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On deferred I-statistical rough convergence of difference sequences in intuitionistic fuzzy normed spaces

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Abstract. In this research article, using the concepts of deferred density and the notion of the ideal I, we extend the idea of rough convergence by introducing the notion of deferred I–statistical rough convergence via difference operators in the framework of intuitionistic fuzzy normed spaces. We define a set of limits of this convergence and prove that the limit set is convex and closed with respect to the intuitionistic fuzzy norm. Furthermore, we develop the concept of deferred I–statistical Δ_r^j–cluster point of a sequence in intuitionistic fuzzy normed spaces and investigate the relations between the set of these cluster points and the limit set of the aforementioned convergence.

1. Introduction

In the last few decades, fuzzy theory has gained vast popularity in the area of research in many branches of mathematics and engineering. In 1965, Zadeh [35] first introduced the theory of fuzzy sets to deal with uncertainty. Based on this theory, Kramosil and Michálek [20] proposed the notion of fuzzy metric space to extend the notion of ordinary metric space. Later, George and Veeramani [16] reformed the definition of fuzzy metric due to [20] and defined a Hausdorff topology on the reformed space. As an extension of fuzzy sets, Atanassov [5] developed the theory of intuitionistic fuzzy sets. In [30], Park generalized the notion of fuzzy metric and put forward the intuitionistic fuzzy metric space concept. Later, Saadati and Park [31] developed the concept of intuitionistic fuzzy normed space.

To generalize the classical notion of convergence, the concept of statistical convergence was put forth by Steinhaus [34] and independently by Fast [15], based only on the convergence criterion for most of the sequence's terms. In summability theory, this concept has recently emerged as one of the most active areas of study. In 2000, Kostyrko et al. [19] extended the notion of statistical convergence and proposed the concept of I–convergence of sequences. The implementation of statistical convergence and I–convergence in intuitionistic fuzzy normed spaces has been studied in [17, 25]. Later, Savaş and Gürdal [32] proposed the notion of I–statistical convergence in intuitionistic fuzzy normed spaces as a variant of statistical convergence. As a new method of convergence, the idea of deferred statistical convergence of sequences was

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studied in [21] by considering the deferred density in the definition of statistical convergence. Since then, many researchers have shown an interest in this topic. Recently, deferred statistical convergence of single sequences [24] and double sequences [26] has been studied in intuitionistic fuzzy normed spaces.

In addition to numerous generalizations of the notion of sequence convergence, rough convergence is also a concept of convergence that deals with the approximate solution from a numerical perspective. Phu [28] first developed the concept of rough convergence of sequences in a finite–dimensional normed linear spaces by specifying the degree of roughness and later introduced the same idea to infinite–dimensional normed linear spaces [29]. Along with the idea of rough convergence, he also looked at analytical characteristics like convexity and the closeness of the set of rough limits. Aytar [6] expanded the concept of rough convergence into rough statistical convergence and investigated the connection between the set of statistical cluster points and the set of rough statistical limit points of a sequence. Furthermore, in [7], Aytar investigated the rough limit set and studied the rough core of a real sequence. The rough convergence served as the inspiration for numerous authors' studies of rough convergence and rough statistical convergence of diverse sequence types. For instance, in [8] and [22, 23], respectively, rough convergence and rough statistical convergence of double and triple sequences were explored. Both Pal et al. [27] and Dündar et al. [10] suggested the concept of rough I–convergence and the set of rough I–limit points of a sequence. Later, Dündar [11] extended the idea of rough I–convergence to the rough I_2–convergence and examined the set of rough I_2–limit points of double sequences. In addition, he discovered two rough I_2–convergence criteria related to this limit set. In 2018, Dündar [12] generalized the idea of rough I_2–convergence to include the rough I_2–lacunary statistical convergence of double sequences and looked at a few characteristics of the rough I_2–lacunary statistical limit set. Furthermore, numerous authors have also explored the notion of rough convergence in various spaces, such as metric spaces [9], 2–normed spaces [4], probabilistic normed spaces [2], etc. In intuitionistic fuzzy normed spaces, Reena et al. [3] recently suggested the idea of rough statistical convergence of sequences by restricting the continuous t–norm to the minimum t–norm.

One of the most intriguing fields of research in mathematics right now is the study of difference operators and related sequence spaces. Using the forward difference operator Δ, the first–order difference sequence space was introduced in [18]. Later, Et and Çolak [13] generalized this concept to the situation of difference sequence spaces with integer order j. The main object of this paper is to develop the concept of deferred I–statistical rough convergence via difference operators of integer order j in intuitionistic fuzzy normed spaces.

2. Preliminaries

Throughout this study, we refer to the collections of all natural and real numbers as $\mathbb{N}$ and $\mathbb{R}$, respectively. For convenience, we recall some definitions as follows:

Let $A \subseteq \mathbb{N}$. The asymptotic (or natural) density of the set A, denoted by $\delta(A)$, is defined as:

$$\delta(A) = \lim_{k\to\infty} \frac{1}{k}\Big|\{n \le k : n \in A\}\Big|,$$

provided the limit exists. Here, $|\{,\}|$ denotes the cardinality of the set $\{,\}$. A sequence (x_k) of numbers is called statistically convergent to l if, for every $\epsilon > 0$,

$$\delta\Big(\{k \in \mathbb{N} : |x_k - l| > \epsilon\}\Big) = 0 \text{ holds.}$$

For this case, we write $x_k \xrightarrow{S} l$ (see [15], [34]).

Definition 2.1. *[14] A real (or complex) valued sequence (x_k) is Δ^j–statistically convergent to l if*

$$\delta\Big(\{k \in \mathbb{N} : |\Delta^j x_k - l| > \epsilon\}\Big) = 0$$

for every $\epsilon > 0$, where $j \in \mathbb{N}$ and

$$\Delta^0 x_k = x_k, \Delta^1 x_k = x_k - x_{k+1}, ..., \Delta^j x_k = \Delta^{j-1}(x_k - x_{k+1})$$

and so that

$$\Delta^j x_k = \sum_{i=0}^{j} (-1)^i \binom{j}{i} x_{k+i} \;\; (k \in \mathbb{N}).$$

Definition 2.2. *[6, 28] A sequence (x_k) in a normed space $(X, \|.\|)$ is rough convergent to $x \in X$ for some $r \geq 0$ if, for every $\mu > 0$, $\exists\, n_0 \in \mathbb{N}$ so that*

$$\|x_k - x\| < r + \mu, \forall\, k \geq n_0.$$

The sequence (x_k) is rough statistically convergent to $x \in X$ for some $r \geq 0$ if, for every $\mu > 0$,

$$\delta\Big(\{k \in \mathbb{N} : \|x_k - x\| \geq r + \mu\}\Big) = 0 \text{ holds.}$$

Definition 2.3. *[33] A binary operation $\star$ on $[0, 1]$ is called continuous t–norm (or* CTN*) if (a) $\star$ is commutative, associative and continuous, (b) $\mu = \mu \star 1$ for any $\mu \in [0, 1]$ and (c) for each $\mu_1, \mu_2, \mu_3, \mu_4 \in [0, 1]$, if $\mu_3 \geq \mu_1$ and $\mu_4 \geq \mu_2$ then $\mu_3 \star \mu_4 \geq \mu_1 \star \mu_2$.*

A binary operation $\circ$ on $[0, 1]$ is called continuous t–conorm (or CTCN*) if (1) $\circ$ is commutative, associative and continuous, (2) $\mu = \mu \circ 0$ for any $\mu \in [0, 1]$ and (3) for each $\mu_1, \mu_2, \mu_3, \mu_4 \in [0, 1]$, if $\mu_3 \geq \mu_1$ and $\mu_4 \geq \mu_2$ then $\mu_3 \circ \mu_4 \geq \mu_1 \circ \mu_2$.*

Definition 2.4. *[31] Assume X is a real vector space, $\star$ and $\circ$ are* CTN *and* CTCN*, respectively and φ, ψ are fuzzy subsets of $X \times (0, \infty)$. The five–tuple $(X, \varphi, \psi, \star, \circ)$ is called an intuitionistic fuzzy normed space (in short,* IFNS*) if, for all $x, y, z \in X$ and $s, t > 0$, the conditions below are met:*

(1) $\varphi(x, s) + \psi(x, s) \leq 1$,

(2) $\varphi(x, s) > 0$ and $\psi(x, s) < 1$,

(3) $\varphi(x, s) = 1$ and $\psi(x, s) = 0 \iff x = 0$,

(4) $\varphi(ax, s) = \varphi\Big(x, \frac{s}{|a|}\Big)$ and $\psi(ax, s) = \psi\Big(x, \frac{s}{|a|}\Big)$ for any $0 \neq a \in \mathbb{R}$,

(5) $\varphi(x, s) \star \varphi(y, t) \leq \varphi(x + y, s + t)$ and $\psi(x, s) \circ \psi(y, t) \geq \psi(x + y, s + t)$,

(6) $\varphi(x, .) : (0, \infty) \longrightarrow (0, 1]$ and $\psi(x, .) : (0, \infty) \longrightarrow (0, 1]$ are continuous,

(7) $\lim_{s\to\infty} \varphi(x, s) = 1$ and $\lim_{s\to 0} \varphi(x, s) = 0$,

(8) $\lim_{s\to\infty} \psi(x, s) = 0$ and $\lim_{s\to 0} \psi(x, s) = 1$.

Here, the tuple (φ, ψ) is known as the intuitionistic fuzzy norm (in short, IFN*) on X.*

Definition 2.5. *Let $(X, \varphi, \psi, \star, \circ)$ be an* IFNS*. The open ball of radius $r > 0$ and center $x \in X$ with regard to $\mu \in (0, 1)$ is the set*

$$\mathcal{B}_x^{(\varphi,\psi)}(r, \mu) = \Big\{y \in X : \varphi(x - y,\; r) > 1 - \mu \text{ and } \psi(x - y,\; r) < \mu\Big\}.$$

Definition 2.6. *[31] Let $(X, \varphi, \psi, \star, \circ)$ be an* IFNS*. A sequence (x_k) in X is convergent to $x \in X$ with regard to (φ, ψ) if*

$$\lim_{k\to\infty} \varphi(x_k - x, t) = 1 \text{ and } \lim_{k\to\infty} \psi(x_k - x, t) = 0$$

for every $t > 0$. In this case, we denote the limit by $x_k \xrightarrow{(\varphi,\psi)} x$.

Definition 2.7. *[19] Let $\Gamma \neq \emptyset$ set and $I \subseteq 2^{\Gamma}$. Then I is called an ideal in Γ if (a) $\emptyset \in I$, (b) $A, B \in I \Rightarrow A \cup B \in I$ and (c) $A \in I, B \subseteq A \Rightarrow B \in I$. An ideal $I \subseteq 2^{\Gamma}$ is nontrivial if $I \neq 2^{\Gamma}$. A nontrivial ideal $I \subseteq 2^{\Gamma}$ is admissible if I contains every singleton subset of X.*

A subset $F \subseteq 2^{\Gamma}$ is called filter on Γ if (c) $\emptyset \notin F$, (d) $A \cap B \in F$ for all $A, B \in F$ and (e) $B \in F$ whenever $A \in F$ and $B \supset A$. For each ideal I in Γ, one can find the filter $F(I)$ associated with ideal I such that $F(I) = \left\{A \subset \Gamma : A^c \in I\right\}$.

Definition 2.8. *[32] Let $(X, \varphi, \psi, \star, \circ)$ be an* IFNS *and I is nontrivial admissible ideal in $\mathbb{N}$. A sequence (x_k) in X is I–statistically convergent to some $x \in X$ with regard to (φ, ψ) if*

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : \varphi(x_k - x, t) \leq 1 - \mu \text{ or } \psi(x_k - x, t) \geq \mu\right\}\right| \geq \epsilon\right\} \in I$$

for every $\epsilon, t > 0$ and $\mu \in (0, 1)$.

For $I = I_f$, the collection of all finite subsets of $\mathbb{N}$, the convergence in Definition 2.8 reduces to the statistical convergence of (x_k) with regard to (φ, ψ) [17].

Agnew [1] in 1932 generalizes the notion of Cesàro mean of real (or complex) sequences and defined deferred Cesàro mean as follows:

Definition 2.9. *For a real (or complex) valued sequence (x_k), the deferred Cesàro mean of (x_k) is defined by*

$$\left(D_p^q(x_k)\right)_n := \frac{1}{q_n - p_n} \sum_{k=p_n+1}^{q_n} x_k, \; n = 1, 2, 3, ...,$$

where $p = (p_n)$ and $q = (q_n)$ are sequence of non–negative integers satisfying

$$p_n < q_n \text{ and } \lim_{n\to\infty} q_n = \infty. \tag{2.1}$$

For $K \subseteq \mathbb{N}$, the deferred density of K is defined by

$$D_p^q(K) = \lim_{n\to\infty} \frac{1}{q_n - p_n}\left|\left\{k \in \mathbb{N} : p_n < k \leq q_n, k \in K\right\}\right|, \tag{2.2}$$

provided the limit exists.

Definition 2.10. *[21] A real (or complex) valued sequence (x_k) is deferred statistically convergent to l if*

$$\lim_{n\to\infty} \frac{1}{q_n - p_n}\left|\left\{k \in \mathbb{N} : p_n < k \leq q_n, \; |x_k - l| \geq \epsilon\right\}\right| = 0$$

for every $\epsilon > 0$.

For $p_n = 0$ and $q_n = n$, the definition coincides with the statistical convergence of (x_k) introduced in [15].

3. $D_p^q(\varphi, \psi)^r_{S(I)}(\Delta^j)$–convergence sequences in IFNS

This section mainly introduces and investigates the notion of $D_p^q(\varphi, \psi)^r_{S(I)}(\Delta^j)$–convergence of sequences. Throughout this study, I stands for the non–trivial admissible ideal in $\mathbb{N}$ and $\Delta^j x_k = \sum_{i=0}^{j}(-1)^i \binom{j}{i} x_{k+i}$ $(j \in \mathbb{N})$ for any sequence (x_k), as well as that (p_n) and (q_n) are sequences of non–negative integers satisfying (2.1). Any other restrictions (if needed) on (p_n), (q_n) and j will be given in the related theorems and examples.

Definition 3.1. *Let $(X, \varphi, \psi, \star, \circ)$ be an* IFNS*. For a sequence (x_k) in X, we say (x_k) is Δ^j–rough convergent to some x with regard to (φ, ψ) for some $r \geq 0$ if, for every $t > 0$ and $\mu \in (0, 1)$, $\exists\, n_0 \in \mathbb{N}$ such that*

$$\varphi(\Delta^j x_k - x, t + r) > 1 - \mu \text{ and } \psi(\Delta^j x_k - x, t + r) < \mu, \forall\, k \geq n_0.$$

We write the limit as $(\varphi, \psi)^r(\Delta^j)$– $\lim x_k = x$.

Definition 3.2. *Let $(X, \varphi, \psi, \star, \circ)$ be an* IFNS*. For a sequence (x_k) in X, we say (x_k) is deferred I–statistically difference rough convergent to some x for some $r \geq 0$ with regard to (φ, ψ) (shortly, $D_p^q(\varphi, \psi)_{S(I)}^r(\Delta^j)$-convergent to x) if, for every $\epsilon, t > 0$ and $\mu \in (0, 1)$,*

$$\left\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\left|\left\{k \in \mathbb{N} : p_n < k \leq q_n,\ \varphi(\Delta^j x_k - x,\ t + r) \leq 1 - \mu \text{ or } \psi(\Delta^j x_k - x,\ t + r) \geq \mu\right\}\right| \geq \epsilon\right\} \in I \quad (3.1)$$

holds. In this case, we denote the limit by $D_p^q(\varphi, \psi)_{S(I)}^r(\Delta^j)\text{–}\lim x_k = x$.

Suppose (x_k) is a sequence in an IFNS $(X, \varphi, \psi, \star, \circ)$.

- For $r = 0$ in (3.1), the deferred I–statistical difference rough convergence of (x_k) is called the deferred I–statistical difference convergence with regard to (φ, ψ).
- If $p_n = 0$ and $q_n = n$ in (3.1), we refer to the deferred I–statistical difference rough convergence of (x_k) as the I–statistical difference rough convergence with regard to (φ, ψ).
- If $I = I_f$, we call the convergence defined in Definition 3.2 the deferred statistical difference rough convergence with regard to (φ, ψ).

Remark 3.3. *Suppose (x_k) is a sequence in an* IFNS *$(X, \varphi, \psi, \star, \circ)$ and $r \geq 0$. Then the limit $(\varphi, \psi)^r(\Delta^j)\text{–}\lim x_k$ as well as the limit $D_p^q(\varphi, \psi)_{S(I)}^r(\Delta^j)\text{–}\lim x_k$ need not be unique for $j \in \mathbb{N}$, provided they exist. We use the notations*

$$(\varphi, \psi)^r(\Delta^j)\text{–}\mathtt{LIM}(x_k) = \left\{x \in X : (\varphi, \psi)^r(\Delta^j)\text{–}\lim x_k = x\right\} \text{ and}$$

$$D_p^q(\varphi, \psi)_{S(I)}^r(\Delta^j)\text{–}\mathtt{LIM}(x_k) = \left\{x \in X : D_p^q(\varphi, \psi)_{S(I)}^r(\Delta^j)\text{–}\lim x_k = x\right\}$$

to denote the collection of all $(\varphi, \psi)^r(\Delta^j)\text{–}\lim x_k$ and $D_p^q(\varphi, \psi)_{S(I)}^r(\Delta^j)\text{–}\lim x_k$, respectively.
We say that a sequence (x_k) is Δ^j–rough convergent with regard to (φ, ψ) if $(\varphi, \psi)^r(\Delta^j)\text{–}\mathtt{LIM}(x_k) \neq \emptyset$ and a sequence (y_k) is $D_p^q(\varphi, \psi)_{S(I)}^r(\Delta^j)$–convergent if $D_p^q(\varphi, \psi)_{S(I)}^r(\Delta^j)\text{–}\mathtt{LIM}(y_k) \neq \emptyset$ for some $r \geq 0$.
If $0 \leq r_1 \leq r_2$, then for any sequence (x_k) in X it is clear that

$$(\varphi, \psi)^{r_1}(\Delta^j)\text{–}\mathtt{LIM}(x_k) \subseteq (\varphi, \psi)^{r_2}(\Delta^j)\text{–}\mathtt{LIM}(x_k)$$

and

$$D_p^q(\varphi, \psi)_{S(I)}^{r_1}(\Delta^j)\text{–}\mathtt{LIM}(x_k) \subseteq D_p^q(\varphi, \psi)_{S(I)}^{r_2}(\Delta^j)\text{–}\mathtt{LIM}(x_k).$$

Example 3.4. *Consider the* IFNS *$(\mathbb{R}, \varphi, \psi, \star, \circ)$, where $(\mathbb{R}, \|.\|)$ is the usual normed space, $\mu_1 \star \mu_2 = \mu_1\mu_2$, $\mu_1 \circ \mu_2 = \min\left\{\mu_1 + \mu_2, 1\right\}$ for all $\mu_1, \mu_2 \in [0, 1]$ and φ, ψ are defined as follows:*

$$\varphi(x, t) = \frac{t}{t + \|x\|} \text{ and } \psi(x, t) = \frac{\|x\|}{t + \|x\|}, \forall x \in X \text{ and } t > 0.$$

Define

$$x_k = \left\{\begin{array}{ll} 2 & \text{if } k = 2n - 1, \\ 1 & \text{otherwise} \end{array}\right\}, n \in \mathbb{N}.$$

Clearly, for $j = 1$ we have

$$\Delta^1 x_k = \left\{\begin{array}{ll} 1 & \text{if } k = 2n - 1, \\ -1 & \text{otherwise} \end{array}\right\}, n \in \mathbb{N}.$$

Then

$$(\varphi,\psi)^r(\Delta^1)\text{–}\mathtt{LIM}(x_k)=\left\{\begin{array}{ll}[1-r,r-1] & \text{if } r\geq 1,\\ \emptyset & \text{otherwise.}\end{array}\right\}$$

Assume $p_n=0$ *and* $q_n=n$ *for all* $n\in\mathbb{N}$ *and take a sequence* (y_k) *in* $\mathbb{R}$ *such that*

$$\Delta^1 y_k=\left\{\begin{array}{ll}k, & \text{if } k=2^n\\ -1, & \text{otherwise}\end{array}\right\}, n\in\mathbb{N}.$$

Then, for any nontrivial admissible ideal I *in* $\mathbb{N}$, *we get*

$$D_p^q(\varphi,\psi)^r_{S(I)}(\Delta^1)\text{–}\mathtt{LIM}(y_k)=\left\{\begin{array}{ll}[-1-r,r-1], & \text{if } r\geq 0\\ \emptyset, & \text{otherwise.}\end{array}\right\}$$

We can notice that both the sequences (x_k) and (y_k) as well as their difference sequences $(\Delta^1 x_k)$ and $(\Delta^1 y_k)$, respectively, are not convergent in the ordinary sense with regard to (φ,ψ). Also, $(\varphi,\psi)^r(\Delta^1)\text{–}\lim y_k$ does not exist.

Note 3.5. *In contrast to the ordinary convergence in an* IFNS $(X,\varphi,\psi,\star,\circ)$, *the* Δ^j*–rough convergence of a sequence* (x_k) *in* X *with regard to* (φ,ψ) *does not generally imply the* Δ^j*–rough convergence of any subsequence of* (x_k) *with regard to the same. For instance, for the sequence* $(x_k)=(k)$ *in the* IFNS *defined in Example 3.4, we have* $(\varphi,\psi)^r(\Delta^1)\text{–}\mathtt{LIM}(x_k)=[1-r,1+r]$ *for all* $r\geq 0$, *but the subsequence* $(x_{2^k})=(k^2)$ *of* (x_k) *is not* Δ^1*–rough convergent for any* $r\geq 0$. *The* $D_p^q(\varphi,\psi)^r_{S(I)}(\Delta^j)$*–convergence of a sequence* (x_k) *in* X *also follows the same reasoning above.*

Example 3.6. *Consider the* IFNS $(\mathbb{R},\varphi,\psi,\star,\circ)$, *where* $(\mathbb{R},||.||)$ *is the usual normed space,* $\mu_1\star\mu_2=\min\{\mu_1,\mu_2\}$, $\mu_1\circ\mu_2=\max\{\mu_1,\mu_2\}$, $\forall\ \mu_1,\mu_2\in[0,1]$ *and* φ, ψ *are defined as*

$$\varphi(x,t)=\frac{t}{t+||x||},\ \psi(x,t)=\frac{||x||}{t+||x||}\text{ for all } x\in\mathbb{R},\ t>0.$$

Take $p_n=0$ *and* $q_n=n$, $\forall\ n\in\mathbb{N}$. *Define*

$$x_k=\left\{\begin{array}{ll}k & \text{if } k=n^2,\\ 1 & \text{otherwise}\end{array}\right\}, n\in\mathbb{N}.$$

Then, for any nontrivial admissible ideal I, *we get*

$$D_p^q(\varphi,\psi)^r_{S(I)}(\Delta^1)\text{–}\mathtt{LIM}(x_k)=[-r,r],\forall\ r\geq 0.$$

On the other hand, for the subsequence (x_{n^2}) *of* (x_k), *we have*

$$D_p^q(\varphi,\psi)^r_{S(I)}(\Delta^1)\text{–}\mathtt{LIM}(x_{n^2})=\emptyset.$$

Lemma 3.7. *Suppose* $(X,\varphi,\psi,\star,\circ)$ *is an* IFNS *and* (x_k) *is a sequence in* X. *Let* $r\geq 0$ *be given. Then, for every* $\epsilon,t>0$ *and* $\mu\in(0,1)$, *the following are equivalent:*

(a) $D_p^q(\varphi,\psi)^r_{S(I)}(\Delta^j)\text{–}\lim x_k=x$.

(b) $\left\{n\in\mathbb{N}:\frac{1}{q_n-p_n}\left|\left\{k\in\mathbb{N}:p_n<k\leq q_n,\ \varphi(\Delta^j x_k-x,\ t+r)\leq 1-\mu\right\}\right|\geq\epsilon\right\}\in I$ *and*

$\left\{n\in\mathbb{N}:\frac{1}{q_n-p_n}\left|\left\{k\in\mathbb{N}:p_n<k\leq q_n,\ \psi(\Delta^j x_k-x,\ t+r)\geq\mu\right\}\right|\geq\epsilon\right\}\in I.$

(c) $\left\{n\in\mathbb{N}:\frac{1}{q_n-p_n}\left|\left\{k\in\mathbb{N}:p_n<k\leq q_n,\ \varphi(\Delta^j x_k-x,\ t+r)\leq 1-\mu \text{ or }\right.\right.\right.$
$\left.\left.\left.\psi(\Delta^j x_k-x,\ t+r)\geq\mu\right\}\right|<\epsilon\right\}\in F(I).$

(d) $\left\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\left|\left\{k \in \mathbb{N} : p_n < k \leq q_n,\ \varphi(\Delta^j x_k - x,\ t + r) \leq 1 - \mu\right\}\right| < \epsilon\right\} \in F(I)$ *and*

$\left\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\left|\left\{k \in \mathbb{N} : p_n < k \leq q_n,\ \psi(\Delta^j x_k - x,\ t + r) \geq \mu\right\}\right| < \epsilon\right\} \in F(I)$.

(e) $I\text{-}\lim_{n\to\infty} \frac{1}{q_n - p_n}\left|\left\{k \in \mathbb{N} : p_n < k \leq q_n,\ \varphi(\Delta^j x_k - x,\ t + r) \leq 1 - \mu \text{ or } \psi(\Delta^j x_k - x,\ t + r) \geq \mu\right\}\right| = 0.$

Proof. Due to its obvious nature, the proof has been omitted. □

Theorem 3.8. *Let $(X, \varphi, \psi, \star, \circ)$ be an* IFNS*. Then, for every sequence (x_k) in X,*

$$(\varphi, \psi)^r(\Delta^j)\text{–LIM}(x_k) \subset D_p^q(\varphi, \psi)_{S(I)}^r(\Delta^j)\text{–LIM}(x_k)$$

holds.

Proof. Assume that $x \in (\varphi, \psi)^r(\Delta^j)\text{–LIM}(x_k)$ for some $r \geq 0$. Then, for every $t > 0$ and $\mu \in (0, 1)$, $\exists\, n_0 \in \mathbb{N}$ so that

$$\varphi(\Delta^j x_k - x, t + r) > 1 - \mu \text{ and } \psi(\Delta^j x_k - x, t + r) < \mu, \forall\, k \geq n_0.$$

Therefore,

$$\left\{k \in \mathbb{N} : \varphi(\Delta^j x_k - x, t + r) \leq 1 - \mu \text{ or } \psi(\Delta^j x_k - x, t + r) \geq \mu\right\} \subseteq \left\{1, 2, ..., n_0 - 1\right\}.$$

Since

$$\lim_{n\to\infty} \frac{1}{q_n - p_n}\left|\left\{k \in \mathbb{N} : p_n < k \leq q_n, k \in \{1, 2, ..., n_0 - 1\}\right\}\right| = 0$$

holds for every $\epsilon > 0$, the set

$$\left\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\left|\left\{k \in \mathbb{N} : p_n < k \leq q_n,\ \varphi(\Delta^j x_k - x,\ t + r) \leq 1 - \mu \text{ or } \psi(\Delta^j x_k - x,\ t + r) \geq \mu\right\}\right| \geq \epsilon\right\}$$

belongs to I_f and hence to I. Therefore $x \in D_p^q(\varphi, \psi)_{S(I)}^r(\Delta^j)\text{–LIM}(x_k)$. As a result, we have

$$(\varphi, \psi)^r(\Delta^j)\text{–LIM}(x_k) \subset D_p^q(\varphi, \psi)_{S(I)}^r(\Delta^j)\text{–LIM}(x_k).$$

□

From Example 3.4, we can see that the above inclusion relation is strict.

Theorem 3.9. *Let $(X, \varphi, \psi, \star, \circ)$ be an* IFNS *and (x_k) be a sequence in X. Then, for any $r > 0$, there are no $x, y \in D_p^q(\varphi, \psi)_{S(I)}^r(\Delta^j)\text{–LIM}(x_k)$ such that $\varphi(x - y, sr) \leq 1 - \mu$ or $\psi(x - y, sr) \geq \mu$ for every $\mu \in (0, 1)$, where $s > 2$.*

Proof. For any given $\mu \in (0, 1)$, $\exists\, \nu \in (0, 1)$ such that $(1 - \nu) \star (1 - \nu) > 1 - \mu$ and $\nu \circ \nu < \mu$. Let on contrary that there exist $x, y \in D_p^q(\varphi, \psi)_{S(I)}^r(\Delta^j)\text{–LIM}(x_k)$ such that for every $\mu \in (0, 1)$,

$$\varphi(x - y, sr) \leq 1 - \mu \text{ or } \psi(x - y, sr) \geq \mu,$$

where $s > 2$. Now, for any $t > 0$, consider the sets

$$N = \left\{k \in \mathbb{N} : p_n < k \leq q_n,\ \varphi\left(\Delta^j x_k - x, \frac{t}{2} + r\right) \leq 1 - \nu \text{ or } \psi\left(\Delta^j x_k - x, \frac{t}{2} + r\right) \geq \nu\right\}$$

and

$$O = \left\{k \in \mathbb{N} : p_n < k \leq q_n, \varphi\left(\Delta^j x_k - y, \frac{t}{2} + r\right) \leq 1 - \nu \text{ or } \psi\left(\Delta^j x_k - y, \frac{t}{2} + r\right) \geq \nu\right\}.$$

Since $x, y \in D_p^q(\varphi,\psi)_{S(I)}^r(\Delta^j)$–$\texttt{LIM}(x_k)$, by Lemma 3.7, we have

$$I\text{–}\lim_{n\to\infty}\frac{1}{q_n-p_n}\left|\left\{k\in\mathbb{N}:p_n<k\le q_n, k\in N\right\}\right|=0$$

and

$$I\text{–}\lim_{n\to\infty}\frac{1}{q_n-p_n}\left|\left\{k\in\mathbb{N}:p_n<k\le q_n, k\in O\right\}\right|=0.$$

Now

$$\begin{aligned}I\text{–}\lim_{n\to\infty}&\frac{1}{q_n-p_n}\left|\left\{k\in\mathbb{N}:p_n<k\le q_n, k\in N\cup O\right\}\right|\\&\le I\text{–}\lim_{n\to\infty}\frac{1}{q_n-p_n}\left|\left\{k\in\mathbb{N}:p_n<k\le q_n, k\in N\right\}\right|\\&\quad+I\text{–}\lim_{n\to\infty}\frac{1}{q_n-p_n}\left|\left\{k\in\mathbb{N}:p_n<k\le q_n, k\in O\right\}\right|\\&=0.\end{aligned}$$

Hence for every $\epsilon>0$,

$$P=\left\{n\in\mathbb{N}:\frac{1}{q_n-p_n}\left|\left\{k\in\mathbb{N}:p_n<k\le q_n, k\in N\cup O\right\}\right|\ge\epsilon\right\}\in I.$$

Let $m\in P^c$ and $\epsilon=\frac{1}{4}$. Then

$$\begin{aligned}&\frac{1}{q_m-p_m}\left|\left\{k\in\mathbb{N}:p_m<k\le q_m, k\in N\cup O\right\}\right|<\frac{1}{4}\\&\implies\frac{1}{q_m-p_m}\left|\left\{k\in\mathbb{N}:p_m<k\le q_m, k\in N^c\cap O^c\right\}\right|\ge1-\frac{1}{4}=\frac{3}{4}.\end{aligned}$$

As a result, we have

$$Q=\left\{k\in\mathbb{N}:p_n<k\le q_n, k\in N^c\cap O^c\right\}\neq\emptyset.$$

Since $s>2$, put $sr=2r+t$ for some $t>0$. If $\varphi(x-y,sr)\le1-\mu$ then for $k\in Q$, we find

$$\begin{aligned}1-\mu&\ge\varphi\left(x-y,t+2r\right)\\&\ge\varphi\left(\Delta^jx_k-x,\frac{t}{2}+r\right)\star\varphi\left(\Delta^jx_k-y,\frac{t}{2}+r\right)\\&>(1-\nu)\star(1-\nu)\\&>1-\mu,\end{aligned}$$

which is absurd. If $\psi(x-y,sr)\ge\mu$ for some $s>2$, then

$$\begin{aligned}\mu&\le\psi\left(x-y,t+2r\right)\\&\le\psi\left(\Delta^jx_k-x,\frac{t}{2}+r\right)\circ\psi\left(\Delta^jx_k-y,\frac{t}{2}+r\right)\\&<\nu\circ\nu\\&<\mu,\end{aligned}$$

which is again absurd. Consequently, each case leads to an absurd result. The proof of our findings is finished with this. □

Proposition 3.10. *Let $(X, , \varphi, \psi, \star, \circ)$ be an* IFNS*. Assume (x_k) and (y_k) are sequences in X with $D_p^q(\varphi,\psi)_{S(I)}^{r_1}(\Delta^j)-\lim x_k = x$ and $D_p^q(\varphi,\psi)_{S(I)}^{r_2}(\Delta^j)-\lim y_k = y$ for some $r_1, r_2 \geq 0$. Then*

$$D_p^q(\varphi,\psi)_{S(I)}^{(r_1+r_2)}(\Delta^j)-\lim[x_k + y_k] = x + y.$$

Proof. For given $\mu \in (0,1)$ choose $\nu \in (0,1)$ with $(1-\nu)\star(1-\nu) > 1-\mu$ and $\nu\circ\nu < \mu$. Suppose $D_p^q(\varphi,\psi)_{S(I)}^{r_1}(\Delta^j)-\lim x_k = x$ and $D_p^q(\varphi,\psi)_{S(I)}^{r_2}(\Delta^j)-\lim y_k = y$ for some $r_1, r_2 \geq 0$. For $t > 0$, consider the sets

$$A = \Big\{k \in \mathbb{N} : p_n < k \leq q_n,\ \varphi\Big(\Delta^j x_k - x, \frac{t}{2} + r_1\Big) \leq 1-\nu \text{ or } \psi\Big(\Delta^j x_k - x, \frac{t}{2} + r_1\Big) \geq \nu\Big\}$$

and

$$B = \Big\{k \in \mathbb{N} : p_n < k \leq q_n,\ \varphi\Big(\Delta^j y_k - y, \frac{t}{2} + r_2\Big) \leq 1-\nu \text{ or } \psi\Big(\Delta^j y_k - y, \frac{t}{2} + r_2\Big) \geq \nu\Big\}.$$

Then

$$\Big\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\Big|\Big\{k \in \mathbb{N} : p_n < k \leq q_n,\ k \in A\Big\}\Big| \geq \epsilon\Big\} \in I \text{ and}$$

$$\Big\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\Big|\Big\{k \in \mathbb{N} : p_n < k \leq q_n,\ k \in B\Big\}\Big| \geq \epsilon\Big\} \in I$$

for each $\epsilon > 0$. Therefore,

$$\Big\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\Big|\Big\{k \in \mathbb{N} : p_n < k \leq q_n,\ k \in A \cup B\Big\}\Big| \geq \epsilon\Big\} \in I.$$

Now, choose $0 < \lambda < 1$ so that $0 < 1-\lambda < \epsilon$. Then

$$P = \Big\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\Big|\Big\{k \in \mathbb{N} : p_n < k \leq q_n,\ k \in A \cup B\Big\}\Big| \geq 1-\lambda\Big\} \in I.$$

Let $m \in P^c$. Then

$$\frac{1}{q_m - p_m}\Big|\Big\{k \in \mathbb{N} : p_m < k \leq q_m,\ k \in A \cup B\Big\}\Big| < 1-\lambda$$

$$\implies \frac{1}{q_m - p_m}\Big|\Big\{k \in \mathbb{N} : p_m < k \leq q_m,\ k \in A^c \cap B^c\Big\}\Big| \geq 1-(1-\lambda) = \lambda.$$

Take $k \in A^c \cap B^c$. Then

$$\begin{aligned}\varphi\Big(\Big(\Delta^j x_k + \Delta^j y_k\Big) - (x+y), t + r_1 + r_2\Big) &\geq \varphi\Big(\Delta^j x_k - x,\ \frac{t}{2} + r_1\Big) \star \varphi\Big(\Delta^j y_k - y,\ \frac{t}{2} + r_2\Big)\\ &\geq (1-\nu)\star(1-\nu)\\ &> 1-\mu\end{aligned}$$

and

$$\begin{aligned}\psi\Big(\Big(\Delta^j x_k + \Delta^j y_k\Big) - (x+y), t + r_1 + r_2\Big) &\leq \psi\Big(\Delta^j x_k - x,\ \frac{t}{2} + r_1\Big) \circ \psi\Big(\Delta^j y_k - y,\ \frac{t}{2} + r_2\Big)\\ &\leq \nu\circ\nu\\ &< \mu.\end{aligned}$$

This implies that

$$A^c \cap B^c \subseteq \Big\{k \in \mathbb{N} : p_n < k \leq q_n,\ \varphi\Big(\Big(\Delta^j x_k + \Delta^j y_k\Big) - (x+y),\ t + r_1 + r_2\Big) > 1 - \mu \text{ and } \psi\Big(\Big(\Delta^j x_k + \Delta^j y_k\Big) - (x+y),\ t + r_1 + r_2\Big) < \mu\Big\}.$$

As a result, for $m \in P^c$, we have

$$\begin{aligned}\lambda &\leq \frac{1}{q_m - p_m}\Big|\Big\{k \in \mathbb{N} : p_m < k \leq q_m,\ k \in A^c \cap B^c\Big\}\Big| \\ &\leq \frac{1}{q_m - p_m}\Big|\Big\{k \in \mathbb{N} : p_m < k \leq q_m,\ \varphi\Big((\Delta^j x_k + \Delta^j y_k) - (x+y),\ t + r_1 + r_2\Big) > 1 - \mu \\ &\qquad \text{and } \psi\Big(\Big(\Delta^j x_k + \Delta^j y_k\Big) - (x+y),\ t + r_1 + r_2\Big) < \mu\Big\}\Big| \\ \implies &\frac{1}{q_m - p_m}\Big|\Big\{k \in \mathbb{N} : p_m < k \leq q_m, \varphi\Big(\Big(\Delta^j x_k + \Delta^j y_k\Big) - (x+y),\ t + r_1 + r_2\Big) \leq 1 - \mu \\ &\qquad \text{or } \psi\Big(\Big(\Delta^j x_k + \Delta^j y_k\Big) - (x+y),\ t + r_1 + r_2\Big) \geq \mu\Big\}\Big| < 1 - \lambda < \epsilon.\end{aligned}$$

Consequently,

$$P^c \subseteq \Big\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\Big|\Big\{k \in \mathbb{N} : p_n < k \leq q_n, \varphi\Big(\Big(\Delta^j x_k + \Delta^j y_k\Big) - (x+y),\ t + r_1 + r_2\Big) \leq 1 - \mu \text{ or } \psi\Big(\Big(\Delta^j x_k + \Delta^j y_k\Big) - (x+y),\ t + r_1 + r_2\Big) \geq \mu\Big\}\Big| < \epsilon\Big\}.$$

Since $P^c \in F(I)$, we have

$$\Big\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\Big|\Big\{k \in \mathbb{N} : p_n < k \leq q_n, \varphi\Big(\Big(\Delta^j x_k + \Delta^j y_k\Big) - (x+y),\ t + r_1 + r_2\Big) \leq 1 - \mu \text{ or } \psi\Big(\Big(\Delta^j x_k + \Delta^j y_k\Big) - (x+y),\ t + r_1 + r_2\Big) \geq \mu\Big\}\Big| < \epsilon\Big\} \in F(I).$$

Hence, by Lemma 3.7, we have $D_p^q(\varphi, \psi)_{S(I)}^{(r_1+r_2)}(\Delta^j)\text{–}\lim[x_k + y_k] = x + y$. □

Remark 3.11. *Proposition 3.10 is not true for r, where $0 < r < r_1 + r_2$, if at least one of r_1 and r_2 is non–zero, i.e., for $r_1 \neq 0$ or $r_2 \neq 0$ if $D_p^q(\varphi, \psi)_{S(I)}^{r_1}(\Delta^j)\text{–}\lim x_k = x$ and $D_p^q(\varphi, \psi)_{S(I)}^{r_2}(\Delta^j)\text{–}\lim y_k = y$ then $D_p^q(\varphi, \psi)_{S(I)}^{r}(\Delta^j)\text{–}\lim[x_k + y_k]$ need not be equal to $x + y$, where $0 < r < r_1 + r_2$.*

Example 3.12. *Consider $(\mathbb{R}, \varphi, \psi, \star, \circ)$, the* IFNS*, defined in Example 3.6. Define*

$$x_k = \left\{\begin{array}{ll} k & \text{if } k = 5^n, \\ 0 & \text{if } k = 2n, \\ 1 & \text{if } k \neq 5^n, 2n \end{array}\right\}, n \in \mathbb{N}$$

and

$$y_k = \left\{\begin{array}{ll} 0 & \text{if } k = 5^n, \\ -1 & \text{if } k = 2n, \\ 1 & \text{if } k \neq 5^n, 2n \end{array}\right\}, n \in \mathbb{N}.$$

Take $p_n = 0$ and $q_n = n$, $\forall\ n \in \mathbb{N}$. Then, for any nontrivial admissible ideal I in $\mathbb{N}$, we get

$$D_p^q(\varphi, \psi)_{S(I)}^{r_1}(\Delta^1)\text{–}\mathtt{LIM}(x_k) = \left\{\begin{array}{ll} [1 - r_1, r_1 - 1] & \text{if } r_1 \geq 1, \\ \emptyset & \text{otherwise.} \end{array}\right\}$$

and

$$D_p^q(\varphi,\psi)^{r_2}_{S(I)}(\Delta^1)\text{–}\mathtt{LIM}(y_k) = \left\{\begin{array}{ll} [2-r_2, r_2-2] & \text{if } r_2 \geq 2, \\ \emptyset & \text{otherwise.} \end{array}\right\}$$

Now,

$$x_k + y_k = \left\{\begin{array}{ll} k & \text{if } k = 5^n, \\ -1 & \text{if } k = 2n, \\ 2 & \text{if } k \neq 5^n, 2n \end{array}\right\}, n \in \mathbb{N}.$$

Then

$$D_p^q(\varphi,\psi)^{r}_{S(I)}(\Delta^1)\text{–}\mathtt{LIM}(x_k + y_k) = \left\{\begin{array}{ll} [3-r, r-3] & \text{if } r \geq 3, \\ \emptyset & \text{otherwise.} \end{array}\right\}$$

Let $r_1 = 1$ and $r_2 = 2$. Then

$$D_p^q(\varphi,\psi)^{r_1}_{S(I)}(\Delta^1)\text{–}\lim x_k = 0$$

and

$$D_p^q(\varphi,\psi)^{r_2}_{S(I)}(\Delta^1)\text{–}\lim y_k = 0.$$

If we take $r < r_1 + r_2 = 3$, then $D_p^q(\varphi,\psi)^{r}_{S(I)}(\Delta^1)\text{–}\mathtt{LIM}[x_k + y_k] = \emptyset$.

Proposition 3.13. *Let $(X, \varphi, \psi, \star, \circ)$ be an IFNS. For a sequence (x_k) in X and some $r \geq 0$, if $D_p^q(\varphi,\psi)^{r}_{S(I)}(\Delta^j)\text{–}\lim x_k = x$ then $D_p^q(\varphi,\psi)^{|a|r}_{S(I)}(\Delta^j)\text{–}\lim ax_k = ax$ for any $a \in \mathbb{R}$.*

Proof. If $a = 0$, there is nothing to prove. Suppose $a \neq 0$. For given $\mu \in (0,1)$, $\exists\ \gamma \in (0,1)$ such that $1-\gamma \geq 1-\mu$. For given $t > 0$, consider

$$P = \Big\{k \in \mathbb{N} : p_n < k \leq q_n,\ \varphi\Big(\Delta^j x_k - x, \frac{t}{2|a|} + r\Big) \leq 1-\gamma \text{ or } \psi\Big(\Delta^j x_k - x, \frac{t}{2|a|} + r\Big) \geq \gamma\Big\}.$$

Since $D_p^q(\varphi,\psi)^{r}_{S(I)}(\Delta^j)\text{–}\lim x_k = x$, the set

$$Q = \Big\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\Big|\Big\{k \in \mathbb{N} : p_n < k \leq q_n,\ k \in P\Big\}\Big| < \epsilon\Big\} \in F(I) \tag{3.2}$$

for each $\epsilon > 0$. Take $m \in Q$. Then

$$\frac{1}{q_m - p_m}\Big|\Big\{k \in \mathbb{N} : p_m < k \leq q_m,\ k \in P\Big\}\Big| < \epsilon$$

$$\Longrightarrow\ \frac{1}{q_m - p_m}\Big|\Big\{k \in \mathbb{N} : p_m < k \leq q_m,\ k \in P^c\Big\}\Big| \geq 1-\epsilon.$$

Now, for $k \in P^c$, we have

$$\begin{aligned} \varphi\Big(a\Delta^j x_k - ax, |a|r + t\Big) &= \varphi\Big(\Delta^j x_k - x, r + \frac{t}{|a|}\Big) \\ &\geq \varphi\Big(\Delta^j x_k - x, r + \frac{t}{2|a|}\Big) \\ &> 1-\gamma \geq 1-\mu \end{aligned}$$

and

$$\psi\Big(a\Delta^j x_k - ax, |a|r + t\Big) = \psi\Big(\Delta^j x_k - x, r + \frac{t}{|a|}\Big) \\ \leq \psi\Big(\Delta^j x_k - x, r + \frac{t}{2|a|}\Big) \\ < \gamma \leq \mu.$$

Consequently,

$$P^c \subseteq \Big\{k \in \mathbb{N} : p_n < k \leq q_n, \varphi\Big(a\Delta^j x_k - ax, |a|r + t\Big) > 1 - \mu \text{ and } \psi\Big(a\Delta^j x_k - ax, |a|r + t\Big) < \mu\Big\}.$$

As a result, for $m \in Q$, it follows that

$$1 - \epsilon \leq \frac{1}{q_m - p_m}\Big|\Big\{k \in \mathbb{N} : p_m < k \leq q_m,\ k \in P^c\Big\}\Big| \\ \leq \frac{1}{q_m - p_m}\Big|\Big\{k \in \mathbb{N} : p_m < k \leq q_m, \varphi\Big(a\Delta^j x_k - ax, |a|r + t\Big) > 1 - \mu \text{ and } \\ \psi\Big(a\Delta^j x_k - ax, |a|r + t\Big) < \mu\Big\}\Big|.$$

This implies that

$$\frac{1}{q_m - p_m}\Big|\Big\{k \in \mathbb{N} : p_m < k \leq q_m, \varphi\Big(a\Delta^j x_k - ax, |a|r + t\Big) \leq 1 - \mu \text{ or } \psi\Big(a\Delta^j x_k - ax, |a|r + t\Big) \geq \mu\Big\}\Big| < \epsilon.$$

Therefore,

$$Q \subseteq \Big\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\Big|\Big\{k \in \mathbb{N} : p_n < k \leq q_n, \varphi\Big(a\Delta^j x_k - ax, |a|r + t\Big) \leq 1 - \mu \text{ or } \\ \psi\Big(a\Delta^j x_k - ax, |a|r + t\Big) \geq \mu\Big\}\Big| < \epsilon\Big\}.$$

From (3.2), it follows that

$$\Big\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\Big|\Big\{k \in \mathbb{N} : p_n < k \leq q_n, \varphi\Big(a\Delta^j x_k - ax, |a|r + t\Big) \leq 1 - \mu \text{ or } \\ \psi\Big(a\Delta^j x_k - ax, |a|r + t\Big) \geq \mu\Big\}\Big| < \epsilon\Big\} \in F(I).$$

Hence, by Lemma 3.7, $D_p^q(\varphi, \psi)_{S(I)}^{|a|r}(\Delta^j)$– $\lim ax_k = ax$. □

Remark 3.14. *For $r > 0$, Proposition 3.13 need not be true for $0 < l < |a|r$, i.e., for some $r > 0$ if $D_p^q(\varphi, \psi)_{S(I)}^{r}(\Delta^j)$– $\lim x_k = x$ then $D_p^q(\varphi, \psi)_{S(I)}^{l}(\Delta^j)$– $\lim ax_k$ need not be equal to ax, where $0 < l < |a|r$ and $a \in \mathbb{R}$.*

Example 3.15. *Consider Example 3.12 and take $a = 2$. Clearly*

$$2x_k = \begin{Bmatrix} 2k & \text{if } k = 5^n, \\ 0 & \text{if } k = 2n, \\ 2 & \text{if } k \neq 5^n, 2n \end{Bmatrix}, n \in \mathbb{N}.$$

and

$$D_p^q(\varphi, \psi)_{S(I)}^{l}(\Delta^1)\text{–LIM}(2x_k) = \begin{Bmatrix} [2 - l, l - 2] & \text{if } l \geq 2, \\ \emptyset & \text{otherwise.} \end{Bmatrix}$$

Let $r_1 = 2$, then

$$D_p^q(\varphi,\psi)_{S(I)}^{2}(\Delta^1)\text{–}\mathtt{LIM}(x_k) = [-1,1]$$

and

$$D_p^q(\varphi,\psi)_{S(I)}^{(2\times 2)}(\Delta^1)\text{–}\mathtt{LIM}(2x_k) = [-2,2] = 2[-1,1].$$

On the other hand, If $2 \le l < 4$, we get

$$D_p^q(\varphi,\psi)_{S(I)}^{l}(\Delta^1)\text{–}\mathtt{LIM}(2x_k) = [2-l, l-2] \neq 2[-1,1].$$

Definition 3.16. *Let $(X,\varphi,\psi,\star,\circ)$ be an IFNS. For a sequence (x_k) in X, we say (x_k) is $\Delta_{\mathtt{IF}}^j$–strongly bounded iff for every $\mu \in (0,1)$, $\exists\, t > 0$ such that $\varphi(\Delta^j x_k, t) > 1-\mu$ and $\psi(\Delta^j x_k, t) < \mu$ for all k.*

Definition 3.17. *Let $(X,\varphi,\psi,\star,\circ)$ be an IFNS. For a sequence (x_k) in X, we say (x_k) is deferred I–statistically $\Delta_{\mathtt{IF}}^j$–strongly bounded iff for every $\mu \in (0,1)$, $\exists\, t > 0$ such that*

$$\left\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\left|\left\{k \in \mathbb{N} : p_n < k \le q_n,\ \varphi(\Delta^j x_k, t) \le 1-\mu \text{ or } \psi(\Delta^j x_k, t) \ge \mu\right\}\right| \ge \epsilon\right\} \in I$$

for any $\epsilon > 0$.

From above definitions, it is evident that if a sequence (x_k) is $\Delta_{\mathtt{IF}}^j$–strongly bounded then $(\varphi,\psi)^r(\Delta^j)$–$\mathtt{LIM}(x_k) \neq \emptyset$ and hence by Theorem 3.8, $D_p^q(\varphi,\psi)_{S(I)}^r(\Delta^j)$–$\mathtt{LIM}(x_k) \neq \emptyset$ for any $r \ge 0$. The converse implication of this result is not true. To overcome this situation, we present the theorem as follows:

Theorem 3.18. *Let $(X,\varphi,\psi,\star,\circ)$ be an IFNS. A sequence (x_k) in X is deferred I–statistically $\Delta_{\mathtt{IF}}^j$–strongly bounded if and only if $D_p^q(\varphi,\psi)_{S(I)}^r(\Delta^j)$–$\mathtt{LIM}(x_k) \neq \emptyset$ for some $r \ge 0$.*

Proof. Assume that (x_k) is deferred I–statistically $\Delta_{\mathtt{IF}}^j$–strongly bounded. Thus, for every $\mu \in (0,1)$, $\exists\, r > 0$ so that

$$\left\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\left|\left\{k \in \mathbb{N} : p_n < k \le q_n,\ \varphi\left(\Delta^j x_k, r\right) \le 1-\mu \text{ or } \psi\left(\Delta^j x_k, r\right) \ge \mu\right\}\right| \ge \epsilon\right\} \in I$$

for any $\epsilon > 0$. Consider

$$C = \left\{k \in \mathbb{N} : p_n < k \le q_n,\ \varphi\left(\Delta^j x_k, r\right) \le 1-\mu \text{ or } \psi\left(\Delta^j x_k, r\right) \ge \mu\right\}.$$

Clearly

$$D = \left\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\left|\left\{k \in \mathbb{N} : p_n < k \le q_n, k \in C\right\}\right| < \epsilon\right\} \in F(I).$$

Now, for $m \in D$, we obtain

$$\frac{1}{q_m - p_m}\left|\left\{k \in \mathbb{N} : p_m < k \le q_m, k \in C\right\}\right| < \epsilon$$

$$\implies \frac{1}{q_m - p_m}\left|\left\{k \in \mathbb{N} : p_m < k \le q_m, k \in C^c\right\}\right| \ge 1-\epsilon.$$

On the other hand, take $k \in C^c$. Then, for any $t > 0$, we have

$$\varphi(\Delta^j x_k, t+r) \ge \varphi(\Delta^j x_k, r) \star \varphi(0,t) > (1-\mu) \star 1 = 1-\mu,$$

and

$$\psi(\Delta^j x_k, t+r) \leq \psi(\Delta^j x_k, r) \circ \psi(0,t) < \mu \circ 0 = \mu.$$

Therefore,

$$C^c \subseteq \Big\{k \in \mathbb{N} : p_n < k \leq q_n, \varphi\Big(\Delta^j x_k, t+r) > 1-\mu \text{ and } \psi(\Delta^j x_k, t+r) < \mu\Big\}.$$

Hence, for $m \in D$, we find

$$\begin{aligned} 1-\epsilon \leq & \frac{1}{q_m - p_m}\Big|\Big\{k \in \mathbb{N} : p_m < k \leq q_m,\ k \in C^c\Big\}\Big| \\ \leq & \frac{1}{q_m - p_m}\Big|\Big\{k \in \mathbb{N} : p_m < k \leq q_m,\ \varphi\Big(\Delta^j x_k, t+r) > 1-\mu \text{ and } \psi(\Delta^j x_k, t+r) < \mu\Big\}\Big| \\ \implies & \frac{1}{q_m - p_m}\Big|\Big\{k \in \mathbb{N} : p_m < k \leq q_m,\ \varphi\Big(\Delta^j x_k, t+r) \leq 1-\mu \text{ or } \psi(\Delta^j x_k, t+r) \geq \mu\Big\}\Big| < \epsilon. \end{aligned}$$

Consequently,

$$\begin{aligned} D \subseteq \Big\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\Big|\Big\{k \in \mathbb{N} : p_n < k\ \leq q_n,\ \varphi\Big(\Delta^j x_k, t+r) \leq 1-\mu \text{ or} \\ \psi(\Delta^j x_k, t+r) \geq \mu\Big\}\Big| < \epsilon\Big\} \in F(I). \end{aligned}$$

By Lemma 3.7, it follows that $0 \in D_p^q(\varphi,\psi)^r_{S(I)}(\Delta^j)\text{–}\mathtt{LIM}(x_k)$.

Conversely, suppose $D_p^q(\varphi,\psi)^r_{S(I)}(\Delta^j)\text{–}\mathtt{LIM}(x_k) \neq \emptyset$ for some $r \geq 0$. Let x be a member of $D_p^q(\varphi,\psi)^r_{S(I)}(\Delta^j)\text{–}\mathtt{LIM}(x_k)$. Then, for every $t>0$, $\epsilon>0$ and $\mu \in (0,1)$, we have

$$\Big\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\Big|\Big\{k \in \mathbb{N} : p_n < k \leq q_n,\ \varphi\Big(\Delta^j x_k - x, t+r\Big) \leq 1-\mu \text{ or } \psi\Big(\Delta^j x_k, t+r\Big) \geq \mu\Big\}\Big| \geq \epsilon\Big\} \in I,$$

which implies that

$$\Big\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\Big|\Big\{k \in \mathbb{N} : p_n < k \leq q_n,\ \Delta^j x_k \notin \mathcal{B}_x^{(\varphi,\psi)}(t+r,\mu)\Big\}\Big| \geq \epsilon\Big\} \in I.$$

Hence, (x_k) is deferred I–statistically $\Delta^j_{\mathtt{IF}}$–strongly bounded. □

We found that the difference rough convergence limits defined above are sets rather than unique points. So we provide some topological as well as geometrical properties of the limit set $D_p^q(\varphi,\psi)^r_{S(I)}(\Delta^j)\text{–}\mathtt{LIM}(x_k)$ as follows:

Theorem 3.19. *Let $(X,\varphi,\psi,\star,\circ)$ be an* IFNS *and (x_k) be a sequence in X. Then $D_p^q(\varphi,\psi)^r_{S(I)}(\Delta^j)\text{–}\mathtt{LIM}(x_k)$ is a closed set for every $r \geq 0$.*

Proof. For given $\mu \in (0,1)$, $\exists\ \nu \in (0,1)$ such that $(1-\nu)\star(1-\nu) > (1-\mu)$ and $\nu \circ \nu < \mu$. Let $x \in cl\Big(D_p^q(\varphi,\psi)^r_{S(I)}(\Delta^j)\text{–}\mathtt{LIM}(x_k)\Big)$, the closure of $D_p^q(\varphi,\psi)^r_{S(I)}(\Delta^j)\text{–}\mathtt{LIM}(x_k)$. Then, there exists a sequence (z_k) in $D_p^q(\varphi,\psi)^r_{S(I)}(\Delta^j)\text{–}\mathtt{LIM}(x_k)$ such that $z_k \xrightarrow{(\varphi,\psi)} x$. Thus, for every $t>0$, $\exists\ n_0 \in \mathbb{N}$ so that

$$\varphi\Big(z_k - x, \frac{t}{2}\Big) > 1-\nu \text{ and } \psi\Big(z_k - x, \frac{t}{2}\Big) < \nu, \forall\ k \geq n_0.$$

Choose $m_0 > n_0$. As a result, we have the set

$$E = \Big\{k \in \mathbb{N} : \varphi\Big(\Delta^j x_k - z_{m_0}, r + \frac{t}{2}\Big) \leq 1-\nu \text{ or } \psi\Big(\Delta^j x_k - z_{m_0}, r + \frac{t}{2}\Big) \geq \nu\Big\}$$

such that

$$F = \Big\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\Big|\Big\{k \in \mathbb{N} : p_n < k \le q_n, k \in E\Big\}\Big| < \epsilon\Big\} \in F(I),$$

for every $\epsilon > 0$. For $m \in F$, we obtain

$$\frac{1}{q_m - p_m}\Big|\Big\{k \in \mathbb{N} : p_m < k \le q_m, k \in E\Big\}\Big| < \epsilon$$

$$\implies \frac{1}{q_m - p_m}\Big|\Big\{k \in \mathbb{N} : p_m < k \le q_m, k \in E^c\Big\}\Big| \ge 1 - \epsilon.$$

Take $k \in E^c$. Then

$$\varphi\Big(\Delta^j x_k - x, t + r\Big) \ge \varphi\Big(\Delta^j x_k - z_{m_0}, r + \frac{t}{2}\Big) \star \varphi\Big(z_{m_0} - x, \frac{t}{2}\Big) > (1-\nu) \star (1-\nu) > 1 - \mu$$

and

$$\psi\Big(\Delta^j x_k - x, t + r\Big) \le \psi\Big(\Delta^j x_k - z_{m_0}, r + \frac{t}{2}\Big) \circ \psi\Big(z_{m_0} - x, \frac{t}{2}\Big) < \nu \circ \nu < \mu.$$

As a result,

$$E^c \subseteq \Big\{k \in \mathbb{N} : p_n < k \le q_n,\ \varphi\Big(\Delta^j x_k - x, t + r\Big) > 1 - \mu \text{ and } \psi\Big(\Delta^j x_k - x, t + r\Big) < \mu\Big\}.$$

Therefore, for $m \in F$, we get

$$\begin{aligned} 1 - \epsilon &\le \frac{1}{q_m - p_m}\Big|\Big\{k \in \mathbb{N} : p_m < k \le q_m,\ k \in E^c\Big\}\Big| \\ &\le \frac{1}{q_m - p_m}\Big|\Big\{k \in \mathbb{N} : p_m < k \le q_m,\ \varphi\Big(\Delta^j x_k - x, t + r) > 1 - \mu \text{ and } \psi(\Delta^j x_k - x, t + r) < \mu\Big\}\Big|. \end{aligned}$$

Hence,

$$\frac{1}{q_m - p_m}\Big|\Big\{k \in \mathbb{N} : p_m < k \le q_m,\ \varphi\Big(\Delta^j x_k - x, t + r) \le 1 - \mu \text{ or } \psi(\Delta^j x_k - x, t + r) \ge \mu\Big\}\Big| < \epsilon.$$

Consequently, we obtain

$$\begin{aligned} F \subseteq \Big\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\Big|\Big\{k \in \mathbb{N} : p_n < k\ \le q_n, \varphi\Big(\Delta^j x_k - x, t + r) \le 1 - \mu \text{ or} \\ \psi(\Delta^j x_k - x, t + r) \ge \mu\Big\}\Big| < \epsilon\Big\} \in F(I). \end{aligned}$$

Therefore, by Lemma 3.7, we have $x \in D_p^q(\varphi, \psi)^r_{S(I)}(\Delta^j)$–$\texttt{LIM}(x_k)$. Hence, $D_p^q(\varphi, \psi)^r_{S(I)}(\Delta^j)$–$\texttt{LIM}(x_k)$ is closed. □

Theorem 3.20. *Let $(X, \varphi, \psi, \star, \circ)$ be an* IFNS *and (x_k) is a sequence in X. Then, for every $r \ge 0$, the set $D_p^q(\varphi, \psi)^r_{S(I)}(\Delta^j)$–$\texttt{LIM}(x_k)$ is convex.*

Proof. Suppose $x, y \in D_p^q(\varphi, \psi)^r_{S(I)}(\Delta^j)$–$\texttt{LIM}(x_k)$ and $\mu \in (0, 1)$ is given. Then, $\exists\, \nu \in (0, 1)$ so that $(1-\nu) \star (1-\nu) > 1 - \mu$ and $\nu \circ \nu < \mu$. We need to show that $\alpha x + (1 - \alpha) y \in D_p^q(\varphi, \psi)^r_{S(I)}(\Delta^j)$–$\texttt{LIM}(x_k)$ for any $\alpha \in [0, 1]$. For $\alpha = 0$ or 1, the result is obvious. Let $\alpha \in (0, 1)$. For every $t > 0$, define

$$G = \Big\{k \in \mathbb{N} : p_n < k \le q_n,\ \varphi\Big(\Delta^j x_k - x, r + \frac{t}{2\alpha}\Big) \le 1 - \nu \text{ or } \psi\Big(\Delta^j x_k - x, r + \frac{t}{2\alpha}\Big) \ge \nu\Big\}$$

and

$$H = \Big\{k \in \mathbb{N} : p_n < k \leq q_n,\ \varphi\Big(\Delta^j x_k - y, r + \frac{t}{2(1-\alpha)}\Big) \leq 1 - \nu \text{ or } \psi\Big(\Delta^j x_k - y, r + \frac{t}{2(1-\alpha)}\Big) \geq \nu\Big\}.$$

Then

$$\Big\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\Big|\Big\{k \in \mathbb{N} : p_n < k \leq q_n,\ k \in G\Big\}\Big| \geq \epsilon\Big\} \in I$$

and

$$\Big\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\Big|\Big\{k \in \mathbb{N} : p_n < k \leq q_n,\ k \in H\Big\}\Big| \geq \epsilon\Big\} \in I$$

for every $\epsilon > 0$. Therefore,

$$\Big\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\Big|\Big\{k \in \mathbb{N} : p_n < k \leq q_n,\ k \in G \cup H\Big\}\Big| \geq \epsilon\Big\} \in I.$$

Choose $0 < \lambda < 1$ so that $0 < 1 - \lambda < \epsilon$. Hence

$$J = \Big\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\Big|\Big\{k \in \mathbb{N} : p_n < k \leq q_n,\ k \in G \cup H\Big\}\Big| \geq 1 - \lambda\Big\} \in I.$$

Let $m \in J^c$. Then

$$\frac{1}{q_m - p_m}\Big|\Big\{k \in \mathbb{N} : p_m < k \leq q_m,\ k \in G \cup H\Big\}\Big| < 1 - \lambda$$

$$\implies \frac{1}{q_m - p_m}\Big|\Big\{k \in \mathbb{N} : p_m < k \leq q_m,\ k \in G^c \cap H^c\Big\}\Big| \geq 1 - (1 - \lambda) = \lambda.$$

Now, take $k \in G^c \cap H^c$. Hence,

$$\begin{aligned}
&\varphi\Big(\Delta^j x_k - [\alpha x + (1-\alpha)y],\ t + r\Big)\\
&\quad = \varphi\Big((1-\alpha)(\Delta^j x_k - y) + \alpha(\Delta^j x_k - x),\ (1-\alpha)r + \alpha r + t\Big)\\
&\quad \geq \varphi\Big((1-\alpha)(\Delta^j x_k - y),\ (1-\alpha)r + \frac{t}{2}\Big) \star \varphi\Big(\alpha(\Delta^j x_k - x),\ \alpha r + \frac{t}{2}\Big)\\
&\quad = \varphi\Big(\Delta^j x_k - y,\ r + \frac{t}{2(1-\alpha)}\Big) \star \varphi\Big(\Delta^j x_k - x,\ r + \frac{t}{2\alpha}\Big)\\
&\quad > (1-\nu) \star (1-\nu)\\
&\quad > 1 - \mu
\end{aligned}$$

and

$$\begin{aligned}
&\psi\Big(\Delta^j x_k - [\alpha x + (1-\alpha)y],\ t + r\Big)\\
&\quad = \psi\Big((1-\alpha)(\Delta^j x_k - y) + \alpha(\Delta^j x_k - x),\ (1-\alpha)r + \alpha r + t\Big)\\
&\quad \leq \psi\Big((1-\alpha)(\Delta^j x_k - y),\ (1-\alpha)r + \frac{t}{2}\Big) \circ \psi\Big(\alpha(\Delta^j x_k - x),\ \alpha r + \frac{t}{2}\Big)\\
&\quad = \psi\Big(\Delta^j x_k - y,\ r + \frac{t}{2(1-\alpha)}\Big) \circ \psi\Big(\Delta^j x_k - x,\ r + \frac{t}{2\alpha}\Big)\\
&\quad < \nu \circ \nu\\
&\quad < \mu.
\end{aligned}$$

As a result, we have

$$G^c \cap H^c \subseteq \Big\{k \in \mathbb{N} : p_n < k \le q_n,\ \varphi\Big(\Delta^j x_k - [\alpha x + (1-\alpha)y],\ t+r\Big) > 1-\mu \text{ and } \psi\Big(\Delta^j x_k - [\alpha x + (1-\alpha)y],\ t+r\Big) < \mu\Big\}.$$

Hence, for $m \in J^c$, we have

$$\begin{aligned}\lambda &\le \frac{1}{q_m - p_m}\Big|\Big\{k \in \mathbb{N} : p_m < k \le q_m,\ k \in G^c \cap H^c\Big\}\Big| \\ &\le \frac{1}{q_m - p_m}\Big|\Big\{k \in \mathbb{N} : p_m < k \le q_m,\ \varphi\Big(\Delta^j x_k - [\alpha x + (1-\alpha)y],\ t+r\Big) > 1-\mu \\ &\qquad \text{and } \psi\Big(\Delta^j x_k - [\alpha x + (1-\alpha)y],\ t+r\Big) < \mu\Big\}\Big| \\ \implies &\frac{1}{q_m - p_m}\Big|\Big\{k \in \mathbb{N} : p_m < k \le q_m,\ \varphi\Big(\Delta^j x_k - [\alpha x + (1-\alpha)y],\ t+r\Big) \le 1-\mu \\ &\qquad \text{or } \psi\Big(\Delta^j x_k - [\alpha x + (1-\alpha)y],\ t+r\Big) \ge \mu\Big\}\Big| < 1-\lambda < \epsilon.\end{aligned}$$

Consequently,

$$J^c \subseteq \Big\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\Big|\Big\{k \in \mathbb{N} : p_n < k \le q_n,\ \varphi\Big(\Delta^j x_k - [\alpha x + (1-\alpha)y],\ t+r\Big) \le 1-\mu \text{ or } \psi\Big(\Delta^j x_k - [\alpha x + (1-\alpha)y],\ t+r\Big) \ge \mu\Big\}\Big| < \epsilon\Big\} \in F(I).$$

Hence, by Lemma 3.7, it follows that $\alpha x + (1-\alpha)y \in D_p^q(\varphi,\psi)^r_{S(I)}(\Delta^j)$–$\mathtt{LIM}(x_k)$. □

Theorem 3.21. *Let $(X,\varphi,\psi,\star,\circ)$ be an* IFNS*. Then $D_p^q(\varphi,\psi)^r_{S(I)}(\Delta^j)$– $\lim x_k = x$ for some $r \ge 0$ if there is a sequence (y_k) in X such that (y_k) is deferred I–statistically difference convergent to x and*

$$\varphi(\Delta^j x_k - \Delta^j y_k, r) > 1-\mu \text{ and } \psi(\Delta^j x_k - \Delta^j y_k, r) < \mu \tag{3.3}$$

hold for every $\mu \in (0,1)$ and for all $k \in \mathbb{N}$.

Proof. For given $\mu \in (0,1)$, choose $\nu \in (0,1)$ so that $(1-\nu)\star(1-\nu) > 1-\mu$ and $\nu \circ \nu < \mu$. Suppose, the sequence (y_k) is deferred I–statistically difference convergent to x and satisfies (3.3). Then, for every $t > 0$ and $\epsilon > 0$, we have

$$L = \Big\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\Big|\Big\{k \in \mathbb{N} : p_n < k \le q_n,\ \varphi(\Delta^j y_k - x, t) \le 1-\nu \text{ or } \psi(\Delta^j y_k - x, t) \ge \nu\Big\}\Big| \ge \epsilon\Big\} \in I.$$

Let $m \in L^c$. Then

$$\begin{aligned}&\frac{1}{q_m - p_m}\Big|\Big\{k \in \mathbb{N} : p_m < k \le q_m,\ \varphi(\Delta^j y_k - x, t) \le 1-\nu \text{ or } \psi(\Delta^j y_k - x, t) \ge \nu\Big\}\Big| < \epsilon \\ \implies &\frac{1}{q_m - p_m}\Big|\Big\{k \in \mathbb{N} : p_m < k \le q_m,\ \varphi(\Delta^j y_k - x, t) > 1-\nu \text{ and } \psi(\Delta^j y_k - x, t) < \nu\Big\}\Big| \ge 1-\epsilon.\end{aligned}$$

Now, define

$$M = \Big\{k \in \mathbb{N} : p_m < k \le q_m,\ \varphi(\Delta^j y_k - x, t) > 1-\nu \text{ and } \psi(\Delta^j y_k - x, t) < \nu\Big\}.$$

For $k \in M$, we get

$$\varphi(\Delta^j x_k - x,\ t+r) \ge \varphi(\Delta^j x_k - \Delta^j y_k, r) \star \varphi(\Delta^j y_k - x, t) > (1-\nu)\star(1-\nu) > 1-\mu$$

and

$$\psi(\Delta^j x_k - x,\ t+r) \leq \psi(\Delta^j x_k - \Delta^j y_k, r) \circ \psi(\Delta^j y_k - x, t) < \nu \circ \nu < \mu.$$

Therefore,

$$M \subseteq \Big\{k \in \mathbb{N} : p_m < k \leq q_m,\ \varphi(\Delta^j x_k - x,\ t+r) > 1-\mu \text{ and } \psi(\Delta^j x_k - x,\ t+r) < \mu\Big\}.$$

This implies that

$$\begin{aligned} 1-\epsilon\ &\leq \frac{1}{q_m - p_m}\Big|\Big\{k \in \mathbb{N} : p_m < k \leq q_m,\ k \in M\Big\}\Big| \\ &\leq \frac{1}{q_m - p_m}\Big|\Big\{k \in \mathbb{N} : p_m < k \leq q_m,\ \varphi\Big(\Delta^j x_k - x,\ t+r\Big) > 1-\mu \\ &\qquad\qquad \text{and } \psi\Big(\Delta^j x_k - x,\ t+r\Big) < \mu\Big\}\Big| \end{aligned}$$

$$\implies \frac{1}{q_m - p_m}\Big|\Big\{k \in \mathbb{N} : p_m < k \leq q_m,\ \varphi\Big(\Delta^j x_k - x,\ t+r\Big) \leq 1-\mu \text{ or } \psi\Big(\Delta^j x_k - x,\ t+r\Big) \geq \mu\Big\}\Big| < \epsilon.$$

Since $m \in L^c$ and $L^c \in F(I)$, we get

$$\begin{aligned} L^c \subseteq \Big\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\Big|\Big\{k \in \mathbb{N} : p_n < k \leq q_n,\ \varphi\Big(\Delta^j x_k - x, t+r\Big) \leq 1-\mu \\ \text{or } \psi\Big(\Delta^j x_k - x,\ t+r\Big) \geq \mu\Big\}\Big| < \epsilon\Big\} \in F(I). \end{aligned}$$

This implies that $D_p^q(\varphi,\psi)_{S(I)}^r(\Delta^j)$– $\lim x_k = x$. □

Theorem 3.22. *Let $(X, \varphi, \psi, \star, \circ)$ be an* IFNS*. If a sequence (x_k) in X is deferred I–statistically difference convergent to x, then there exists $\nu \in (0,1)$ such that*

$$cl\Big(\mathcal{B}_x^{(\varphi,\psi)}(r,\nu)\Big) \subseteq D_p^q(\varphi,\psi)_{S(I)}^r(\Delta^j)\text{–}\mathtt{LIM}(x_k) \textit{ for some } r > 0.$$

Proof. For given $\mu \in (0,1)$, $\exists\, \nu \in (0,1)$ so that $(1-\nu) \star (1-\nu) > 1-\mu$ and $\nu \circ \nu < \mu$. Suppose (x_k) is deferred I–statistically difference convergent to x. Then

$$R = \Big\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\Big|\Big\{k \in \mathbb{N} : p_n < k \leq q_n,\ \varphi(\Delta^j x_k - x, t) \leq 1-\nu \text{ or } \psi(\Delta^j x_k - x, t) \geq \nu\Big\}\Big| \geq \epsilon\Big\} \in I$$

for every $\epsilon, t > 0$. For $m \in R^c$, we have

$$\frac{1}{q_m - p_m}\Big|\Big\{k \in \mathbb{N} : p_m < k \leq q_m,\ \varphi(\Delta^j x_k - x, t) \leq 1-\nu \text{ or } \psi(\Delta^j x_k - x, t) \geq \nu\Big\}\Big| < \epsilon$$

$$\implies \frac{1}{q_m - p_m}\Big|\Big\{k \in \mathbb{N} : p_m < k \leq q_m,\ \varphi(\Delta^j x_k - x, t) > 1-\nu \text{ and } \psi(\Delta^j x_k - x, t) < \nu\Big\}\Big| \geq 1-\epsilon.$$

Now, let $w \in cl\Big(\mathcal{B}_x^{(\varphi,\psi)}(r,\nu)\Big)$ for some $r > 0$. Then

$$\varphi(x - w, r) \geq 1-\nu \text{ and } \psi(x - w, r) \leq \nu.$$

Define

$$S = \Big\{k \in \mathbb{N} : p_m < k \leq q_m,\ \varphi(\Delta^j x_k - x, t) > 1-\nu \text{ and } \psi(\Delta^j x_k - x, t) < \nu\Big\}.$$

Thus for $k \in S$, similarly to above, we have

$$\varphi(\Delta^j x_k - w, t + r) > 1 - \mu \text{ and } \psi(\Delta^j x_k - w, t + r) < \mu.$$

Therefore,

$$S \subseteq \Big\{k \in \mathbb{N} : p_m < k \leq q_m,\ \varphi(\Delta^j x_k - w, t + r) > 1 - \mu \text{ and } \psi(\Delta^j x_k - w, t + r) < \mu\Big\}$$

and hence

$$\begin{aligned} 1 - \epsilon \ &\leq \frac{1}{q_m - p_m}\Big|\Big\{k \in \mathbb{N} : p_m < k \leq q_m,\ k \in S\Big\}\Big| \\ &\leq \frac{1}{q_m - p_m}\Big|\Big\{k \in \mathbb{N} : p_m < k \leq q_m,\ \varphi\Big(\Delta^j x_k - w,\ t + r\Big) > 1 - \mu \\ &\qquad\qquad \text{and } \psi\Big(\Delta^j x_k - w,\ t + r\Big) < \mu\Big\}\Big|. \end{aligned}$$

This implies that

$$\frac{1}{q_m - p_m}\Big|\Big\{k \in \mathbb{N} : p_m < k \leq q_m,\ \varphi\Big(\Delta^j x_k - w,\ t + r\Big) \leq 1 - \mu \text{ or } \psi\Big(\Delta^j x_k - w,\ t + r\Big) \geq \mu\Big\}\Big| < \epsilon.$$

Since $m \in R^c$ and $R^c \in F(I)$, we obtain

$$\begin{aligned} R^c \subseteq \Big\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\Big|\Big\{k \in \mathbb{N} : p_n < k \leq q_n,\ \varphi\Big(\Delta^j x_k - w, t + r\Big) \leq 1 - \mu \\ \text{or } \psi\Big(\Delta^j x_k - w,\ t + r\Big) \geq \mu\Big\}\Big| < \epsilon\Big\} \in F(I). \end{aligned}$$

Consequently, $w \in D^q_p(\varphi, \psi)^r_{S(I)}(\Delta^j)\text{–LIM}(x_k)$. As a result $cl\Big(\mathcal{B}^{(\varphi,\psi)}_x(r, \nu)\Big) \subseteq D^q_p(\varphi, \psi)^r_{S(I)}(\Delta^j)\text{–LIM}(x_k)$. □

Definition 3.23. *Let $(X, \varphi, \psi, \star, \circ)$ be an IFNS and (x_k) be a sequence in X. A point $z \in X$ is called deferred I–statistical Δ^j_r–cluster point of (x_k) with regard to (φ, ψ) for some $r \geq 0$ if, for any $t > 0$ and $\mu \in (0, 1)$, we have*

$$\Big\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\Big|\Big\{k \in \mathbb{N} : p_n < k \leq q_n,\ \varphi(\Delta^j x_k - z,\ t + r) \leq 1 - \mu \text{ or } \psi(\Delta^j x_k - z,\ t + r) \geq \mu\Big\}\Big| < \epsilon\Big\} \notin I.$$

We denote the set of all deferred I–statistical Δ^j_r–cluster point of (x_k) by $\Gamma^{(\varphi,\psi)^r_{S(I)}}_{D^p_q}(\Delta^j x_k)$.

Note 3.24. *For $r = 0$, the deferred I–statistical Δ^j_r–cluster point of (x_k) is known as the deferred I–statistical Δ^j–cluster point of (x_k) and the collections of such cluster points is denoted by $\Gamma^{(\varphi,\psi)_{S(I)}}_{D^p_q}(\Delta^j x_k)$.*

Theorem 3.25. *Let $(X, \varphi, \psi, \star, \circ)$ be an IFNS. Then the set $\Gamma^{(\varphi,\psi)^r_{S(I)}}_{D^p_q}(\Delta^j x_k)$ is closed for every sequence (x_k) in X and each $r \geq 0$.*

Proof. For given $\mu \in (0, 1)$, $\exists\ \nu \in (0, 1)$ so that $(1 - \nu) \star (1 - \nu) > 1 - \nu$ and $\nu \circ \nu < \mu$. Let $z \in cl\Big(\Gamma^{(\varphi,\psi)^r_{S(I)}}_{D^p_q}(\Delta^j x_k)\Big)$. Then, there is a sequence (z_k) in $\Gamma^{(\varphi,\psi)^r_{S(I)}}_{D^p_q}(\Delta^j x_k)$ such that $z_k \xrightarrow{(\varphi,\psi)} z$. Thus, for every $t > 0$, $\exists\ n_0 \in \mathbb{N}$ so as

$$\varphi\Big(z_k - z, \frac{t}{2}\Big) > 1 - \nu \text{ and } \psi\Big(z_k - z, \frac{t}{2}\Big) < \nu, \forall\ k \geq n_0.$$

Fix $m_0 > n_0$ and set

$$T = \Big\{k \in \mathbb{N} : p_n < k \leq q_n, \varphi\Big(\Delta^j x_k - z_{m_0}, r + \frac{t}{2}\Big) \leq 1 - \nu \text{ or } \psi\Big(\Delta^j x_k - z_{m_0}, r + \frac{t}{2}\Big) \geq \nu\Big\}.$$

As a result, for every $\epsilon > 0$, we obtain

$$U = \Big\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\Big|\Big\{k \in \mathbb{N} : p_n < k \leq q_n, k \in T\Big\}\Big| < \epsilon\Big\} \notin I.$$

Similarly, as the proof of Theorem 3.19, we get

$$U \subseteq \Big\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\Big|\Big\{k \in \mathbb{N} : p_n < k \leq q_n,\ \varphi\Big(\Delta^j x_k - z, t + r) \leq 1 - \mu \text{ or } \psi(\Delta^j x_k - z, t + r) \geq \mu\Big\}\Big| < \epsilon\Big\}.$$

Since $U \notin I$, we have

$$\Big\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\Big|\Big\{k \in \mathbb{N} : p_n < k \leq q_n,\ \varphi\Big(\Delta^j x_k - z, t + r) \leq 1 - \mu \text{ or } \psi(\Delta^j x_k - z, t + r) \geq \mu\Big\}\Big| < \epsilon\Big\} \notin I,$$

i.e., $z \in \Gamma^{(\varphi,\psi)^r_{S(I)}}_{D^p_q}(\Delta^j x_k)$. Hence the set $\Gamma^{(\varphi,\psi)^r_{S(I)}}_{D^p_q}(\Delta^j x_k)$ is closed. □

Theorem 3.26. *Let (x_k) be a sequence in the* IFNS $(X, \varphi, \psi, \star, \circ)$ *and $r \geq 0$ be given. If $\varphi(z - y, r) > 1 - \mu$ and $\psi(z - y, r) < \mu$ hold for an arbitrary $z \in \Gamma^{(\varphi,\psi)_{S(I)}}_{D^p_q}(\Delta^j x_k)$ and $\mu \in (0, 1)$, then $y \in \Gamma^{(\varphi,\psi)^r_{S(I)}}_{D^p_q}(\Delta^j x_k)$.*

Proof. The result is direct, so the proof is omitted. □

Theorem 3.27. *Let $(X, \varphi, \psi, \star, \circ)$ be an* IFNS *and (x_k) be a sequence in X. Then, for some $r > 0$ and $\nu \in (0, 1)$, we have*

$$\Gamma^{(\varphi,\psi)^r_{S(I)}}_{D^p_q}(\Delta^j x_k) = \bigcup_{x \in \Gamma^{(\varphi,\psi)_{S(I)}}_{D^p_q}(\Delta^j x_k)} cl\Big(\mathcal{B}^{(\varphi,\psi)}_x(r, \nu)\Big).$$

Proof. For any given $\mu \in (0, 1)$, $\exists\ \nu \in (0, 1)$ such that $(1 - \nu) \star (1 - \nu) > 1 - \mu$ and $\nu \circ \nu < \mu$. Let

$$z \in \bigcup_{x \in \Gamma^{(\varphi,\psi)_{S(I)}}_{D^p_q}(\Delta^j x_k)} cl\Big(\mathcal{B}^{(\varphi,\psi)}_x(r, \nu)\Big), r > 0.$$

Then, $\exists\ x \in \Gamma^{(\varphi,\psi)_{S(I)}}_{D^p_q}(\Delta^j x_k)$ so that $z \in cl\Big(\mathcal{B}^{(\varphi,\psi)}_x(r, \nu)\Big)$, i.e., $\varphi(x - z, r) \geq 1 - \nu$ and $\psi(x - z, r) \leq \nu$. Since $x \in \Gamma^{(\varphi,\psi)_{S(I)}}_{D^p_q}(\Delta^j x_k)$, we have

$$X = \Big\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\Big|\Big\{k \in \mathbb{N} : p_n < k \leq q_n,\ \varphi(\Delta^j x_k - x,\ t) \leq 1 - \nu \text{ or } \psi(\Delta^j x_k - x,\ t) \geq \nu\Big\}\Big| < \epsilon\Big\} \notin I$$

for every $\epsilon, t > 0$. Consider

$$X_1 = \Big\{k \in \mathbb{N} : p_n < k \leq q_n,\ \varphi(\Delta^j x_k - x,\ t) \leq 1 - \nu \text{ or } \psi(\Delta^j x_k - x,\ t) \geq \nu\Big\}.$$

Then, we have

$$X_1^c \subseteq \Big\{k \in \mathbb{N} : p_n < k \leq q_n,\ \varphi(\Delta^j x_k - z, t + r) > 1 - \mu \text{ and } \psi(\Delta^j x_k - z, t + r) < \mu\Big\}. \tag{3.4}$$

Now take $m \in X$. Then

$$\frac{1}{q_m - p_m}\Big|\Big\{k \in \mathbb{N} : p_m < k \leq q_m,\ k \in X_1\Big\}\Big| < \epsilon$$

$$\implies \frac{1}{q_m - p_m}\Big|\Big\{k \in \mathbb{N} : p_m < k \le q_m,\ k \in X_1^c\Big\}\Big| \ge 1 - \epsilon.$$

Hence, by (3.4), it follows that

$$\frac{1}{q_m - p_m}\Big|\Big\{k \in \mathbb{N} : p_m < k \le q_m,\ \varphi(\Delta^j x_k - z, t + r) > 1 - \mu \text{ and } \psi(\Delta^j x_k - z, t + r) < \mu\Big\}\Big| \ge 1 - \epsilon.$$

$$\implies \frac{1}{q_m - p_m}\Big|\Big\{k \in \mathbb{N} : p_m < k \le q_m,\ \varphi(\Delta^j x_k - z, t + r) \le 1 - \mu \text{ or } \psi(\Delta^j x_k - z, t + r) \ge \mu\Big\}\Big| < \epsilon.$$

As a result, we get

$$X \subseteq \Big\{n \in \mathbb{N} : \frac{1}{q_m - p_m}\Big|\Big\{k \in \mathbb{N} : p_n < k \le q_n,\ \varphi(\Delta^j x_k - z, t + r) \le 1 - \mu \text{ or } \psi(\Delta^j x_k - z, t + r) \ge \mu\Big\}\Big| < \epsilon\Big\}.$$

Since $X \notin I$, it follows that

$$\Big\{n \in \mathbb{N} : \frac{1}{q_m - p_m}\Big|\Big\{k \in \mathbb{N} : p_n < k \le q_n,\ \varphi(\Delta^j x_k - z, t + r) \le 1 - \mu \text{ or } \psi(\Delta^j x_k - z, t + r) \ge \mu\Big\}\Big| < \epsilon\Big\} \notin I.$$

Hence $z \in \Gamma_{D_q^p}^{(\varphi,\psi)^r_{S(I)}}(\Delta^j x_k)$. Consequently,

$$\Gamma_{D_q^p}^{(\varphi,\psi)^r_{S(I)}}(\Delta^j x_k) \supseteq \bigcup_{x \in \Gamma_{D_q^p}^{(\varphi,\psi)_{S(I)}}(\Delta^j x_k)} cl\Big(\mathcal{B}_x^{(\varphi,\psi)}(r, \nu)\Big). \tag{3.5}$$

Conversely, assume that $y \in \Gamma_{D_q^p}^{(\varphi,\psi)^r_{S(I)}}(\Delta^j x_k)$. Then $y \in \bigcup_{x \in \Gamma_{D_q^p}^{(\varphi,\psi)_{S(I)}}(\Delta^j x_k)} cl\Big(\mathcal{B}_x^{(\varphi,\psi)}(r, \nu)\Big)$. Otherwise $y \notin cl\Big(\mathcal{B}_x^{(\varphi,\psi)}(r, \nu)\Big)$ for any $x \in \Gamma_{D_q^p}^{(\varphi,\psi)_{S(I)}}(\Delta^j x_k)$, i.e.,

$$\varphi(x - y, r) < 1 - \nu \text{ or } \psi(x - y, r) > \nu.$$

Hence, by Theorem 3.26, it follows that $y \notin \Gamma_{D_q^p}^{(\varphi,\psi)^r_{S(I)}}(\Delta^j x_k)$, which contradicts our assumption. Therefore,

$$\Gamma_{D_q^p}^{(\varphi,\psi)^r_{S(I)}}(\Delta^j x_k) \subseteq \bigcup_{x \in \Gamma_{D_q^p}^{(\varphi,\psi)_{S(I)}}(\Delta^j x_k)} cl\Big(\mathcal{B}_x^{(\varphi,\psi)}(r, \nu)\Big). \tag{3.6}$$

From (3.5) and (3.6), the result follows. □

Corollary 3.28. *Let $(X, \varphi, \psi, \star, \circ)$ be an* IFNS*. If a sequence (x_k) in X is deferred I–statistically difference convergent to x, then $\Gamma_{D_q^p}^{(\varphi,\psi)^r_{S(I)}}(\Delta^j x_k) \subseteq D_p^q(\varphi, \psi)^r_{S(I)}(\Delta^j)$–*LIM*$(x_k)$ for some $r > 0$.*

Proof. Suppose (x_k) is deferred I–statistically difference convergent to x. Hence $x \in \Gamma_{D_q^p}^{(\varphi,\psi)_{S(I)}}(\Delta^j x_k)$. Therefore, by Theorem 3.27, for some $r > 0$ and $\mu \in (0, 1)$, we have

$$\Gamma_{D_q^p}^{(\varphi,\psi)^r_{S(I)}}(\Delta^j x_k) = cl\Big(\mathcal{B}_x^{(\varphi,\psi)}(r, \mu)\Big). \tag{3.7}$$

Also, from Theorem 3.22, it follows that

$$cl\Big(\mathcal{B}_x^{(\varphi,\psi)}(r, \mu)\Big) \subseteq D_p^q(\varphi, \psi)^r_{S(I)}(\Delta^j)\text{–}\mathtt{LIM}(x_k). \tag{3.8}$$

Hence, by (3.7) and (3.8), we have

$$\Gamma_{D_q^p}^{(\varphi,\psi)^r_{S(I)}}(\Delta^j x_k) \subseteq D_p^q(\varphi, \psi)^r_{S(I)}(\Delta^j)\text{–}\mathtt{LIM}(x_k).$$

□

4. Conclusions

Rough convergence and difference operators have been the subjects of extensive investigation in many areas of mathematics. Since ideals and deferred densities have become increasingly significant in sequence convergence theory in recent years, in this paper, we extend the intriguing idea of rough convergence to the context of intuitionistic fuzzy norm spaces via difference operators by incorporating both ideals and deferred density. Furthermore, we examine some characteristics of the limit set of this new notion of convergence.

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