slant distribution $\mathfrak{D}^1 \oplus \langle\xi\rangle$ defines a totally geodesic foliation if and only if*

$$A_{FT_2X_2}X_1 - A_{FX_2}T_1X_1 + A_{FT_1X_1}X_2 - A_{FX_1}T_2X_2 \in \mathfrak{D}^2,$$

for any $X_1 \in \Gamma\left(\mathfrak{D}^1 \oplus \langle\xi\rangle\right)$, $X_2 \in \Gamma\left(\mathfrak{D}^2\right)$.

Proof. For any $X_1, Y_1 \in \Gamma\left(\mathfrak{D}^1 \oplus \langle\xi\rangle\right)$, $X_2, Y_2 \in \Gamma\left(\mathfrak{D}^2\right)$, from (2.1), (2.5) and (2.6), we have

$$g(\nabla_{Y_1}X_1, X_2) = g\left(\varphi\widetilde{\nabla}_{Y_1}X_1, \varphi X_2\right) = g\left(\widetilde{\nabla}_{Y_1}\varphi X_1, \varphi X_2\right) - g\left(\left(\widetilde{\nabla}_{Y_1}\varphi\right)X_1, \varphi X_2\right).$$

By (2.3) and (2.10), we derive

$$\begin{aligned} g(\nabla_{Y_1}X_1, X_2) &= g\left(\widetilde{\nabla}_{Y_1}\varphi X_1, \varphi X_2\right) \\ &= g\left(\widetilde{\nabla}_{Y_1}T_1X_1, T_2X_2\right) + g\left(\widetilde{\nabla}_{Y_1}FX_1, T_2X_2\right) + g\left(\widetilde{\nabla}_{Y_1}\varphi X_1, FX_2\right). \end{aligned}$$

Using the Gauss and Weingarten formulas (2.6), we obtain

$$g(\nabla_{Y_1}X_1, X_2) = g(\nabla_{Y_1}T_1X_1, T_2X_2) - g(A_{FX_1}Y_1, T_2X_2) - g\left(\widetilde{\nabla}_{Y_1}FX_2, \varphi X_1\right).$$

Then from (2.2), (2.3), (2.5), (2.10) and (2.15), we get

$$\begin{aligned} g(\nabla_{Y_1}X_1, X_2) &= g(\nabla_{Y_1}T_1X_1, T_2X_2) - g(A_{FX_1}Y_1, T_2X_2) \\ &\quad - sin^2\theta_2 g\left(\widetilde{\nabla}_{Y_1}X_2, X_1\right) - g\left(\widetilde{\nabla}_{Y_1}FTX_2, X_1\right). \end{aligned}$$

Again by using (2.6), and the symmetry of the shape operator A, we arrive at

$$cos^2\theta_2 g(\nabla_{Y_1}X_1, X_2) = g(\nabla_{Y_1}T_1X_1, T_2X_2) + g(A_{FT_2X_2}X_1 - A_{FX_1}T_2X_2, Y_1). \tag{3.2}$$

We note that $T_1X_1 \in \Gamma\left(\mathfrak{D}^1\right)$, $T_2X_2 \in \Gamma\left(\mathfrak{D}^2\right)$, then, replacing X_1 by T_1X_1, X_2 by T_2X_2, by using (2.13), (2.4) and the orthogonality of the vector fields, we obtain

$$\begin{aligned}
&cos^2\theta_2 g\left(\nabla_{Y_1}T_1X_1, T_2X_2\right)\\
=&g\left(\nabla_{Y_1}T_1^2X_1, T_2^2X_2\right) + g\left(A_{FT_2^2X_2}T_1X_1 - A_{FT_1X_1}T_2^2X_2, Y_1\right)\\
=&cos^2\theta_1 cos^2\theta_2 g\left(\nabla_{Y_1}X_1, X_2\right) - cos^2\theta_2 g\left(A_{FX_2}T_1X_1 - A_{FT_1X_1}X_2, Y_1\right).
\end{aligned}$$

Since M is proper, then $cos^2\theta_2 \neq 0$, from the above relation we get

$$g\left(\nabla_{Y_1}T_1X_1, T_2X_2\right) = cos^2\theta_1 g\left(\nabla_{Y_1}X_1, X_2\right) - g\left(A_{FX_2}T_1X_1 - A_{FT_1X_1}X_2, Y_1\right). \tag{3.3}$$

Adding equations (3.2) and (3.3), we find that

$$\begin{aligned}
&\left(cos^2\theta_2 - cos^2\theta_1\right) g\left(\nabla_{Y_1}X_1, X_2\right)\\
=&g\left(A_{FT_2X_2}X_1 - A_{FX_2}T_1X_1 + A_{FT_1X_1}X_2 - A_{FX_1}T_2X_2, Y_1\right).
\end{aligned} \tag{3.4}$$

Hence, according to definition 2.1, the proof of the Lemma is completed. □

Lemma 3.3. *Let M be a proper bi-slant submanifold of a Kenmotsu manifold $\left(\widetilde{M}, \varphi, \xi, \eta, g\right)$ with bi-slant angle $\{\theta_1, \theta_2\}$. Then, the slant distribution $\mathfrak{D}^2$ defines a totally geodesic foliation if and only if*

$$A_{FT_1X_1}X_2 - A_{FX_1}T_2X_2 + A_{FT_2X_2}X_1 - A_{FX_2}T_1X_1 + \left(cos^2\theta_2 - cos^2\theta_1\right)\eta\left(X_1\right)X_2 \in \mathfrak{D}^2$$

for any $X_1 \in \Gamma\left(\mathfrak{D}^1 \oplus \langle\xi\rangle\right)$, $X_2 \in \Gamma\left(\mathfrak{D}^2\right)$.

Proof. For any $X_1, Y_1 \in \Gamma\left(\mathfrak{D}^1 \oplus \langle\xi\rangle\right)$, $X_2, Y_2 \in \Gamma\left(\mathfrak{D}^2\right)$, from (2.1), (2.5) and (2.6), we have

$$\begin{aligned}
g\left(\nabla_{Y_2}X_2, X_1\right) &= g\left(\varphi\widetilde{\nabla}_{Y_2}X_2, \varphi X_1\right) + \eta\left(\widetilde{\nabla}_{Y_2}X_2\right)\eta\left(X_1\right)\\
&= g\left(\widetilde{\nabla}_{Y_2}\varphi X_2, \varphi X_1\right) - g\left(\left(\widetilde{\nabla}_{Y_2}\varphi\right)X_2, \varphi X_1\right) - \eta\left(X_1\right)g\left(\widetilde{\nabla}_{Y_2}\xi, X_2\right).
\end{aligned}$$

By (2.1), (2.4) and the orthogonality of $\mathfrak{D}^2$ and $\langle\xi\rangle$, we derive

$$g\left(\nabla_{Y_2}X_2, X_1\right) = g\left(\widetilde{\nabla}_{Y_2}\varphi X_2, \varphi X_1\right) - \eta\left(X_1\right)g\left(X_2, Y_2\right).$$

Using (2.10) and the Gauss and Weingarten formulas (2.6), we deduce

$$\begin{aligned}
g\left(\nabla_{Y_2}X_2, X_1\right) &= g\left(\nabla_{Y_2}T_2X_2, T_1X_1\right) - g\left(A_{FX_2}Y_2, T_1X_1\right)\\
&\quad - g\left(\widetilde{\nabla}_{Y_2}FX_1, \varphi X_2\right) - \eta\left(X_1\right)g\left(X_2, Y_2\right).
\end{aligned} \tag{3.5}$$

Consider the third term in the right hand side of the above relation by using (2.2), (2.3), (2.4), (2.5), (2.6) and (2.15), we find that

$$\begin{aligned}
&- g\left(\widetilde{\nabla}_{Y_2}FX_1, \varphi X_2\right)\\
=&g\left(\varphi\widetilde{\nabla}_{Y_2}FX_1, X_2\right) = g\left(\widetilde{\nabla}_{Y_2}\varphi FX_1, X_2\right) - g\left(\left(\widetilde{\nabla}_{Y_2}\varphi\right)FX_1, X_2\right)\\
=&g\left(\widetilde{\nabla}_{Y_2}tFX_1, X_2\right) + g\left(\widetilde{\nabla}_{Y_2}fFX_1, X_2\right)\\
=&- sin^2\theta_1 g\left(\nabla_{Y_2}X_1, X_2\right) + sin^2\theta_1\eta\left(X_1\right)g\left(\widetilde{\nabla}_{Y_2}\xi, X_2\right) - g\left(\widetilde{\nabla}_{Y_2}FT_1X_1, X_2\right)\\
=&- sin^2\theta_1 g\left(\nabla_{Y_2}X_1, X_2\right) + sin^2\theta_1\eta\left(X_1\right)g\left(X_2, Y_2\right) + g\left(A_{FT_1X_1}Y_2, X_2\right).
\end{aligned} \tag{3.6}$$

Plugging (3.6) into (3.5), we have the following by using the symmetry of the shape operator A

$$\begin{aligned} cos^2\theta_1 g(\nabla_{Y_2}X_2, X_1) =& g(\nabla_{Y_2}T_2X_2, T_1X_1) + g(A_{FT_1X_1}X_2 - A_{FX_2}T_1X_1, Y_2) \\ &- cos^2\theta_1\eta(X_1)g(X_2, Y_2). \end{aligned} \tag{3.7}$$

Replacing X_1 by T_1X_1, X_2 by T_2X_2, then by using (2.1), (2.13), (2.4) and the orthogonality of the vector fields, we obtain

$$\begin{aligned} & cos^2\theta_1 g(\nabla_{Y_2}T_2X_2, T_1X_1) \\ =& g\left(\nabla_{Y_2}T_2^2X_2, T_1^2X_1\right) + g\left(A_{FT_1^2X_1}T_2X_2 - A_{FT_2X_2}T_1^2X_1, Y_2\right) \\ =& cos^2\theta_1 cos^2\theta_2 g(\nabla_{Y_2}X_2, X_1) + cos^2\theta_1 cos^2\theta_2\eta(X_1)g\left(\widetilde{\nabla}_{Y_2}\xi, X_2\right) \\ &- cos^2\theta_1 g(A_{FX_1}T_2X_2 - A_{FT_2X_2}X_1, Y_2) - cos^2\theta_1\eta(X_1)g\left(\widetilde{\nabla}_{Y_2}\xi, FT_2X_2\right) \\ =& cos^2\theta_1 cos^2\theta_2 g(\nabla_{Y_2}X_2, X_1) + cos^2\theta_1 cos^2\theta_2\eta(X_1)g(Y_2, X_2) \\ &- cos^2\theta_1 g(A_{FX_1}T_2X_2 - A_{FT_2X_2}X_1, Y_2). \end{aligned}$$

Since M is proper, then $cos^2\theta_1 \neq 0$, from the above relation we get

$$\begin{aligned} g(\nabla_{Y_2}T_2X_2, T_1X_1) =& cos^2\theta_2 g(\nabla_{Y_2}X_2, X_1) - g(A_{FX_1}T_2X_2 - A_{FT_2X_2}X_1, Y_2) \\ &+ cos^2\theta_2\eta(X_1)g(Y_2, X_2). \end{aligned} \tag{3.8}$$

Adding equations (3.7) and (3.8), we arrive at

$$\begin{aligned} \left(cos^2\theta_1 - cos^2\theta_2\right)g(\nabla_{Y_2}X_2, X_1) =& g(A_{FT_1X_1}X_2 - A_{FX_1}T_2X_2 + A_{FT_2X_2}X_1 - A_{FX_2}T_1X_1, Y_2) \\ &+ \left(cos^2\theta_2 - cos^2\theta_1\right)\eta(X_1)g(Y_2, X_2). \end{aligned} \tag{3.9}$$

Thus, similarly to Lemma 3.2, we complete the proof of our lemma. □

4. Warped product bi-slant submanifolds

In [6], Bishop and O'Neill introduced the notion of warped product manifolds as follows: Let (M_1, g_1) and (M_1, g_1) be two Riemannian manifolds, respectively, and f a positive differentiable function on M_1. Consider the Riemannian product manifold $M_1 \times M_2$ with its canonical projection maps $\pi_1 : M_1 \times M_2 \to M_1$ and $\pi_2 : M_1 \times M_2 \to M_2$. Then their warped product manifold $M = \left(M_1 \times_f M_2, g\right)$ is the Riemannian product manifold $M_1 \times M_2$ equipped with the Riemannian structure such that

$$g(X, Y) = g_1(\pi_{1*}X, \pi_{1*}Y) + (f \circ \pi_1)^2 g_2(\pi_{2*}X, \pi_{2*}Y), \tag{4.1}$$

for any vector field $X, Y \in \Gamma(TM)$, where $*$ is the symbol for the tangent maps.

Remark 4.1. *A warped product manifold $M = \left(M_1 \times_f M_2, g\right)$ is said to be trivial if the warping function f is constant. In this case, the warped product manifold is a Riemannian product manifold. From [6] and [10], we have the following facts for a warped product manifold:*

(i) $\nabla_{X_1}X_2 = \nabla_{X_2}X_1 = (X_1 lnf)X_2$, *for any $X_1 \in \Gamma(TM_1)$ and $X_2 \in \Gamma(TM_2)$, where ∇ is the Levi-Civita connection on M.*

(ii) M_1 and M_2 are totally geodesic and totally umbilical submanifolds of M, respectively.

In this section, we study the warped product bi-slant submanifold of a Kenmotsu manifold. We defined these submanifolds as follows.

Definition 4.2. *A warped product $M = M_1 \times_f M_2$ of a θ_1-slant submanifold M_1 and a θ_2-slant submanifold M_2 of a Kenmotsu manifold $\left(\widetilde{M}, \varphi, \xi, \eta, g\right)$ is called a warped product bi-slant submanifold.*

A warped product bi-slant submanifold $M_1 \times_f M_2$ is called proper if both M_1 and M_2 are proper slant submanifolds with slant angle $\theta_1, \theta_2 \neq 0, \frac{\pi}{2}$ of $\widetilde{M}$. Otherwise, a warped product bi-slant submanifold $M_1 \times_f M_2$ is a contact CR-warped product of the form if $\theta_1 = 0, \theta_2 = \frac{\pi}{2}$ or $\theta_2 = 0, \theta_1 = \frac{\pi}{2}$ discussed in [4, 25]. Also, the warped product pseudo-slant and the warped product semi-slant submanifolds were discussed in [3] and [22, 27], respectively. It is known that from [22], if the structure vector field ξ is tangent to M_2, then the warped product is trivial. Since we do not study the non-existence of warped products, throughout we consider ξ is tangent to M_1, in this case, we have the following facts

$$\xi ln f = 1, \qquad \sigma(X_2, \xi) = 0, \tag{4.2}$$

for any $X_2 \in \Gamma(TM_2)$.

Definition 4.3. *A warped product bi-slant submanifold $M = M_1 \times_f M_2$ of a Kenmotsu manifold $\left(\widetilde{M}, \varphi, \xi, \eta, g\right)$ is said to be mixed totally geodesic if $\sigma(X_1, X_2) = 0$, for each vector field $X_1 \in \Gamma(TM_1)$ and $X_2 \in \Gamma(TM_2)$.*

First, we provide a non-trivial example of warped product bi-slant submanifols.

Example 4.4. *Let $\mathbb{R}^{11}$ be an Euclidean 11-space endowed with the standard metric and cartesian coordinates $(x_1, y_1, \cdots, x_5, y_5, t)$ and with the canonical structure (φ, ξ, η, g) defined by*

$$\varphi\left(\frac{\partial}{\partial x_i}\right) = -\frac{\partial}{\partial y_i}, \quad \varphi\left(\frac{\partial}{\partial y_j}\right) = \frac{\partial}{\partial x_j}, \quad \varphi\left(\frac{\partial}{\partial t}\right) = 0$$
$$\xi = \frac{\partial}{\partial t}, \quad \eta = dt, \quad g = \langle \quad , \quad \rangle, \quad 1 \leq i, j \leq 5.$$

If we denote by any vector field $X = a_i \frac{\partial}{\partial x_i} + b_j \frac{\partial}{\partial y_j} + c\frac{\partial}{\partial t}$ $\quad (1 \leq i, j \leq 5)$ tangent to $\mathbb{R}^{11}$, it can be easily proved that (φ, ξ, η, g) is an almost contact metric structure on $\mathbb{R}^{11}$ by using (2.1). Consider a 5-dimensional submanifold M of $\mathbb{R}^{11}$ given by

$$\begin{aligned}\chi(u, v, w, s, t) = (&u, v cos\theta, 0, v sin\theta, uvw cos\theta,\\ &uvs cos\theta, uvw sin\theta, uvs sin\theta, uvs - uvw, uvs + uvw, t),\end{aligned}$$

for $\theta \in \left(0, \frac{\pi}{2}\right)$, whose tangent space is spanned by the orthonormal vector fields

$$e_1 = \frac{\partial}{\partial x_1}, \quad e_2 = cos\theta\frac{\partial}{\partial y_1} + sin\theta\frac{\partial}{\partial y_2},$$
$$e_3 = uvcos\theta\frac{\partial}{\partial x_3} + uvsin\theta\frac{\partial}{\partial x_4} - uv\frac{\partial}{\partial x_5} + uv\frac{\partial}{\partial y_5},$$
$$e_4 = uvcos\theta\frac{\partial}{\partial y_3} + uvsin\theta\frac{\partial}{\partial y_4} + uv\frac{\partial}{\partial x_5} + uv\frac{\partial}{\partial y_5}, \quad e_5 = \frac{\partial}{\partial t}.$$

Clearly, we have

$$\varphi e_1 = -\frac{\partial}{\partial y_1}, \quad \varphi e_2 = cos\theta\frac{\partial}{\partial x_1} + sin\theta\frac{\partial}{\partial x_2},$$
$$\varphi e_3 = -uvcos\theta\frac{\partial}{\partial y_3} - uvsin\theta\frac{\partial}{\partial y_4} + uv\frac{\partial}{\partial y_5} + uv\frac{\partial}{\partial x_5},$$
$$\varphi e_4 = uvcos\theta\frac{\partial}{\partial x_3} + uvsin\theta\frac{\partial}{\partial x_4} - uv\frac{\partial}{\partial y_5} + uv\frac{\partial}{\partial x_5}, \quad e_5 = \frac{\partial}{\partial t}.$$

Define the distributions $\mathfrak{D}^1 = Span\{e_1, e_2\}$ *and* $\mathfrak{D}^2 = Span\{e_3, e_4\}$, *respectively, then it is clear that* $\mathfrak{D}^1$ *is a* θ_1*-slant distribution with* $\theta_1 = \theta$ *and* $\mathfrak{D}^2$ *is a* θ_2*-slant distribution with* $\theta_2 = \arccos\left(\frac{1}{3}\right)$, *such that* ξ *is tangent to* M. *Hence,* M *is a proper bi-slant submanifold with bi-slant angle* $\{\theta_1, \theta_2\}$ *of* $\mathbb{R}^{11}$. *Furthermore, it is easy to verify that both the distributions* $\mathfrak{D}^1 \oplus \langle\xi\rangle$ *and* $\mathfrak{D}^2$ *are integrable. We denote the integral manifolds of* $\mathfrak{D}^1 \oplus \langle\xi\rangle$ *and* $\mathfrak{D}^2$ *by* M_1 *and* M_2, *respectively. Then the metric tensor* g *of the product manifold* $M = M_1 \times M_2$ *is given by*

$$g = du^2 + dv^2 + dt^2 + 3u^2v^2\left(dw^2 + ds^2\right) = g_{M_1} + \left(\sqrt{3}uv\right)^2 g_{M_2}.$$

Thus, M *is a proper warped product bi-slant submanifold of* $\mathbb{R}^{11}$ *of the form* $M_1 \times_f M_2$ *with warping function* $f = \sqrt{3}uv$, *such that* ξ *is tangent to* M_1.

Now, we give the following lemma, which plays a crucial role in our main results.

Lemma 4.5. *Let* $M = M_1 \times_f M_2$ *be a warped product bi-slant submanifold of a Kenmotsu manifold* $\left(\widetilde{M}, \varphi, \xi, \eta, g\right)$ *such that* ξ *is tangent to* M_1, *where* M_1 *and* M_2 *are a* θ_1*-slant submanifold and a* θ_2*-slant submanifold of* $\widetilde{M}$, *respectively, then we have*

(i) $g(\sigma(X_2, Y_2), FX_1) - g(\sigma(X_1, Y_2), FX_2) = (X_1 lnf - \eta(X_1))\, g(T_2X_2, Y_2) + T_1X_1 lnf g(X_2, Y_2)$.

(ii) $g(\sigma(X_1, Y_1), FX_2) = g(\sigma(Y_1, X_2), FX_1)$.

For any $X_1, Y_1 \in \Gamma(TM_1)$, $X_2, Y_2 \in \Gamma(TM_2)$.

Proof. For any $X_1, Y_1 \in \Gamma(TM_1)$, $X_2, Y_2 \in \Gamma(TM_2)$, from (2.6) and (2.10), we can write

$$\begin{aligned} g(\sigma(X_1, Y_2), FX_2) &= g\left(\widetilde{\nabla}_{Y_2}X_1, FX_2\right) \\ &= g\left(\widetilde{\nabla}_{Y_2}X_1, \varphi X_2\right) - g(\nabla_{Y_2}X_1, TX_2). \end{aligned}$$

By (2.2), (2.5) and the (i) of Remark 4.1, we derive

$$\begin{aligned} g(\sigma(X_1, Y_2), FX_2) &= -\, g\left(\varphi\widetilde{\nabla}_{Y_2}X_1, X_2\right) - X_1 lnf g(Y_2, T_2X_2) \\ &= g\left(\left(\widetilde{\nabla}_{Y_2}\varphi\right)X_1, X_2\right) - g\left(\widetilde{\nabla}_{Y_2}\varphi X_1, X_2\right) - X_1 lnf g(Y_2, T_2X_2). \end{aligned}$$

Using (2.3), (2.6), (2.10) and by the orthogonality of the vector field ξ, we deduce

$$\begin{aligned} g(\sigma(X_1, Y_2), FX_2) = &-\eta(X_1)\, g(T_2Y_2, X_2) - g(\nabla_{Y_2}T_1X_1, X_2) \\ &+ g(A_{FX_1}Y_2, X_2) - X_1 lnf g(Y_2, T_2X_2). \end{aligned}$$

Then from (2.11), (2.7), and again using the (i) of Remark 4.1, we obtain

$$\begin{aligned} g(\sigma(X_1, Y_2), FX_2) = &\,\eta(X_1)\, g(Y_2, T_2X_2) - T_1X_1 lnf g(X_2, Y_2) \\ &+ g(\sigma(X_2, Y_2), FX_1) - X_1 lnf g(Y_2, T_2X_2). \end{aligned}$$

Hence, the proof of the first assertion is complete. Next, we prove the second assertion. Using (2.2), (2.6) and (2.10), we get

$$g(\sigma(Y_1, X_2), FX_1) = -g\left(\varphi\widetilde{\nabla}_{Y_1}X_2, X_1\right) - g(\nabla_{Y_1}X_2, T_1X_1).$$

Then from (2.5) and the (i) of Remark 4.1, we arrive at

$$g(\sigma(Y_1, X_2), FX_1) = g\left(\left(\widetilde{\nabla}_{Y_1}\varphi\right)X_2, X_1\right) - g\left(\widetilde{\nabla}_{Y_1}\varphi X_2, X_1\right) - Y_1 lnf g(X_2, T_1X_1).$$

Since the orthogonality of the vector fields, by (2.3), we find that the first term and the last term in the right hand side of the above relation vanishes, thus we have

$$g\left(\sigma\left(Y_1, X_2\right), FX_1\right) = -g\left(\widetilde{\nabla}_{Y_1}\varphi X_2, X_1\right).$$

Therefore, again using (2.6) and (2.10), we have

$$g\left(\sigma\left(Y_1, X_2\right), FX_1\right) = -g\left(\nabla_{Y_1} T_2 X_2, X_1\right) + g\left(A_{FX_2} Y_1, X_1\right).$$

Thus, by (2.7) and the (i) of Remark 4.1, the above reduced to

$$g\left(\sigma\left(Y_1, X_2\right), FX_1\right) = g\left(\sigma\left(Y_1, X_1\right), FX_2\right),$$

which is the required result together with the symmetry of σ. Hence, the proof of the Lamma is complete. □

Here, we recall the following lemma for later use.

Lemma 4.6 (Hiepko's Theorem). *[30] Let $\mathfrak{D}^1$ and $\mathfrak{D}^2$ be two orthogonal distributions on a Riemannian manifold M. Suppose that $\mathfrak{D}^1$ and $\mathfrak{D}^2$ both are involutive such that $\mathfrak{D}^1$ is a totally geodesic foliation and $\mathfrak{D}^2$ is a spherical foliation. Then M is locally isometric to a non trivial warped product $M_1 \times_f M_2$, where M_1 and M_2 are integral manifolds of $\mathfrak{D}^1$ and $\mathfrak{D}^2$, respectively.*

Theorem 4.7. *Let M be a proper bi-slant submanifold of a Kenmotsu manifold $\left(\widetilde{M}, \varphi, \xi, \eta, g\right)$ with bi-slant angle $\{\theta_1, \theta_2\}$. Then, M is locally a warped product submanifold of the form $M_1 \times_f M_2$ if and only if the shape operator satisfies*

$$A_{FX_1} X_2 - A_{FX_2} X_1 = \left(X_1(\mu) - \eta\left(X_1\right)\right) T_2 X_2 + T_1 X_1(\mu) X_2, \tag{4.3}$$

for any $X_1 \in \mathfrak{D}^1 \oplus \langle\xi\rangle$, $X_2 \in \mathfrak{D}^2$ and for some smooth function μ on M such that $Y_2(\mu) = 0$, for any $Y_2 \in \mathfrak{D}^2$.

Proof. If $M = M_1 \times_f M_2$ is a warped product bi-slant submanifold, then for any $X_1, Y_1 \in \Gamma(TM_1)$, $X_2, Y_2 \in \Gamma(TM_2)$, from the (ii) of Lemma 4.5, we find that $g(A_{FX_1}X_2 - A_{FX_2}X_1,$
$Y_1) = 0$, which means that $A_{FX_1}X_2 - A_{FX_2}X_1$ has no componenet in TM_1, i.e., it is lies in TM_2 only. Then, the relation (4.3) follows from the (i) of Lemma 4.5 with $\mu = lnf$.

Conversely, if M is a proper $\{\theta_1, \theta_2\}$-bi-slant submanifold of a Kenmotsu manifold $\widetilde{M}$ with two proper slant distributions $\mathfrak{D}^1 \oplus \langle\xi\rangle$ and $\mathfrak{D}^2$ such that (4.3) holds, for any $X_1, Y_1 \in \Gamma\left(\mathfrak{D}^1 \oplus \langle\xi\rangle\right)$, $X_2, Y_2 \in \Gamma\left(\mathfrak{D}^2\right)$, replacing X_1 and X_2 by T_1X_1 and T_2X_2, respectively, then the given condition (4.3) becomes the following by (2.1) and (2.13).

$$\begin{aligned} A_{FT_1X_1} X_2 - A_{FX_2} T_1 X_1 &= -cos^2\theta_1 \left(X_1(\mu) - \eta\left(X_1\right)\right) X_2 + T_1 X_1(\mu) T_2 X_2, \\ A_{FX_1} T_2 X_2 - A_{FT_2X_2} X_1 &= -cos^2\theta_2 \left(X_1(\mu) - \eta\left(X_1\right)\right) X_2 + T_1 X_1(\mu) T_2 X_2. \end{aligned} \tag{4.4}$$

First, we consider the distribution $\mathfrak{D}^1 \oplus \langle\xi\rangle$, plugging (4.4) into the (3.4) of Lemma 3.2, we find that $\left(cos^2\theta_2 - cos^2\theta_1\right) g\left(\nabla_{Y_1} X_1, X_2\right) = 0$, since M is proper, we derive $g\left(\nabla_{Y_1} X_1, X_2\right) = 0$, and also, it is clear that $g\left(\left[X_1, Y_1\right], X_2\right) = 0$, which means that the leaves of the distribution $\mathfrak{D}^1 \oplus \langle\xi\rangle$ are totally geodesic in M and the distribution $\mathfrak{D}^1 \oplus \langle\xi\rangle$ is integrable. On the other hand, for the distribution $\mathfrak{D}^2$, combining the relations (4.4) and (3.9) of Lemma 3.3, we conclude that

$$\begin{aligned} \left(cos^2\theta_1 - cos^2\theta_2\right) g\left(\nabla_{Y_2} X_2, X_1\right) = &\left(cos^2\theta_2 - cos^2\theta_1\right)\left(X_1(\mu) - \eta\left(X_1\right)\right) g\left(X_2, Y_2\right) \\ &+ \left(cos^2\theta_2 - cos^2\theta_1\right) \eta\left(X_1\right) g\left(Y_2, X_2\right). \end{aligned}$$

Since M is proper, the above equality reduced to

$$g\left(\nabla_{Y_2} X_2, X_1\right) = -X_1(\mu) g\left(X_2, Y_2\right). \tag{4.5}$$

Also, from the above relation we conclude that $g([X_2, Y_2], X_1) = 0$, that is, the slant distribution $\mathfrak{D}^2$ is integrable. If we denote by $\hat{\sigma}$ the second fundamental form of a leaf M_2 of $\mathfrak{D}^2$ in M, then (4.5) yields

$$g(\hat{\sigma}(X_2, Y_2), X_1) = -X_1(\mu) g(X_2, Y_2).$$

Therefore, from the definition of gradient (2.12), we get

$$\hat{\sigma}(X_2, Y_2) = -\nabla \mu g(X_2, Y_2),$$

which means that M_2 is totally umbilical in M with the non vanishing mean curvature vector $H = -\nabla\mu$. Now we prove that H is parallel corresponding to the normal connection $\hat{\nabla}^{\perp}$ of a leaf M_2 of $\mathfrak{D}^2$ in M, from (2.6) and (2.12), we obtain

$$\begin{aligned} g\left(\hat{\nabla}^{\perp}_{X_2} \nabla \mu, X_1\right) &= g(\nabla_{X_2} \nabla \mu, X_1) = X_2 g(\nabla\mu, X_1) - g(\nabla\mu, \nabla_{X_2} X_1) \\ &= X_2(X_1\mu) - [X_2, X_1]\mu + g(\nabla_{X_1} \nabla \mu, X_2) \\ &= X_1(X_2\mu) + g(\nabla_{X_1} \nabla \mu, X_2) = 0. \end{aligned}$$

Since $X_2(\mu) = 0$ and $\mathfrak{D}^1 \oplus \langle\xi\rangle$ is a totally geodesic foliation in M, for any $X_1 \in \Gamma\left(\mathfrak{D}^1 \oplus \langle\xi\rangle\right)$, $X_2 \in \Gamma\left(\mathfrak{D}^2\right)$. Thus, the leaves of the distribution $\mathfrak{D}^2$ are totally umbilical in M with non-vanishing parallel mean curvature vector H, i.e., M_2 is an extrinsic sphere in M. Then from Hiepko's Theorem, M is a warped product submanifold, the proof of Theorem 4.7 is complete. □

The following corollary is given by a similar proof procedure to the above theorem.

Corollary 4.8. *Let M be a proper mixed totally geodesic bi-slant submanifold of a Kenmotsu manifold $\left(\widetilde{M}, \varphi, \xi, \eta, g\right)$ with bi-slant angle $\{\theta_1, \theta_2\}$. Then, M is locally a mixed totally geodesic warped product submanifold of the form $M_1 \times_f M_2$ if and only if the shape operator satisfies*

$$\begin{aligned} &A_{FT_1X_1}X_2 - A_{FX_1}T_2X_2 = \left(cos^2\theta_2 - cos^2\theta_1\right)(X_1(\mu) - \eta(X_1))X_2, \\ &A_{FT_2X_2}X_1 - A_{FX_2}T_1X_1 = 0, \end{aligned}$$

for any $X_1 \in \Gamma\left(\mathfrak{D}^1 \oplus \langle\xi\rangle\right)$, $X_2 \in \Gamma\left(\mathfrak{D}^2\right)$ and for some smooth function μ on M such that $Y_2(\mu) = 0$, for any $Y_2 \in \mathfrak{D}^2$.

Theorem 4.9. *Let $M = M_1 \times_f M_2$ be a mixed totally geodesic warped product bi-slant submanifold of a Kenmotsu manifold $\left(\widetilde{M}, \varphi, \xi, \eta, g\right)$ such that ξ is tangent to M_1, where M_1 and M_2 are a θ_1-slant submanifold and a θ_2-slant submanifold of $\widetilde{M}$, respectively, then one of the following two cases must occur:*

(i) $\theta_2 = \frac{\pi}{2}$.
(ii) $\nabla ln f = \xi$.

Proof. For any $X_1, Y_1 \in \Gamma(TM_1)$, $X_2, Y_2 \in \Gamma(TM_2)$, since M is mixed totally geodesic, the (i) of Lemma 4.5 becomes

$$g(\sigma(X_2, Y_2), FX_1) = (X_1 ln f - \eta(X_1)) g(T_2X_2, Y_2) + T_1X_1 ln f g(X_2, Y_2). \tag{4.6}$$

Interchanging X_2 by Y_2 in (4.6) and using (2.11), we obtain

$$g(\sigma(X_2, Y_2), FX_1) = -(X_1 ln f - \eta(X_1)) g(T_2X_2, Y_2) + T_1X_1 ln f g(X_2, Y_2). \tag{4.7}$$

Then from (4.6), (4.7), we arrive at

$$(X_1 ln f - \eta(X_1)) g(T_2X_2, Y_2) = 0.$$

Replacing X_2 by T_2X_2, we conclude that

$$-cos^2\theta_2 (X_1 ln f - \eta(X_1)) g(X_2, Y_2) = 0.$$

Hence, the proof of the theorem is complete by using the the definition of gradient (2.12). □

5. General improved Chen's inequality for Warped product bi-slant submanifolds

In [28], Uddin et al. established the following Chen's inequality for a proper mixed totally geodesic warped product bi-slant submanifold $M = M_1 \times_f M_2$ of a Kenmotsu manifold $\widetilde{M}$ such that ξ is tangent to M_1, where M_1 and M_2 are a θ_1-slant submanifold and a θ_2-slant submanifold of $\widetilde{M}$, respectively

$$\|\sigma\|^2 \geqslant 2q csc^2\theta_1 \left(cos^2\theta_1 + cos^2\theta_2\right)\left(\left\|\nabla ln f\right\|^2 - 1\right), \tag{5.1}$$

where $2q = \mathrm{dimM_2}$. From Theorem 4.9, we find that the inequality (5.1) becomes $\|\sigma\|^2 \geqslant 0$ since M is proper, which means that the study of this Chen's inequality is meaningless. Thus, in this section, we establish the general improved Chen's inequality for a non-mixed totally geodesic proper warped product bi-slant submanifold.

Let $M = M_1 \times_f M_2$ be an m-dimensional proper warped product bi-slant submanifold of a $(2n+1)$-dimensional Kenmotsu manifold $\widetilde{M}$ such that M_1 and M_2 are proper slant submanifolds with slant angles θ_1 and θ_2, repectively. We denote the tangent bundles of M_1 and M_2 by $\mathfrak{D}^1$ and $\mathfrak{D}^2$ with their real dimensions $2p+1$ and $2q$, respectively, such that $\xi \in \Gamma\left(\mathfrak{D}^1\right)$. Based on the following remark

Remark 5.1. *If $M = M_1 \times_f M_2$ is a warped product bi-slant submanifold of a Kenmotsu manifold $\widetilde{M}$, then the tangent bunble TM and the normal bunble $T^\perp M$ of M are respectively decomposed as*

(i) $TM = \mathfrak{D}^1 \oplus \mathfrak{D}^2$,
(ii) $T^\perp M = F\mathfrak{D}^1 \oplus F\mathfrak{D}^2 \oplus \nu$,

where ν is the φ-invariant normal subbundle of $T^\perp M$, and $\xi \in \Gamma\left(\mathfrak{D}^1\right)$.

We set that

(1) $\mathfrak{D}^1 = \mathrm{Span}\left\{e_1, \cdots, e_p, e_{p+1} = \sec\theta_1 T_1 e_1, \cdots, e_{2p} = \sec\theta_1 T_1 e_p, \xi\right\}$,
(2) $\mathfrak{D}^2 = \mathrm{Span}\left\{e_{2p+2} = \hat{e}_1, \cdots, e_{2p+q+1} = \hat{e}_q, e_{2p+q+2} = \hat{e}_{q+1} = \sec\theta_2 T_2 \hat{e}_1, \cdots, e_{2p+2q+1} = \hat{e}_{2q} = \sec\theta_2 T_2 \hat{e}_q\right\}$,
(3) $F\mathfrak{D}^1 = \mathrm{Span}\left\{e_{m+1} = \tilde{e}_1 = \csc\theta_1 F e_1, \cdots, e_{m+p} = \tilde{e}_p = \csc\theta_1 F e_p, e_{m+p+1} = \tilde{e}_{p+1} = \csc\theta_1 \sec\theta_1 F T_1 e_1, \cdots, e_{m+2p} = \tilde{e}_{2p} = \csc\theta_1 \sec\theta_1 F T_1 e_p\right\}$,
(4) $F\mathfrak{D}^2 = \mathrm{Span}\left\{e_{m+2p+1} = \tilde{e}_{2p+1} = \csc\theta_2 F\hat{e}_1, \cdots, e_{m+2p+q} = \tilde{e}_{2p+q} = \csc\theta_2 F\hat{e}_q, e_{m+2p+q+1} = \tilde{e}_{2p+q+1} = \csc\theta_2 \sec\theta_2 F T_2 \hat{e}_1, \cdots, e_{m+2p+2q} = \tilde{e}_{2p+2q} = \csc\theta_2 \sec\theta_2 F T_2 \hat{2}_q\right\}$,
(5) $\nu = \mathrm{Span}\left\{e_{m+2p+2q+1} = \tilde{e}_{2p+2q+1}, \cdots, e_{2n+1} = \tilde{e}_{2n+1-m}\right\}$.

Theorem 5.2. *Let $M^m = M_1 \times_f M_2$ be a proper warped product bi-slant submanifold of a Kenmotsu manifold $\left(\widetilde{M}^{2n+1}, \varphi, \xi, \eta, g\right)$ such that ξ is tangent to M_1, where M_1 and M_2 are a $(2p+1)$-dimensional θ_1-slant submanifold and a $2q$-dimensional θ_2-slant submanifold of $\widetilde{M}$, respectively, then we have the following:*

(i) The squared norm of the second fundamental form σ of M satisfies

$$\|\sigma\|^2 \geqslant \left(4q csc^2\theta_2 cos^2\theta_1 + 4q cot^2\theta_2\right)\left(\left\|\nabla ln f\right\|^2 - 1\right). \tag{5.2}$$

(ii) If the equality holds in (33), then M_1 is a totally geodesic submanifold of $\widetilde{M}$ and M_2 is a totally umbilical submanifold of $\widetilde{M}$. In other words, M is a minimal submanifold of $\widetilde{M}$.

Proof. For the proper warped product bi-slant submanifold M^m with dimension m, using the constructed frame fields, we can write

$$\|\sigma\|^2 = \left\|\sigma\left(\mathfrak{D}^1, \mathfrak{D}^1\right)\right\|^2 + 2\left\|\sigma\left(\mathfrak{D}^1, \mathfrak{D}^2\right)\right\|^2 + \left\|\sigma\left(\mathfrak{D}^2, \mathfrak{D}^2\right)\right\|^2. \tag{5.3}$$

Then, using (2.8), we consider the each term in the right hand side of (5.3) as follows:

$$\left\|\sigma\left(\mathfrak{D}^1, \mathfrak{D}^1\right)\right\|^2 = g\left(\sigma\left(\mathfrak{D}^1, \mathfrak{D}^1\right), F\mathfrak{D}^1\right)^2 + g\left(\sigma\left(\mathfrak{D}^1, \mathfrak{D}^1\right), F\mathfrak{D}^2\right)^2 + g\left(\sigma\left(\mathfrak{D}^1, \mathfrak{D}^1\right), \nu\right)^2. \tag{5.4}$$

As we have no relations for the warped products of the first and the third ν-component terms in (5.4), by dropping these positive terms, we obtain

$$\left\|\sigma\left(\mathfrak{D}^1, \mathfrak{D}^1\right)\right\|^2 \geq csc^2\theta_2 \sum_{i,j=1}^{2p+1} \sum_{r=1}^{2q} g\left(\sigma\left(e_i, e_j\right), F\hat{e}_r\right)^2.$$

Since $e_{2p+1} = \xi$, and from (4.2), it is known that $\sigma(\xi, X) = 0$, for any $X \in TM$. Then from (ii) of the Lemma 4.5,we get

$$\left\|\sigma\left(\mathfrak{D}^1, \mathfrak{D}^1\right)\right\|^2 \geq csc^2\theta_2 \sum_{i,j=1}^{2p} \sum_{r=1}^{2q} g\left(\sigma\left(e_i, \hat{e}_r\right), Fe_j\right)^2. \tag{5.5}$$

Similarly, by dropping the ν-component positive term, using (i) of Lemma 4.5, we derive

$$\begin{aligned}
&\left\|\sigma\left(\mathfrak{D}^1, \mathfrak{D}^2\right)\right\|^2 \\
=&g\left(\sigma\left(\mathfrak{D}^1, \mathfrak{D}^2\right), F\mathfrak{D}^1\right)^2 + g\left(\sigma\left(\mathfrak{D}^1, \mathfrak{D}^2\right), F\mathfrak{D}^2\right)^2 + g\left(\sigma\left(\mathfrak{D}^1, \mathfrak{D}^2\right), \nu\right)^2 \\
\geq&csc^2\theta_1 \sum_{i,j=1}^{2p} \sum_{r=1}^{2q} g\left(\sigma\left(e_i, \hat{e}_r\right), Fe_j\right)^2 + csc^2\theta_2 \sum_{i=1}^{2p} \sum_{k,r=1}^{2q} g\left(\sigma\left(e_i, \hat{e}_k\right), F\hat{e}_r\right)^2 \\
=&csc^2\theta_1 \sum_{i,j=1}^{2p} \sum_{r=1}^{2q} g\left(\sigma\left(e_i, \hat{e}_r\right), Fe_j\right)^2 + csc^2\theta_2 \sum_{i=1}^{2p} \sum_{k,r=1}^{2q} \{g\left(\sigma\left(\hat{e}_k, \hat{e}_r\right), Fe_i\right) \\
&+ \left(\eta\left(e_i\right) - e_i lnf\right) g\left(\hat{e}_k, T_2\hat{e}_r\right) - T_1 e_i lnf\, g\left(\hat{e}_k, \hat{e}_r\right)\}^2.
\end{aligned} \tag{5.6}$$

If we denote $\left(\eta\left(e_i\right) - e_i lnf\right) g\left(\hat{e}_k, T_2\hat{e}_r\right) - T_1 e_i lnf\, g\left(\hat{e}_k, \hat{e}_r\right)$ by h^i_{kr}, then we arrive at

$$\begin{aligned}
&\left\|\sigma\left(\mathfrak{D}^1, \mathfrak{D}^2\right)\right\|^2 \\
\geq&csc^2\theta_1 \sum_{i,j=1}^{2p} \sum_{r=1}^{2q} g\left(\sigma\left(e_i, \hat{e}_r\right), Fe_j\right)^2 + csc^2\theta_2 \sum_{i=1}^{2p} \sum_{k,r=1}^{2q} g\left(\sigma\left(\hat{e}_k, \hat{e}_r\right), Fe_i\right)^2 \\
&+ 2csc^2\theta_2 \sum_{i=1}^{2p} \sum_{k,r=1}^{2q} g\left(\sigma\left(\hat{e}_k, \hat{e}_r\right), Fe_i\right) h^i_{kr} + csc^2\theta_2 \sum_{i=1}^{2p} \sum_{k,r=1}^{2q} \left(h^i_{kr}\right)^2.
\end{aligned} \tag{5.7}$$

Since $e_{2p+1} = \xi$, we note that $\eta\left(e_i\right) = 0$, $i = 1, \cdots, 2p$. Then, using (2.1), (2.11), (2.13) and the orthogonality of the adopted frame fields, we compute the third term in the right hand side of (5.7) as follows

$$
\begin{aligned}
&2csc^2\theta_2 \sum_{i=1}^{2p}\sum_{k,r=1}^{2q} g\left(\sigma\left(\hat{e}_k,\hat{e}_r\right),Fe_i\right)h^i_{kr}\\
=&2csc^2\theta_2 \sum_{i=1}^{2p}\sum_{k,r=1}^{q} \{g\left(\sigma\left(\hat{e}_k,\hat{e}_r\right),Fe_i\right)\left(-e_i lnfg\left(\hat{e}_k,T_2\hat{e}_r\right)-T_1e_i lnfg\left(\hat{e}_k,\hat{e}_r\right)\right)\\
&+sec^2\theta_2 g\left(\sigma\left(T_2\hat{e}_k,\hat{e}_r\right),Fe_i\right)\left(-e_i lnfg\left(T_2\hat{e}_k,T_2\hat{e}_r\right)-T_1e_i lnfg\left(T_2\hat{e}_k,\hat{e}_r\right)\right)\\
&+sec^2\theta_2 g\left(\sigma\left(\hat{e}_k,T_2\hat{e}_r\right),Fe_i\right)\left(-e_i lnfg\left(\hat{e}_k,T_2^2\hat{e}_r\right)-T_1e_i lnfg\left(\hat{e}_k,T_2\hat{e}_r\right)\right)\\
&+sec^4\theta_2 g\left(\sigma\left(T_2\hat{e}_k,T_2\hat{e}_r\right),Fe_i\right)\left(-e_i lnfg\left(T_2\hat{e}_k,T_2^2\hat{e}_r\right)-T_1e_i lnfg\left(T_2\hat{e}_k,T_2\hat{e}_r\right)\right)\Big\}\\
=&2csc^2\theta_2 \sum_{i=1}^{2p}\sum_{k=1}^{q} \{-T_1e_i lnfg\left(\sigma\left(\hat{e}_k,\hat{e}_k\right),Fe_i\right)-e_i lnfg\left(\sigma\left(T_2\hat{e}_k,\hat{e}_k\right),Fe_i\right)\\
&+e_i lnfg\left(\sigma\left(\hat{e}_k,T_2\hat{e}_k\right),Fe_i\right)-sec^2\theta_2 T_1e_i lnfg\left(\sigma\left(T_2\hat{e}_k,T_2\hat{e}_k\right),Fe_i\right)\Big\}\\
=&-2csc^2\theta_2 \sum_{i=1}^{2p}\sum_{k=1}^{2q} T_1e_i lnfg\left(\sigma\left(\hat{e}_k,\hat{e}_k\right),Fe_i\right)=0.
\end{aligned}
\tag{5.8}
$$

Similarly, applying the constructed frame fields, the forth term in the right hand side of (5.7) can be decomposed as

$$
\begin{aligned}
csc^2\theta_2 \sum_{i=1}^{2p}\sum_{k,r=1}^{2q}\left(h^i_{kr}\right)^2 =& \sum_{i=1}^{2p}\sum_{k,r=1}^{q} csc^2\theta_2\left\{\left(-e_i lnfg\left(\hat{e}_k,T_2\hat{e}_r\right)-T_1e_i lnfg\left(\hat{e}_k,\hat{e}_r\right)\right)^2\right.\\
&\left.+sec^2\theta_2\left(-e_i lnfg\left(T_2\hat{e}_k,T_2\hat{e}_r\right)-T_1e_i lnfg\left(T_2\hat{e}_k,\hat{e}_r\right)\right)^2\right\}\\
&+csc^2\theta_2 sec^2\theta_2\left\{\left(-e_i lnfg\left(\hat{e}_k,T_2^2\hat{e}_r\right)-T_1e_i lnfg\left(\hat{e}_k,T_2\hat{e}_r\right)\right)^2\right.\\
&\left.+sec^2\theta_2\left(-e_i lnfg\left(T_2\hat{e}_k,T_2^2\hat{e}_r\right)-T_1e_i lnfg\left(T_2\hat{e}_k,T_2\hat{e}_r\right)\right)^2\right\}\\
=&\sum_{i=1}^{2p} 2qcsc^2\theta_2\left(T_1e_i lnf\right)^2+2qcot^2\theta_2\left(e_i lnf\right)^2\\
=&\sum_{i=1}^{p}\left(2qcsc^2\theta_2+2qcot^2\theta_2 sec^2\theta_1\right)\left(T_1e_i lnf\right)^2\\
&+\left(2qcsc^2\theta_2 cos^2\theta_1+2qcot^2\theta_2\right)\left(e_i lnf\right)^2\\
=&\sum_{i=1}^{2p+1}\left(2qcsc^2\theta_2+2qcot^2\theta_2 sec^2\theta_1\right)\left(T_1e_i lnf\right)^2.
\end{aligned}
$$

Using (2.11), (2.14), and the fact that $\xi lnf=1$, the above expression reduces to

$$
csc^2\theta_2 \sum_{i=1}^{2p}\sum_{k,r=1}^{2q}\left(h^i_{kr}\right)^2=\left(2qcsc^2\theta_2 cos^2\theta_1+2qcot^2\theta_2\right)\left(\left\|\nabla lnf\right\|^2-1\right).
\tag{5.9}
$$

Plugging (5.8) and (5.9) into the (5.7), we get

$$\begin{aligned}&\left\|\sigma\left(\mathfrak{D}^1,\mathfrak{D}^2\right)\right\|^2\\&\geq csc^2\theta_1\sum_{i,j=1}^{2p}\sum_{r=1}^{2q}g\left(\sigma\left(e_i,\hat{e}_r\right),Fe_j\right)^2+csc^2\theta_2\sum_{i=1}^{2p}\sum_{k,r=1}^{2q}g\left(\sigma\left(\hat{e}_k,\hat{e}_r\right),Fe_i\right)^2\\&\quad+\left(2qcsc^2\theta_2cos^2\theta_1+2qcot^2\theta_2\right)\left(\left\|\nabla lnf\right\|^2-1\right).\end{aligned}\tag{5.10}$$

Furthermore, we consider the third term of (5.3)

$$\left\|\sigma\left(\mathfrak{D}^2,\mathfrak{D}^2\right)\right\|^2=g\left(\sigma\left(\mathfrak{D}^2,\mathfrak{D}^2\right),F\mathfrak{D}^1\right)^2+g\left(\sigma\left(\mathfrak{D}^2,\mathfrak{D}^2\right),F\mathfrak{D}^2\right)^2+g\left(\sigma\left(\mathfrak{D}^2,\mathfrak{D}^2\right),\nu\right)^2.\tag{5.11}$$

Dropping the no relations for the warped products of the second and third positive terms in (5.11), then we have

$$\left\|\sigma\left(\mathfrak{D}^2,\mathfrak{D}^2\right)\right\|^2\geq csc^2\theta_1\sum_{i=1}^{2p}\sum_{k,r=1}^{2q}g\left(\sigma\left(\hat{e}_k,\hat{e}_r\right),Fe_i\right)^2.\tag{5.12}$$

Combining (5.5), (5.10) and (5.12), we conclude that

$$\begin{aligned}\|\sigma\|^2\geq&\left(csc^2\theta_2+2csc^2\theta_1\right)\sum_{i,j=1}^{2p}\sum_{r=1}^{2q}g\left(\sigma\left(e_i,\hat{e}_r\right),Fe_j\right)^2\\&+\left(2csc^2\theta_2+csc^2\theta_1\right)\sum_{i=1}^{2p}\sum_{k,r=1}^{2q}g\left(\sigma\left(\hat{e}_k,\hat{e}_r\right),Fe_i\right)^2\\&+\left(4qcsc^2\theta_2cos^2\theta_1+4qcot^2\theta_2\right)\left(\left\|\nabla lnf\right\|^2-1\right).\end{aligned}\tag{5.13}$$

Since M is proper, the inequality (5.2) follows from (5.11) by leaving the first and second positive terms.

If the equality holds in (5.2), then from all the dropped terms in the right hand side of (5.4), (5.6), (5.11) and (5.13), we deduce to

$$\sigma\left(\mathfrak{D}^1,\mathfrak{D}^1\right)=0,\quad\sigma\left(\mathfrak{D}^2,\mathfrak{D}^2\right)=0,\quad\sigma\left(\mathfrak{D}^1,\mathfrak{D}^2\right)\subset F\mathfrak{D}^2.\tag{5.14}$$

If we denote the second fundamental forms of M_1 or M_2 in M and in $\widetilde{M}$ by $\hat{\sigma}$ and $\widetilde{\sigma}$, respectively, then we have $\widetilde{\sigma}=\sigma+\hat{\sigma}$, using this fact with (ii) of Remark 4.1, we conclude that M_1 is a totally geodesic submanifold and M_2 is a totally umbilical submanifold of $\widetilde{M}$. Also, from the definition 2.1, it is clear that M is minimal. Hence, we complete the proof of Theorem 5.2. □

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