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Stabilization by the discrete observations feedback control and intermittent control

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Abstract. The stabilization of (partial) differential equations by (stochastic) feedback control based on discrete-time state observations and intermittent control is considered in this paper. It is the first time to obtain the stabilization of differential equations with nonlocal delays based on discrete-time state observations and obtain the uniform bound for nonlinear differential equations. Some examples are given to illustrate our results.

1. Introduction

In the real world, the method that by using the discrete-time state observations or intermittent control to stabilize the solutions is often used in control theory. Recently, Mao [1, 2] obtained the stabilization by discrete observation. Then there are a lot of authors considering the similar question: You et al. [3] obtained the stabilization of hybrid systems by feedback control based on discrete-time state observations and they considered many kinds of stability including H_∞ stability and asymptotic stability; Dong et al. [4] obtained the almost sure exponential stabilization by stochastic feedback control based on discrete-time observations; Li-Mao [5] obtained the stabilisation of highly nonlinear hybrid stochastic differential delay equations by delay feedback control; Fei et al. [6] considered the stabilization of highly nonlinear hybrid systems by feedback control based on discrete-time state observations, also see [7]; Liu-Wu [8] obtained the intermittent stochastic stabilization based on discrete-time observation with time delay; Shen et al. [9, 10] obtained the stabilization for hybrid stochastic systems by aperiodically intermittent control and stabilization of stochastic differential equations driven by G-Levy process with discrete-time feedback control; Mao et al. [11] obtained the stabilization by intermittent control for hybrid stochastic differential delay equations. Guo et al. [12] generalized the results of [1, 2] to the polynomial case similar to [13]. Just recently, Zhou et al. [22] considered the consensus of NMASs with MSTs subjected to DoS attacks under event-triggered control.

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However, we find there are some cases which are not considered. The previous results focus on the stabilization, i.e., adding some control term make the unstable solution to become stable. *For nonlocal delayed differential equations, there is no result about the stabilization by feedback control. In addition, there is only a small amount of research on this issue: adding some control term make the finite time blowup solution to become global exist.* In this paper, we give positive answer for deterministic case. The method we used here is borrowing from [14, 15]: by introducing new variable, we translate the differential equations with nonlocal delays into system without delay, and then by using the time-homogeneous property, we obtain the desired results.

On the other hand, we notice that Azouani et al. [16] introduced a new method to study the data assimilation problem. Based on ideas that have been developed for designing finite-dimensional feedback controls for dissipative dynamical systems, a new continuous data assimilation algorithm is introduced, see [17] for similar method. In [16, 17], the authors used finite-dimensional feedback control scheme to stabilize solutions of infinite-dimensional dissipative evolution equations including reaction-diffusion systems. More precisely, they considered the two-dimensional Navier-Stokes system

$$\begin{cases} \frac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla q = f + \mu(I_h(v) - I_h(u)), \\ \nabla\cdot u = 0, \end{cases} \tag{1}$$

where v is a solution of (1) with $\mu = 0$ and I_h is a linear interpolants. The no-slip Dirichlet boundary conditions and periodic boundary conditions are included. They obtained that under some assumptions on h, μ, ν it holds that $\lim_{t\to\infty} \|u - v\|_{L^2(D)} = 0$. Later, Foias et al. [18] considered a discrete data assimilation scheme for the solutions of the two-dimensional Navier-Stokes equations. In this paper, we reconsider the reaction-diffusion equations in stochastic sense. Our aim is to stabilize the solutions by adding discrete-time observations. More precisely, we consider

$$\begin{cases} \frac{\partial u}{\partial t} - \nu\Delta u = \alpha u - ku(\delta(t), x)\chi_{\delta(t)}(t), \\ \nabla\cdot u = 0, \end{cases} \tag{2}$$

where $\chi_t(s) = 1$ when $t = s$ and $\chi = 0$ otherwise. There is a big difference from [16, 17] because the method used here is different. We first translate (2) into ODEs and obtain the asymptotic behavior by using the results of differential equations. In addition, we consider the stochastic cases and want to show the difference between the discrete-time observations and intermittent control.

The rest of this paper is organized as follows. In Section 2, we consider the deterministic cases. Section 3 is concerned with stochastic cases. Some examples are given in Section 4.

2. Deterministic Cases

We first consider the deterministic case. Consider

$$\begin{cases} x'(t) = \alpha x(t), \quad t > 0, \\ x|_{t=0} = x_0, \end{cases} \tag{3}$$

where $\alpha > 0$. It is easy to see that the solution of (3) is $x(t) = x_0 e^{\alpha t}$, which implies that $x(t) \to \infty$ as $t \to \infty$. We want to stabilize the solution of (3), i.e., the solution $x(t)$ decays to zero as time goes to infinity. Inspired by Mao [2], we first prove that the discrete time observations can stabilize the solution. Now, we consider

$$\begin{cases} x'(t) = \alpha x(t) - \sigma x(\delta(t)), \quad t > 0, \\ x|_{t=0} = x_0, \end{cases} \tag{4}$$

where $\sigma > \alpha > 0$, $\delta(t) = [\frac{t}{\tau}]\tau$, $[\frac{t}{\tau}]$ is the integer part of $\frac{t}{\tau}$ and $\tau > 0$. Let $t_k = k\tau$ for $k = 0, 1, 2, \dots$ and $x_k = x(t_k)$. It is easy to see that when $t \in [t_k, t_{k+1}]$,

$$x(t) = e^{\alpha(t-t_k)}(1 - \frac{\sigma}{\alpha})x_k + \frac{\sigma}{\alpha}x_k,$$

which implies that

$$x_{k+1} = e^{\alpha\tau}(1 - \frac{\sigma}{\alpha})x_k + \frac{\sigma}{\alpha}x_k. \tag{5}$$

Note that

$$(1 - \frac{\sigma}{\alpha})(e^{\alpha\tau} - 1) < 0, \quad for \;\; \tau > 0, \tag{6}$$

so there exists a positive number ε such that

$$e^{\alpha\tau}(1 - \frac{\sigma}{\alpha}) + \frac{\sigma}{\alpha} < e^{-\varepsilon\tau}. \tag{7}$$

Submitting (7) into (5), we get

$$\frac{x_{k+1}}{x_k} \le e^{-\varepsilon\tau}. \tag{8}$$

In fact, we note that $e^{\alpha\tau}(1 - \frac{\sigma}{\alpha}) + \frac{\sigma}{\alpha} < 1$ for any $\tau > 0$. Thus when $e^{\alpha\tau}(1 - \frac{\sigma}{\alpha}) + \frac{\sigma}{\alpha} > 0$, we can find $\varepsilon > 0$ such that

$$e^{\alpha\tau}(1 - \frac{\sigma}{\alpha}) + \frac{\sigma}{\alpha} = e^{-\varepsilon\tau}. \tag{9}$$

Consequently, we have

$$x(t) \le e^{-\varepsilon\tau}x_k, \quad \forall t \in [t_k, t_{k+1}].$$

By induction, for any $t > 0$ there exists $k \in \mathbb{N}$ such that $t \in [k\tau, (k+1)\tau]$ and then by using (8), we get

$$x(t) \le e^{-\varepsilon\tau}x_k \le e^{-2\varepsilon\tau}x_{k-1} \le \cdots \le e^{-k\varepsilon\tau}x_0 \le x_0 e^{\varepsilon\tau}e^{-\varepsilon t}, \quad \forall t \in [t_k, t_{k+1}].$$

Combining the above discussions, we obtain the following result.

Theorem 2.1. *Let $\sigma > \alpha$. The solution of (4) will decay exponentially with the rate ε provided that (7) holds.*

We remark that the discrete time observations can make the solution $x(t)$ of (3) stable, i.e., in order to get $x(t) \le Ce^{-\varepsilon t}$, we only need take $\tau > 0$ satisfying (9). ***Comparing Theorem 2.1 with the stochastic case [2, Theorem 2.1], we have that in our Theorem 2.1 the solution has exact decay rate. Moreover, we can get any decay rate ε.*** But for the stochastic case, it is hard to obtain those.

We can interpret the result of Theorem 2.1: if $t \in [0, \tau]$, (4) becomes

$$\begin{cases} x'(t) = \alpha x(t) - \sigma x_0, & 0 \le t \le \tau, \\ x|_{t=0} = x_0. \end{cases}$$

Obviously, $x'(0) < 0$ implies that $x(t)$ decreases at time 0. And we can conclude that $x(t)$ is a decreasing function, but it is hard to get the exponential decay.

Now, we study the nonlocal differential equation:

$$\begin{cases} x'(t) = \alpha x(t) + \int_{-\infty}^{t} \beta e^{-\beta(t-s)}x(s)ds, & t > 0, \\ x(t) = x_0(t) \ge 0, & t \le 0, \end{cases} \tag{10}$$

where $\alpha, \beta > 0$. Obviously, the solutions of (10) will go to infinity as time goes to infinity. In the following, we use the discrete-time observations to control the solutions.

$$\begin{cases} x'(t) = \alpha x(t) + \int_{-\infty}^{t} \beta e^{-\beta(t-s)}x(s)ds - kx(\delta(t)), & t > 0, \\ x(t) = x_0(t) \ge 0, & t \le 0. \end{cases} \tag{11}$$

Due to the solution of (10) does not satisfy the time-homogeneous property, the method of [2] is not suitable directly. Fortunately, if we let

$$y(t) = \int_{-\infty}^{t} \beta e^{-\beta(t-s)} x(s)ds, \tag{12}$$

we can translate (11) into the following system

$$\begin{cases} x'(t) = \alpha x(t) + y(t) - kx(\delta(t)), & t > 0, \\ y'(t) = \beta(x(t) - y(t)), & t > 0, \\ x|_{t=0} = x_0(0), \;\; y|_{t=0} = \int_{-\infty}^{0} \beta e^{\beta s} x(s)ds. \end{cases} \tag{13}$$

We will use a similar method [2] to deal with (13). The reference equations for (13) read

$$\begin{cases} \hat{x}'(t) = -(k-\alpha)\hat{x}(t) + \hat{y}(t), & t > 0, \\ \hat{y}'(t) = \beta(\hat{x}(t) - \hat{y}(t)), & t > 0, \\ x|_{t=0} = x_0(0), \;\; y|_{t=0} = \int_{-\infty}^{0} \beta e^{\beta s} x(s)ds. \end{cases} \tag{14}$$

We first obtain the properties of reference equations (14).

Lemma 2.2. *Assume that $k > \alpha + \beta + 1$. It holds that*

$$|\hat{x}(t)| + |\hat{y}(t)| \le 4(|x_0(0)| + |y_0|)e^{-\gamma t}, \quad t \ge 0, \tag{15}$$

where $\gamma = \beta(1 - \frac{1}{k-\alpha-\beta})$.

Proof. From the first equation of (10), we have that if there exists some point $t_0 > 0$ such that $x(t_0) = 0$, then we get $x'(t_0) > 0$. Consequently, we have $x(t) \ge 0$ for $t \ge 0$. It follows from constant variation method that

$$\begin{aligned} \hat{x}(t) &= e^{-(k-\alpha)t}x_0 + \int_0^t e^{-(k-\alpha)(t-s)} \int_{-\infty}^{s} \beta e^{-\beta(s-r)}\hat{x}(r)drds, \\ &= e^{-(k-\alpha)t}x_0 + e^{-(k-\alpha)t} \int_0^t e^{-(k-\alpha-\beta)s} \int_{-\infty}^{s} \beta e^{\beta r}\hat{x}(r)drds \qquad (16) \\ &\le e^{-\beta t}\left(x_0 + \int_{-\infty}^{0} \beta e^{\beta s} x_0(s)ds\right) + \frac{\beta}{k-\alpha-\beta} e^{-\beta t} \int_0^t e^{\beta s}\hat{x}(s)ds. \qquad (17) \end{aligned}$$

Consequently, by Gronwall's inequality, we get

$$|\hat{x}(t)| \le 2(|x_0(0)| + |y_0|)e^{-\beta\left(1-\frac{1}{k-\alpha-\beta}\right)t}, \;\; t \ge 0.$$

Submitting this into $\hat{y}$, we get

$$|\hat{y}(t)| \le 2(|x_0(0)| + |y_0|)e^{-\beta\left(1-\frac{1}{k-\alpha-\beta}\right)t}, \;\; t \ge 0.$$

The proof is complete. □

The next lemma is concerned with the difference between the solutions of (11) and (13).

Lemma 2.3. *It holds that*

$$|x(t) - \hat{x}(t)| + |y(t) - \hat{y}(t)| \le K(\tau, k)(|x_0(0)| + |y_0|)\left[e^{\left(\frac{\beta}{\beta+\alpha}+k+\alpha+\beta+1\right)t} - 1\right], \quad t \ge 0,$$

where $K(\tau, k)$ is defined in (29).

Proof. We first get that

$$x(t) = e^{\alpha t}x_0 + e^{\alpha t}\int_0^t e^{-\alpha s}\int_{-\infty}^s \beta e^{-\beta(s-r)}x(r)drds - ke^{\alpha t}\int_0^t e^{\alpha s}x(\delta(s))ds,$$

which implies that

$$\begin{aligned}
\max_{0\le s\le t}|x(s)| &\le e^{\alpha t}|x_0(0)| + e^{\alpha t}\int_0^t e^{-\alpha s}\int_{-\infty}^s \beta e^{-\beta(s-r)}\max_{0\le z\le r}|x(z)|drds \\
&\quad + ke^{\alpha t}\int_0^t e^{-\alpha s}\max_{0\le z\le s}|x(z)|ds \qquad (18)\\
&\le e^{\alpha t}\left(|x_0(0)| + \frac{|y_0|}{\alpha+\beta}\right) + e^{\alpha t}\int_0^t \beta e^{\beta r}\max_{0\le z\le r}|x(z)|dr\int_r^t e^{-(\alpha+\beta)s}ds \qquad (19)\\
&\quad + ke^{\alpha t}\int_0^t e^{-\alpha s}\max_{0\le z\le s}|x(z)|ds \qquad (20)\\
&\le e^{\alpha t}(1\vee\frac{1}{\alpha+\beta})(|x_0(0)|+|y_0|) \qquad (21)\\
&\qquad + \left(\frac{\beta}{\beta+\alpha}+k\right)e^{\alpha t}\int_0^t \beta e^{-\alpha r}\max_{0\le z\le r}|x(z)|dr. \qquad (22)
\end{aligned}$$

By using Gronwall's inequality again, we get

$$\max_{0\le s\le t}|x(s)| \le 2(1\vee\frac{1}{\alpha+\beta})(|x_0(0)|+|y_0|)e^{\left(\frac{\beta}{\beta+\alpha}+k+\alpha\right)t},\ \ t\ge 0. \qquad (23)$$

Submitting the above inequality into the definition of y, we get

$$\max_{0\le s\le t}|y(s)| \le 2(1\vee\frac{1}{\alpha+\beta})(|x_0(0)|+|y_0|)e^{\left(\frac{\beta}{\beta+\alpha}+k+\alpha\right)t},\ \ t\ge 0. \qquad (24)$$

Then we obtain

$$\begin{aligned}
|x(t)-x(\delta(t))| &= \Big|\int_{\delta(t)}^t dx(s)\Big| = \Big|\int_{\delta(t)}^t [\alpha x(s) + \int_0^s \beta e^{-\beta(s-r)}x(r)dr - kx(\delta(s))]ds\Big| \\
&\le 2(\alpha+1+k)(1\vee\frac{1}{\alpha+\beta})(|x_0(0)|+|y_0|)\tau e^{\left(\frac{\beta}{\beta+\alpha}+k+\alpha\right)t}. \qquad (25)
\end{aligned}$$

It follows from (13) and (14) that

$$\begin{aligned}
(x(t)-\hat{x}(t))' &= -(k-\alpha)(x(t)-\hat{x}(t)) + y(s) - \hat{y}(s) + k(x(t)-x(\delta(t))), \\
(y(t)-\hat{y}(t))' &= \beta[(x(t)-\hat{x}(t)) - (y(t)-\hat{y}(t))], \qquad (26)
\end{aligned}$$

which implies that

$$\begin{aligned}
|x(t)-\hat{x}(t)| + |y(t)-\hat{y}(t)| &\le (\beta+1)\int_0^t [|x(s)-\hat{x}(s)| + |y(s)-\hat{y}(s)|] \\
&\quad + \int_0^t k|x(s)-x(\delta(s))|ds. \qquad (27)
\end{aligned}$$

The Gronwall inequality yields that

$$|x(t)-\hat{x}(t)| + |y(t)-\hat{y}(t)| \le ke^{(\beta+1)t}\int_0^t |x(s)-x(\delta(s))|ds.$$

Using (25), we obtain

$$
\begin{aligned}
&|x(t)-\hat{x}(t)|+|y(t)-\hat{y}(t)| \\
&\le K(\tau,k)(|x_0(0)|+|y_0|)\left[e^{\left(\frac{\beta}{\beta+\alpha}+k+\alpha+\beta+1\right)t}-1\right],
\end{aligned} \tag{28}
$$

where

$$
K(\tau,k):=2k(\alpha+1+k)(1\vee\frac{1}{\alpha+\beta})\tau. \tag{29}
$$

The proof is complete. □

Now the main result is as followings.

Theorem 2.4. *Assume that $k>\alpha+\beta+1$. Then there exists a $\tau^*>0$ such that for any $\tau\in(0,\tau^*)$, the solution of (13) will decay exponentially, where τ^* is the unique root of*

$$
K(\tau,k)\left[e^{\left(\frac{\beta}{\beta+\alpha}+k+\alpha+\beta+1\right)\left(\tau+\log(\frac{1}{4\varepsilon})/\gamma\right)}-1\right]=1-\varepsilon, \tag{30}
$$

and $\varepsilon\in(0,1)$, γ is defined as in Lemma 2.2.

Proof. It is easy to see that the right hand side of (30) is a continuously increasing function of $\tau\ge 0$ and equals to zero when $\tau=0$. Thus (30) must have a unique root τ^*. Let $\tau\in(0,\tau^*)$. Choose a positive integer $\bar{k}$ such that

$$
\frac{\log(\frac{1}{4\varepsilon})}{\gamma\tau}\le\bar{k}<1+\frac{\log(\frac{1}{4\varepsilon})}{\gamma\tau},
$$

where γ is defined as in Lemma 2.2. Thus we have $4e^{-\gamma\bar{k}\tau}\le\varepsilon$. By Lemma 2.2, we have

$$
|\hat{x}(\bar{k}\tau)|+|\hat{y}(\bar{k}\tau)|\le 4(|x_0(0)|+|y_0|)e^{-\gamma\bar{k}\tau},\quad t\ge 0. \tag{31}
$$

Note that

$$
\begin{aligned}
|x(\bar{k}\tau)|+|y(\bar{k}\tau)| &\le |\hat{x}(\bar{k}\tau)|+|\hat{y}(\bar{k}\tau)|+|x(\bar{k}\tau)-\hat{x}(\bar{k}\tau)|+|y(\bar{k}\tau)-\hat{y}(\bar{k}\tau)| \\
&\le (|x_0(0)|+|y_0|)\left[\varepsilon+K(\tau,k)\left(e^{\left(\frac{\beta}{\beta+\alpha}+k+\alpha+\beta+1\right)\bar{k}\tau}-1\right)\right].
\end{aligned} \tag{32}
$$

It follows from the definition of $\bar{k}$ that

$$
\begin{aligned}
&\varepsilon+K(\tau,k)\left(e^{\left(\frac{\beta}{\beta+\alpha}+k+\alpha+\beta+1\right)\bar{k}\tau}-1\right) \\
&\le \varepsilon+K(\tau,k)\left(e^{\left(\frac{\beta}{\beta+\alpha}+k+\alpha+\beta+1\right)\left(\tau+\log(\frac{1}{4\varepsilon})/\gamma\right)}-1\right)<1.
\end{aligned} \tag{33}
$$

We may therefore write

$$
\varepsilon+K(\tau,k)\left(e^{\left(\frac{\beta}{\beta+\alpha}+k+\alpha+\beta+1\right)\bar{k}\tau}-1\right)=e^{-\lambda\bar{k}\tau}.
$$

Consequently, we get

$$
|x(\bar{k}\tau)|+|y(\bar{k}\tau)|\le e^{-\lambda\bar{k}\tau}.
$$

Due to the time-homogeneous property of (13), we therefore see easily that

$$
|x(i\bar{k}\tau)|+|y(i\bar{k}\tau)|\le|x((i-1)\bar{k}\tau)|+|y((i-1)\bar{k}\tau)|e^{-\lambda\bar{k}\tau}\le e^{-\lambda i\bar{k}\tau},\ \ \forall i=1,2,\ldots.
$$

It follows from (23) and (24) that

$$\max_{0\le s\le \bar{k}\tau}[|x(s)|+|y(s)|]\le N(|x_0(0)|+|y_0|),\ \ t\ge 0.$$

where $N=2(1\vee\frac{1}{\alpha+\beta})e^{\left(\frac{\beta}{\beta+\alpha}+k+\alpha\right)\bar{k}\tau}$. Similarly, we can get

$$\max_{i\bar{k}\tau\le s\le (i+1)\bar{k}\tau}[|x(s)|+|y(s)|]\le N(|x(i\bar{k}\tau)|+|y(i\bar{k}\tau)|),\ \ t\ge 0.$$

Therefore, for any $t>0$, there exists i such that $i\bar{k}\tau\le t\le (i+1)\bar{k}\tau$ and

$$[|x(t)|+|y(t)|]\le Ne^{-\lambda t},\ \ t\ge 0.$$

Due to that (11) is equivalent to (13), the proof is complete. □

Remark 2.5. *The reason why the deterministic ordinary differential equations with nonlocal delays can be stabilized by discrete-time observations is that we only consider the absolute value of $x(t)$. Comparing with stochastic case, we will consider $x^2(t)$ because the second variation of Brownian motion is finite. And we will not get the similar result to Theorem 2.6.*

Next, we consider the intermittent control. One can add the intermittent control to (3) to get the exponential decay, and here we omit it. We study another problem: the intermittent control can hinder blowup in finite time. In order to show that, we consider the following equation

$$\begin{cases} x'(t)=x^{\alpha}(t),\ \ t>0,\\ x|_{t=0}=x_0>0, \end{cases} \tag{34}$$

where $\alpha>1$. It is easy to see that the solution of (34) is

$$x(t)=\left(x_0^{1-\alpha}-(\alpha-1)t\right)^{-\frac{1}{\alpha-1}},$$

which blows up in finite time. Due to that we can not get the exact solution similar to (4), we consider the intermittent control. Consider

$$\begin{cases} x'(t)=x^{\alpha}(t)-kh(t)x^{\alpha}(t),\ \ t>0,\\ x|_{t=0}=x_0, \end{cases} \tag{35}$$

where $k>0$ and

$$h(t)=\begin{cases} 1, & t\in[t_i,s_i),\\ 0, & t\in[s_i,t_{i+1}),\quad i=0,1,2,\cdots. \end{cases}$$

Here we assume $0=t_0<s_0<t_1<s_1<t_2<s_2<\ldots$. The feedback controller only works at time span $[t_i,s_i)$ and will vanish in time span $[s_i,t_{i+1})$, $i=0,1,2,\cdots$. Now we need some notations on the intermittent control strategy. Let $\inf_i(s_i-t_i)=\varphi>0$, $\sup_i(t_{i+1}-t_i)=\omega>0$, $\psi=\limsup_{i\to\infty}\psi_i>0$, where $\psi_i=(t_{i+1}-s_i)(t_{i+1}-t_i)^{-1}$ and ψ is the maximum proportion of the rest width $t_{i+1}-s_i$ in the time span $t_{i+1}-t_i$. It is easy to see that the solution of (35) is

$$x(t)=\begin{cases} \left(x_0^{1-\alpha}+(\alpha-1)(k-1)t\right)^{-\frac{1}{\alpha-1}}, & t\in[0,s_0],\\ \left(x_0^{1-\alpha}+k(\alpha-1)s_0-(\alpha-1)t\right)^{-\frac{1}{\alpha-1}}, & t\in[s_0,t_1],\\ \left(x_0^{1-\alpha}+k(\alpha-1)s_0-k(\alpha-1)t_1+(\alpha-1)(k-1)t\right)^{-\frac{1}{\alpha-1}}, & t\in[t_1,s_1],\\ \left(x_0^{1-\alpha}+k(\alpha-1)s_0-k(\alpha-1)t_1+k(\alpha-1)s_1-(\alpha-1)t\right)^{-\frac{1}{\alpha-1}}, & t\in[s_1,t_2],\\ \ldots. \end{cases}$$

If we want to get the solution $x(t)$ of (35) does not blow up in finite time, we need assume that

$$\begin{cases} ks_0 > t_1, \\ k(s_0 + s_1 - t_1) > t_2, \\ k(s_0 + s_1 - t_1 + s_2 - t_2) > t_3, \\ \dots . \end{cases}$$

Thus we obtain the following result.

Theorem 2.6. *Let $k > 1$ and $\omega < k\varphi$. The solution of (35) will exist globally.*

Theorem 2.6 shows that under the condition that the total work-time $\sum_{i=1}^{n}(s_i - t_i)$ before t times k is larger than t, the solution will exist. That is to say, the total rest time should be not too large. For equation (34), the discrete observations feedback control is not suitable because we can not get the exact solution of (34) with discrete observations feedback control. But in the next section, we will find the difference between the discrete observations feedback control and intermittent control.

Lastly, we consider the stabilization of reaction-diffusion equations

$$\begin{cases} \partial_t u = \Delta u + \alpha u - k u(\delta(t), x)\chi_{\delta(t)}(t), & t > 0,\ x \in D, \\ u(x, 0) = u_0(x), & x \in D, \\ u(x, t) = 0, & t > 0,\ x \in \partial D, \end{cases} \tag{36}$$

where $\alpha, k > 0$. Obviously, if $k = 0$, the well-posedness of (36) was established by many authors and when $\alpha > \lambda_1$, we can not conclude that the solution decay exponentially, where λ_1 is the first eigenvalue corresponding to elliptic equation:

$$\begin{cases} -\Delta\phi = \lambda\phi, & \text{in } D, \\ \phi = 0, & \text{on } \partial D. \end{cases}$$

Then, all the eigenvalues are strictly positive, increasing and the eigenfunction ϕ_1 corresponding to the smallest eigenvalue λ_1 does not change sign in domain D, as shown in [19]. In the followings, we will show that if $k > (\alpha - \lambda_1)$, then $\|u\|_{L^2(D)}$ will decay exponentially. When $k > 0$, the well-posedness of (36) can be established by using the results of the delayed reaction-diffusion equations and here we focus on the stabilization by discrete-time observations. Multiplying u on both sides of (36), integrating over D and using Poincare inequality, we get

$$\begin{aligned} \frac{1}{2}\partial_t v(t) &= (\Delta u, u) + \alpha v - k v(\delta(t)) \\ &\le (\alpha - \lambda_1) v - k v(\delta(t)) \end{aligned} \tag{37}$$

with initial data $v(0) = (u_0, u_0)$, where $v = (u, u)$. Now we can deal with (37) similar to (4) and obtain the following result.

Theorem 2.7. *Let $k > \alpha - \lambda_1$. The solutions of (37) will decay exponentially with the rate ε provided that*

$$e^{(\alpha-\lambda_1)\tau}\left(1 - \frac{k}{\alpha - \lambda_1}\right) + \frac{k}{\alpha - \lambda_1} < e^{-\varepsilon\tau}.$$

In other words, the solution u of (36) will decay exponentially in the norm of $L^2(D)$.

3. Stochastic Cases

In this section, we consider the stochastic case under the condition that $\alpha - \lambda_1 > 0$. We first consider the stochastic reaction-diffusion equation

$$\begin{cases} du = (\Delta u + \alpha u)dt + \sigma u(\delta(t), x)\chi_{\delta(t)}(t)dB_t, & t > 0,\ x \in D, \\ u(x, 0) = u_0(x), & x \in D, \\ u(x, t) = 0, & t > 0,\ x \in \partial D, \end{cases} \tag{38}$$

where $\alpha > 0, \sigma \in \mathbb{R}$, B_t is a one-dimensional Brownian motion. In fact, equation (38) can be regarded as a stochastic differential delay equation if one define $\delta : [0, \infty) \to [0, \tau)$ by $\delta(t) = t - k\tau$ for $t \in [k\tau, (k + 1)\tau)$, $k = 0, 1, 2, \ldots$. The well-posedness of (38) can be obtained by the abstract result in [20]. Similar to (36), we can translate (38) into the following equation

$$\begin{aligned} \frac{1}{2}dv(t) &= [(\Delta u, u) + \alpha v]dt + \sigma v(\delta(t))dB_t \\ &\leq [(\alpha - \lambda_1)v]dt + \sigma v(\delta(t))dB_t \end{aligned} \tag{39}$$

with initial data $v(0) = (u_0, u_0)$. In order to do so, we recall the first result of [2]. Consider the scalar linear stochastic equation

$$dX(t) = \alpha X(t) + \sigma X\left(\left[\frac{t}{\tau}\right]\tau\right)dB(t) \tag{40}$$

on $t \geq 0$ with initial value $x(0) = x_0 \in \mathbb{R}$, where τ is a positive constant. Noting that $v = (u, u) \geq 0$ almost surely, so the method used in [2] is suitable for (39). Because the proof is highly similar to [2, Theorem 2.1], we omit the details. But we only interpret the difference. It is easy to see that

$$v_{k+1} \leq v_k(e^{(\alpha-\lambda_1)\tau} + \hat{\sigma}Z_k),\ \ Z_k \sim N(0, 1),\ \ \hat{\sigma} = \sqrt{\frac{\sigma^2}{2(\alpha - \lambda_1)}(e^{2(\alpha-\lambda_1)\tau} - 1)}.$$

Due to $v_k \geq 0$ for all $k \geq 0$, we can take absolute value on both sides and the rest of proof is just same as that of Theorem in [2].

Proposition 3.1. *If $\alpha - \lambda_1 - \frac{\sigma^2}{2} < 0$, then there is a positive number τ^* such that for any initial value u_0, the solution of (38) satisfies $\|u(t)\|_{L^2(D)} \to 0$ almost surely as $t \to 0$ provided $\tau \in (0, \tau^*)$.*

In the above Proposition 3.1, we did not give a concrete bound for τ^*. In fact, τ^* is a unique solution of some equation similar to [2, Theorem 2.1].

As for stochastic reaction-diffusion equations on bounded domain

$$\begin{cases} du = (\Delta u + \alpha u)dt + \sigma u(\delta(t), x)dB_t, & t > 0,\ x \in D, \\ u(x, 0) = u_0(x), & x \in D, \\ u(x, t) = 0, & t > 0,\ x \in \partial D, \end{cases} \tag{41}$$

and

$$\begin{cases} du = (\Delta u + \alpha u)dt - ku(\delta(t), x) + \sigma u dB_t, & t > 0,\ x \in D, \\ u(x, 0) = u_0(x), & x \in D, \\ u(x, t) = 0, & t > 0,\ x \in \partial D, \end{cases} \tag{42}$$

we obtain the similar decay property. In fact, letting $v = (u, \phi_1)$, we have (41) and (42) are equivalent to

$$\begin{cases} dv = (\alpha - \lambda_1)vdt + \sigma v(\delta(t))dB_t, & t > 0, \\ v(0) = (u_0, \phi_1), \end{cases} \tag{43}$$

and

$$\begin{cases} dv = (\alpha - \lambda_1)vdt - kv(\delta(t)) + \sigma v dB_t, & t > 0, \\ v(0) = (u_0, \phi_1), \end{cases} \tag{44}$$

respectively. However, the method has a disadvantage: we can not prove that the solutions of (41) and (42) are positive almost surely, and so we only get the decay of $|(u, \phi_1)|$ and can not get the decay of u.

But for intermittent control, we can prove the positive of solutions. More precisely, we consider

$$\begin{cases} du = (\Delta u + \alpha u - kh(t)u)dt + \sigma u dB_t, & t > 0, \ x \in D, \\ u(x, 0) = u_0(x) \geq 0, & x \in D, \\ u(x, t) = 0, & t > 0, \ x \in \partial D, \end{cases} \tag{45}$$

where $h(t)$ is defined as in Section 2. It follows from [21, Theorem 3.1] that the solution of (45) keeps positive almost surely. So, similar to (41), letting $v(t) = (u, \phi_1)$, we can translate (45) into

$$\begin{cases} dv = (\alpha - \lambda_1 - kh(t))vdt + \sigma v dB_t, & t > 0, \\ v(0) = (u_0, \phi_1). \end{cases} \tag{46}$$

Similar to [9, 11], we can obtain the stabilization result. It is easy to see that the solution of (46) can be written as

$$v(t) = \begin{cases} v_0 \exp\Big((\alpha - \lambda_1 - k - \frac{1}{2}\sigma^2)t + \sigma B_t\Big), & t \in [0, s_0], \\ v_0 \exp\Big((\alpha - \lambda_1 - \frac{1}{2}\sigma^2)t - ks_0 + \sigma B_t\Big), & t \in [s_0, t_1], \\ v_0 \exp\Big((\alpha - \lambda_1 - k - \frac{1}{2}\sigma^2)t - ks_0 + kt_1 + \sigma B_t\Big), & t \in [t_1, s_1], \\ v_0 \exp\Big((\alpha - \lambda_1 - \frac{1}{2}\sigma^2)t - ks_0 + kt_1 - ks_1 + \sigma B_t\Big), & t \in [s_1, t_2], \\ \dots. \end{cases}$$

Note that $\|u(t)\|_{L^1(D)} \leq (\inf_D \phi_1)^{-1} v(t)$ and

$$\limsup_{t \to \infty} \frac{B_t}{\sqrt{2t \log \log t}} = 1 \quad a.s.,$$

we have the following result.

Theorem 3.2. *If $(\alpha - \lambda_1 - \frac{1}{2}\sigma^2)\omega < k\varphi$, then $v(t) \to 0$ almost surely as $t \to 0$. That is to say, the solution of (45) u satisfies*

$$\|u(t)\|_{L^1(D)} \to 0, \quad \text{as } t \to 0, \ \text{almost surely.}$$

4. Examples

Example 1: Consider

$$\begin{cases} x'(t) = x(t) - 2x(\delta(t)), & t > 0, \\ x|_{t=0} = x_0. \end{cases} \tag{47}$$

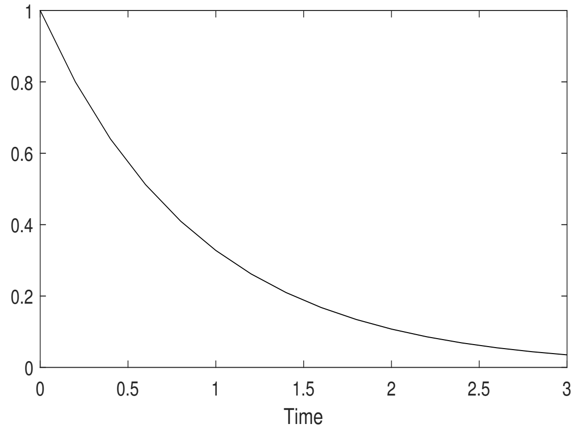

Figure 1: Plot of the computed result $x(t)$.

Take $\varepsilon = 4/7$ and $\tau = \ln(7/4)$, $x_0 = 1$, and then (7) holds. Fig.1 shows that the solution of (47) decays exponentially with parameter $\varepsilon = 4/7$ by Theorem 2.1.

Example 2: Consider

$$\begin{cases} x'(t) = x(t) + \int_{-\infty}^{t} e^{-(t-s)}x(s)ds - 4x(\delta(t)), & t > 0, \\ x(t) = x_0(t) \geq 0, & t \leq 0. \end{cases} \tag{48}$$

Let $\tau > 0$ satisfy

$$48\tau \left[e^{\frac{15}{2}(\tau + \log 16)} - 1 \right] = \frac{15}{16}.$$

we take $x_0 = 0$, Fig 2. shows that the solution of (48) decays exponentially with parameter $\varepsilon = 4/7$ by Theorem 2.4.

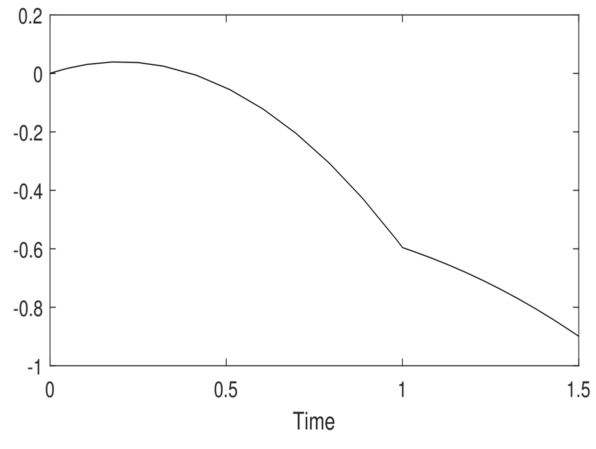

Figure 2: Plot of the computed result $x(t)$.

Example 3: Consider

$$\begin{cases} x'(t) = x(t)^{\alpha} - 4h(t)x^{\alpha}(t), & t > 0, \\ x|_{t=0} = x_0. \end{cases} \tag{49}$$

Set $\omega = 1$, $\varphi = 1/2$, $\alpha = 1.5$ and $x_0 = 0$. Fig 3. shows that the solution of (49) keeps bounded for all time by Theorem 2.6.

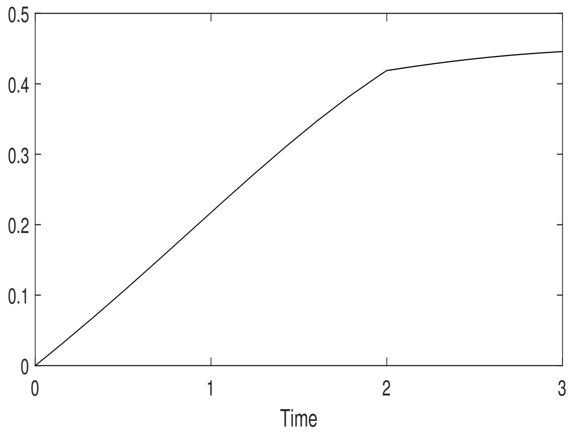

Figure 3: Plot of the computed result $x(t)$.

Example 4: Consider the following stochastic heat equation

$$\begin{cases} du = (\Delta u + \alpha u)dt - ku(\delta(t), x) + \sigma u dB_t, & t > 0,\ x \in (0,1), \\ u(x,0) = u_0(x), & x \in (0,1), \\ u(x,t) = 0, & t > 0,\ x = 0 \text{ or } 1. \end{cases} \tag{50}$$

Let $v(t) = (u(t), \phi_1)$, then $v(t)$ satisfies

$$v(t)' = (\alpha - \lambda_1)v(t) - kv(\delta(t)) + \sigma v(t)dB(t), \quad v(0) = (u_0, \phi_1). \tag{51}$$

Take $\alpha - \lambda_1 = 0.1$, $v_0 = 0.5$ and $k = 0$, then the solutions of (51) will not decay to zero, see Fig 4(a). But if we take $k = 0.3$, then the solutions of (51) will decay to zero, see Fig 4(b).

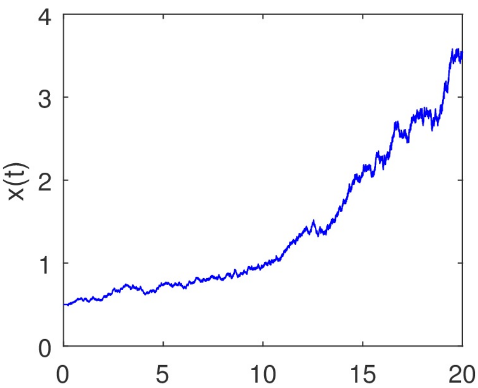

Fig 4a: v does not go to zero

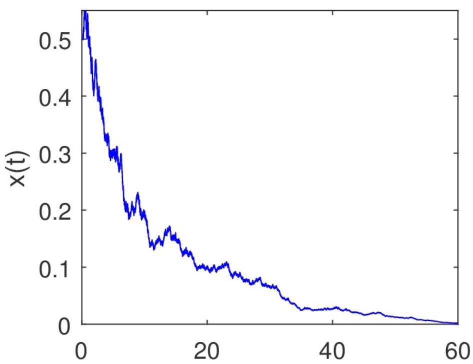

Fig 4b: v does go to zero

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