Filomat 38:18 (2024), 6581–6586
https://doi.org/10.2298/FIL2418581C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Total outer-independent domination in regular graphs

Abel Cabrera-Martínez[a]

[a]*Universidad de Córdoba, Departamento de Matemáticas, Campus de Rabanales, 14071, Córdoba, Spain*

Abstract. Let G be a connected graph of order n. A set $D \subseteq V(G)$ is a total outer-independent dominating set of G if $N(v) \cap D \neq \emptyset$ for every $v \in D$ and $N(v) \subseteq D$ for every $v \in V(G) \setminus D$. The total outer-independent domination number of G, denoted by $\gamma_t^{oi}(G)$, is the minimum cardinality among all total outer-independent dominating sets of G. We show that if G is a k-regular graph with $k \geq 3$, then $\left(\frac{k}{2k-1}\right)n \leq \gamma_t^{oi}(G) \leq \left(\frac{k}{k+1}\right)n$. In addition, we characterize the k-regular graphs satisfying the above bounds, except for the case of cubic graphs attaining the upper bound. Finally, we obtain improved bounds (with respect to the previous ones) on $\gamma_t^{oi}(G)$ for the case in which G is a claw-free regular graph.

1. Introduction

Let G be a finite and undirected graph with vertex set $V(G)$ and edge set $E(G)$. Given a vertex $v \in V(G)$, the *open neighbourhood* of v, denoted by $N(v)$, is the set of neighbours of v; that is, $N(v) = \{x \in V(G) : xv \in E(G)\}$. The values $\delta(G) = \min\{|N(x)| : x \in V(G)\}$ and $\Delta(G) = \max\{|N(x)| : x \in V(G)\}$ denote the *minimum* and *maximum degrees* of G, respectively. If $\delta(G) = \Delta(G) = k$, then we say that G is a k-regular graph. Given a set $D \subseteq V(G)$, $\overline{D}$ denotes the complement of D; that is, $\overline{D} = V(G) \setminus D$. In addition, $G[D]$ denotes the subgraph of G induced by D. On the other hand, we say that D is a *packing* of G if its vertices are pairwise at distance at least three apart in G. The set of packings of G will be denoted as $\mathcal{P}(G)$. As usual, we use the notation K_n, N_n and $K_{1,n-1}$ for complete graphs, edgeless graphs and star graphs of order n, respectively. For $k \geq 1$ an integer, we use the standard notation $[k] = \{1, \dots, k\}$.

A *dominating set* of G is a set $D \subseteq V(G)$ that satisfies that $N(v) \cap D \neq \emptyset$ for every $v \in \overline{D}$. The set of dominating sets of G will be denoted as $\mathcal{D}(G)$. The *domination number* of G is defined as $\gamma(G) = \min\{|D| : D \in \mathcal{D}(G)\}$. This parameter was formally defined by Berge in 1958, although it has roots in many sources, including defense strategies, games such as chess, computer communication networks, and network surveillance and security [7].

The principal variations of the dominating sets in a graph G are based on conditions that are imposed on the subgraphs $G[D]$ and/or $G[\overline{D}]$, with $D \in \mathcal{D}(G)$. We next define one of the variants of dominating sets well studied in the last decade.

A *total outer-independent dominating set* of G is a set $D \subseteq V(G)$ which satisfies that $G[D]$ has no isolated vertex and $G[\overline{D}]$ is isomorphic to an edgeless graph; that is, $N(v) \cap D \neq \emptyset$ for every $v \in D$ and $N(v) \subseteq D$ for

2020 *Mathematics Subject Classification*. 05C69.
Keywords. Total outer-independent domination; Domination; Regular graphs.
Received: 17 September 2023; Revised: 18 January 2024; Accepted: 23 February 2024
Communicated by Paola Bonacini
Email address: acmartinez@uco.es (Abel Cabrera-Martínez)

every $v \in \overline{D}$. The set of total outer-independent dominating sets of G will be denoted as $\mathcal{D}_t^{oi}(G)$. The *total outer-independent domination number* of G is defined to be,

$$\gamma_t^{oi}(G) = \min\{|D| : D \in \mathcal{D}_t^{oi}(G)\}.$$

A $\gamma_t^{oi}(G)$*-set* is a total outer-independent dominating set of G of cardinality $\gamma_t^{oi}(G)$. This concept was introduced by Soner et al. [11] and studied further, for example, in [3–5, 9, 10].

In this article we study the total outer-independent domination number of regular graphs. In particular, we show that if G is a connected k-regular graph of order n with $k \geq 3$, then $\left(\frac{k}{2k-1}\right)n \leq \gamma_t^{oi}(G) \leq \left(\frac{k}{k+1}\right)n$. In addition, we characterize the k-regular graphs satisfying the above bounds, except for the case of cubic graphs attaining the upper bound. Finally, we obtain improved bounds (with respect to the previous ones) on $\gamma_t^{oi}(G)$ for the case in which G is a claw-free regular graph.

2. Results

We will begin this section by presenting general lower and upper bounds on the total outer-independent domination number of a k-regular graph. For this purpose, we shall need the following useful lemma.

Lemma 2.1. *Let G be a connected k-regular graph with $k \geq 3$. Let D be a $\gamma_t^{oi}(G)$-set and $D_i = \{v \in D : |N(v) \cap D| = i\}$ for every $i \in [k]$. Then $|D_k| \leq |D_1|$.*

Proof. If $D_k = \emptyset$, then we are done. Let us assume that $D_k \neq \emptyset$, and let $x \in D_k$. By definition of D_k, it follows that $N(x) \subseteq D$. If $N(x) \subseteq D \setminus D_1$, then $D \setminus \{x\} \in \mathcal{D}_t^{oi}(G)$, a contradiction. Hence, $N(x) \cap D_1 \neq \emptyset$. Therefore, every vertex in D_k has a private neighbour in D, which is clearly in D_1. As a consequence, we obtain that $|D_k| \leq |D_1|$, which completes the proof. □

Theorem 2.2. *If G is a connected k-regular graph of order n with $k \geq 3$, then*

$$\left(\frac{k}{2k-1}\right)n \leq \gamma_t^{oi}(G) \leq \left(\frac{k}{k+1}\right)n.$$

Proof. Let D be a $\gamma_t^{oi}(G)$-set and $D_i = \{v \in D : |N(v) \cap D| = i\}$ for every $i \in [k]$. Observe that $D = \cup_{i\in[k]} D_i$ and $D_i \cap D_j \neq \emptyset$ for any different subscripts $i, j \in [k]$. The following chain of equalities arise from counting argument on the number of edges joining D with $\overline{D}$.

$$\sum_{i\in[k-1]} (k-i)|D_i| = k|\overline{D}| = k(n - |D|) = kn - k|D|. \quad (1)$$

By equality (1) and the fact that $\sum_{i\in[k-1]} |D_i| = |D| - |D_k|$ we deduce that

$$|D| - |D_k| + \sum_{i\in[k-2]} (k-i-1)|D_i| = kn - k|D|,$$

which leads to the following chain of equalities.

$$\begin{aligned} |D| &= \left(\frac{k}{k+1}\right)n - \frac{\sum_{i\in[k-2]}(k-i-1)|D_i| - |D_k|}{k+1} \\ &= \left(\frac{k}{k+1}\right)n - \frac{\sum_{i\in[k-2]\setminus\{1\}}(k-i-1)|D_i|}{k+1} - \frac{(k-2)|D_1| - |D_k|}{k+1}. \end{aligned} \quad (2)$$

By Lemma 2.1 we have that $|D_k| \leq |D_1|$. So, $(k-2)|D_1| - |D_k| \geq 0$ due to the fact that $k \geq 3$. Combining this previous bound with the chain of equalities (2), it is easy to deduce that $|D| \leq \left(\frac{k}{k+1}\right)n$.

Now, by (1) and the fact that $\sum_{i\in[k-1]}(k-i)|D_i| = (k-1)|D| - \sum_{i\in[k]\setminus\{1\}}(i-1)|D_i|$ we deduce that

$$(k-1)|D| - \sum_{i\in[k]\setminus\{1\}}(i-1)|D_i| = kn - k|D|,$$

which leads to the following equality.

$$|D| = \left(\frac{k}{2k-1}\right)n + \frac{\sum_{i\in[k]\setminus\{1\}}(i-1)|D_i|}{2k-1}. \tag{3}$$

So, $|D| \geq \left(\frac{k}{2k-1}\right)n$ is an immediate consequence of equality (3). Therefore, the proof is complete. □

The complete graph K_4 and the cube graph Q_3 are examples of cubic graphs satisfying the equality associated with the upper bound given in Theorem 2.2. Hence, this upper bound is the best possible for the case $k = 3$. We next proceed to characterize the k-regular graphs satisfying the bounds given in Theorem 2.2, except for the case of cubic graphs attaining the upper bound. Before, we shall need the following useful lemma.

Lemma 2.3. *Let G be a connected k-regular graph of order n with $k \geq 3$. Let D be a $\gamma_t^{oi}(G)$-set and $D_i = \{v \in D : |N(v) \cap D| = i\}$ for every $i \in [k]$. The following statements hold.*

(i) *If $k \geq 4$ and $|D| = \left(\frac{k}{k+1}\right)n$, then $D = D_{k-1}$, and as a consequence, $\overline{D} \in \mathcal{P}(G)$.*

(ii) *If $|D| = \left(\frac{k}{2k-1}\right)n$, then $D = D_1$.*

Proof. We first assume that $k \geq 4$ and $|D| = \left(\frac{k}{k+1}\right)n$. In the proof of Theorem 2.2 we deduce the chain of equalities (2). In particular, we obtain that

$$|D| = \left(\frac{k}{k+1}\right)n - \frac{\sum_{i\in\{2,\ldots,k-2\}}(k-i-1)|D_i|}{k+1} - \frac{(k-2)|D_1| - |D_k|}{k+1}. \tag{4}$$

By Lemma 2.1 it follows that $|D_k| \leq |D_1|$. If $D_k \neq \emptyset$, then $(k-2)|D_1| - |D_k| > 0$ due to the fact that $k \geq 4$. Combining this previous bound with the equality (4) we obtain that $|D| < \left(\frac{k}{k+1}\right)n$, a contradiction. Hence $D_k = \emptyset$, which implies that $D_1 = \cdots = D_{k-2} = \emptyset$. Therefore, $D = D_{k-1}$. Now, we proceed to prove that $\overline{D} \in \mathcal{P}(G)$. If $|\overline{D}| = 1$, then $\overline{D} \in \mathcal{P}(G)$. Let us assume that $|\overline{D}| \geq 2$ and let $x, y \in \overline{D}$ be any two different vertices. By definition, it follows that $xy \notin E(G)$. If there exists a vertex $v \in N(x) \cap N(y)$, then $v \in D = D_{k-1}$, which is a contradiction. Therefore, $N(x) \cap N(y) = \emptyset$. As a consequence, $\overline{D} \in \mathcal{P}(G)$, which completes the proof of (i).

Now, we proceed to prove (ii). In the proof of Theorem 2.2 we deduce the equality (3), which establishes the following.

$$|D| = \left(\frac{k}{2k-1}\right)n + \frac{\sum_{i\in[k]\setminus\{1\}}(i-1)|D_i|}{2k-1}.$$

If we assume that $|D| = \left(\frac{k}{2k-1}\right)n$, then it is straightforward that $D_i = \emptyset$ for every $i \in [k] \setminus \{1\}$. Therefore $D = D_1$, which completes the proof. □

Theorem 2.4. *Let G be a connected k-regular graph of order n with $k \geq 4$. Then $\gamma_t^{oi}(G) = \left(\frac{k}{k+1}\right)n$ if and only if $G \cong K_{k+1}$.*

Proof. If $G \cong K_{k+1}$, then $\gamma_t^{oi}(G) = k = \left(\frac{k}{k+1}\right)n$, as required. On the other hand, let us assume that G satisfies that $\gamma_t^{oi}(G) = \left(\frac{k}{k+1}\right)n$. Let D be a $\gamma_t^{oi}(G)$-set. By Lemma 2.3-(i) we have that $D = D_{k-1}$ and that $\overline{D} \in \mathcal{P}(G)$. Let $x \in \overline{D}$. Now, suppose that $|\overline{D}| \geq 2$. Observe that $n = |D| + |\overline{D}| \geq |N(x)| + 2 = k + 2$. This implies that $G[N(x) \cup \{x\}]$ is not a complete graph K_{k+1}. Let $x_1, x_2 \in N(x) \subseteq D = D_{k-1}$ such that $x_1x_2 \notin E(G)$, and let $D' = (D \setminus \{x_1, x_2\}) \cup \{x\}$. Since $k \geq 4$ and $\overline{D} \setminus \{x\} \in \mathcal{P}(G)$, it is easy to verify that $D' \in \mathcal{D}_t^{oi}(G)$, which is a contradiction. Hence $\overline{D} = \{x\}$, and as a consequence of the fact that $D = D_{k-1}$, it follows that $V(G) = N(x) \cup \{x\}$. Therefore $G \cong K_{k+1}$, which completes the proof. □

In order to characterize the connected k-regular graph G with $\gamma_t^{oi}(G) = \left(\frac{k}{2k-1}\right)|V(G)|$, we need to define the next family of graphs.

Definition 2.5. *Let $k \geq 3$ be an integer. We say that a connected k-regular graph G belongs to the family $\mathcal{G}_k$ whenever $|V(G)| = (2k-1)r$ for some $r \geq 1$ and its vertex set $V(G)$ can be partitioned into two vertex sets A and B such that $G[A] \cong (\frac{kr}{2})K_2$ and $G[B] \cong N_{r(k-1)}$.*

It is easy to check that $\mathcal{G}_k \neq \emptyset$ for any $k \geq 3$. For example, let F_k be a k-regular graph of order $n = 4k-2$ defined as follows:

- $V(F_k) = \left(\cup_{i\in[k]}\{x_i, y_i\}\right) \cup \left(\cup_{i\in[k-1]}\{v_i, w_i\}\right)$.
- $E(F_k) = \left(\cup_{i\in[k]} x_i y_i\right) \cup \left(\cup_{i\in[k]}\left(\cup_{j\in[k-1]}\{x_i v_j, y_i w_j\}\right)\right)$.

Observe that $F_k \in \mathcal{G}_k$. In this case, and following the parameters used in the Definition 2.5, we have that $r = 2$, $A = \cup_{i\in[k]}\{x_i, y_i\}$ and $B = \cup_{i\in[k-1]}\{v_i, w_i\}$. Figure 1 shows the graph $F_4 \in \mathcal{G}_4$.

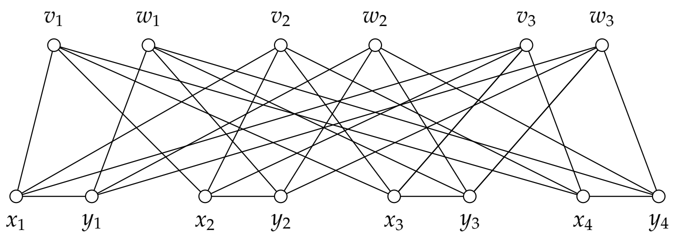

Figure 1: The graph F_4.

Theorem 2.6. *Let G be a connected k-regular graph of order n with $k \geq 3$. Then $\gamma_t^{oi}(G) = \left(\frac{k}{2k-1}\right)n$ if and only if $G \in \mathcal{G}_k$.*

Proof. We first assume that G satisfies that $\gamma_t^{oi}(G) = \left(\frac{k}{2k-1}\right)n$. Let D be a $\gamma_t^{oi}(G)$-set. By Lemma 2.3-(ii) we have that $D = D_1$, which implies that $G[D] \cong sK_2$. In addition, we have that $G[\overline{D}] \cong N_{n-2s}$ due to the fact that $\overline{D}$ is an independent set of G of cardinality $|\overline{D}| = n - 2s$. Since $|D| = \left(\frac{k}{2k-1}\right)n$, it follows that $n \equiv 0 \pmod{2k-1}$. Hence, $n = (2k-1)r$ for some $r \geq 1$ and so, $s = \frac{|D|}{2} = \frac{kr}{2}$. As a consequence of the previous equalities, it follows that $V(G)$ is partitioned into two vertex sets D and $\overline{D}$ where $G[D] \cong (\frac{kr}{2})K_2$ and $G[\overline{D}] \cong N_{r(k-1)}$. Therefore $G \in \mathcal{G}_k$, as required.

On the other hand, we assume that $G \in \mathcal{G}_k$; that is, G is a connected k-regular graph of order $n = (2k-1)r$ (for some $r \geq 1$) with a vertex partition (A, B) such that $G[A] \cong (\frac{kr}{2})K_2$ and $G[B] \cong N_{r(k-1)}$. It is easy to observe that $A \in \mathcal{D}_t^{oi}(G)$. Hence, $\gamma_t^{oi}(G) \leq |A| = kr = \left(\frac{k}{2k-1}\right)n$. By the lower bound given in Theorem 2.2 it follows that $\gamma_t^{oi}(G) = \left(\frac{k}{2k-1}\right)n$, which completes the proof. □

2.1. Claw-free regular graphs

A graph is *claw-free* if it contains no induced $K_{1,3}$. In this subsection we obtain lower and upper bounds on the total outer-independent domination number of claw-free regular graphs.

Theorem 2.7. *Let $k \geq 3$ be an integer. If G is a connected claw-free k-regular graph of order $n > k+1$, then*

$$\left(\frac{k}{k+2}\right)n \leq \gamma_t^{oi}(G) \leq \left(\frac{k-1}{k}\right)n.$$

Proof. Cabrera-Martínez et al. [6, Theorem 8-(ii)] showed that if G is a connected claw-free graph of order n with $\delta(G) \geq 3$, then $\gamma_t^{oi}(G) = n - \alpha(G)$, where $\alpha(G)$ denotes the classical independence number of G; that is, the maximum cardinality among all independent sets of G. Brooks' theorem [1, Theorem V.1.6] implies that if $G \not\cong K_{\Delta+1}$ is a connected graph of order n with maximum degree Δ, then $\alpha(G) \geq n/\Delta$. Hence, if $k \geq 3$ and G is a connected claw-free k-regular graph of order $n > k+1$, then $\gamma_t^{oi}(G) = n - \alpha(G) \leq n - \frac{n}{k} = \left(\frac{k-1}{k}\right)n$, as desired.

Finally, we proceed to prove the lower bound. If $k = 3$, then we are done by Theorem 2.2. From now on, we assume that $k \geq 4$. Let D be a $\gamma_t^{oi}(G)$-set. Observe that $D = \cup_{i\in[k]} D_i$ and $D_i \cap D_j \neq \emptyset$ for any different subscripts $i, j \in [k]$. Let us suppose that there exists a vertex $x \in D_k$. Since G is claw-free, it is easy to see that $N(x) \subseteq D \setminus D_1$. This implies that $D \setminus \{x\} \in \mathcal{D}_t^{oi}(G)$, a contradiction. Hence, $D_k = \emptyset$. If there exists a vertex $x \in \cup_{j\in[k-3]} D_j$, then there exist three vertices $x_1, x_2, x_3 \in N(x) \setminus D$, which is a contradiction because the subgraph induced by the set $\{x, x_1, x_2, x_3\}$ is isomorphic to $K_{1,3}$. Therefore, $D_j = \emptyset$ for every $j \in [k-3]$, which implies that $D = D_{k-1} \cup D_{k-2}$. Now, from a counting argument on the number of edges joining D with $\overline{D}$, we deduce the following.

$$2|D_{k-2}| + |D_{k-1}| = k|\overline{D}| = k(n - |D|) = kn - k|D|. \tag{5}$$

By equality (5) and the fact that $|D| = |D_{k-2}| + |D_{k-1}|$ we deduce that $2|D| - |D_{k-1}| = kn - k|D|$, which leads to the following equality.

$$|D| = \left(\frac{k}{k+2}\right)n + \frac{|D_{k-1}|}{k+2}. \tag{6}$$

So, $|D| \geq \left(\frac{k}{k+2}\right)n$ is an immediate consequence of equality (6). Therefore, the proof is complete. □

The following result show that the lower bound given in Theorem 2.7 is tight for any even integer $k \geq 4$.

Proposition 2.8. *Let $k \geq 4$ be any even integer. If G is a connected k-regular graph of order $n = k+2$, then $\gamma_t^{oi}(G) = k$.*

Proof. We first observe that G is a claw-free graph. Let u, v be any two non-adjacent vertices of G. It is easy to see that $V(G) \setminus \{u, v\} \in \mathcal{D}_t^{oi}(G)$. Hence $\gamma_t^{oi}(G) \leq |V(G) \setminus \{u, v\}| = k$. On the other hand, Theorem 2.7 leads to $\gamma_t^{oi}(G) \geq \left(\frac{k}{k+2}\right)n = k$. Therefore, $\gamma_t^{oi}(G) = k$, which completes the proof. □

We conclude this subsection by showing that every connected {claw,diamond}-free cubic graph achieves the equality associated with the upper bound given in Theorem 2.7. Recall that a *diamond* is the graph $K_4 - e$, where e denotes an arbitrary edge of the complete graph K_4.

Theorem 2.9. *If G is a connected {claw,diamond}-free cubic graph of order $n \geq 6$, then*

$$\gamma_t^{oi}(G) = \frac{2n}{3}.$$

Proof. First, we proceed to prove that any two triangles of G are vertex-disjoint. Let T_1 and T_2 be two different triangles of G. Suppose that there exists a vertex $x \in V(T_1) \cap V(T_2)$. Since $|N(x)| = 3$ and $|N(x) \cap V(T_i)| \geq 2$ for every $i \in [2]$, it follows that there exists a vertex $y \in N(x) \cap V(T_1) \cap V(T_2)$. This implies that $xy \in E(T_1) \cap E(T_2)$. By the fact that T_1 and T_2 are two different triangles of G and $n \geq 6$, it is easy to deduce that $G[V(T_1) \cup V(T_2)]$ is isomorphic to a diamond, which is a contradiction. Hence $V(T_1) \cap V(T_2) = \emptyset$, as required. Moreover, we observe that every vertex of G is contained in a triangle because G is a claw-free graph. From the two previous statements, it follows that every vertex of G is contained in a unique triangle, which implies that G is a vertex-disjoint union of triangles. Therefore $V(G) = \cup_{i\in[n/3]} V(T_i)$, where $T_1, \ldots, T_{n/3}$ are pairwise different triangles of G.

Let D be a $\gamma_t^{oi}(G)$-set. By definition, it follows that $|D \cap V(T_i)| \geq 2$ for every $i \in [n/3]$. Hence, $|D| = \sum_{i\in[n/3]} |D \cap V(T_i)| \geq \frac{2n}{3}$. Now, by Theorem 2.7 we have that $|D| \leq \frac{2n}{3}$. Therefore $|D| = \frac{2n}{3}$, which completes the proof. □

3. Open problems

We conclude our article with the following problems that we have yet to settle.

Problem 3.1. *Characterize the connected cubic graphs that achieve equality in the upper bound given in Theorem 2.2; that is, characterize the connected cubic graphs G of order n satisfying* $\gamma_t^{oi}(G) = \frac{3n}{4}$.

Problem 3.2. *For* $k \geq 4$*, Theorem 2.4 establishes that the equality in the upper bound of Theorem 2.2 is achieved if and only if* $G \cong K_{k+1}$*. In such a sense, it is interesting to determine the smallest positive constant* $c(k) < \frac{k}{k+1}$ *such that every k-regular graph G of order* $n > k+1$ *satisfies that* $\gamma_t^{oi}(G) \leq c(k) \cdot n$.

Problem 3.3. *Characterize the connected claw-free k-regular graphs that achieve equalities in the lower and upper bounds given in Theorem 2.7; that is, characterize the claw-free k-regular graphs G of order* $n > k+1$ *satisfying either* $\gamma_t^{oi}(G) = \left(\frac{k}{k+2}\right)n$ *or* $\gamma_t^{oi}(G) = \left(\frac{k-1}{k}\right)n$.

References

[1] B. Bollobás, *Extremal Graph Theory*, Academic Press, New York, 1978.
[2] A. Cabrera Martínez, *Double outer-independent domination number of graphs*, Quaest. Math. **44** (2021), 1835–1850.
[3] A. Cabrera Martínez, S. Cabrera García, I. Peterin, I.G. Yero, *The total co-independent domination number of some graph operations*, Rev. de la Union Mat. Argentina. **63** (2022), 153–168.
[4] A. Cabrera Martínez, F. A. Hernández Mira, J. M. Sigarreta Almira, I.G. Yero, *A note on total co-independent domination in trees*, Contrib. Discrete Math. **19(3)** (2024), 184-195.
[5] A. Cabrera Martínez, F.A. Hernández Mira, J.M. Sigarreta Almira, I.G. Yero, *On computational and combinatorial properties of the total co-independent domination number of graphs*, Computer J. **62** (2019), 97–108.
[6] A. Cabrera-Martínez, J.L. Sánchez, J.M. Sigarreta Almira, *On the total domination number of total graphs*, Discuss. Math. Graph Theory **44** (2024), 933-951.
[7] T.W. Haynes, S.T. Hedetniemi, M.A. Henning, *Domination in Graphs: Core Concepts*, Springer Monographs in Mathematics, Cham, 2023.
[8] M. Krzywkowski, *A lower bound on the total outer-independent domination number of a tree*, Comptes Rendus Math. **349** (2011), 7–9.
[9] M. Krzywkowski, *An upper bound on the total outer-independent domination number of a tree*, Opus. Math. **32** (2012), 153–158.
[10] D.A. Mojdeh, I. Peterin, B. Samadi, I.G. Yero, *On three outer-independent domination related parameters in graphs*, Discrete Appl. Math. **294** (2021), 115–124.
[11] N.D. Soner, B.V. Dhananjaya Murthy, G. Deepak, *Total co-independent domination in graphs*, Appl. Math. Sci. **6** (2012), 6545–6551.