Filomat 38:18 (2024), 6621–6637
https://doi.org/10.2298/FIL2418621N

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Approximation by GBS associated properties of Szász-Mirakjan-Jakimovski-Leviatan-Kantorovich operators

Md. Nasiruzzaman[a]

[a]*Department of Mathematics, Faculty of Science, University of Tabuk, PO Box 4279, Tabuk-71491, Saudi Arabia*

Abstract. Our motive in the present article is to study the bivariate and GBS associated properties of the Szász-Mirakjan-Jakimovski-Leviatan-Kantorovich operators. We design the operators and obtain the weighted approximation properties by mixed modulus of continuity and convergence of Lipschitz class bivariate function. Finally, we design GBS type bivariate mixed difference operators and calculate the approximation properties in Bögel continuous functions space.

1. Introduction and Preliminaries

Szász-Mirakjan Positive linear operators [26] for the continuous function f on $[0,\infty)$ were developed by mathematician Szász-Mirakjan in 1950 and were extensively explored in place of Bernstein operators [5]. Finally, for all $z\in[0,\infty)$ and $f\in C[0,\infty)$ Szász introduced the operators follows:

$$S_r(f;z)=e^{-rz}\sum_{k=0}^{\infty}\frac{(rz)^k}{k!}f\left(\frac{k}{r}\right), \tag{1}$$

where $f\in C[0,\infty)$ is the continuous function space on $[0,\infty)$. In recent years, Sucu [25] proposed an exponential function on the Dunkl generalization by inserting a non-negative number suppose be $\eta\geq 0$, which led to the development of Szász-Mirakjan-operators by

$$\mathcal{S}_r^*(f;z)=\frac{1}{e_\eta(rz)}\sum_{\kappa=0}^{\infty}\frac{(rz)^\kappa}{\gamma_\eta(\kappa)}f\left(\frac{\kappa+2\eta\theta_\kappa}{r}\right), \tag{2}$$

where $e_\eta(z)=\sum_{\kappa=0}^{\infty}\frac{z^\kappa}{\gamma_\eta(\kappa)}$ and a recursion formula for $\kappa=0,1,2,\ldots$

$$\frac{\gamma_\eta(\kappa+1)}{(\kappa+1+2\eta\theta_{\kappa+1})}=\gamma_\eta(\kappa),$$

$$\theta_\kappa=\begin{cases}0 & \text{for } \kappa=2r,\quad r\in\mathbb{N}\cup\{0\},\\ 1 & \text{for } \kappa=2r+1,\quad r\in\mathbb{N}\cup\{0\}.\end{cases} \tag{3}$$

2020 *Mathematics Subject Classification*. Primary 41A36; Secondary 441A25, 33C45.
Keywords. Bernstein-Stancu type polynomials; q-calculus; modulus of continuity; Peetre's K-functional; Korovkin's theorem; Voronovskaja-type theorem.
Received: 04 October 2023; Revised: 01 December 2023; Accepted: 16 December 2023
Communicated by Miodrag Spalević
Email address: nasir3489@gmail.com, mfarooq@ut.edu.sa (Md. Nasiruzzaman)

By utilizing the Appell polynomials [3], Jakimovski and Leviatan established the series of Szász-Mirakjan positive linear operators in 1969, called it Szász-Mirakjan-Leviatan operators as follows:

$$J_r(h;z) = \frac{e^{-rz}}{L(1)} \sum_{\kappa=0}^{\infty} H_\kappa(rz) f\Big(\frac{\kappa}{r}\Big), \tag{4}$$

where the Appell polynomials be $L(u)e^{uz} = \sum_{\kappa=0}^{\infty} H_\kappa(z)u^\kappa$ and $L(1) \neq 0, L(u) = \sum_{\kappa=0}^{\infty} b_\kappa u^\kappa, H_\kappa(z) = \sum_{j=0}^{\kappa} b_j \frac{z^{\kappa-j}}{(\kappa-j)!}$ ($\kappa \in \mathbb{N}$). For $L(1) = 1$ the equality (4) reduced to the Szász-Mirakjan-operator by (1). Recently, in [17], by use of generated exponential function Nasiruzzaman and Aljohani studied the approximation properties of Szász-Mirakjan-Jakimovski-Leviatan operators. Most recent, A. Alotaibi studied the bivariate type approximation properties of operators [17] by applying the unbounded sequences of positive function (see [2]).

Nasiruzzaman and Aljohani have also construct the Kantorovich operators which involving the Appell polynomials by generalized exponential function [18]. Thus for every $f \in C_\vartheta[0,\infty) = \{f \in C[0,\infty) : f(t) = O(t^\vartheta)\}$ as $t \to \infty$, and all $z \in [0,\infty)$, $\vartheta > n$, $n \in \mathbb{N}$, $L(1) \neq 0$, $\eta \geq 0$ operators define by

$$\mathcal{R}^*_{r,\eta}(f;z) = \frac{r}{L(1)e_\eta(rz)} \sum_{\kappa=0}^{\infty} H_\kappa(rz) \int_{\frac{\kappa+2\eta\theta_\kappa}{r}}^{\frac{\kappa+1+2\eta\theta_\kappa}{r}} f(t)\mathrm{d}t. \tag{5}$$

Lemma 1.1. *[18] Let basic test function $\gamma_j = t^j$, for $j = 0,1,2,3,4$, operators $\mathcal{R}^*_{r,\eta}$ have $\mathcal{R}^*_{r,\eta}(\gamma_0;z) = 1$, $\mathcal{R}^*_{r,\eta}(\gamma_1;z) = z + \frac{1}{2r}\left(\frac{2L'(1)}{L(1)} + 4\eta + 1\right)$, and*

$$\begin{aligned}
(1)\quad \mathcal{R}^*_{r,\eta}(\gamma_2;z) &= z^2 + \frac{1}{r}\left(2\frac{L'(1)}{L(1)} + 4\eta + 2\right)z \\
&+ \frac{1}{3r^2}\left(3\frac{L''(1)}{L(1)} + 6(1+2\eta)\frac{L'(1)}{L(1)} + 12\eta^2 + 6\eta + 1\right); \\
(2)\quad \mathcal{R}^*_{r,\eta}(\gamma_3;z) &= z^3 + \frac{3}{2r}\left(2\frac{L'(1)}{L(1)} + 4\eta + 3\right)z^2 \\
&+ \frac{3}{2r^2}\left(2\frac{L''(1)}{L(1)} + 2(3+4\eta)\frac{L'(1)}{L(1)} + 8\eta^2 + 8\eta + 3\right)z \\
&+ \frac{1}{4r^3}\Big(12\frac{L'''(1)}{L(1)} + 6(3+4\eta)\frac{L''(1)}{L(1)} + 6(8\eta^2 + 8\eta + 3)\frac{L'(1)}{L(1)} \\
&+ 32\eta^3 + 8\eta^2 + 8\eta + 1\Big); \\
(3)\quad \mathcal{R}^*_{r,\eta}(\gamma_4;z) &= z^4 + \frac{1}{r}\left(4\frac{L'(1)}{L(1)} + 8\eta + 8\right)z^3 \\
&+ \frac{1}{r^2}\left(6\frac{L''(1)}{L(1)} + (14+24\eta)\frac{L'(1)}{L(1)} + 24\eta^2 + 36\eta + 9\right)z^2 \\
&+ \frac{1}{r^3}\Big(4\frac{L'''(1)}{L(1)} + (14+24\eta)\frac{L''(1)}{L(1)} + (38 + 72\eta + 48\eta^2)\frac{L'(1)}{L(1)} \\
&+ 32\eta^3 + 48\eta^2 + 36\eta + 13\Big)z \\
&+ \frac{1}{r^4}\Big(\frac{L'^{v}(1)}{L(1)} + 4(3+2\eta)\frac{L'''(1)}{L(1)} + (19 + 36\eta + 24\eta^2)\frac{L''(1)}{L(1)} \\
&+ (13 + 36\eta + 48\eta^2 + 32\eta^3)\frac{L'(1)}{L(1)} + 16\eta^4 + 16\eta^3 + 8\eta^2 + 2\eta + 1\Big).
\end{aligned}$$

There are numerous articles about these works that have been published, however we prefer those by Jakimovski-Leviatan-Beta integral operators [1], family of Bernstein-Kantorovich operators [11], Stancu type Bernstein-Kantorovich operators [12], Korovkin and Voronovskaya types approximation theorems [13], bivariate type generalized Bernstein–Schurer operators [14], modified q-Bernstein–Kantorovich operators [15], q-analogue of Jakimovski-Leviatan operators [16], Szász operators to bivariate functions [19], Baskakov-Durrmeyer operators [20], Beta Jakimovski-Leviatan operators [21], Jakimovski-Leviatan-Beta operators [22], univariate and bivariate λ-Kantorovich operators [23], B-differentiable functions by GBS operators [24], Szász-gamma operators [27].

2. Operators and its Basic Estimates

In this section we focus on the operators $\mathcal{R}^{*}_{r,\eta}$ defined by (5) and constructed the bivariate form for the operators by taking into account the operators introduced by [4, 10] and we then examine the convergence outcomes.

We take $\mathbb{R}^2_+ = \mathbb{R}_+ \times \mathbb{R}_+$ where $\mathbb{R}_+ = [0,\infty)$ and $\mathcal{I}^2 = \{(z_1,z_2) : 0 \le z_1 < \infty,\ 0 \le z_2 < \infty\}$, and $C\left(\mathcal{I}^2\right)$ be the set of all continuous functions defined on $\mathcal{I}^2$ and which satisfies the norm equipped by $\|f\|_{C(\mathcal{I}^2)} = \sup_{(z_1,z_2)\in\mathcal{I}^2} |f(z_1,z_2)|$. Then for all $f \in C(\mathcal{I}^2)$ considering the operators $\mathcal{K}^{\eta_1,\eta_2}_{r_1,r_2}$ which acting $C(\mathcal{I}^2) \to C(\mathcal{I}^2)$ by

$$\mathcal{K}^{\eta_1,\eta_2}_{r_1,r_2}(f;z_1,z_2) = \sum_{\kappa_1=0}^{\infty}\sum_{\kappa_2=0}^{\infty} L^{\eta_1,\eta_2}_{r_1,r_2}(z_1,z_2) H^{r_2,\kappa_2}_{r_1,\kappa_1}(z_1,z_2) \int_{\frac{\kappa_1+2\eta_1\theta_{\kappa_1}}{r_1}}^{\frac{\kappa_1+1+2\eta_1\theta_{\kappa_1}}{r_1}} \int_{\frac{\kappa_2+2\eta_2\theta_{\kappa_2}}{r_2}}^{\frac{\kappa_2+1+2\eta_2\theta_{\kappa_2}}{r_2}} f(t_1,t_2)\mathrm{d}t_1\mathrm{d}t_2, \tag{6}$$

where $L^{\eta_1,\eta_2}_{r_1,r_2}(z_1,z_2) = \frac{r_1}{L_1(1)e_{\eta_1}(r_1z_1)}\frac{r_2}{L_2(1)e_{\eta_2}(r_2z_2)}$ and $H^{r_2,\kappa_2}_{r_1,\kappa_1}(z_1,z_2) = H_{\kappa_1}(r_1z_1)H_{\kappa_2}(r_2z_2)$ with $L_1(1), L_2(1) \neq 0$, and $r_1, r_2 \in \mathbb{R}$, $\eta_1,\eta_2 \ge 0$.

Lemma 2.1. *For all $i = 1,2$ and $z_i \in [0,\infty)$, $H_{\kappa_i}(z_i) \ge 0$, $\eta_i \ge 0$ and $L_i(1) \neq 0$, if we define*

$$L_i(\alpha)e_{\eta_i}(\alpha z_i) = \sum_{\kappa_i=0}^{\infty} H_{\kappa_i}(z_i)\alpha^{\kappa_i}. \tag{7}$$

Then we have

$$\sum_{\kappa_i=0}^{\infty} H_{\kappa_i}(r_iz_i) = L_i(1)e_{\eta_i}(r_iz_i), \tag{8}$$

$$\sum_{\kappa_i=0}^{\infty} \kappa_i H_{\kappa_i}(r_iz_i) = \Big(L_i'(1) + r_iz_iL_i(1)\Big)e_{\eta_i}(r_iz_i), \tag{9}$$

$$\sum_{\kappa_i=0}^{\infty} \kappa_i^2 H_{\kappa_i}(r_iz_i) = \Big(L_i''(1) + (2r_iz_i+1)L_i'(1) + r_iz_i(r_iz_i+1)L_i(1)\Big)e_{\eta_i}(r_iz_i), \tag{10}$$

$$\begin{aligned}\sum_{\kappa_i=0}^{\infty} \kappa_i^3 H_{\kappa_i}(r_iz_i) \quad &= \quad \Big(L_i'''(1) + 3(r_iz_i+1)L_i''(1) + (3r_i^2z_i^2 + 6r_iz_i + 2)L_i'(1) \\ &+ \quad r_iz_i(r_i^2z_i^2 + 3r_iz_i + 2)L_i(1)\Big)e_{\eta_i}(r_iz_i),\end{aligned}$$

$$
\begin{aligned}
\sum_{\kappa_i=0}^{\infty}\kappa_i^4 H_{\kappa_i}(r_i z_i) &= \Big(L_i^{\prime v}(1)+(4r_iz_i+6)L_i'''(1)+(6r_i^2z_i^2+18r_iz_i+11)L_i''(1)\\
&+ (4r_i^3z_i^3+18r_i^2z_i^2+22r_iz_i+6)L_i'(1)\\
&+ r_iz_i(r_i^3z_i^3+6r_i^2z_i^2+11r_iz_i+6)L_i(1)\Big)e_{\eta_i}(r_iz_i).
\end{aligned}
$$

Lemma 2.2. *Let we define the auxiliary operators from the equality* (6) *by*

$$
\mathcal{P}_{r_1}^{\eta_1}(f;z_1,z_2)=\frac{r_1}{L_1(1)e_{\eta_1}(r_1z_1)}\sum_{\kappa_1=0}^{\infty}H_{\kappa_1}(r_1z_1)\int_{\frac{\kappa_1+2\eta_1\theta_{\kappa_1}}{r_1}}^{\frac{\kappa_1+1+2\eta_1\theta_{\kappa_1}}{r_1}} f(t_1,t_2)\mathrm{d}t_1 \tag{11}
$$

$$
\mathcal{Q}_{r_2}^{\eta_2}(f;z_1,z_2)=\frac{r_2}{L_2(1)e_{\eta_2}(r_2z_2)}\sum_{\kappa_2=0}^{\infty}H_{\kappa_2}(r_2z_2)\int_{\frac{\kappa_2+2\eta_2\theta_{\kappa_2}}{r_2}}^{\frac{\kappa_2+1+2\eta_2\theta_{\kappa_2}}{r_2}} f(t_1,t_2)\mathrm{d}t_2, \tag{12}
$$

then we get the equality

$$
\mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}(f;z_1,z_2)=\mathcal{P}_{r_1}^{\eta_1}\left(\mathcal{Q}_{r_2}^{\eta_2}(f;z_1,z_2)\right)=\mathcal{Q}_{r_2}^{\eta_2}\left(\mathcal{P}_{r_1}^{\eta_1}(f;z_1,z_2)\right).
$$

Proof. It is obvious to write that

$$
\begin{aligned}
\mathcal{P}_{r_1}^{\eta_1}\left(\mathcal{Q}_{r_2}^{\eta_2}(f;z_1,z_2)\right) &= \mathcal{P}_{r_1}^{\eta_1}\Big(\frac{r_2}{L_2(1)e_{\eta_2}(r_2z_2)}\sum_{\kappa_2=0}^{\infty}H_{\kappa_2}(r_2z_2)\int_{\frac{\kappa_2+2\eta_2\theta_{\kappa_2}}{r_2}}^{\frac{\kappa_2+1+2\eta_2\theta_{\kappa_2}}{r_2}} f(t_1,t_2)\mathrm{d}t_2\Big)\\
&= \frac{r_2}{L_2(1)e_{\eta_2}(r_2z_2)}\sum_{\kappa_2=0}^{\infty}H_{\kappa_2}(r_2z_2)\mathcal{P}_{r_1}^{\eta_1}\Big(\int_{\frac{\kappa_2+2\eta_2\theta_{\kappa_2}}{r_2}}^{\frac{\kappa_2+1+2\eta_2\theta_{\kappa_2}}{r_2}} f(t_1,t_2)\mathrm{d}t_2\Big)\\
&= \sum_{\kappa_1=0}^{\infty}\sum_{\kappa_2=0}^{\infty}L_{r_1,r_2}^{\eta_1,\eta_2}(z_1,z_2)H_{r_1,\kappa_1}^{r_2,\kappa_2}(z_1,z_2)\int_{\frac{\kappa_1+2\eta_1\theta_{\kappa_1}}{r_1}}^{\frac{\kappa_1+1+2\eta_1\theta_{\kappa_1}}{r_1}}\int_{\frac{\kappa_2+2\eta_2\theta_{\kappa_2}}{r_2}}^{\frac{\kappa_2+1+2\eta_2\theta_{\kappa_2}}{r_2}} f(t_1,t_2)\mathrm{d}t_1\mathrm{d}t_2\\
&= \mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}(f;z_1,z_2).
\end{aligned}
$$

Similarly, we prove $\mathcal{Q}_{r_2}^{\eta_2}\left(\mathcal{P}_{r_1}^{\eta_1}(f;z_1,z_2)\right)=\mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}(f;z_1,z_2)$. □

Lemma 2.3. *Let $\psi_{m,n}=t_1^m t_2^n$ be the test function then we have $\mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}(\psi_{0,0};z_1,z_2)=1$ and for $m=1,2,3,4$ and $n=0$, operators* (6) *have the following basic estimates:*

$$
\begin{aligned}
(1)\quad \mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}(\psi_{1,0};z_1,z_2) &= z_1+\frac{1}{2r_1}\left(2\frac{L_1'(1)}{L_1(1)}+4\eta_1+1\right);\\
(2)\quad \mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}(\psi_{2,0};z_1,z_2) &= z_1^2+\frac{1}{r_1}\left(2\frac{L_1'(1)}{L_1(1)}+4\eta_1+2\right)z_1\\
&+ \frac{1}{3r_1^2}\left(3\frac{L_1''(1)}{L_1(1)}+6(1+2\eta_1)\frac{L_1'(1)}{L_1(1)}+12\eta_1^2+6\eta_1+1\right);\\
(3)\quad \mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}(\psi_{3,0};z_1,z_2) &= z_1^3+\frac{3}{2r_1}\left(2\frac{L_1'(1)}{L_1(1)}+4\eta_1+3\right)z_1^2
\end{aligned}
$$

$$+ \frac{3}{2r_1^2}\left(2\frac{L_1''(1)}{L_1(1)} + 2(3+4\eta_1)\frac{L_1'(1)}{L_1(1)} + 8\eta_1^2 + 8\eta_1 + 3\right)z_1$$
$$+ \frac{1}{4r_1^3}\left(12\frac{L_1'''(1)}{L_1(1)} + 6(3+4\eta_1)\frac{L_1''(1)}{L_1(1)} + 6(8\eta_1^2 + 8\eta_1 + 3)\frac{L_1'(1)}{L_1(1)}\right.$$
$$\left. + 32\eta_1^3 + 8\eta_1^2 + 8\eta_1 + 1\right);$$

(4)
$$\mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}(\psi_{4,0};z_1,z_2) = z_1^4 + \frac{1}{r_1}\left(4\frac{L_1'(1)}{L_1(1)} + 8\eta_1 + 8\right)z_1^3$$
$$+ \frac{1}{r_1^2}\left(6\frac{L_1''(1)}{L_1(1)} + (14+24\eta_1)\frac{L_1'(1)}{L_1(1)} + 24\eta_1^2 + 36\eta_1 + 9\right)z_1^2$$
$$+ \frac{1}{r_1^3}\left(4\frac{L_1'''(1)}{L_1(1)} + (14+24\eta_1)\frac{L_1''(1)}{L_1(1)} + (38 + 72\eta_1 + 48\eta_1^2)\frac{L_1'(1)}{L_1(1)}\right.$$
$$\left. + 32\eta_1^3 + 48\eta_1^2 + 36\eta_1 + 13\right)z_1$$
$$+ \frac{1}{r_1^4}\left(\frac{L_1^{iv}(1)}{L_1(1)} + 4(3+2\eta_1)\frac{L_1'''(1)}{L_1(1)} + (19 + 36\eta_1 + 24\eta_1^2)\frac{L_1''(1)}{L_1(1)}\right.$$
$$\left. + (13 + 36\eta_1 + 48\eta_1^2 + 32\eta_1^3)\frac{L_1'(1)}{L_1(1)} + 16\eta_1^4 + 16\eta_1^3 + 8\eta_1^2 + 2\eta_1 + 1\right).$$

Proof. For the test function $\psi_{m,n} = t_1^m t_2^n$, from article [18] it is very easy to see that for all $n = 0$ and $m = 0, 1, 2, 3, 4$

$$\mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}(\psi_{m,n};z_1,z_2) = \mathcal{R}_{r,\eta}^*(t_1^m;z_1).$$

□

Lemma 2.4. *Take test function $\psi_{m,n} = t_1^m t_2^n$ for any $n = 1, 2, 3, 4$ and $m = 0$ then operators $\mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}(f;z_1,z_2)$ have the following identities:*

(1)
$$\mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}(\psi_{0,1};z_1,z_2) = z_2 + \frac{1}{2r_2}\left(2\frac{L_2'(1)}{L_2(1)} + 4\eta_2 + 1\right);$$

(2)
$$\mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}(\psi_{0,2};z_1,z_2) = z_2^2 + \frac{1}{r_2}\left(2\frac{L_2'(1)}{L_2(1)} + 4\eta_2 + 2\right)z_2$$
$$+ \frac{1}{3r_2^2}\left(3\frac{L_2''(1)}{L_2(1)} + 6(1+2\eta_2)\frac{L_2'(1)}{L_2(1)} + 12\eta_2^2 + 6\eta_2 + 1\right);$$

(3)
$$\mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}(\psi_{0,3};z_1,z_2) = z_2^3 + \frac{3}{2r_2}\left(2\frac{L_2'(1)}{L_2(1)} + 4\eta_2 + 3\right)z_2^2$$
$$+ \frac{3}{2r_2^2}\left(2\frac{L_2''(1)}{L_2(1)} + 2(3+4\eta_2)\frac{L_2'(1)}{L_2(1)} + 8\eta_2^2 + 8\eta_2 + 3\right)z_2$$
$$+ \frac{1}{4r_2^3}\left(12\frac{L_2'''(1)}{L_2(1)} + 6(3+4\eta_2)\frac{L_2''(1)}{L_2(1)} + 6(8\eta_2^2 + 8\eta_2 + 3)\frac{L_2'(1)}{L_2(1)}\right.$$
$$\left. + 32\eta_2^3 + 8\eta_2^2 + 8\eta_2 + 1\right);$$

(4)
$$\mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}(\psi_{0,4};z_1,z_2) = z_2^4 + \frac{1}{r_2}\left(4\frac{L_2'(1)}{L_2(1)} + 8\eta_2 + 8\right)z_2^3$$

$$
\begin{aligned}
&+ \frac{1}{r_2^2}\left(6\frac{L_2''(1)}{L_2(1)} + (14+24\eta_2)\frac{L_2'(1)}{L_2(1)} + 24\eta_2^2 + 36\eta_2 + 9\right)z_2^2\\
&+ \frac{1}{r_2^3}\Big(4\frac{L_2'''(1)}{L_2(1)} + (14+24\eta_2)\frac{L_2''(1)}{L_2(1)} + (38+72\eta_2+48\eta_2^2)\frac{L_2'(1)}{L_2(1)}\\
&+ 32\eta_2^3 + 48\eta_2^2 + 36\eta_2 + 13\Big)z_2\\
&+ \frac{1}{r_2^4}\Big(\frac{L_2^{iv}(1)}{L_2(1)} + 4(3+2\eta_2)\frac{L_2'''(1)}{L_2(1)} + (19+36\eta_2+24\eta_2^2)\frac{L_2''(1)}{L_2(1)}\\
&+ (13+36\eta_2+48\eta_2^2+32\eta_2^3)\frac{L_2'(1)}{L_2(1)} + 16\eta_2^4 + 16\eta_2^3 + 8\eta_2^2 + 2\eta_2 + 1\Big).
\end{aligned}
$$

Proof. For the test function $\psi_{m,n} = t_1^m t_2^n$, from article [18] it is very easy to see that for all $m = 0$ and $n = 0, 1, 2, 3, 4$

$$\mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}(\psi_{m,n}; z_1, z_2) = \mathcal{R}_{r,\eta}^{*}(t_2^n; z_2).$$

□

Lemma 2.5. *Operators $\mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}(.\,;\,.)$ have the following central moments:*

$$
\begin{aligned}
(1)\quad & \mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}((t_1-z_1); z_1,z_2) = \frac{1}{2r_1}\left(2\frac{L_1'(1)}{L_1(1)} + 4\eta_1 + 1\right);\\
(2)\quad & \mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}((t_2-z_2); z_1,z_2) = \frac{1}{2r_2}\left(2\frac{L_2'(1)}{L_2(1)} + 4\eta_2 + 1\right);\\
(3)\quad & \mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}((t_1-z_1)^2; z_1,z_2) = \frac{z_1}{r_1} + \frac{1}{3r_1^2}\left(3\frac{L_1''(1)}{L_1(1)} + 6(1+2\eta_1)\frac{L_1'(1)}{L_1(1)} + 12\eta_1^2 + 6\eta_1 + 1\right);\\
(4)\quad & \mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}((t_2-z_2)^2; z_1,z_2) = \frac{z_2}{r_2} + \frac{1}{3r_2^2}\left(3\frac{L_2''(1)}{L_2(1)} + 6(1+2\eta_2)\frac{L_2'(1)}{L_2(1)} + 12\eta_2^2 + 6\eta_2 + 1\right);\\
(5)\quad & \mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}((t_1-z_1)^4; z_1,z_2) = \frac{-7}{r_1^2}z_1^2 + \frac{1}{r_1^3}\Big(-8\frac{L_1'''(1)}{L_1(1)} - 4\frac{L_1''(1)}{L_1(1)} + 4(5+6\eta_1)\frac{L_1'(1)}{L_1(1)}\\
&+ 4\left(10\eta_1^2 + 7\eta_1 + 3\right)\Big)z_1\\
&+ \frac{1}{r_1^4}\Big(\frac{L_1^{iv}(1)}{L_1(1)} + 4(3+2\eta_1)\frac{L_1'''(1)}{L_1(1)} + (19+36\eta_1+24\eta_1^2)\frac{L_1''(1)}{L_1(1)}\\
&+ (13+36\eta_1+48\eta_1^2+32\eta_1^3)\frac{L_1'(1)}{L_1(1)} + 16\eta_1^4 + 16\eta_1^3 + 8\eta_1^2 + 2\eta_1 + 1\Big);\\
(6)\quad & \mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}((t_2-z_2)^4; z_1,z_2) = \frac{-7}{r_2^2}z_2^2 + \frac{1}{r_2^3}\Big(-8\frac{L_2'''(1)}{L_2(1)} - 4\frac{L_2''(1)}{L_2(1)} + 4(5+6\eta_2)\frac{L_2'(1)}{L_2(1)}\\
&+ 4\left(10\eta_2^2 + 7\eta_2 + 3\right)\Big)z_2\\
&+ \frac{1}{r_2^4}\Big(\frac{L_2^{iv}(1)}{L_2(1)} + 4(3+2\eta_2)\frac{L_2'''(1)}{L_2(1)} + (19+36\eta_2+24\eta_2^2)\frac{L_2''(1)}{L_2(1)}\\
&+ (13+36\eta_2+48\eta_2^2+32\eta_2^3)\frac{L_2'(1)}{L_2(1)} + 16\eta_2^4 + 16\eta_2^3 + 8\eta_2^2 + 2\eta_2 + 1\Big).
\end{aligned}
$$

Lemma 2.6. *Let $z_1, z_2 \in \mathcal{I}^2$ then for sufficiently large $r_1, r_2 \in \mathbb{N}$ we can observe the following inequalities:*

$$(1)\quad \mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}((t_1-z_1)^2; z_1,z_2) \leq O\left(\frac{1}{r_1^2}\right)(z_1+1)^2 \leq M_1(z_1+1)^2 \text{ as } r_1, r_2 \to \infty;$$

$$(2)\quad \mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}((t_2-z_2)^2; z_1,z_2) \leq O\left(\frac{1}{r_2^2}\right)(z_2+1)^2 \leq C_1(z_2+1)^2 \text{ as } r_1, r_2 \to \infty;$$

$$(3)\quad \mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}((t_1-z_1)^4; z_1,z_2) \leq O\left(\frac{1}{r_1^4}\right)(z_1+1)^4 \leq M_2(z_1+1)^4 \text{ as } r_1, r_2 \to \infty;$$

$$(4)\quad \mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}((t_2-z_2)^4; z_1,z_2) \leq O\left(\frac{1}{r_2^4}\right)(z_2+1)^4 \leq C_2(z_2+1)^4 \text{ as } r_1, r_2 \to \infty.$$

Remark 2.7. *For the operators $\mathcal{P}_{r_1}^{\eta_1}$ and $\mathcal{Q}_{r_1}^{\eta_2}$ be defined by* (11) *and* (12) *then operators we get $\mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}(\psi_{0,0}; z_1,z_2) = \mathcal{P}_{r_1}^{\eta_1}(\psi_{0,0}; z_1,z_2) = \mathcal{Q}_{r_2}^{\eta_2}(\psi_{0,0}; z_1,z_2)$ and for all $i, j = 1,2,3,4$, we have*

$$(1)\quad \mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}(\psi_{i,0}; z_1,z_2) = \mathcal{P}_{r_1}^{\eta_1}(\psi_{i,0}; z_1,z_2);$$
$$(2)\quad \mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}(\psi_{0,j}; z_1,z_2) = \mathcal{Q}_{r_2}^{\eta_2}(\psi_{0,j}; z_1,z_2).$$

3. Degree of convergence and weighted approximation properties

Suppose $\phi \in C(\mathcal{I}^2)$, the set of all continuous functions on $\mathcal{I}^2$ and $\delta_1,\ \delta_2 > 0$, then one has the properties of mixed modulus of continuity for order two

$$\omega\left(\phi;\delta_1,\delta_2\right) = \sup\{|\ \phi(t,s)-\phi(z_1,z_2)\ |: (t_1,t_2),(z_1,z_2) \in \mathcal{I}^2\}$$

where $|\ t_1 - z_1\ | \leq \delta_1,\ |\ t_2 - z_2\ | \leq \delta_2$ and the partial modulus of continuity given by:

$$\omega_1\left(\phi;\delta_1\right) = \sup_{0\leq z_1\leq 1}\ \sup_{|y_1-y_2|\leq\delta_1} \{|\ \varphi(z_1,y_1)-\varphi(z_1,y_2)\ |\},$$

$$\omega_2\left(\phi;\delta_2\right) = \sup_{0\leq z_2\leq 1}\ \sup_{|x_1-x_2|\leq\delta_2} \{|\ \varphi(x_1,z_2)-\varphi(x_2,z_2)\ |\}.$$

Theorem 3.1. *Suppose $(z_1,z_2) \in \mathcal{I}^2$, then for all $\phi \in C(\mathcal{I}^2)$ we get the inequality*

$$|\ \mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}(\phi; z_1,z_2) - \phi(z_1,z_2)\ | \leq 2\omega_1(\phi;\delta_{r_1}(z_1)) + 2\omega_2(\phi;\delta_{r_2}(z_2)).$$

Proof. We get our desired results by use of well-known Cauchy-Schwarz inequality, thus easy to get here

$$\begin{aligned}
|\ \mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}(\phi; z_1,z_2) - \phi(z_1,z_2)\ | &\leq \mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}\left(|\ \phi(t_1,t_2)-\phi(z_1,z_2)\ |; z_1,z_2\right) \\
&\leq \mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}\left(|\ \phi(t_1,t_2)-\phi(z_1,t_2)\ |; z_1,z_2\right) \\
&+ \mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}\left(|\ \phi(z_1,t_2)-\phi(z_1,z_2)\ |; z_1,z_2\right) \\
&\leq \mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}\left(\omega_1(\phi;|\ t_1-z_1\ |); z_1,z_2\right) + \mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}\left(\omega_2(\phi;|\ t_2-z_2\ |); z_1,z_2\right) \\
&\leq \omega_1(\phi;\delta_1)\Big(\mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}(\psi_{0,0}; z_1,z_2) + \delta_1^{-1}\mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}(|\ t_1-z_1\ |; z_1,z_2)\Big) \\
&+ \omega_2(\phi;\delta_2)\Big(\mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}(\psi_{0,0}; z_1,z_2) + \delta_2^{-1}\mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}(|\ t_2-z_2\ |; z_1,z_2)\Big)
\end{aligned}$$

$$
\begin{aligned}
&\leq \omega_1(\phi;\delta_1)\Big\{\mathcal{P}^{\eta_1}_{r_1}(\psi_{0,0}(t_1,t_2);z_1,z_2)\\
&+ \frac{1}{\delta_1}\sqrt{\mathcal{K}^{\eta_1,\eta_2}_{r_1,r_2}((t_1-z_1)^2;z_1,z_2)}\sqrt{\mathcal{P}^{\eta_1}_{r_1}(\psi_{0,0}(t,s);z_1,z_2)}\Big\}\\
&+ \omega_2(\phi;\delta_2)\Big\{\mathcal{Q}^{\eta_2}_{r_2}(\psi_{0,0}(t_1,t_2);z_1,z_2)\\
&+ \frac{1}{\delta_2}\sqrt{\mathcal{K}^{\eta_1,\eta_2}_{r_1,r_2}((t_2-z_2)^2;z_1,z_2)}\sqrt{\mathcal{Q}^{\eta_2}_{r_2}(\psi_{0,0}(t_1,t_2);z_1,z_2)}\Big\}.
\end{aligned}
$$

If we put

$$\delta_1 = \sqrt{\mathcal{K}^{\eta_1,\eta_2}_{r_1,r_2}((t_1-z_1)^2;z_1,z_2)} = \delta_{r_1}(z_1)$$

$$\delta_2 = \sqrt{\mathcal{K}^{\eta_1,\eta_2}_{r_1,r_2}((t_2-z_2)^2;z_1,z_2)} = \delta_{r_2}(r_2),$$

then easy to get desired inequality. □

We suppose $C_B(\mathcal{I}^2)$ be the set of all continuous and bounded function on $\mathcal{I}^2$, then for any $\varphi \in C_B(\mathcal{I}^2)$ and $\chi_1,\chi_2 \in [0,1]$, one has the Lipschitz class bivariate function for the sets $\mathcal{J}_1 \times \mathcal{J}_2 \subset \mathcal{I}^2$ by:

$$
\begin{aligned}
\mathcal{L}_{\chi_1,\chi_2}(\mathcal{J}_1\times\mathcal{J}_2) &= \Big\{\varphi : \sup(1+t_1)^{\chi_1}(1+t_2)^{\chi_2}\left(\varphi_{\chi_1,\chi_2}(t_1,t_2)-\varphi_{\chi_1,\chi_2}(z_1,z_2)\right)\Big\}\\
&\leq C\frac{1}{(1+z_1)^{\chi_1}}\frac{1}{(1+z_2)^{\chi_2}},
\end{aligned}
$$

$$\varphi_{\chi_1,\chi_2}(t_1,t_2)-\varphi_{\chi_1,\chi_2}(z_1,z_2) = \frac{|\,\varphi(t_1,t_2)-\varphi(z_1,z_2)\,|}{|\,t_1-z_1\,|^{\chi_1}|\,t_2-z_2\,|^{\chi_2}};\quad (t_1,t_2),(z_1,z_2)\in\mathcal{I}^2. \tag{13}$$

Theorem 3.2. *For all $\varphi \in \mathcal{L}_{\chi_1,\chi_2}(\mathcal{J}_1\times\mathcal{J}_2)$, we get the inequality*

$$
\begin{aligned}
&|\,\mathcal{K}^{\eta_1,\eta_2}_{r_1,r_2}(\varphi;z_1,z_2)-\varphi(z_1,z_2)\,|\\
&\quad\leq C\Big\{\left(\delta^2_{r_1}(z_1)\right)^{\frac{\chi_1}{2}}\left(\delta^2_{r_2}(z_2)\right)^{\frac{\chi_2}{2}} + (d(z_1,\mathcal{J}_1))^{\chi_1}\left(\delta^2_{r_2}(z_2)\right)^{\chi_2}\\
&\quad+ (d(z_2,\mathcal{J}_2))^{\frac{\chi_2}{2}}\left(\delta^2_{r_1}(z_1)\right)^{\frac{\chi_1}{2}} + 2\,(d(z_1,\mathcal{J}_1))^{\chi_1}\,(d(z_2,\mathcal{J}_2))^{\chi_2}\Big\},
\end{aligned}
$$

where $\chi_1,\chi_2 \in [0,1]$, $C > 0$ and $\delta_{r_1}(z_1)$, $\delta_{r_2}(z_2)$ are given by Theorem 3.1.

Proof. We take $|\,z_1-x_0\,| = d(z_1,\mathcal{J}_1)$ and $|\,z_2-y_0\,| = d(z_2,\mathcal{J}_2)$. For any $(z_1,z_2)\in\mathcal{I}^2$, and $(x_0,y_0)\in\mathcal{J}_1\times\mathcal{J}_2$ we let $d(z_1,\mathcal{J}_1) = \inf\{|\,z_1-z_2\,| : z_2\in\mathcal{J}_2\}$. Thus we can write here

$$|\,\varphi(t_1,t_2)-\varphi(z_1,z_2)\,| \leq C\,|\,\varphi(t_1,t_2)-\varphi(x_0,y_0)\,| + |\,\varphi(x_0,y_0)-\varphi(z_1,z_2)\,|\,. \tag{14}$$

Apply operators $\mathcal{K}^{\eta_1,\eta_2}_{r_1,r_2}$, then easy to obtain

$$
\begin{aligned}
|\,\mathcal{K}^{\eta_1,\eta_2}_{r_1,r_2}(\varphi;z_1,z_2)-\varphi(z_1,z_2)\,| &\leq \mathcal{K}^{\eta_1,\eta_2}_{r_1,r_2}\left(|\,\varphi(z_1,z_2)-\varphi(x_0,y_0)\,| + |\,\varphi(x_0,y_0)-\varphi(z_1,z_2)\,|\right)\\
&\leq C\mathcal{K}^{\eta_1,\eta_2}_{r_1,r_2}\left(|\,t_1-x_0\,|^{\chi_1}|\,t_2-y_0\,|^{\chi_2};z_1,z_2\right)\\
&+ C\,|\,z_1-x_0\,|^{\chi_1}|\,z_2-y_0\,|^{\chi_2}\,.
\end{aligned}
$$

For all $a,b\geq 0$ and $\chi\in[0,1]$ we know inequality $(a+b)^{\chi}\leq a^{\chi}+b^{\chi}$, thus

$$|\,t_1-x_0\,|^{\chi_1} \leq |\,t_1-z_1\,|^{\chi_1} + |\,z_1-x_0\,|^{\chi_1},$$

$$| t_2 - y_0 |^{\chi_1} \leq | t_2 - z_2 |^{\chi_2} + | z_2 - y_0 |^{\chi_2} .$$

Therefore

$$\begin{aligned} | \mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}(\varphi; z_1, z_2) - \varphi(z_1, z_2) | &\leq C\mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2} (| t_1 - z_1 |^{\chi_1} | t_2 - z_2 |^{\chi_2}; z_1, z_2) \\ &+ C | z_1 - x_0 |^{\chi_1} \mathcal{Q}_{r_2}^{\eta_2} (| t_2 - z_2 |^{\chi_2}; z_1, z_2) \\ &+ C | z_2 - y_0 |^{\chi_2} \mathcal{P}_{r_1}^{\eta_1} (| t_1 - z_1 |^{\chi_1}; z_1, z_2) \\ &+ 2C | z_1 - x_0 |^{\chi_1} | z_2 - y_0 |^{\chi_2} \mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2} (\psi_{0,0}; z_1, z_2) . \end{aligned}$$

Apply the Hölder inequality on $\mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}$, we get

$$\begin{aligned} \mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2} (| t_1 - z_1 |^{\chi_1} | t_2 - z_2 |^{\chi_2}; z_1, z_2) &= \mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2} (| t_1 - z_1 |^{\chi_1}; z_1, z_2) \mathcal{Q}_{r_2}^{\eta_2} (| t_2 - z_2 |^{\chi_2}; z_1, z_2) \\ &\leq \left(\mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}(| t_1 - z_1 |^2; z_1, z_2)\right)^{\frac{\chi_1}{2}} \left(\mathcal{P}_{r_1}^{\eta_1}(\psi_{0,0}; z_1, z_2)\right)^{\frac{2-\chi_1}{2}} \\ &\times \left(\mathcal{Q}_{r_2}^{\eta_2}(| t_2 - z_2 |^2; z_1, z_2)\right)^{\frac{\chi_2}{2}} \left(\mathcal{Q}_{r_2}^{\eta_2}(\psi_{0,0}; z_1, z_2)\right)^{\frac{2-\chi_2}{2}} . \end{aligned}$$

Thus we can obtain

$$\begin{aligned} &| \mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}(\varphi; z_1, z_2) - \varphi(z_1, z_2) | \\ &\leq C\left(\delta_{r_1}^2(z_1)\right)^{\frac{\chi_1}{2}} \left(\mathcal{P}_{r_1}^{\eta_1}(\psi_{0,0}; z_1, z_2)\right)^{\frac{2-\chi_1}{2}} \left(\delta_{r_2}^2(z_2)\right)^{\frac{z_2}{2}} \left(\mathcal{Q}_{r_2}^{\eta_2}(\psi_{0,0}; z_1, z_2)\right)^{\frac{2-\lambda_2}{2}} \\ &+ C\,(d(z_1, \mathcal{J}_1))^{\chi_1} \left(\delta_{r_2}^2(z_2)\right)^{\frac{\chi_2}{2}} \left(\mathcal{Q}_{r_2}^{\eta_2}(\psi_{0,0}; z_1, z_2)\right)^{\frac{\chi_2}{2}} \\ &+ C\,(d(z_2, \mathcal{J}_2))^{\chi_2} \left(\delta_{r_1}^2(z_1)\right)^{\frac{\chi_1}{2}} \left(\mathcal{P}_{r_1}^{\eta_1}(\psi_{0,0}; z_1, z_2)\right)^{\frac{\lambda_1}{2}} \\ &+ 2C\,(d(z_1, \mathcal{J}_1))^{\chi_1} (d(z_2, \mathcal{J}_2))^{\chi_2} . \end{aligned}$$

Thus we complete the desired proof. □

Theorem 3.3. *If $\phi' \in C(\mathcal{I}^2)$, then for all $(z_1, z_2) \in \mathcal{I}^2$, operators $\mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}$ satisfying*

$$| \mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}(\phi; z_1, z_2) - \phi(z_1, z_2) | \leq \| \phi_{z_1} \|_{C(\mathcal{I}^2)} \left(\delta_{r_1}^2(z_1)\right)^{\frac{1}{2}} + \| \phi_{z_2} \|_{C(\mathcal{I}^2)} \left(\delta_{r_2}^2(z_2)\right)^{\frac{1}{2}} ,$$

where $\delta_{r_1}(z_1)$ and $\delta_{r_2}(z_2)$ are defined by Theorem 3.1 and ϕ' defined for the first order set of all continuous function on $\mathcal{I}^2$.

Proof. For any fixed $(z_1, z_2) \in \mathcal{I}^2$ and for all $\phi' \in C(\mathcal{I}^2)$, we get the following equality

$$\phi(t_1, t_2) - \phi(z_1, z_2) = \int_{z_1}^{t_1} \phi_{\varrho}(\varrho, t_2) \mathrm{d}\varrho + \int_{z_2}^{t_2} \phi_{\mu}(z_1, \mu) \mathrm{d}\mu.$$

On apply $\mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}$

$$\mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2} \left(\phi(t_1, t_2); z_1, z_2\right) - \phi(z_1, z_2) = \mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2} \left(\int_{z_1}^{t_1} \phi_{\varrho}(\varrho, t_2) \mathrm{d}\varrho; z_1, z_2\right) + \mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2} \left(\int_{z_2}^{t_2} \phi_{\mu}(z_1, \mu) \mathrm{d}\mu; z_1, z_2\right). \quad (15)$$

Apply the sup-norm on $\mathcal{I}^2$ then easy to get

$$\left|\int_{z_1}^{t_1}\phi_\varrho(\varrho,t_2)\mathrm{d}\varrho\right|\leq\int_{z_1}^{t_1}|\,\phi_\varrho(\varrho,t_2)\mathrm{d}\varrho\,|\leq|\,t_1-z_1\,|\|\,\phi_{z_1}\,\|_{C(\mathcal{I}^2)} \tag{16}$$

$$\left|\int_{z_2}^{t_2}\phi_\mu(z_1,\mu)\mathrm{d}\mu\right|\leq\int_{z_2}^{t_2}|\,\phi_\mu(z_1,\mu)\mathrm{d}\mu\,|\leq|\,t_2-z_2\,|\|\,\phi_{z_2}\,\|_{C(\mathcal{I}^2)}\,. \tag{17}$$

In the view of (15), (16) and (17) we can obtain

$$\begin{aligned}
|\,\mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}\left(\phi(z_1,z_2);z_1,z_2\right)-\phi(z_1,z_2)\,| &\leq \mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}\left(\left|\int_{z_1}^{t}\phi_\varrho(\varrho,t_2)\mathrm{d}\varrho\right|;z_1,z_2\right)\\
&+ \mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}\left(\left|\int_{z_2}^{t_2}\varphi_\mu(z_1,\mu)\mathrm{d}\mu\right|;z_1,z_2\right)\\
&\leq \|\,\phi_{z_1}\,\|_{C(\mathcal{I}^2)}\,\mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}\left(|\,t_1-z_1\,|;z_1,z_2\right)\\
&+ \|\,\phi_{z_2}\,\|_{C(\mathcal{I}^2)}\,\mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}\left(|\,t_2-z_2\,|;z_1,z_2\right)\\
&\leq \|\,\phi_{z_1}\,\|_{C(\mathcal{I}^2)}\left(\mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}((t_1-z_1)^2;z_1,z_2)\mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}(\psi_{0,0};z_1,z_2)\right)^{\frac{1}{2}}\\
&+ \|\,\phi_{z_2}\,\|_{C(\mathcal{I}^2)}\left(\mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}((t_2-z_2)^2;z_1,z_2)\mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}(\psi_{0,0};z_1,z_2)\right)^{\frac{1}{2}}\\
&= \|\,\phi_{z_1}\,\|_{C(\mathcal{I}^2)}\left(\delta_{r_1}^2(z_1)\right)^{\frac{1}{2}}+\|\,\phi_{z_2}\,\|_{C(\mathcal{I}^2)}\left(\delta_{r_2}^2(z_2)\right)^{\frac{1}{2}}.
\end{aligned}$$

□

Take weight function $\Theta(z_1,z_2)=1+z_1^2+z_2^2$, then one satisfies the property $B_\Theta\left(\mathbb{R}_+^2\right)=\{f:|\,f(z_1,z_2)\,|\leq M_f\Theta(z_1,z_2)\}$, where $M_f>0$. Let $C^{(r)}\left(\mathbb{R}_+^2\right)$ be the set of r-times continuous and differentiable functions on $\mathbb{R}_+^2=\{(z_1,z_2)\in\mathbb{R}^2:z_1,z_2\in[0,\infty)\}$. In addition, suppose

$$C_\Theta\left(\mathbb{R}_+^2\right)=\{f:f\in B_\Theta\cap C_\Theta\left(\mathbb{R}_+^2\right)\};$$

$$C_\Theta^k\left(\mathbb{R}_+^2\right)=\{f:f\in C_\Theta\left(\mathbb{R}_+^2\right);\quad\text{such that}\ \lim_{z_1,z_2\to\infty}\frac{f(z_1,z_2)}{\Theta(z_1,z_2)}=k_f<\infty\};$$

$$C_\Theta^0\left(\mathbb{R}_+^2\right)=\{f\ \text{such that}\ f\in C_\Theta^k\left(\mathbb{R}_+^2\right);\quad\text{satisfying}\ \lim_{z_1,z_2\to\infty}\frac{f(z_1,z_2)}{\Theta(z_1,z_2)}=0\}.$$

The norm on B_Θ be defined as $\|\,f\,\|_\Theta=\sup_{z_1,z_2\in\mathbb{R}_+^2}\frac{|f(z_1,z_2)|}{\Theta(z_1,z_2)}$.

For all $f\in C_\Theta^0\left(\mathbb{R}_+^2\right)$ and $\delta_1,\delta_2>0$, the modulus of continuity in weighted space be

$$\omega_\Theta(f;\delta_1,\delta_2)=\sup_{z_1,z_2\in[0,\infty)}\ \sup_{0\leq|h_1|\leq\delta_1,\ 0\leq|h_2|\leq\delta_2}\frac{|\,f(z_1+h_1,z_2+h_2)-h(z_1,z_2)\,|}{\Theta(h_1,h_2)\Theta(z_1,z_2)} \tag{18}$$

and for any $r_1,r_2>0$ satisfying the inequality

$$\omega_\Theta(f;r_1\delta_1,r_2\delta_2)\leq4(r_1+1)(r_2+1)(1+\delta_1^2)(1+\delta_2^2)\omega_\Theta(f;\delta_1,\delta_2).$$

It also follows that

$$\begin{aligned}
|\,f(t_1,v)-f(z_1,z_2)\,| &\leq \Theta\left(|\,t_1-z_1\,|,|\,t_2-z_2\,|\right)\Theta(z_1,z_2)\omega_\Theta\left(f;|\,t_1-z_1\,|,|\,t_2-z_2\,|\right)\\
&\leq (1+z_1^2+z_2^2)(1+(t_1-z_1)^2)(1+(t_2-z_2)^2)\omega_\Theta\left(f;|\,t_1-z_1\,|,|\,t_2-z_2\,|\right).
\end{aligned}$$

Theorem 3.4. *Let $f \in C^0_\Theta\left(\mathbb{R}^2_+\right)$ then for sufficiently large any positive integers r_1 and r_2 we deduce that*

$$\frac{\mid \mathcal{K}^{\eta_1,\eta_2}_{r_1,r_2}(f;z_1,z_2) - f(z_1,z_2) \mid}{(1+z_1^2+z_2^2)^3} \leq M\left(O(r_1^{-2})\right)\left(O(r_2^{-2}\right)\omega_\Theta\left(f;O(r_1^{-2}),O(r_2^{-2})\right),$$

where $\delta_1 = \sqrt{\mathcal{K}^{\eta_1,\eta_2}_{r_1,r_2}((t_1-z_1)^2;z_1,z_2)} \leq O\left(\frac{1}{r_1}\right)$ *and* $\delta_2 = \sqrt{\mathcal{K}^{\eta_1,\eta_2}_{r_1,r_2}((t_2-z_2)^2;z_1,z_2)} \leq O\left(\frac{1}{r_2}\right)$.

Proof. We use above inequalities then for all $\delta_1, \delta_2 > 0$ easy to deduce that

$$\begin{aligned}
\mid f(t_1,t_2) - f(z_1,z_2) \mid &\leq 4(1+z_1^2+z_2^2)\left(1+(t_1-z_1)^2\right)\left(1+(t_2-z_2)^2\right)\omega_\Theta(f;\delta_1,\delta_2)\\
&\times \left(1+\frac{\mid t_1-z_1 \mid}{\delta_1}\right)\left(1+\frac{\mid t_2-z_2\mid}{\delta_m}\right)(1+\delta_1^2)(1+\delta_2^2)\\
&= 4(1+z_1^2+z_2^2)(1+\delta_1^2)(1+\delta_2^2)\\
&\times \left(1+\frac{\mid t_1-z_1\mid}{\delta_1}+(t_1-z_1)^2+\frac{\mid t_1-z_1\mid}{\delta_1}(t_1-z_1)^2\right)\\
&\times \left(1+\frac{\mid t_2-z_2\mid}{\delta_2}+(t_2-z_2)^2+\frac{\mid t_2-z_2\mid}{\delta_2}(t_2-z_2)^2\right)\omega_\Theta(f;\delta_1,\delta_2).
\end{aligned}$$

Applying operators $\mathcal{K}^{\eta_1,\eta_2}_{r_1,r_2}$, then get

$$\begin{aligned}
\mid \mathcal{K}^{\eta_1,\eta_2}_{r_1,r_2}(f;z_1,z_2) - f(z_1,z_2) \mid &\leq \mathcal{K}^{\eta_1,\eta_2}_{r_1,r_2}(\mid f(t_1,t_2)-f(z_1,z_2)\mid;z_1,z_2)\,4(1+z_1^2+z_2^2)\\
&\times \mathcal{K}^{\eta_1,\eta_2}_{r_1,r_2}\left(1+\frac{\mid t_1-z_1\mid}{\delta_1}+(t_1-z_1)^2+\frac{\mid t_1-z_1\mid}{\delta_1}(t_1-z_1)^2;z_1,z_2\right)\\
&\times \mathcal{K}^{\eta_1,\eta_2}_{r_1,r_2}\left(1+\frac{\mid t_2-z_2\mid}{\delta_2}+(t_2-z_2)^2+\frac{\mid t_2-z_2\mid}{\delta_2}(t_2-z_2)^2;z_1,z_2\right)\\
&\times (1+\delta_1^2)(1+\delta_2^2)\omega_\Theta(f;\delta_1,\delta_2)\\
&= 4(1+z_1^2+z_2^2)(1+\delta_1^2)(1+\delta_2^2)\omega_\Theta(f;\delta_1,\delta_2)\\
&\times \left(1+\frac{1}{\delta_1}\mathcal{K}^{\eta_1,\eta_2}_{r_1,r_2}(\mid t_1-z_1\mid;z_1,z_2)+\mathcal{K}^{\eta_1,\eta_2}_{r_1,r_2}((t_1-z_1)^2;z_1,z_2)\right.\\
&+ \left.\frac{1}{\delta_1}\mathcal{K}^{\eta_1,\eta_2}_{r_1,r_2}(\mid t_1-z_1\mid(t_1-z_1)^2;z_1,z_2\right)\\
&\times \left(1+\frac{1}{\delta_2}\mathcal{K}^{\eta_1,\eta_2}_{r_1,r_2}(\mid t_2-z_2\mid;z_1,z_2)+\mathcal{K}^{\eta_1,\eta_2}_{r_1,r_2}((t_2-z_2)^2;z_1,z_2)\right.\\
&+ \left.\frac{1}{\delta_2}\mathcal{K}^{\eta_1,\eta_2}_{r_1,r_2}(\mid t_2-z_2\mid(t_2-z_2)^2;z_1,z_2\right).
\end{aligned}$$

Applying Cauchy-Schwarz inequality, we get

$$\begin{aligned}
\mid \mathcal{K}^{\eta_1,\eta_2}_{r_1,r_2}(f;z_1,z_2) - f(z_1,z_2) \mid &\leq 4(1+z_1^2+z_2^2)(1+\delta_1^2)(1+\delta_2^2)\omega_\Theta(f;\delta_1,\delta_2)\\
&\times \left[1+\frac{1}{\delta_1}\sqrt{\mathcal{K}^{\eta_1,\eta_2}_{r_1,r_2}((t_1-z_1)^2;z_1,z_2)}+\mathcal{K}^{\eta_1,\eta_2}_{r_1,r_2}((t_1-z_1)^2;z_1,z_2)\right.\\
&+ \left.\frac{1}{\delta_1}\sqrt{\mathcal{K}^{\eta_1,\eta_2}_{r_1,r_2}((t_1-z_1)^2;z_1,z_2)}\sqrt{\mathcal{K}^{\eta_1,\eta_2}_{r_1,r_2}((t_1-z_1)^4;z_1,z_2)}\right]\\
&\times \left[1+\frac{1}{\delta_2}\sqrt{\mathcal{K}^{\eta_1,\eta_2}_{r_1,r_2}((t_2-z_2)^2;z_1,z_2)}+\mathcal{K}^{\eta_1,\eta_2}_{r_1,r_2}((t_2-z_2)^2;z_1,z_2)\right.\\
&+ \left.\frac{1}{\delta_2}\sqrt{\mathcal{K}^{\eta_1,\eta_2}_{r_1,r_2}((t_2-z_2)^2;z_1,z_2)}\sqrt{\mathcal{K}^{\eta_1,\eta_2}_{r_1,r_2}((t_2-z_2)^4;z_1,z_2)}\right].
\end{aligned}$$

In the view of Lemma 2.6 we get

$$\begin{aligned} \mid \mathcal{K}^{\eta_1,\eta_2}_{r_1,r_2}(f;z_1,z_2)-f(z_1,z_2)\mid &\leq 4(1+z_1^2+z_2^2)(1+\delta_1^2)(1+\delta_2^2)\omega_\Theta(f;\delta_1,\delta_2) \\ &\times \Big[2+M_1(z_1+1)^2+M_2(z_1+1)^2\Big]\Big[2+C_1(z_2+1)^2+C_2(z_2+1)^2\Big]. \end{aligned}$$

By choosing $\delta_1=\sqrt{\mathcal{K}^{\eta_1,\eta_2}_{r_1,r_2}((t_1-z_1)^2;z_1,z_2)}\leq O\left(\frac{1}{r_1}\right)$ and $\delta_2=\sqrt{\mathcal{K}^{\eta_1,\eta_2}_{r_1,r_2}((t_2-z_2)^2;z_1,z_2)}\leq O\left(\frac{1}{r_2}\right)$, we led to get our results. □

Theorem 3.5 ([8, 9]). *Let the weight function $\Theta(z_1,z_2)=1+z_1^2+z_2^2$, then for all $(z_1,z_2)\in\mathbb{R}^+\times\mathbb{R}^+$ the sequence of positive linear operators $\{L_{n,m}\}_{n,m\geq 1}:C_\Theta\to B_\Theta$ satisfying*

$$\lim_{n,m\to\infty}\| L_{n,m}(\Theta;z_1,z_2)-1\|_\Theta\leq C,$$

where C be a positive constant.

Theorem 3.6 ([8, 9]). *The sequence of positive linear operators $\{L_{n,m}\}_{n,m\geq 1}$ which acting $C_\Theta\to B_\Theta$ satisfying the following conditions*

1. $\lim_{n,m\to\infty}\| L_{n,m}(1;z_1,z_2)-1\|_\Theta=0,$
2. $\lim_{n,m\to\infty}\| L_{n,m}(t_1;z_1,z_2)-z_1\|_\Theta=0,$
3. $\lim_{n,m\to\infty}\| L_{n,m}(t_2;z_1,z_2)-z_2\|_\Theta=0,$
4. $\lim_{n,m\to\infty}\| L_{n,m}(t_1^2+t_2^2;z_1,z_2)-(z_1^2+z_2^2\|_\Theta=0,$

then for each $f\in C^0_\Theta$,

$$\lim_{n,m\to\infty}\| L_{n,m}(f)-f\|_\Theta=0,$$

and for any $g\in C_\Theta\setminus C^0_\Theta$, it follows that

$$\lim_{n,m\to\infty}\| L_{n,m}(g)-g\|_\Theta\geq 1.$$

Theorem 3.7. *For all $f\in C^0_\Theta\left(\mathbb{R}^2_+\right)$ the operators $\mathcal{K}^{\eta_1,\eta_2}_{r_1,r_2}$ by (6) satisfying the equality*

$$\| \mathcal{K}^{\eta_1,\eta_2}_{r_1,r_2}f-f\|_\Theta=0.$$

Proof. $\| \mathcal{K}^{\eta_1,\eta_2}_{r_1,r_2}(\Theta;z_1,z_2)\|_\Theta$

$$\begin{aligned} &= \sup_{(z_1,z_2)\in\mathbb{R}^2_+}\frac{\mid \mathcal{K}^{\eta_1,\eta_2}_{r_1,r_2}(1+t_1^2+t_2^2;z_1,z_2)\mid}{1+z_1^2+z_2^2} \\ &\leq 1+\sup_{(z_1,z_2)\in\mathbb{R}^2_+}\left[\frac{1}{1+z_1^2+z_2^2}\left|\left(\mathcal{K}^{\eta_1,\eta_2}_{r_1,r_2}(t_1^2;z_1,z_2)+\mathcal{K}^{\eta_1,\eta_2}_{r_1,r_2}(t_2^2;z_1,z_2)\right)\right|\right] \\ &= 1+\sup_{(z_1,z_2)\in\mathbb{R}^2_+}\frac{1}{1+z_1^2+z_2^2}\left[z_1^2+\frac{1}{r_1}\left(2\frac{L_1'(1)}{L_1(1)}+4\eta_1+2\right)z_1+z_2^2+\frac{1}{r_2}\left(2\frac{L_2'(1)}{L_2(1)}+4\eta_2+2\right)z_2\right] \\ &+ \sup_{(z_1,z_2)\in\mathbb{R}^2_+}\frac{1}{1+z_1^2+z_2^2}\left[1+\frac{1}{3r_1^2}\left(3\frac{L_1''(1)}{L_1(1)}+6(1+2\eta_1)\frac{L_1'(1)}{L_1(1)}+12\eta_1^2+6\eta_1+1\right)\right. \\ &+ \left.\frac{1}{3r_2^2}\left(3\frac{L_2''(1)}{L_2(1)}+6(1+2\eta_2)\frac{L_2'(1)}{L_2(1)}+12\eta_2^2+6\eta_2+1\right)\right] \end{aligned}$$

$$\leq \quad 1 + \max_{(z_1,z_2)\in\mathbb{R}_+^2} \left|\xi_{r_1,r_2}(z_1,z_2)\right| + \max_{(z_1,z_2)\in\mathbb{R}_+^2} \left|\gamma_{r_1,r_2}(z_1,z_2)\right|$$

where

$$\begin{aligned}
\xi_{r_1,r_2}(z_1,z_2) &= \frac{1}{1+z_1^2+z_2^2}\left[z_1^2 + \frac{1}{r_1}\left(2\frac{L_1'(1)}{L_1(1)} + 4\eta_1 + 2\right)z_1 + z_2^2 + \frac{1}{r_2}\left(2\frac{L_2'(1)}{L_2(1)} + 4\eta_2 + 2\right)z_2\right] \\
\gamma_{r_1,r_2}(z_1,z_2) &= \frac{1}{1+z_1^2+z_2^2}\frac{1}{3r_1^2}\left(3\frac{L_1''(1)}{L_1(1)} + 6(1+2\eta_1)\frac{L_1'(1)}{L_1(1)} + 12\eta_1^2 + 6\eta_1 + 1\right) \\
&+ \frac{1}{1+z_1^2+z_2^2}\frac{1}{3r_2^2}\left(3\frac{L_2''(1)}{L_2(1)} + 6(1+2\eta_2)\frac{L_2'(1)}{L_2(1)} + 12\eta_2^2 + 6\eta_2 + 1\right).
\end{aligned}$$

Therefore, if $r_1, r_2 \to \infty$, and $\max_{(z_1,z_2)\in\mathbb{R}_+^2}$ then there exists positive number M such that $\xi_{r_1,r_2}(z_1,z_2) + \gamma_{r_1,r_2}(z_1,z_2) < M$. Finally, we conclude that

$$\| \mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}(\Theta; z_1,z_2) \|_\Theta \leq M.$$

In the view of Theorem 3.5 we can see $\mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2} : C_\Theta\left(\mathbb{R}_+^2\right) \to B_\Theta\left(\mathbb{R}_+^2\right)$. Therefore, if we can show the conditions of Theorem 3.6 are satisfied then we complete the proof of Theorem 3.7. Thus by use of Lemma 2.3 and Lemma 2.4 we can see $\lim_{r_1,r_2\to\infty} \| \mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}(1; z_1,z_2) - 1 \|_\Theta = 0$, $\lim_{r_1,r_2\to\infty} \| \mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}(t_1; z_1,z_2) - z_1 \|_\Theta = 0$, $\lim_{r_1,r_2\to\infty} \| \mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}(t_2; z_1,z_2) - z_2 \|_\Theta = 0$ and

$$\begin{aligned}
&\lim_{r_1,r_2\to\infty} \| \mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}(t_1^2+t_2^2; z_1,z_2) - (z_1^2+z_2^2) \|_\Theta \\
&\leq \sup_{(z_1,z_2)\in\mathbb{R}_+^2} \frac{1}{1+z_1^2+z_2^2}\left[\left|\mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}(t_1^2; z_1,z_2) + \mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}(t_2^2; z_1,z_2) - (z_1^2+z_2^2)\right|\right] \\
&\leq \max_{(z_1,z_2)\in\mathbb{R}_+^2} \frac{1}{1+z_1^2+z_2^2}\left|\mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}(t_1^2; z_1,z_2) - z_1^2\right| + \max_{(z_1,z_2)\in\mathbb{R}_+^2} \frac{1}{1+z_1^2+z_2^2}\left|\mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}(t_2^2; z_1,z_2) - z_2^2\right|,
\end{aligned}$$

if $r_1, r_2 \to \infty$, and $\max_{(z_1,z_2)\in\mathbb{R}_+^2}$, then obvious we get that

$$\frac{1}{1+z_1^2+z_2^2}\left\{\left|\mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}(t_1^2; z_1,z_2) - z_1^2\right| + \left|\mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}(t_2^2; z_1,z_2) - z_2^2\right|\right\} = 0,$$

therefore, we obvious get that $\lim_{r_1,r_2\to\infty} \| \mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}(t_1^2+t_2^2; z_1,z_2) - (z_1^2+z_2^2) \|_\Theta = 0$, which completes the proof of Theorem 3.7. □

Theorem 3.8 ([8, 9]). *Let $\Theta_1(z_1,z_2) \geq 1$ be another set of continuous function such that $\lim_{z_1,z_2\to\infty} \frac{\Theta(z_1,z_2)}{\Theta_1(z_1,z_2)} = 0$. Then for all $f \in C_\Theta\left(\mathbb{R}_+^2\right)$ the operators $\{L_{n,m}\}_{n,m\geq 1} : C_{\Theta_1} \to B_{\Theta_1}$ satisfying all the conditions of Theorem 3.6, we get the equality*

$$\| L_{n,m}(f) - f \|_{\Theta_1} = 0.$$

Theorem 3.9. *For any $\Theta^*(z_1,z_2) \geq 1$ the set of all continuous functions such that $\lim_{|z_1,z_2|\to\infty} \frac{\Theta(z_1,z_2)}{\Theta^*(z_1,z_2)} = 0$. Then for all $f \in C_{\Theta^*}\left(\mathbb{R}_+^2\right)$ the sequence operators $\mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2} : C_{\Theta^*}\left(\mathbb{R}_+^2\right) \to B_{\Theta^*}\left(\mathbb{R}_+^2\right)$ satisfying the equality*

$$\| \mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}(f) - f \|_{\Theta^*} = 0.$$

Proof. We take in accounts the Theorem 3.7 and Theorem 3.8, then easy to write that the operators $\mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}$ acting from $C_{\Theta^*}\left(\mathbb{R}_+^2\right) \to B_{\Theta^*}\left(\mathbb{R}_+^2\right)$. Now from the Theorem 3.5 we can write here

$$\begin{aligned}
\| \mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}(\Theta; z_1,z_2) \|_{\Theta^*} &\leq 1 + \sup_{(z_1,z_2)\in\mathcal{I}^2} \frac{z_1^2}{\Theta^*(z_1,z_2)} + \frac{1}{r_1}\left(2\frac{L_1'(1)}{L_1(1)} + 4\eta_1 + 2\right) \sup_{(z_1,z_2)\in\mathcal{I}^2} \frac{z_1}{\Theta^*(z_1,z_2)} \\
&+ \frac{1}{3r_1^2}\left(3\frac{L_1''(1)}{L_1(1)} + 6(1+2\eta_1)\frac{L_1'(1)}{L_1(1)} + 12\eta_1^2 + 6\eta_1 + 1\right) \sup_{(z_1,z_2)\in\mathcal{I}^2} \frac{1}{\Theta^*(z_1,z_2)} \\
&+ \sup_{(z_1,z_2)\in\mathcal{I}^2} \frac{z_2^2}{\Theta^*(z_1,z_2)} + \frac{1}{r_2}\left(2\frac{L_2'(1)}{L_2(1)} + 4\eta_2 + 2\right) \sup_{(z_1,z_2)\in\mathcal{I}^2} \frac{z_2}{\Theta^*(z_1,z_2)} \\
&+ \frac{1}{3r_2^2}\left(3\frac{L_2''(1)}{L_2(1)} + 6(1+2\eta_2)\frac{L_2'(1)}{L_2(1)} + 12\eta_2^2 + 6\eta_2 + 1\right) \sup_{(z_1,z_2)\in\mathcal{I}^2} \frac{1}{\Theta^*(z_1,z_2)} \\
&\leq 1 + \max_{\substack{0\leq z_1\leq\mathbb{R}_+ \\ 0\leq z_2\leq\mathbb{R}_+}} \frac{z_1^2}{\Theta^*(z_1,z_2)} + \max_{\substack{0\leq z_1\leq\mathbb{R}_+ \\ 0\leq z_2\leq\mathbb{R}_+}} \frac{z_2^2}{\Theta^*(z_1,z_2)} \\
&= 1 + \alpha(z_1,z_2) + \beta(z_1,z_2), \quad r_1, r_2 \to \infty
\end{aligned}$$

where clearly, $\alpha(z_1,z_2) = 1 + \max_{\substack{0\leq z_1\leq\mathbb{R}_+ \\ 0\leq z_2\leq\mathbb{R}_+}} \frac{z_1^2}{\Theta^*(z_1,z_2)}$ and $\beta(z_1,z_2) = \max_{\substack{0\leq z_1\leq\mathbb{R}_+ \\ 0\leq z_2\leq\mathbb{R}_+}} \frac{z_2^2}{\Theta^*(z_1,z_2)}$, thus there exists a positive constant C such that $\alpha(z_1,z_2) + \beta(z_1,z_2) < C$. Therefore we can write

$$\| \mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}(\Theta; z_1,z_2) \|_{\Theta_1} \leq 1 + C.$$

In the view of Theorem 3.5 we can see $\mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2} : C_{\Theta^*}\left(\mathbb{R}_+^2\right) \to B_{\Theta^*}\left(\mathbb{R}_+^2\right)$. Therefore, from the conditions of Theorem 3.6 and Lemma 2.3, and Lemma 2.4 we can get $\| \mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}(1; z_1,z_2) - 1 \|_{\Theta^*} = 0$, $\| \mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}(t_1; z_1,z_2) - z_1 \|_{\Theta^*} = 0$, $\| \mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}(t_2; z_1,z_2) - z_2 \|_{\Theta^*} = 0$ and

$$\| \mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}(t_1^2 + t_2^2; z_1,z_2) - (z_1^2 + z_2^2) \|_{\Theta^*} = 0.$$

Therefore, operators $\mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}$ satisfies all the conditions of Theorem 3.6, thus we can write $\| \mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}(f) - f \|_{\Theta^*} = 0$, which completes the proof of Theorem 3.9. □

4. Approximation properties in Bögel space

For any $(u,v),(z_1,z_2) \in \mathcal{I}^2$, we define the bivariate type mixed difference operator $\tilde{\Delta}_{(u,v)}f(z_1,z_2)$ such that

$$\tilde{\Delta}_{(u,v)}f(z_1,z_2) = f(u,v) - f(u,z_2) - f(z_1,v) + f(z_1,z_2),$$

where $f : \mathcal{I}^2 \to \mathcal{R}$ which defined on real compact intervals of $\mathcal{I}^2$. For $\lim_{(u,v)\to(z_1,z_2)} \tilde{\Delta}_{(u,v)}f(z_1,z_2) = 0$, at any point $(z_1,z_2) \in \mathcal{I}^2$ then we call the function $f : \mathcal{I}^2 \to \mathcal{R}$ is Bögel-continuous function defined on $\mathcal{I}^2$. Let we define the space of all Bögel-continuous (B-continuous) on $(z_1,z_2) \in \mathcal{I}^2$ by $C_B(\mathcal{I}^2)$ and given that $C_B(\mathcal{I}^2) = \{f$ such that $f : \mathcal{I}^2 \to \mathcal{R}$ be f, B-bounded on $\mathcal{I}^2\}$. Moreover, the function $f : \mathcal{I}^2 \to \mathcal{R}$ is known as the Bögel-differentiable function defined for all $(z_1,z_2) \in \mathcal{I}^2$ which the limit exits finite and be finite given by

$$\lim_{(u,v)\to(z_1,z_2)} \frac{\tilde{\Delta}_{(u,v)}f(z_1,z_2)}{(u-z_1)(v-z_2)} = D_B f(z_1,z_2) < \infty. \tag{19}$$

Let we define the space of all Bögel-differentiable function by $D_\varphi f(z_1,z_2)$ such that $D_\varphi(\mathcal{I}^2) = \{f$ such that $f : \mathcal{I}^2 \to \mathcal{R}$ known B-differentiable defined on $\mathcal{I}^2\}$. If $f : \mathcal{I}^2 \to \mathcal{R}$ be B-bounded, then for any $(u,v),(z_1,z_2) \in \mathcal{I}^2$ there exits a real positive number M such that $|\tilde{\Delta}_{(u,v)}f(z_1,z_2)| \leq M$. Clearly, for any compact subset of $\mathcal{I}^2$

the B-continuous function is also be the B-bounded on $\mathcal{I}^2$. Suppose $B_\varphi(\mathcal{I}^2)$ be the space of all B-bounded function defined on $\mathcal{I}^2$ and which the norm on B equipped by $\| f \|_B = \sup_{(u,v),(z_1,z_2)\in\mathcal{I}^2} | \tilde{\Delta}_{(u,v)} f(z_1, z_2) |$. Let the mixed modulus of continuity be $\omega_B : \mathcal{I}^2 \to \mathcal{R}$, then for any $f \in B_\varphi(\mathcal{I}_{\alpha_n})$ and all $(u, v) \in \mathcal{I}^2$, $(z_1, z_2) \in \mathcal{I}^2$ it defined by:

$$\omega_B(f; \delta_1, \delta_2) = \sup\{\tilde{\Delta}_{(u,v)} f(z_1, z_2) :| u - z_1 |\leq \delta_1,\ | v - z_2 |\leq \delta_2\}. \tag{20}$$

Moreover, let $C_\varphi(\mathcal{I}^2)$ be the set of all B-continuous function on $\mathcal{I}^2$, where $\mathcal{I}^2 = [0,\infty) \times [0,\infty)$. For more related article on Bögel continuous functions space we see [6, 7].

Let $(z_1, z_2) \in \mathcal{I}^2$ and $r_1, r_2 > 0$ be an integer then for each $f \in C(\mathcal{I}^2)$ we define the GBS type an auxiliary operators by

$$\mathcal{G}^{\eta_1,\eta_2}_{r_1,r_2}(f(t_1, t_2); z_1, z_2) = \mathcal{K}^{\eta_1,\eta_2}_{r_1,r_2}(f(t_1, z_2) + f(z_1, t_2) - f(t_1, t_2); z_1, z_2). \tag{21}$$

More precise we can write

$$\mathcal{G}^{\eta_1,\eta_2}_{r_1,r_2}(f(t_1, t_2); z_1, z_2) = \sum_{\kappa_1,\kappa_2=0}^{\infty} L^{\eta_1,\eta_2}_{r_1,r_2}(z_1, z_2) H^{r_2,\kappa_2}_{r_1,\kappa_1}(z_1, z_2) \int_{\frac{\kappa_1+2\eta_1\theta_{\kappa_1}}{r_1}}^{\frac{\kappa_1+1+2\eta_1\theta_{\kappa_1}}{r_1}} \int_{\frac{\kappa_2+2\eta_2\theta_{\kappa_2}}{r_2}}^{\frac{\kappa_2+1+2\eta_2\theta_{\kappa_2}}{r_2}} g(t_1, t_2) \mathrm{d}t_1 \mathrm{d}t_2 \tag{22}$$

where, $g(t_1, t_2) = (f(t_1, z_2) + f(z_1, t_2) - f(t_1, t_2))$.

Theorem 4.1. *Let $(z_1, z_2) \in \mathcal{I}^2$, then for all $f \in C_\varphi(\mathcal{I}^2)$, we get*

$$| \mathcal{G}^{\eta_1,\eta_2}_{r_1,r_2}(f(t_1, t_2); z_1, z_2) - f(z_1, z_2) |\leq 4\omega_B(f; \delta_{r_1}(z_1), \delta_{r_2}(z_2)),$$

where $\delta_{r_1}(z_1)$ and $\delta_{r_1}(z_2)$ are defined by Theorem 3.1.

Proof. Let $(z_1, z_2) \in \mathcal{I}^2$, then clearly for all $(t_1, t_2) \in \mathcal{I}^2$ and $\delta_1 > 0$, $\delta_2 > 0$, we see

$$| \tilde{\Delta}_{(z_1,z_2)} f(t_1, t_2) |\leq \omega_B(f; | t_2 - z_2 | | t_1 - z_1 |) \leq \left(1 + \frac{| t_2 - z_2 |}{\delta_2}\right)\left(1 + \frac{| t_1 - z_1 |}{\delta_1}\right)\omega_B(f; \delta_1, \delta_2).$$

Applying the monotonicity property, linearity and Cauchy-Schwarz inequality then easy to obtain

$$\begin{aligned}
&| \mathcal{G}^{\eta_1,\eta_2}_{r_1,r_2}(f(t_1, t_2); z_1, z_2) - f(z_1, z_2) | \\
&\quad\leq \mathcal{K}^{\eta_1,\eta_2}_{r_1,r_2}\left(| \tilde{\Delta}_{(z_1,z_2)} f(t_1, t_2) |; z_1, z_2\right) \\
&\quad\leq \left(\mathcal{K}^{\eta_1,\eta_2}_{r_1,r_2}(\psi_{0,0}; z_1, z_2) + \frac{1}{\delta_1}\left(\mathcal{K}^{\eta_1,\eta_2}_{r_1,r_2}((t_1 - z_1)^2; z_1, z_2)\right)^{\frac{1}{2}}\right. \\
&\quad + \frac{1}{\delta_2}\left(\mathcal{K}^{\eta_1,\eta_2}_{r_1,r_2}((t_2 - z_2)^2; z_1, z_2)\right)^{\frac{1}{2}} \\
&\quad + \left.\frac{1}{\delta_1}\left(\mathcal{K}^{\eta_1,\eta_2}_{r_1,r_2}((t_1 - z_1)^2; z_1, z_2)\right)^{\frac{1}{2}} \frac{1}{\delta_2}\left(\mathcal{K}^{\eta_1,\eta_2}_{r_1,r_2}((t_2 - z_2)^2; z_1, z_2)\right)^{\frac{1}{2}}\right)\omega_B(f; \delta_1, \delta_2).
\end{aligned}$$

From Theorem 3.1, easy to see $\delta_1^2 = \delta_{r_1}^2(z_1) = \mathcal{K}^{\eta_1,\eta_2}_{r_1,r_2}((t_1 - z_1)^2; z_1, z_2)$ and $\delta_2^2 = \delta_{r_2}^2(z_2) = \mathcal{K}^{\eta_1,\eta_2}_{r_1,r_2}((t_2 - z_2)^2; z_1, z_2)$. □

Let $t = (t_1, t_2)$, $s = (z_1, z_2) \in \mathcal{I}^2$, then for a positive L and any number $0 < \chi \leq 1$, the Lipschitz class of maximal functions for B-continuous functions defined by

$$Lip_\chi = \left\{f \in C(\mathcal{I}^2) :| \tilde{\Delta}_{(z_1,z_2)} f(t, s) |\leq L \| t - s \|^\chi \right\} \tag{23}$$

where $\| t - s \| = \sqrt{(t_1 - z_1)^2 + (t_2 - z_2)^2}$ is the Euclidean norm.

Theorem 4.2. *For all $f \in Lip_\chi$ we get*

$$| \mathcal{G}_{r_1,r_2}^{\eta_1,\eta_2}(f(t,s);z_1,z_2) - f(z_1,z_2) | \leq L\{\delta_{r_1}^2(z_1) + \delta_{r_2}^2(z_2)\}^{\frac{\chi}{2}},$$

where $\delta_{r_1}(z_1)$ and $\delta_{r_2}(z_2)$ are defined by Theorem 3.1 and $L > 0$, $0 < \chi \leq 1$.

Proof. For the operators $\mathcal{G}_{r_1,r_2}^{\eta_1,\eta_2}$, we can write

$$\begin{aligned} \mathcal{G}_{r_1,r_2}^{\eta_1,\eta_2}(f(t,s);z_1,z_2) &= \mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}(f(z_1,s) + f(t,z_2) - f(t,s);z_1,z_2) \\ &= \mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}\left(f(z_1,z_2) - \tilde{\Delta}_{(z_1,z_2)}f(t,s);z_1,z_2\right) \\ &= f(z_1,z_2)\mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}(\psi_{0,0};z_1,z_2) - \mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}\left(\tilde{\Delta}_{(z_1,z_2)}f(t,s);z_1,z_2\right). \end{aligned}$$

Which implies that

$$\begin{aligned} | \mathcal{G}_{r_1,r_2}^{\eta_1,\eta_2}(f(t,s);z_1,z_2) - f(z_1,z_2) | &\leq \mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}\left(| \tilde{\Delta}_{(z_1,z_2)}f(t,s) |;z_1,z_2\right) \\ &\leq L\mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}(\| t-s \|^\chi;z_1,z_2) \\ &\leq L\{\mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}\left(\| t-s \|^2;z_1,z_2\right)\}^{\frac{\chi}{2}} \\ &\leq L\{\mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}\left((t_1-z_1)^2;z_1,z_2\right) + \mathcal{K}_{r_1,r_2}^{\eta_1,\eta_2}\left((t_2-z_2)^2;z_1,z_2\right)\}^{\frac{\chi}{2}}. \end{aligned}$$

□

5. Conclusion and Observation

Purpose of this manuscript is to study the bivariate and GBS associated properties of the Szász-Jakimovski-Leviatan-Kantorovich operators. We finally obtain the mixed modulus of continuity and more generalized appropriate approximation properties in Bögel continuous function spaces rather than the recent published article [18].

Availability of data and material
No data were used to support of this manuscript.

References

[1] A. Alotaibi, M. Mursaleen: *Approximation of Jakimovski-Leviatan-Beta type integral operators via q-calculus*, AIMS Mathematics, **5**(4), 3019–3034 (2020)

[2] A. Alotaibi: *On the Approximation by Bivariate Szász–Jakimovski–Leviatan-Type Operators of Unbounded Sequences of Positive Numbers*, Mathematics, **11** article id: 1009 (2023), https://doi.org/10.3390/math11041009

[3] P. Appell: *Une classe de polynômes*, Ann. Sci. École Norm. Sup. **9**, 119–144 (1880)

[4] D. Bărbosu: *Kantorovich-Stancu type operators*, J. Inequal. Pure Appl. Math., **5**(3) (2004), Article number 53.

[5] S.N. Bernstein: *Démonstration du théorème de Weierstrass fondée sur le calcul des probabilités*, Commun. Kharkov Math. Soc. **13**, 1–2 (1912/1913)

[6] K. Bögel:*Mehrdimensionale differentiation von Funktionen mehrerer veränderlichen*, J. Reine Angew. Math., **170** 197–217 (1934)

[7] K. Bögel: *Über die mehrdimensionale differentiation*, Jahresber. Dtsch. Math.Ver., **65**, 45–71 (1935)

[8] A.D. Gadžiev: *Positive linear operators in weighted spaces of functions of several variables*, Izv. Akad. Nauk Azerbaidzhan. SSR Ser. Fiz.-Tekhn. Mat. Nauk, no. **4**(1), 32–37 (1980)

[9] A.D. Gadžiev, H. Hacisalihoglu: *Convergence of the Sequences of Linear Positive Operators*, Ankara University, Yenimahalle (1995)

[10] L.V. Kantorovich: *Sur certaines dévelopments suivant les polynômes de la forme de S. Bernstein I–II*, C.R. Acad. Sci. USSR A, **20** (1930) 563–568.

[11] S.A. Mohiuddine, T. Acar, A. Alotaibi: *Construction of a new family of Bernstein-Kantorovich operators*. Math. Meth. Appl. Sci. **40**, 7749–7759 (2017)

[12] S.A. Mohiuddine, F. Özger: *Approximation of functions by Stancu variant of Bernstein-Kantorovich operators based on shape parameter α*, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales, Serie A. RACSAM, **114**, Article number:70 (2020)

[13] S.A. Mohiuddine, B. Hazarika, M.A. Alghamdi: *Ideal relatively uniform convergence with Korovkin and Voronovskaya types approximation theorems* Filomat **33** 4549–4560 2019

[14] S.A. Mohiuddine: *Approximation by bivariate generalized Bernstein–Schurer operators and associated GBS operators*, Adv Differ Equ., 2020, 676 (2020). https://doi.org/10.1186/s13662-020-03125-7

[15] M. Mursaleen, F. Khan, A. Khan: *Approximation properties for modified q-Bernstein–Kantorovich operators*, Numer. Funct. Anal. Optim. **36** (9) 1178–1197 (2015)

[16] M. Mursaleen, K.J. Ansari, M. Nasiruzzaman: *Approximation by q-analogue of Jakimovski-Leviatan operators involving q-Appell polynomials*, Iranian Jour. Sci. Tech. A., **41** 891–900 (2017)

[17] M. Nasiruzzaman, A.F. Aljohani: *Approximation by Szász-Jakimovski-Leviatan type operators via aid of Appell polynomials*, Journal of function spaces, Article ID 9657489. https://doi.org/10.1155/2020/9657489

[18] M. Nasiruzzaman, A.F. Aljohani: *Approximation by Parametric Extension of Szász-Mirakjan-Kantorovich Operators Involving the Appell Polynomials*, Journal of function spaces, 2020(1):1-11, Article ID 8863664. https://doi.org/10.1155/2020/8863664

[19] M. Nasiruzzaman: *Approximation properties by Szász operators to bivariate functions via Dunkl analogue*, Iran. J. Sci. Technol.Trans. Sci., **45** 259–269 (2021)

[20] M. Nasiruzzaman, N. Rao, M. Kumar, R. Kumar: *Approximation on bivariate parametric-extension of Baskakov-Durrmeyer operators*. Filomat, **35**, 2783–2800 (2021)

[21] M. Nasiruzzaman, Mohammed A.O. Toma, S. S. Capizzano, N. Rao, M. A. Mursaleene: *Approximation results for Beta Jakimovski-Leviatan type operators via q-analogue*, Filomat textbf37 (24) 8389–8404 (2023)

[22] M. Nasiruzzaman, M. Mursaleen: *Approximation by Jakimovski-Leviatan-Beta operators in weighted space*, AAdv Differ Equ., **2020** Article id:393, (2020)

[23] F. Özger: *Weighted statistical approximation properties of univariate and bivariate λ-Kantorovich operators*, Filomat **33**(11) 3473–3486 (2019)

[24] O.T. Pop: *Approximation of B-differentiable functions by GBS operators*, An. Univ. Oradea, Fasc. Mat. **14** 15–31 (2007)

[25] S. Sucu: *Dunkl analogue of Szász operators*, Appl. Math. Comput., **244**, 42–48 (2014)

[26] O. Szász: *Generalization of S. Bernstein's polynomials to the infinite interval*, J. Res. Natl. Bur. Stand., **45**, 239–245 (1950)

[27] A. Wafi, N. Rao: *Szász-gamma operators based on Dunkl analogue* Iran. J. Sci. Technol. Trans. A, Sci. **43** (1) 213–223 (2019)