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Existence and uniqueness of ϱ-mild solutions for integrodifferential equations with state-dependent nonlocal conditions

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Abstract. In this paper, we present some results related to the existence and uniqueness of mild solutions for a class of integrodifferential equations with state-dependent nonlocal conditions using resolvent operators. We assume the noncompactness condition for both the resolvent operator and the state-dependent function in the nonlocal condition. Our results were derived through the application of the ϱ-norm, measures of noncompactness, and fixed-point theory. In the final section, we illustrate our obtained results with an example to demonstrate their applicability.

1. Introduction

In this work, we focus on the existence and uniqueness of solutions for the following integrodifferential equation with state-dependent nonlocal conditions:

$$\begin{aligned} \zeta'(\delta) &= A\zeta(\delta) + \int_0^\delta \Upsilon(\delta - s)\zeta(s)ds + \Psi(\delta, \zeta(\delta)), \quad \delta \in J := [0, a], \\ \zeta(0) &= \zeta_0 + H(\sigma(\zeta), \zeta), \end{aligned} \tag{1}$$

where the state variable $\zeta(\cdot)$ takes values within the Banach space X. $A : \mathcal{D}(A) \subset X \to X$ denotes the closed and linear operator, which infinitesimally generates an analytical semigroup on X, denoted as $(S(\delta))_{\delta\geq 0}$. Additionally, $(\Upsilon(\delta))_{\delta\geq 0}$ represents a family of operators that are closed and linear, with domain $\mathcal{D}(A) \subset \mathcal{D}(\Upsilon(\delta))$. The functions $\Psi : J \times X_\varrho \to X$, $H : [0, a] \times C([0, a]; X_\varrho) \to X_\varrho$, and $\sigma : C([0, a]; X_\varrho) \to [0, a]$ are given and will be determined later. Here, X_ϱ denotes the domain where the operator A^ϱ is defined for each $0 < \varrho \leq 1$, and it is equipped with an appropriate norm described later.

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In recent years, partial integrodifferential equations (PIDEs) have become an active research topic due to their ability to accurately model complex systems in various fields, such as chemical engineering, image processing, fluid dynamics, and so on. One notable example is the heat conduction model that describes the behavior of materials demonstrating fading memory effects and viscoelasticity (see [23, 31, 32, 36]). PIDEs combine integral equations with partial differential equations (PDEs), resulting in improved practical significance and accuracy compared to classical differential equations. For further investigation of PIDE models and their applications, refer to [1, 27, 37].

The theory of the resolvent operator has emerged as a powerful and significantly important tool for investigating integrodifferential evolution equations. Grimmer [24] applied the theory of resolvent operator to investigate the existence results for a class of evolution equations of the following form:

$$\begin{aligned} \zeta'(\delta) &= A\zeta(\delta) + \int_0^\delta \Upsilon(\delta - s)\zeta(s)\, ds + g(\delta), \quad \delta \geq 0, \\ \zeta(0) &= \zeta_0 \in X, \end{aligned} \tag{2}$$

where $g : \mathbb{R}^+ \to X$ represents a continuous function. Further, Grimmer used resolvent operators to represent the solutions of (2) in [25, 26]. The resolvent operator is crucial in resolving problem (2) in both strict and weak senses, effectively substituting the need for a C_0-semigroup in the context of evolution equations [10]. Up to now, through the application of resolvent operator theory, extensive research has been done regarding various topics related to integrodifferential evolution equations, see the works [4, 7, 18, 28, 39].

However, the Cauchy initial condition $\zeta(0) = \zeta_0$ seems to exhibit limitations in the description of real-world scenarios. To address this issue, Byszewski [8] introduced the concept of nonlocal conditions in 1991 and initiated the nonlocal Cauchy problems of evolution equations. Nonlocal conditions have a crucial impact on practical physical problems. For example, they provide a more accurate description of the diffusion phenomenon compared to the classical initial conditions [13]. So far, the study of evolution equations with nonlocal initial conditions has drawn extensive attention, and many interesting results have been achieved, see [6, 9, 12, 17, 30, 34]. For instance, Hernández and O'Regan in [29] established some results on the existence and uniqueness of solutions for the following abstract differential equation under the state-dependent nonlocal conditions:

$$\begin{aligned} \zeta'(\delta) &= A\zeta(\delta) + F(\delta, \zeta(x(\delta))), \quad \delta \in [0, a], \\ \zeta(0) &= H(\sigma(\zeta), \zeta). \end{aligned} \tag{3}$$

In [16], Ezzinbi and Ghnimi considered the problem of the existence and regularity of solutions of the following integrodifferential equation with nonlocal conditions:

$$\begin{aligned} \zeta'(\delta) &= A\zeta(\delta) + \int_0^\delta \Upsilon(\delta - s)\zeta(s)\, ds + \Psi(\delta, \zeta(\delta)), \quad \delta \in [0, a], \\ \zeta(0) &= \zeta_0 + g(\zeta). \end{aligned} \tag{4}$$

Further, El Matloub and Ezzinbi [19] addressed the existence of mild solutions to problem (4) using the fractional power operator under the assumptions of compactness of the resolvent operator and g. Recent research on PIDEs under nonlocal conditions has been carried out by many researchers, see works [2, 9, 40] and references therein. Hence, the study and investigation of PIDEs subject to state-dependent nonlocal conditions hold great practical significance, which motivates our present research.

Inspired by the aforementioned works, our primary focus in this paper is to investigate the problem of the existence and uniqueness of mild solutions for the equation (1) with state-dependent nonlocal conditions. These nonlocal conditions in problem (1) are state-dependent, involving different types of nonlocal conditions, thus generalizing many conditions studied in the literature [12, 19]. We mainly address the problem in the space X_ϱ (for each $0 < \varrho \leq 1, X_\varrho \subset X$), for which we apply the theory of fractional power operators and ϱ-norms. We establish the existence and uniqueness of mild solutions for problem (1) using the Banach contraction theorem. Then, we prove the existence of at least one solution by applying Monch's fixed point theorem, all without assuming the compactness of the resolvent operator or imposing conditions on the compactness of the function $H(\cdot,\cdot)$ in the nonlocal condition. Our approach mainly involves the application of the resolvent operator theory introduced by Grimmer, the measures of noncompactness in the sense of Kuratowski, and fixed point theory.

This article is organized briefly as follows. In Section 2, we review some concepts and preliminary facts used in the main results. In Section 3, we present our main results, where we investigate the existence and uniqueness of mild solutions of problem (1). In the last section, we provide an illustrative example to demonstrate the applicability of our proposed methodology.

2. Preliminaries

In this section, we first present some definitions and initial results that will be necessary for the subsequent discussion.

Throughout this paper, let $A : \mathcal{D}(A) \subset X \to X$ the closed and linear operator, which infinitesimally generates an analytical semigroup $(S(\delta))_{\delta\geq 0}$ on X. Let $\rho(A)$ represent the resolvent set of A under the assumption that it contains spectral eigenvalue 0, i.e. $0 \in \rho(A)$. Thus, for $\varrho \in (0,1]$, the fractional power A^ϱ can be defined as a linear, invertible, and closed operator on its dense set $\mathcal{D}(A^\varrho)$, and its corresponding norm on $\mathcal{D}(A^\varrho)$ is defined as

$$\|\zeta\|_\varrho = \|A^\varrho\zeta\|, \text{ for } \zeta \in \mathcal{D}(A^\varrho).$$

Let X_ϱ denote the Banach space $\left(\mathcal{D}(A^\varrho), \|\cdot\|_\varrho\right)$, for each $\varrho \in (0,1]$. Moreover, $X_\varrho \hookrightarrow X_\beta$ for $0 < \beta < \varrho \leq 1$ and the imbedding is compact whenever the resolvent of A denoted by $\mathfrak{R}(\lambda, A) = (\lambda I - A)^{-1}$ is compact. Let $\mathcal{B}(X_\varrho; X_\beta)$ stands for the space of linear operators that are bounded from $X_\varrho \hookrightarrow X_\beta$ and we use the abbreviation $\mathcal{B}(X)$ for this notation for $X_\varrho = X_\beta$. In addition, $C([0,a], X_\varrho)$ denotes the Banach space of all continuous functions from $[0,a]$ to X_ϱ equipped with the norm

$$\|\zeta\|_C = \sup_{\delta\in[0,a]} \|A^\varrho\zeta(\delta)\|, \ \zeta \in C([0,a], X_\varrho).$$

We refer [35] for further details on fractional power operators.

Next, we review some basic theory of resolvent operators introduced in [11, 24, 25]. The role of the resolvent operator is crucial for examining the existence of solutions for (1). This theory will serve as the main tool in the subsequent sections.

Definition 2.1. *([24]) A family $(\mathfrak{R}(\delta))_{\delta\geq 0}$ of bounded linear operators on X is referred to as a resolvent operator for*

$$\begin{aligned} \zeta'(\delta) &= A\zeta(\delta) + \int_0^\delta \Upsilon(\delta - s)\zeta(s)\, ds, \quad \delta \geq 0, \\ \zeta(0) &= \zeta_0 \in X, \end{aligned} \tag{5}$$

if it satisfies the following properties:

(a) For some constants $M \geq 1$ and $b \in \mathbb{R}$, $\mathfrak{R}(0) = I$ and $\|\mathfrak{R}(\delta)\| \leq Me^{bt}$.

(b) For each $\zeta \in X$ and $\delta \geq 0$, $\mathfrak{R}(\delta)\zeta$ is continuous.

(c) For $\zeta \in Y$, $\mathfrak{R}(\cdot)\zeta \in C^1([0,a]; X) \cap C([0,a]; Y)$ such that for $\delta \geq 0$, we have

$$\mathfrak{R}'(\delta)\zeta = A\mathfrak{R}(\delta)\zeta + \int_0^\delta \Upsilon(\delta - s)\mathfrak{R}(s)\zeta\, ds$$

$$= \quad \mathfrak{R}(\delta)A\zeta + \int_0^\delta \mathfrak{R}(\delta - s)\Upsilon(s)\zeta\, ds.$$

In what follows, we denote Υ^* the Laplace transform of Υ, and we shall assume the following hypotheses on the operators A and $(\Upsilon(\delta))_{\delta\geq 0}$ associated to (5) [26]:

($\mathcal{H}_1'$) The operator A generates an analytic semigroup on X. $(\Upsilon(\delta))_{\delta\geq 0}$ is a family of closed and linear operators on X with a domain at least $D(A)$ for a.e. $\delta \geq 0$, with $\Upsilon(\delta)y$ is strongly measurable for all $y \in D(A)$, and $\delta \geq 0$ and $\|\Upsilon(\delta)\zeta\| \leq \eta(\delta)\|\zeta\|$ for $\eta \in L^1_{loc}(0, +\infty)$ with $\eta^*(\lambda)$ absolutely convergent for $Re\lambda > 0$.

($\mathcal{H}_2'$) There exists $\rho(\lambda) := (\lambda I - A - \Upsilon^*(\lambda))^{-1}$ a bounded operator on X which is analytic for λ in the region Λ defined as

$$\Lambda = \left\{\lambda \in \mathbb{C} : |\arg(\lambda)| < \frac{\pi}{2} + b\right\},$$

for $b \in (0, \pi/2)$. In Λ, if $0 < \epsilon \leq |\lambda|$, there exists a constant $M = M(\epsilon) > 0$ such that $\|\rho(\lambda)\| \leq M|\lambda|^{-1}$.

($\mathcal{H}_3'$) $A\rho(\lambda) \in \mathcal{B}(X)$, for some $\lambda \in \Lambda$ are analytic on Λ into $\mathcal{B}(X)$. Further, for $\lambda \in \Lambda$, $\Upsilon^*(\lambda)$ belongs to $\mathcal{B}(Y, X)$ and $\Upsilon^*(\lambda)\rho(\lambda) \in \mathcal{B}(Y, X)$. Given $\epsilon > 0$, there exists $M = M(\epsilon) > 0$ so that for $\lambda \in \Lambda$ with $|\lambda| \geq \epsilon$,

$$\|A\rho(\lambda)\|_{1,0} + \|\Upsilon^*(\lambda)\rho(\lambda)\|_{1,0} \leq M|\lambda|^{-1},$$

and

$$\|\Upsilon^*(\lambda)\|_{1,0} \to 0 \text{ as } |\lambda| \to \infty \text{ in } \Lambda.$$

In addition, $\|A\rho(\lambda)\| \leq M|\lambda|^{-n}$ for some $n > 0$, and $\lambda \in \Lambda$ with $|\lambda| \geq \epsilon$. Further, there exists $D \subset \mathcal{D}(A^2)$ which is dense in Y such that $A(D)$ and $\Upsilon^*(\lambda)(D)$ are contained in $Y$and $\|\Upsilon^*(\lambda)\zeta\|_1$ is bounded for each $\zeta \in D$ and $\lambda \in \Lambda$ with $|\lambda| \geq \epsilon$.

Then, based on [26], it follows that the above mentioned hypotheses ($\mathcal{H}_1'$) − ($\mathcal{H}_3'$) are satisfied, and hence there exists a resolvent operator $(\mathfrak{R}(\delta))_{\delta\geq 0}$ for system (5) which is defined by

$$\mathfrak{R}(\delta)\zeta = \begin{cases} (2\pi i)^{-1} \int_\Gamma e^{\lambda\delta}\rho(\lambda)\zeta\, d\lambda, & \delta > 0, \\ I, & \delta = 0. \end{cases}$$

where $\rho(\lambda) = (\lambda I - A - \Upsilon^*(\lambda))^{-1}$ and Γ denotes the a contour type employed to acquire an analytic semigroup. The selection of a contour Γ to be included in the regions Λ formed by Γ_i, $i = 1, 2, 3$ that are given respectively by

$$\begin{aligned} \Gamma_1 &= \left\{re^{i\Phi} : r \geq 1\right\}, \\ \Gamma_2 &= \left\{e^{i\theta} : -\phi \leq \theta \leq \phi\right\}, \\ \Gamma_3 &= \left\{re^{-i\Phi} : r \geq 1\right\}, \end{aligned}$$

for $\phi \in \left(\frac{\pi}{2}, \frac{\pi}{2} + b\right)$. The above curves are oriented in such a way that the imaginary part of λ is increasing on Γ_1 and Γ_2. Furthermore, the operator $\mathfrak{R}(\delta)$ is an analytic and there exists some constants $M, M_\varrho > 0$ such that for $\delta \in (0, a]$ and $\varrho \in [0, 1]$,

$$\|\mathfrak{R}(\delta)\| \leq M \text{ and } \|A^\varrho\mathfrak{R}(\delta)\| \leq M_\varrho\delta^{-\varrho}.$$

Lemma 2.2. *([21]) For $\delta > 0$, $\mathfrak{R}(\delta)$ is continuous in the uniform operator topology of $\mathcal{B}(X)$.*

Lemma 2.3. *([20]) For $\delta > 0$, $A\mathfrak{R}(\delta)$ is continuous in the uniform operator topology of $\mathcal{B}(X)$.*

Remark 2.4. *Generally speaking, A^ϱ and $\mathfrak{R}(\delta)$ do not necessarily commute. However, their commutativity can be achieved under the following assumption:*

$(\mathcal{H}_4')$ *If* $\Upsilon^*(\lambda)A^{-\varrho}\zeta = A^{-\varrho}\Upsilon(\lambda)^*\zeta$, *for any* $\zeta \in \mathcal{D}(A)$, *then for* $\zeta \in \mathcal{D}(A^{\varrho})$,

$$A^{\varrho}\mathfrak{R}(\delta)\zeta = \mathfrak{R}(\delta)A^{\varrho}\zeta,$$

see [15] for further details. For simplicity, we always assume that this condition is valid.

In the following, we recall some fundamental aspects of the Kuratowski measure of noncompactness that are necessary to establish our main results. We denote Ω_X as the family of all bounded subsets of a Banach space X.

Definition 2.5. *([5]) The Kuratowski measure of noncompactness of a subset* Θ *of* X *is a function* $\mu : \Omega_X \to \mathbb{R}^+$ *defined as*

$$\mu(\Theta) = \inf\Big\{\epsilon > 0 : \Theta \subseteq \cup_{i=1}^{n}\Theta_i \text{ with } diam(\Theta_i) \leq \epsilon\Big\}, \ \ \forall \Theta \in \Omega_X.$$

The following lemma defines certain properties of the Kuratowski measure of noncompactness, which can be demonstrated using the definition provided above:

Lemma 2.6. *([5]) Let* $\Theta_1, \Theta_2 \in \Omega_X$. *Then:*

(p_1) $\mu(\Theta_1) = 0$ *if and only if* Θ_1 *is relatively compact (i.e.,* $\overline{\Theta_1}$ *is compact);*

(p_2) $\mu(\Theta_1) = \mu(\overline{\Theta_1})$;

(p_3) $\Theta_1 \subset \Theta_2$ *implies* $\mu(\Theta_1) \leq \mu(\Theta_2)$;

(p_4) $\mu(\Theta_1 + \Theta_2) \leq \mu(\Theta_1) + \mu(\Theta_2)$.

(p_5) $\mu(\Theta_1 \cup \Theta_2) = max\{\mu(\Theta_1), \mu(\Theta_2)\}$;

(p_6) *for any* $c \in \mathbb{R}$, $\mu(\Theta_1) = |c|\mu(\Theta_1)$;

(p_7) *for any* $r \in \mathbb{R}$, $\mu(\{r\} \cup \Theta_1) = \mu(\Theta_1)$;

(p_8) $\mu(\overline{Conv}(\Theta_1)) = \mu(\Theta_1)$, *where* $\overline{Conv}(\Theta_1)$ *represents the closed convex hull of a set* Θ_1.

Theorem 2.7. *([3, 33]) Let* X *be a Banach space, and let* Θ *be a closed, bounded, and convex subset of* X *containing the origin* $(0 \in \Theta)$. *Consider a continuous mapping* $\nabla : \Theta \to \Theta$. *If, for every subset* $\mathfrak{I}$ *of* Θ, *the following implication holds:*

$$\mathfrak{I} = \overline{Conv}\nabla(\mathfrak{I}) \ \ \text{or} \ \ \mathfrak{I} = \nabla(\mathfrak{I}) \cup \{0\} \ \ \text{implies} \ \ \mu(\mathfrak{I}) = 0,$$

then, the mapping ∇ *possesses a fixed point.*

Lemma 2.8. *([22]) If* Θ *is a bounded and equicontinuous subset of* $C([0, a], X)$, *then*

(i) $\mu(\Theta) = \sup_{\delta \in [0,a]} \mu(\Theta(\delta))$.

(ii) For $\delta \in J$, $\mu\left(\left\{\int_0^a \zeta(s)ds : \zeta \in \Theta\right\}\right) \leq \int_0^a \mu(\Theta(s))ds$
with

$$\Theta(s) = \{\zeta(s) : \zeta \in \Theta\}, \ s \in J.$$

3. Main results

In this section, we first examine the existence and uniqueness of ϱ-mild solutions of the nonlocal problem (1). These solutions are expressed based on the resolvent operator, as defined below.

Definition 3.1. *A function $\zeta \in C([0,a], X_\varrho)$ is said to be a mild solution of problem* (1) *if*

$$\zeta(\delta) = \mathfrak{R}(\delta)\left(\zeta_0 + H(\sigma(\zeta),\zeta)\right) + \int_0^\delta \mathfrak{R}(\delta - s)\Psi(s,\zeta(s))ds, \quad \textit{for } \delta \in [0,a]. \tag{6}$$

In order to formulate our first existence and uniqueness results, we need the following hypothesis:

($\mathcal{H}_1$) The function $\Psi : [0,a]\times X_\varrho \to X$ satisfies the following conditions:

- (i) $\Psi(\cdot,\zeta)$ is measurable for $\zeta \in X_\varrho$ and $\Psi(\delta,\cdot)$ is continuous for a. e. $\delta \in [0,a]$;
- (ii) there exists $p(\cdot) \in L^\infty([0,a])$ such that

$$\|\Psi(\delta,\zeta_1) - \Psi(\delta,\zeta_2)\| \le p(\delta)\|\zeta_1 - \zeta_2\|_\varrho, \ \ \zeta_1,\zeta_2 \in X_\varrho;$$

- (iii) the function $p_1 : \delta \to p_1(\delta) = \|\Psi(\delta,0)\|$ belongs to $L^\infty([0,a])$.

($\mathcal{H}_2$) The functions $H(\cdot,\cdot) : [0,a]\times C([0,a],X_\varrho) \to X_\varrho$ and $\sigma(\cdot) : C([0,a],X_\varrho) \to [0,a]$ are continuous, and there exist $q, q^* > 0$ such that

$$\|H(\sigma(\zeta_1),\zeta_1) - H(\sigma(\zeta_2),\zeta_2)\|_\varrho \le q\|\zeta_1 - \zeta_2\|_C, \ \ \zeta_1,\zeta_2 \in C([0,a],X_\varrho)$$

and

$$\|H(\sigma(\zeta),0)\|_\varrho \le q^*.$$

Set $p^* = \|p\|_{L^\infty}$.

($\mathcal{H}_3$) $\Upsilon(\delta) \in \mathcal{B}(X_\beta, X)$ for some $0 < \beta < 1$, a. e. $\delta \ge 0$ and $\|\Upsilon(\delta)y\| \le b(\theta)\|y\|_\beta$

$$y \in X_\beta, \text{ with } b \in L^r_{loc}(0;\infty), \text{ where } r > \frac{1}{1-\beta}.$$

Theorem 3.2. *Assume that the conditions* ($\mathcal{H}_1$) – ($\mathcal{H}_3$) *are satisfied. If*

$$Mq + p^* M_\varrho \left(\frac{a^{-\varrho+1}}{-\varrho+1}\right) < 1, \tag{7}$$

then the problem (1) *has a unique mild solution $\zeta(\cdot) \in C([0,a],X_\varrho)$.*

Proof. We define the operator ∇ on $C([0,a],X_\varrho)$ as

$$(\nabla\zeta)(\delta) = \mathfrak{R}(\delta)(\zeta_0 + H(\sigma(\zeta),\zeta)) + \int_0^\delta \mathfrak{R}(\delta - s)\Psi(s,\zeta(s))\,ds, \ \text{ for } \ \delta \in [0,a]. \tag{8}$$

It is clear that the fixed point of the operator ∇ is exactly a mild solution to the problem (1). Thus, in what follows, we shall prove that the operator ∇ admits a unique fixed point. To do so, we have to show that the operator ∇ is well-defined and contractive.

First, we show that the operator ∇ is well-defined on $C([0,a],X_\varrho)$. In fact, let $\zeta \in C([0,a],X_\varrho)$, then for any $\delta \in [0,a]$, we get

$$\|\nabla\zeta(\delta)\|_\varrho \le \left\|\mathfrak{R}(\delta)\left(\zeta_0 + H(\sigma(\zeta),\zeta)\right) + \int_0^\delta \mathfrak{R}(\delta - s)\Psi(s,\zeta(s))\,ds\right\|_\varrho$$

$$\leq \|\Re(\delta)\|\Big(\|\zeta_0\|_\varrho + \|H(\sigma(\zeta),\zeta)\|_\varrho\Big) + M_\varrho \int_0^\delta \frac{1}{(\delta-s)^\varrho}\|\Psi(s,\zeta(s))\|\, ds.$$

From assumptions $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$, it yields that for $\delta \in [0,a]$

$$\begin{aligned}
\|\nabla\zeta(\delta)\|_\varrho &\leq M\Big(\|\zeta_0\|_\varrho + \|H(\sigma(\zeta),\zeta) - H(\sigma(\zeta),0) + H(\sigma(\zeta),0)\|_\varrho\Big)\\
&\quad + M_\varrho \int_0^\delta \frac{1}{(\delta-s)^\varrho}\,\|\Psi(s,\zeta(s)) - \Psi(s,0) + \Psi(s,0)\|\ \, ds\\
&\leq M\Big(\|\zeta_0\|_\varrho + \|H(\sigma(\zeta),\zeta) - H(\sigma(\zeta),0)\|_\varrho + \|H(\sigma(\zeta),0)\|_\varrho\Big)\\
&\quad + M_\varrho \int_0^\delta \frac{1}{(\delta-s)^\varrho}\|\Psi(s,\zeta(s)) - \Psi(s,0)\|\, ds\\
&\quad + M_\varrho \int_0^\delta \frac{1}{(\delta-s)^\varrho}\|\Psi(s,0)\|\, ds\\
&\leq M\left(\|\zeta_0\|_\varrho + q\|\zeta\|_C + q^*\right) + M_\varrho \int_0^\delta \frac{1}{(\delta-s)^\varrho} p(s)\|\zeta(s)\|_\varrho\, ds\\
&\quad + M_\varrho \int_0^\delta \frac{1}{(\delta-s)^\varrho} p_1(s)\, ds\\
&\leq M\left(\|\zeta_0\|_\varrho + q\|\zeta\|_C + q^*\right) + M_\varrho p^* \frac{a^{1-\varrho}}{1-\varrho}\|\zeta\|_C + M_\varrho p_1^* \frac{a^{1-\varrho}}{1-\varrho}\\
&\leq M\left(\|\zeta_0\|_\varrho + q^*\right) + M_\varrho p_1^* \frac{a^{1-\varrho}}{1-\varrho} + \Big(Mq + M_\varrho p^* \frac{a^{1-\varrho}}{1-\varrho}\Big)\|\zeta\|_C,
\end{aligned}$$

then,

$$\begin{aligned}
\|\nabla\zeta\|_C &\leq M\left(\|\zeta_0\|_\varrho + q^*\right) + M_\varrho p_1^* \frac{a^{1-\varrho}}{1-\varrho} + \Big(Mq + M_\varrho p^* \frac{a^{1-\varrho}}{1-\varrho}\Big)\|\zeta\|_C\\
&< +\infty.
\end{aligned}$$

This means that the function $\nabla\zeta \in C([0,a], X_\varrho)$.

Next, we prove that ∇ is a contraction mapping. Indeed, let $\zeta, \xi \in C([0,a], X_\varrho)$; then, for any $\delta \in [0,a]$, by applying assumptions $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$, we obtain

$$\begin{aligned}
\|(\nabla\zeta)(\delta) - (\nabla\xi)(\delta)\|_\varrho &= \Big\|\Re(\delta)\Big(H(\sigma(\zeta),\zeta) - H(\sigma(\xi),\xi)\Big)\\
&\qquad + \int_0^\delta \Re(\delta-s)\Big(\Psi(s,\zeta(s)) - \Psi(s,\xi(s))\Big)\, ds\Big\|_\varrho\\
&\leq \|\Re(\delta)\|\|H(\sigma(\zeta),\zeta)) - H(\sigma(\xi),\xi))\|_\varrho\\
&\quad + \Big\|\int_0^\delta \Re(\delta-s)\Big(\Psi(s,\zeta(s)) - \Psi(s,\xi(s))\Big)\, ds\Big\|_\varrho\\
&\leq Mq\|\zeta-\xi\|_C + M_\varrho \int_0^\delta \frac{1}{(\delta-s)^\varrho} p(s)\|\zeta(s)-\xi(s)\|_\varrho\, ds\\
&\leq Mq\|\zeta-\xi\|_C + M_\varrho p^* \frac{a^{1-\varrho}}{1-\varrho}\|\zeta-\xi\|_C\\
&\leq \Big(Mq + M_\varrho p^* \frac{a^{1-\varrho}}{1-\varrho}\Big)\|\zeta-\xi\|_C.
\end{aligned}$$

Therefore,

$$\|(\nabla\zeta)-(\nabla\xi)\|_C \leq \left(Mq + M_\varrho p^* \left(\frac{a^{1-\varrho}}{1-\varrho}\right)\right)\|\zeta-\xi\|_C,$$

for all $\zeta, \xi \in C([0,a], X_\varrho)$. Hence, from (7), ∇ is a contraction. By the Banach fixed point theorem, ∇ has a unique fixed point ζ, which corresponds to the unique mild solution of problem (1) in $C([0,a], X_\varrho)$. □

To get the next result, we apply the Mönch fixed point theorem to problem (1). For this, we impose the following assumptions:

($\mathcal{H}_4$) $\{\mathfrak{R}(\delta)\}_{\delta\geq 0}$ is uniformly continuous.

($\mathcal{H}_5$) There exists a function $p(\cdot) \in L^\infty(J)$ such that

$$\|\Psi(\delta,\zeta)\| \leq p(\delta)(\|\zeta\|_\varrho + 1) \quad \text{for a.e. } \delta \in [0,a] \text{ and } \zeta \in X_\varrho.$$

($\mathcal{H}_6$) There exists a function $g(\cdot) \in L^1([0,a], \mathbb{R}^+)$ such that, for any bounded set $D_1 \subset X_\varrho$,

$$\mu(\Psi(\delta, D_1)) \leq g(\delta)\mu(D_1) \text{ for a.e. } \delta \in [0,a].$$

($\mathcal{H}_7$) $Mq + M_\varrho p^* \dfrac{a^{1-\varrho}}{1-\varrho} < 1$.

($\mathcal{H}_8$) There exists $0 < K < \dfrac{1}{M}$ such that

$$\mu(H(\sigma(D_2), D_2)) \leq K \sup_{\theta\in[0,a]} \mu(D_2(\theta)),$$

for any bounded $D_2 \subset C([0,a], X_\varrho)$, where μ represents the measure of noncompactness on the Banach space X_ϱ.

We consider the Kuratowski measure of noncompactness, denoted by μ_c, defined on the family of bounded subsets of the space $C([0,a], X_\varrho)$ as follows:

$$\mu_c(D) = m(D) + \sup_{\delta\in[0,a]} e^{-\tau\int_0^\delta g(\xi)\,d\xi}\mu(D(\delta)),$$

where $m(D)$ is the modulus of continuity, and $\tau > \dfrac{M}{1-KM}$.

Remark 3.3. *Notice that $m(D) = 0$ when the bounded set D is equicontinuous.*

Theorem 3.4. *Assume that the above conditions $(\mathcal{H}_1)(i)$, $(\mathcal{H}_2)-(\mathcal{H}_8)$ hold. If*

$$M\left(\frac{1}{\tau} + K\right) < 1, \tag{9}$$

then the problem (1) *has at least one mild solution.*

Proof. Consider the same operator $\nabla : C([0,a], X_\varrho) \to C([0,a], X_\varrho)$ as defined in the proof of Theorem 3.2. It is evident that the solutions to problem (1) correspond to the fixed points of this operator. By utilizing the Mönch fixed point theorem, we establish the existence of at least one fixed point for ∇. We will demonstrate that the conditions of Mönch fixed point theorem (Theorem 2.7) are satisfied.

Consider the ball

$$B_r = \Big\{\zeta(\cdot) \in C([0,a], X_\varrho) : \|\zeta\|_{C=C([0,a],X_\varrho)} = \sup_{\delta\in[0,a]} \|\zeta(\delta)\|_\varrho \le r\Big\}.$$

It is obvious that B_r is a bounded, closed, and convex set.

Initially, we show that the operator ∇ is well-defined. We proceed in the same way as in the proof of Theorem 3.2.

Next, we prove that there exists a positive r such that $\nabla(B_r) \subset B_r$. Suppose the contrary, that this is not true. Then, for each $r > 0$, there exists $\zeta_r \in B_r$, and $\delta_r \in (0, a]$ such that $\|(\nabla\zeta_r)(\delta_r)\|_\varrho > r$. By assumptions $(\mathcal{H}_2)$ and $(\mathcal{H}_5)$, one has

$$\begin{aligned} r < \|(\nabla\zeta_r)(\delta_r)\|_\varrho &= \Big\|\mathfrak{R}(\delta_r)\Big(\zeta_0 + H(\sigma(\zeta_r),\zeta_r)\Big) + \int_0^\delta \mathfrak{R}(\delta_r - s)\Psi(s,\zeta_r(s))\,ds\Big\|_\varrho \\ &\le M\|\zeta_0 + H(\sigma(\zeta_r),\zeta_r)\|_\varrho + \int_0^{\delta_r} \|A^\varrho\mathfrak{R}(\delta_r - s)\|\|\Psi(s,\zeta_r(s))\|\,ds \\ &\le M\Big(\|\zeta_0\|_\varrho + q\|\zeta_r\|_C + q^*\Big) + M_\varrho\int_0^{\delta_r} \frac{1}{(\delta_r - s)^\varrho} p(s)\Big(\|\zeta_r(s)\|_\varrho + 1\Big)\,ds \\ &\le M\Big(\|\zeta_0\|_\varrho + qr + q^*\Big) + M_\varrho p^* \frac{a^{1-\varrho}}{1-\varrho}\Big(r+1\Big) \\ &\le M\Big(\|\zeta_0\|_\varrho + q^*\Big) + M_\varrho p^*\frac{a^{1-\varrho}}{1-\varrho} + \Big(Mq + M_\varrho p^*\frac{a^{1-\varrho}}{1-\varrho}\Big)r. \end{aligned} \tag{10}$$

Dividing both sides of the inequality (10) by r and taking the lower limit as $r \to +\infty$, we obtain that

$$1 \le Mq + M_\varrho p^* \frac{a^{1-\varrho}}{1-\varrho},$$

which contradicts the assumption $(\mathcal{H}_7)$. Therefore, there exists $r > 0$ such that $\nabla(B_r) \subset B_r$.

Further, we prove that ∇ is continuous on B_r. In fact, let $(\zeta^{(n)})_{n\in\mathbb{N}}$ be a sequence in B_r such that $\zeta^{(n)} \to \zeta$ as $n \to +\infty$ in $C = C([0,a], X_\varrho)$, for some $\zeta \in B_r$, that is

$$\|\zeta^{(n)} - \zeta\|_C = \sup_{t\in[0,a]} \|A^\varrho\Big(\zeta^{(n)}(\delta) - \zeta(\delta)\Big)\| \to 0 \text{ as } n \to +\infty.$$

From assumptions $(\mathcal{H}_1)(i)$ and $(\mathcal{H}_2)$, we have

$$\Psi(s,\zeta^{(n)}(s)) \to \Psi(s,\zeta(s)) \text{ as } n \to +\infty, \text{ for a.e. } s \in [0,a], \tag{11}$$

and

$$H(\sigma(\zeta^{(n)}),\zeta^{(n)}) \to H(\sigma(\zeta),\zeta) \text{ as } n \to +\infty. \tag{12}$$

Note that, by assumption $(\mathcal{H}_5)$, we have

$$\Big\|\Psi(s,\zeta^{(n)}(s)) - \Psi(s,\zeta(s))\Big\| \le 2p(s)\Big(\|\zeta\|_\varrho + 1\Big) \le 2p(s)(r+1). \tag{13}$$

Thus,

$$\begin{aligned} \Big\|(\nabla\zeta^{(n)})(\delta) - (\nabla\zeta)(\delta)\Big\|_\varrho &\le M\|H(\sigma(\zeta^{(n)}),\zeta^{(n)}) - H(\sigma(\zeta),\zeta)\|_\varrho \\ &\quad + \int_0^\delta \Big\|\mathfrak{R}(\delta - s)\Big[\Psi(s,\zeta^{(n)}(s)) - \Psi(s,\zeta(s))\Big]\Big\|_\varrho\,ds \end{aligned}$$

$$\leq M\|H(\sigma(\zeta^{(n)}),\zeta^{(n)})-H(\sigma(\zeta),\zeta)\|_{\varrho} +M_{\varrho}\int_0^{\delta}\frac{1}{(\delta-s)^{\varrho}}\left\|\Psi(s,\zeta^{(n)}(s))-\Psi(s,\zeta(s))\right\|ds.$$

Then, from (11), (12), (13), and by using the Lebesgue dominated convergence theorem, we get that

$$\left\|\nabla\zeta^{(n)}-\nabla\zeta\right\|_C=\sup_{\delta\in[0,a]}\left\|(\nabla\zeta^{(n)})(\delta)-(\nabla\zeta)(\delta)\right\|_{\varrho}\to 0 \text{ as } n\to+\infty,$$

i.e. ∇ is continuous.

Now, we show that $\nabla(B_r)$ is equicontinuous. Let $\zeta\in B_r$, $\delta_1,\delta_2\in[0,a]$ such that $\delta_1<\delta_2$. Thus, we have

$$\begin{aligned}
&\|(\nabla\zeta)(\delta_2)-(\nabla\zeta)(\delta_1)\|_{\varrho}\\
&\leq \|\big(\mathfrak{R}(\delta_2)-\mathfrak{R}(\delta_1)\big)\big(\zeta_0+H(\sigma(\zeta),\zeta)\big)\|_{\varrho}+\int_{\delta_1}^{\delta_2}\|\mathfrak{R}(\delta_2-s)\Psi(s,\zeta(s))\|_{\varrho}\,ds\\
&\quad+\int_0^{\delta_1}\|[\mathfrak{R}(\delta_2-s)-\mathfrak{R}(\delta_1-s)]\Psi(s,\zeta(s))\|_{\varrho}\,ds\\
&\leq \|[\mathfrak{R}(\delta_2)-\mathfrak{R}(\delta_1)]\,\zeta_0\|_{\varrho}+\|[\mathfrak{R}(\delta_2)-\mathfrak{R}(\delta_1)]\,H(\sigma(\zeta),\zeta))\|_{\varrho}\\
&\quad+\int_{\delta_1}^{\delta_2}\|\mathfrak{R}(\delta_2-s)\Psi(s,\zeta(s))\|_{\varrho}\,ds\\
&\quad+\int_0^{\delta_1}\|[\mathfrak{R}(\delta_2-s)-\mathfrak{R}(\delta_1-s)]\Psi(s,\zeta(s))\|_{\varrho}\,ds.
\end{aligned}$$

By using the assumptions $(\mathcal{H}_2)$ and $(\mathcal{H}_5)$, we obtain

$$\begin{aligned}
\|(\nabla\zeta)(\delta_2)-(\nabla\zeta)(\delta_1)\|_{\varrho} &\leq \Big(\|\zeta_0\|_{\varrho}+q^*+qr\Big)\|\mathfrak{R}(\delta_2)-\mathfrak{R}(\delta_1)\|_{\varrho}+\frac{M_{\varrho}p^*(1+r)}{1-\varrho}(\delta_2-\delta_1)^{1-\varrho}\\
&\quad+p^*(1+r)\int_0^{\delta_1}\|\mathfrak{R}(\delta_2)-\mathfrak{R}(\delta_1)\|_{\varrho}\,ds.
\end{aligned}$$

As $(\mathfrak{R}(\delta))_{\delta>0}$ and $(A^{\varrho}\mathfrak{R}(\delta))_{\delta>0}$ are continuous in δ in the uniform operator topology, the right-hand side of the above inequality tends to zero as $\delta_2-\delta_1\to 0$. Hence, $\nabla(B_r)$ is equicontinuous on $[0,a]$.

Finally, it remains to be proved that the Mönch condition is satisfied. Let $\mathfrak{I}$ be a subset of B_r, such that $\mathfrak{I}\subset\nabla(\mathfrak{I})\cup\{0\}$. $\mathfrak{I}$ is bounded and equicontinuous, and therefore, the function $\delta\to\mu(\mathfrak{I}(\delta))$ is continuous on J. By Lemma 2.8, $(\mathcal{H}_6)$, $(\mathcal{H}_8)$, and the properties of the measure μ, for each $\delta\in J$, we have

$$\begin{aligned}
\mu(\mathfrak{I}(\delta)) &\leq \mu((\nabla\mathfrak{I})(\delta)\cup\{0\})\leq\mu((\nabla\mathfrak{I})(\delta))\\
&\leq MK\sup_{\delta\in[0,a]}\mu(\mathfrak{I}(\delta))+M\int_0^{\delta}e^{\tau\int_0^s g(\xi)d\xi}e^{-\tau\int_0^s g(\xi)d\xi}g(s)\mu(\mathfrak{I}(s))\,ds\\
&\leq MK\sup_{\delta\in[0,a]}\mu(\mathfrak{I}(\delta))+M\int_0^{\delta}g(s)e^{\tau\int_0^s g(\xi)d\xi}e^{-\tau\int_0^s g(\xi)d\xi}\mu(\mathfrak{I}(s))\,ds\\
&\leq MK\sup_{\delta\in[0,a]}\mu(\mathfrak{I}(\delta))+M\sup_{\delta\in[0,a]}e^{-\tau\int_0^s g(\xi)d\xi}\mu(\mathfrak{I}(\delta))\int_0^{\delta}g(s)e^{\tau\int_0^s g(\xi)d\xi}\,ds\\
&\leq MK\sup_{\delta\in[0,a]}\mu(\mathfrak{I}(\delta))+\frac{M}{\tau}e^{\tau\int_0^{\delta}g(\xi)d\xi}e^{-\tau\int_0^{\delta}g(\xi)d\xi}\mu(\mathfrak{I}(\delta)).
\end{aligned}$$

This inequality can be reduced to

$$\mu(\mathfrak{I}(\delta)) \quad \leq \quad M\Big(\frac{1}{\tau}+K\Big)e^{\tau\int_0^\delta g(\xi)d\xi}\sup_{\delta\in[0,a]} e^{-\tau\int_0^\delta g(\xi)d\xi}\mu(\mathfrak{I}(\delta)).$$

Thus, for any $\delta \in [0, a]$, we get

$$e^{-\tau\int_0^\delta g(\xi)d\xi}\mu(\mathfrak{I}(\delta)) \quad \leq \quad M\Big(\frac{1}{\tau}+K\Big)\sup_{\delta\in[0,a]} e^{-\tau\int_0^\delta g(\xi)d\xi}\mu(\mathfrak{I}(\delta)).$$

That means that

$$\mu_c(\mathfrak{I}) \quad \leq \quad M\Big(\frac{1}{\tau}+K\Big)\mu_c(\mathfrak{I}).$$

Therefore, from Mönch's condition and (9), we get

$$\mu_c(\mathfrak{I}) = 0.$$

Then $\mathfrak{I}$ is relatively compact in $C([0,a], X_\varrho)$. In view of the Ascoli-Arzelà theorem, $\mathfrak{I}$ is relatively compact in B_r. Hence, using the Theorem 2.7, ∇ has a fixed point which is a mild solution of our problem (1). □

4. Application

To illustrate the application of our main results, we consider the following nonlocal partial integrodifferential equation:

$$\begin{cases} \dfrac{\partial z(\delta,x)}{\partial\delta} = \dfrac{\partial^2 z(\delta,x)}{\partial x^2} + \displaystyle\int_0^\delta b(\delta-s)\frac{\partial^2 z(s,x)}{\partial x^2}\,dx \\ \qquad\qquad +c(\delta)\ell_1\Big(\dfrac{\partial z(\delta,x)}{\partial x}\Big), \quad (\delta,x)\in[0,a]\times[0,\pi], \\ z(\delta,0) = z(\delta,\pi) = 0, \quad \delta\in[0,a] \\ z(0,x) + \displaystyle\int_0^t h(s)\ell_2\Big(z(\delta,x)\Big)\,ds = z_0(x), \quad 0\le x\le\pi, \end{cases} \tag{14}$$

where $z_0(x) \in X := L^2[0,\pi]$, $b : [0,a] \to \mathbb{R}$, $c : [0,a] \to \mathbb{R}$, $h : [0,a] \to \mathbb{R}$ are continuous functions. Furthermore, $\ell_i : \mathbb{R} \to \mathbb{R}$ is continuous, and there are constants $k_i > 0$ for $i = 1, 2$, such that for any $(\zeta_1, \zeta_2) \in \mathbb{R}^2$,

$$|\ell_i(\zeta_1) - \ell_i(\zeta_2)| \le k_i|\zeta_1 - \zeta_2|. \tag{15}$$

We need to rewrite the system (14) in the abstract form. To do so, we define the operator $A : \mathcal{D}(A) \subset X \to X$ by

$$\mathcal{D}(A) = \{z \in X : z \text{ and } z' \text{ are absolutely continuous on } [0,\pi], z'' \in X, z(0) = z(\pi) = 0\}.$$

For each $z \in \mathcal{D}(A)$,

$$Az = z''.$$

It is well known that A infinitesimally generates a compact, analytic, and self-adjoint C_0-semigroup $(S(\delta))_{\delta\geq 0}$ on X [35]. Moreover, A has a discrete spectrum given by its eigenvalues, which are $\{-n^2,\ n\in\mathbb{N}\}$. The related normalized eigenvectors are $e_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx)$, for $n \in \mathbb{N}$. Then, the following properties holds for the fractional power of A and the analytic semigroup $(S(\delta))_{\delta\geq 0}$:

(p_1) For $z \in D(A)$, it holds $(-A)z = \sum_{n=1}^{\infty} n^2 < z, e_n > e_n$. For $\varrho = \frac{1}{2}$ and $z \in X$, $(-A)^{-\frac{1}{2}}z = \sum_{n=1}^{\infty} \frac{1}{n} < z, e_n > e_n$. Particularly, $\|A^{-1}\| \leq 1$ and $\|A^{-\frac{1}{2}}\| \leq 1$.

(p_2) For $z \in X$, it holds $S(\delta)z = \sum_{n=1}^{\infty} e^{-n^2\delta} < z, e_n > e_n$, and thus $\|S(\delta)\| \leq e^{-\delta}$.

(p_3) The operator $A^{\frac{1}{2}}$ is defined as

$$(-A)^{\frac{1}{2}}z = \sum_{n=1}^{\infty} n < z, e_n > e_n, \;\; z \in X \text{ on } D(A^{\frac{1}{2}}),$$

with

$$\begin{aligned} D(A^{\frac{1}{2}}) &= \left\{ z(\cdot) \in X : \sum_{n=1}^{\infty} n < z, e_n > e_n \in X \right\} \\ &= \{ z(\cdot) \in X, z' \in X, z(0) = z(\pi) = 0 \}. \end{aligned}$$

In this section, we need the following Lemma to prove our main result.

Lemma 4.1. *([41]) If $\zeta \in X_{\frac{1}{2}}$, then ζ is absolutely continuous with $\zeta' \in X$, and*

$$\|\zeta\|_{\frac{1}{2}} = \|\zeta'\| = \|A^{\frac{1}{2}}\zeta\|.$$

Now, we define

$$\begin{cases} \zeta(\delta)(x) = z(\delta, x), & \text{for } \delta \in [0, a] \text{ and } x \in [0, \pi], \\ \zeta(0)(x) = z(0, x), & \text{for } x \in [0, \pi], \\ \zeta'(\delta)(x) = \dfrac{\partial}{\partial \delta} z(\delta, x), & \text{for } \delta \in [0, a] \text{ and } x \in [0, \pi]. \end{cases}$$

We define also $\Upsilon(\cdot)$ as follows

$$\begin{cases} \mathcal{D}(\Upsilon) = \mathcal{D}(A), \\ (\Upsilon(\delta)y)(x) = b(\delta)y''(x), & \text{for } \delta \in [0, a] \text{ and } x \in [0, \pi], \end{cases}$$

the function b is continuous on $[0, a]$, thus it is bounded. According to [15], if b satisfies $g_1(\lambda) := 1 + b^*(\lambda) \neq 0$ with $\lambda g_1(\lambda) \in \Lambda$, for $\lambda \in \Lambda$, and moreover, if $b^*(\lambda) \to 0$, as $|\lambda| \to +\infty$, $\lambda \in \Lambda$, then, the linear system (5) has a resolvent operator $(\mathfrak{R}(\delta))_{\delta \geq 0}$.

Now, we define the function $\Psi : [0, a] \times X_{\frac{1}{2}} \to X$ by

$$\Psi(\delta, \zeta)(x) = c(\delta)\ell_1(\zeta'(x)).$$

Let $H : [0, a] \times C([0, a], X_{\frac{1}{2}}) \to X_{\frac{1}{2}}$, and $\sigma : C([0, a]; X_{\frac{1}{2}}) \to [0, a]$ defined by

$$H(\sigma(\zeta), \zeta)(x) = \int_0^t \sigma(\zeta)(s)\ell_2(\zeta(x))\, ds, \;\; z \in C([0, a], X_{\frac{1}{2}}),$$

where

$$\sigma(\zeta)(\delta) = h(\delta).$$

Then, with those notations, we can transform the system (14) into the form of (1). Now, we check that the assumptions of Theorem 3.2 are fulfilled for this system.

By the definition of function Ψ, we see that Ψ is well defined and satisfies $(\mathcal{H}_1)(i)$. Thanks to (15), we can also prove that Ψ satisfies $(\mathcal{H}_1)(ii)$. In fact, let $\delta \in [0, a]$, and $(\zeta_1, \zeta_2) \in X_{\frac{1}{2}} \times X_{\frac{1}{2}}$, then we have

$$\begin{aligned} \|\Psi(\delta,\zeta_1) - \Psi(\delta,\zeta_2)\|^2 &= \int_0^\pi \left|c(\delta)\ell_1(\zeta_1'(x)) - \ell_1(\zeta_2'(x))\right|^2 dx \\ &\le |c(\delta)|^2 \int_0^\pi k_1^2|\zeta_1'(x) - \zeta_2'(x)|^2\, dx \\ &\le |c(\delta)|^2 k_1^2 \int_0^\pi |\zeta_1'(x) - \zeta_2'(x)|^2\, dx \\ &\le |c(\delta)|^2 k_1^2 \|\zeta_1 - \zeta_2\|_{\frac{1}{2}}^2, \end{aligned}$$

which leads to

$$\|\Psi(\delta,\zeta_1) - \Psi(\delta,\zeta_2)\| \le |c(\delta)||k|\|\zeta_1 - \zeta_2\|_{\frac{1}{2}}.$$

By the definition of H and (15), for any $\zeta_1, \zeta_2 \in C([0, a]; X_{\frac{1}{2}})$, we have

$$\begin{aligned} \|H(\sigma(\zeta_1),\zeta_1) - H(\sigma(\zeta_2),\zeta_2)\|_{\frac{1}{2}}^2 &= \int_0^\pi \left|\frac{\partial}{\partial x}\Big(\int_0^t h(s)[\ell_2(\zeta_1(x)) - \ell_2(\zeta_2(x))]\, ds\Big)\right|^2 dx \\ &\le \int_0^\pi \left|\frac{\partial}{\partial x}\Big(\int_0^t h(s)k_2|\zeta_1(x) - \zeta_2(x)|\, ds\Big)\right|^2 dx \\ &\le \int_0^\pi \Big(\int_0^t |h(s)|^2 k_2^2 |\zeta_1'(x) - \zeta_2'(x)|^2\, ds\Big)\, dx \\ &\le \lambda k_2^2 \|\zeta_1 - \zeta_2\|_{\frac{1}{2}}^2 \\ &\le K\|\zeta_1 - \zeta_2\|_{C([0,a];X_{\frac{1}{2}})}^2, \end{aligned}$$

which shows that $(\mathcal{H}_2)$ is hold. Therefore, it can be concluded from Theorem 3.2 that the system (14) has a unique mild solution on $[0, a]$.

5. Conclusion

In the present paper, we first study the existence and uniqueness of mild solutions of the PIDE with a state-dependent nonlocal condition in space X_ϱ using analytic resolvent operators. The results are obtained under the assumptions of noncompactness for both the resolvent operator and the state-dependent function H in the nonlocal condition. Our methodology relies on the theory of fractional power operators, fixed point theory, and the measures of noncompactness in the sense of Kuratowski. It is clear that the results in this paper expand and extend the aforementioned literature. It would be interesting to extend this study in the case of time scales, which include continuous and discrete problems (see [38] and references therein).

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