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Inequalities involving extreme eigenvalues and positive linear maps

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Abstract. In this paper, we discuss some inequalities involving unital positive linear maps and extreme eigenvalues of positive semidefinite (or positive definite) matrices. We also obtain a lower bound for the condition number; and a lower bound for Kantorovich ratio. In addition, some inequalities involving traces and extreme eigenvalues of a given $n \times n$ complex matrix are obtained when all of its eigenvalues are nonnegative.

1. Introduction

Throughout this paper, let $\mathbb{M}_k$ be the set of all $k \times k$ complex matrices and $\operatorname{tr}(A)$ be the trace of $A \in \mathbb{M}_k$. Recall that a linear map $\Phi : \mathbb{M}_k \to \mathbb{M}_n$ is positive (or strictly positive) if $\Phi(A)$ is positive semidefinite (or positive definite) whenever A is positive semidefinite (or positive definite), and it is said to be unital if $\Phi(I_k) = I_n$, where I_n stands for the identity matrix of $n \times n$ order. A positive linear functional $\varphi : \mathbb{M}_k \to \mathbb{C}$ is a special case of such a map; see [3]. Beginning with Kadison, several authors have studied unital positive linear maps, see [2–6, 8, 9, 12, 18]. In [8], Kadison showed that for any Hermitian matrix $A \in \mathbb{M}_k$, we have

$$\Phi\left(A^2\right) \geq (\Phi(A))^2, \tag{1}$$

where $\Phi : \mathbb{M}_k \to \mathbb{M}_n$ is any unital positive linear map. A complementary inequality of (1) was obtained by Bhatia and Davis [2] which states that if A is any Hermitian element of $\mathbb{M}_k$ whose spectrum lies in the interval $[m, M]$, then

$$\Phi\left(A^2\right) - (\Phi(A))^2 \leq (MI_n - \Phi(A))(\Phi(A) - mI_n) \leq \left(\frac{M-m}{2}\right)^2 I_n. \tag{2}$$

A refinement in (2) for unital positive linear functionals was obtained by Sharma and Kumari [18]:

$$\varphi\left(A^2\right) - (\varphi(A))^2 \leq \left(\frac{\varphi\left(A^2\right)}{2\varphi(A)}\right)^2 \leq \left(\frac{M-m}{2}\right)^2, \tag{3}$$

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for $\varphi\left(A^2\right) \geq 2\left(\varphi\left(A\right)\right)^2 > 0$ and A is any positive semidefinite matrix. Suppose that $A \in \mathbb{M}_n$ has real eigenvalues. In literature, several authors have obtained bounds for extreme eigenvalues; see [10, 12, 17, 20, 22] and references therein. In particular, Wolkowicz and Styan [22] proved that if $A \in \mathbb{M}_n$ has real eigenvalues contained in the interval $[m, M]$, then

$$M \geq \frac{\operatorname{tr}(A)}{n} + \sqrt{\frac{\operatorname{tr}\left(B^2\right)}{n(n-1)}}. \tag{4}$$

Inequality (4) was sharpened by Sharma et al. [21]. They proved that

$$M \geq \frac{\operatorname{tr}(A)}{n} + \left(\frac{n^2-3n+3}{n^3(n-1)^3}\right)^{\frac{1}{4}} \frac{\operatorname{tr}\left(B^2\right)}{\left(\operatorname{tr}\left(B^4\right)\right)^{\frac{1}{4}}}, \tag{5}$$

where $B = A - \frac{\operatorname{tr}(A)}{n} I_n$.
In [17], Sharma et al. showed that for every positive definite matrix $A = \left(a_{ij}\right) \in \mathbb{M}_n$, $A^2 = \left(c_{ij}\right) \in \mathbb{M}_n$, we have

$$M \geq \frac{\operatorname{tr}\left(A^2\right) \pm 2\beta}{\operatorname{tr}(A) \pm 2\alpha}, \tag{6}$$

where $\alpha = \Re a_{ij}\left(\text{or } \Im a_{ij}\right)$ and $\beta = \Re c_{ij}\left(\text{or } \Im c_{ij}\right)$, and $\Re(A)$ and $\Im(A)$ are the real and imaginary parts of A, respectively.
Let $A \in \mathbb{M}_n$ be a positive definite matrix whose spectrum lies in the interval $[m, M]$. Then the quantity $k(A) = \dfrac{M-m}{M+m}$ is known as the Kantorovich ratio. It is important in the study of positive definite as it governs the rate of convergence for solving the linear system of equation $Ax = b$. see [1]. One lower bound for $k(A)$ was obtained by Barnes and Hoffmann [1]; for any indices i and j,

$$(k(A))^2 \geq \frac{B_{ij}(A)}{\left(a_{ii}+a_{jj}\right)^2 + B_{ij}(A)}, \tag{7}$$

where $B_{ij} = \left|a_{ii} - a_{jj}\right|^2 + 2\sum_{k\neq i} |a_{ik}|^2 + 2\sum_{k\neq j} |a_{jk}|^2$.
For a positive definite matrix $A \in \mathbb{M}_n$, we denote $c(A)$ by the condition number of A, and it is defined as the ratio of the largest eigenvalue M of A to the smallest eigenvalue m of A, that is, $c(A) = \frac{M}{m}$. In this context, Bhatia and Sharma [4] proved that

$$c(A) \geq \left(\sqrt{\frac{\varphi\left(A^2\right) - (\varphi(A))^2}{(\varphi(A))^2}} + \sqrt{1 + \frac{\left(\varphi\left(A^2\right) - (\varphi(A))^2\right.}{(\varphi(A))^2}}\right)^2, \tag{8}$$

where φ is any unital strictly positive linear functional defined on $\mathbb{M}_n$.
Let $A \in \mathbb{M}_n, (n \geq 3)$, and let $\lambda_1(A), \lambda_2(A), ..., \lambda_n(A)$ be the eigenvalues of A. The spread of A denoted $spd(A)$, is defined by $spd(A) = \max_{i,j} |\lambda_i(A) - \lambda_j(A)|$. This quantity was first introduced by Mirsky [13]. In literature, inequalities for spreads have been studied by several authors; see [1, 4, 5, 10, 11, 13, 16, 20–22] and references therein.
In this paper, we mainly focus on inequalities involving unital positive linear maps and extreme eigenvalues of positive semidefinite (or positive definite) matrices.

2. Main Results

We begin with the following theorem which provides us an inequality involving unital positive linear maps and extreme eigenvalues of a positive semidefinite matrix.

Theorem 2.1. *Let $A \in \mathbb{M}_k$ be a positive semidefinite matrix with $O \leq mI_k \leq A \leq MI_k$. Let $\Phi : \mathbb{M}_k \to \mathbb{M}_n$ be a unital positive linear map. Then for any positive integer r,*

$$(M+m)\,\Phi\left(A^r\right) \geq \Phi\left(A^{r+1}\right). \tag{9}$$

Proof. For any positive integer r, the matrices A^r and $(M+m)\,I_k - A$ are positive semidefinite and commute. So, $A^r\left((M+m)\,I_k - A\right) \geq O$; that is,

$$(M+m)\,A^r \geq A^{r+1}.$$

Applying Φ to both sides of the above inequality yields (9). □

We now present a lemma which will be used in the proof of Theorem 2.3.

Lemma 2.2. *Let $A \in \mathbb{M}_k$ be a positive semidefinite matrix with $O \leq mI_k \leq A \leq MI_k$, and let $\varphi : \mathbb{M}_k \to \mathbb{C}$ be a unital strictly positive linear functional. If $\varphi\left(A^2\right) \geq 2\left(\varphi\left(A\right)\right)^2$, then*

$$M+m \geq 2\varphi\left(A\right). \tag{10}$$

Proof. Applying Theorem 2.1, for a unital positive linear functional φ and taking $r=1$, we get that

$$M+m \geq \frac{\varphi\left(A^2\right)}{\varphi\left(A\right)}. \tag{11}$$

Inequality (10) follows from (11) since $\varphi\left(A^2\right) \geq 2\left(\varphi\left(A\right)\right)^2$. □

Theorem 2.3. *Let $A \in \mathbb{M}_n$ be a positive semidefinite matrix with $O \leq mI_n \leq A \leq MI_n$, and let $\varphi : \mathbb{M}_n \to \mathbb{C}$ be a unital positive linear functional with $\varphi\left(A\right) > 0$. If $\varphi\left(A^2\right) \geq 2\left(\varphi\left(A\right)\right)^2$, then for every positive integer r*

$$M \geq \varphi\left(A\right) + \varphi\left(B^{2r}\right)^{\frac{1}{2r}}, \tag{12}$$

where $B = A - \varphi\left(A\right) I_n$.

Proof. By the spectral theorem of Hermitian matrices, for any positive integer r, we have

$$A = \textstyle\sum_{i=1}^{n} \lambda_i\left(A\right) P_i \quad \text{and} \quad B^{2r} = \sum_{i=1}^{n} \left(\lambda_i\left(A\right) - \varphi\left(A\right)\right)^{2r} P_i, \tag{13}$$

where $\lambda_i(A)$ are the eigenvalue of A and P_i's are the corresponding orthogonal projections with $\sum_{i=1}^{n} P_i = I_n$. Applying φ, we deduce from (13) that

$$\varphi\left(A\right) = \textstyle\sum_{i=1}^{n} \lambda_i\left(A\right) \varphi\left(P_i\right) \quad \text{and} \quad \varphi\left(B^{2r}\right) = \sum_{i=1}^{n} \left(\lambda_i\left(A\right) - \varphi\left(A\right)\right)^{2r} \varphi\left(P_i\right),$$

with $\sum_{i=1}^{n} \varphi\left(P_i\right) = 1$.
By using Lemma 2.2, we have

$$M - \varphi\left(A\right) \geq \varphi\left(A\right) - m. \tag{14}$$

Now combining (14) with $m - \varphi\left(A\right) \leq \lambda_i\left(A\right) - \varphi\left(A\right) \leq M - \varphi\left(A\right)$, we get $\varphi\left(A\right) - M \leq \lambda_i\left(A\right) - \varphi\left(A\right) \leq M - \varphi\left(A\right)$. Thus, we observe that

$$\left(\lambda_i\left(A\right) - \varphi\left(A\right)\right)^{2r} \leq \left(M - \varphi\left(A\right)\right)^{2r}, \tag{15}$$

holds for each positive integer r.
Summing (15) from 1 to n, we have

$$\varphi\left(B^{2r}\right) = \textstyle\sum_{i=1}^{n} \left(\lambda_i\left(A\right) - \varphi\left(A\right)\right)^{2r} \varphi\left(P_i\right) \leq \left(M - \varphi\left(A\right)\right)^{2r} \sum_{i=1}^{n} \varphi\left(P_i\right). \tag{16}$$

Inequality (12) now follows immediately from (16) because $\sum_{i=1}^{n} \varphi(P_i) = 1$. □

Example 2.4. *Let*

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

From (5) *and* (6), *we have* $M \geq 3.12$, $M \geq 4.33$, *respectively while our bound* (12) *provides a better estimate* $M \geq 4.46$ *when* $r = 11$ *for the choice of a unital positive linear functional* $\varphi(A) = a_{11}$.

We now present a lower bound for spectral radius $\rho(N)$ of a nonnegative symmetric matrix $N \in \mathbb{M}_k$.

Corollary 2.5. *Let* $N \in \mathbb{M}_k$ *be a nonnegative symmetric matrix. If* $\varphi\left(N^4\right) \geq 2\left(\varphi\left(N^2\right)\right)^2$, *then*

$$\rho(N) \geq \sqrt{\varphi(N^2) + (\varphi(M^{2r}))^{\frac{1}{2r}}}, \tag{17}$$

where r is any positive integer and $M = N^2 - \varphi\left(N^2\right) I_k$.

Proof. We know that for a nonnegative matrix N, $\rho(N)$ is an eigenvalue of N, see [7]. Since $\lambda_{max}\left(N^2\right) = (\lambda_{max}(N))^2 = (\rho(N))^2$ and $N^2 \geq 0$, the assertion immediately follows from Theorem 2.3. □

Remark 2.6. *Let* $N = \left(n_{ij}\right) \in \mathbb{M}_k$ *be nonnegative (not necessarily symmetric). Let* $x_{ij} = \min\left\{n_{ij}, n_{ji}\right\}$ *for all i, j and* $X = (x_{ij}) \in \mathbb{M}_k$. *Then for nonnegative matrices N, X, and* $N - X$ *we get that* $\rho(N) \geq \rho(X)$. *Also, see [7].*

We present the following lemma, which we will used in Theorem 2.8.

Lemma 2.7. *Let A be any Hermitian element of* $\mathbb{M}_k$ *whose spectrum lies in the interval* $[m, M]$, *and let* $\varphi : \mathbb{M}_k \to \mathbb{C}$ *be a unital positive linear map. Then*

$$m \leq \varphi(A) - \sqrt{\varphi(A^2) - (\varphi(A))^2} \quad \text{for} \quad \varphi\left((A - \varphi(A) I_k)^3\right) \leq 0. \tag{18}$$

Proof. Since $\varphi(A^r) = \mu'_r$, where μ'_r is the r-th order moment about origin; therefore, we can write (2.17) of [15] in the following equivalent form:

$$\frac{\left(\varphi\left(A^2\right) - (\varphi(A))^2\right)^2 - (\varphi(A) - m)^2\left(\varphi\left(A^2\right) - (\varphi(A))^2\right)}{\varphi(A) - m} \leq \varphi\left((A - \varphi(A) I_k)^3\right). \tag{19}$$

Inequality (18) now follows from (19), because $\varphi\left((A - \varphi(A) I_k)^3\right) \leq 0$. □

Our next result gives a lower bound for the condition number of a positive definite matrix in terms of unital positive linear functionals.

Theorem 2.8. *Let* $A \in \mathbb{M}_k$ *be a positive definite matrix with* $mI_k \leq A \leq MI_k$, *and let* $\varphi_1, \varphi_2 : \mathbb{M}_k \to \mathbb{C}$ *be unital positive linear functionals. Then*

$$c(A) \geq 1 + \frac{2\sqrt{\varphi_1(A^2) - (\varphi_1(A))^2}}{\varphi_2(A) - \sqrt{\varphi_2(A^2) - (\varphi_2(A))^2}} \quad \text{for} \quad \varphi_2\left((A - \varphi_2(A) I_k)^3\right) \leq 0. \tag{20}$$

Proof. By Lemma 2.7, we have

$$m \leq \varphi_2(A) - \sqrt{\varphi_2(A^2) - (\varphi_2(A))^2},$$

or equivalently

$$M - m \leq \left(\frac{M}{m} - 1\right)\left(\varphi_2(A) - \sqrt{\varphi_2(A^2) - (\varphi_2(A))^2}\right). \tag{21}$$

Using inequality (2) *for the unital positive linear functional* φ_1, *we have*

$$2\sqrt{\varphi_1(A^2) - (\varphi_1(A))^2} \leq M - m. \tag{22}$$

Combining (21) *and* (22), *we get*

$$2\sqrt{\varphi_1(A^2) - (\varphi_1(A))^2} \leq \left(\frac{M}{m} - 1\right)\left(\varphi_2(A) - \sqrt{\varphi_2(A^2) - (\varphi_2(A))^2}\right). \tag{23}$$

Inequality (20) *now follows from* (23) . □

We present the following lemma which will used to show improvement (conditional) of (8) (see Remark 2.10).

Lemma 2.9. *Let* $A \in \mathbb{M}_k$ *be a positive definite matrix with* $mI_k \leq A \leq MI_k$, *and let* $\varphi : \mathbb{M}_k \to \mathbb{C}$ *be a unital positive linear functional. Then*

$$\varphi\left(A^2\right) < 2(\varphi(A))^2 \quad \text{for} \quad \varphi\left((A - \varphi(A) I_k)^3\right) \leq 0. \tag{24}$$

Proof. By combining (19) with $\varphi\left((A - \varphi(A) I_k)^3\right) \leq 0$, we find that

$$\left(\varphi\left(A^2\right) - (\varphi(A))^2\right)^2 - (\varphi(A) - m)^2\left(\varphi\left(A^2\right) - (\varphi(A))^2\right) \leq 0,$$

which gives

$$\varphi\left(A^2\right) - (\varphi(A))^2 - (\varphi(A) - m)^2 \leq 0. \tag{25}$$

Since $m > 0$, therefore we find that

$$\varphi\left(A^2\right) - 2(\varphi(A))^2 < \varphi\left(A^2\right) - (\varphi(A))^2 - (\varphi(A) - m)^2. \tag{26}$$

Inequality (24) now follows immediately (25) and (26). □

Remark 2.10. *In this remark, we will show an improvement (conditional) of* (8). *For this, in* (20), *choosing* $\varphi_1 = \varphi_2 = \varphi$ *with* $\varphi\left((A - \varphi(A) I_k)^3\right) \leq 0$, *we find that*

$$c(A) \geq 1 + \frac{2\sqrt{\varphi(A^2) - (\varphi(A))^2}}{\varphi(A) - \sqrt{\varphi(A^2) - (\varphi(A))^2}}, \tag{27}$$

or equivalently

$$c(A) \geq \frac{1 + \sqrt{\kappa - 1}}{1 - \sqrt{\kappa - 1}},$$

where $\kappa = \frac{\varphi(A^2)}{(\varphi(A))^2}$. *Also, we can write* (8) *in the following equivalent form*

$$c(A) \geq 2\kappa - 1 + 2\sqrt{\kappa^2 - \kappa}. \tag{28}$$

From (1) *and* (24), *we conclude that* $1 \leq \kappa < 2$. *For this* κ, (27) *provides improvement (conditionally) over* (28) *and hence over* (8) *because*

$$\frac{1 + \sqrt{\kappa - 1}}{1 - \sqrt{\kappa - 1}} \geq 2\kappa - 1 + 2\sqrt{\kappa^2 - \kappa}.$$

In the next theorem, we present a refinement of (3).

Theorem 2.11. *Let $A \in \mathbb{M}_n$ be a positive definite matrix with $mI_n \leq A \leq MI_n$, and let $\varphi : \mathbb{M}_n \to \mathbb{C}$ be a unital strictly positive linear functional. If $\varphi\left(A^2\right) \geq 2\left(\varphi(A)\right)^2$, then*

$$\left(\frac{\varphi\left(A^2\right)}{2\varphi(A)}\right)^2 \leq \varphi\left(A^2\right) - \left(\varphi(A)\right)^2 + \left(\frac{\varphi\left(A^3\right)}{2\varphi(A^2)} - \varphi(A)\right)^2 \leq \frac{(M-m)^2}{4}. \tag{29}$$

Proof. We will prove the first inequality in (29). We can write

$$\left(\frac{\varphi\left(A^2\right)}{2\varphi(A)}\right)^2 = \varphi\left(A^2\right) - \left(\varphi(A)\right)^2 + \left(\frac{\varphi\left(A^2\right)}{2\varphi(A)} - \varphi(A)\right)^2.$$

The left-hand side of (29) holds if and only if

$$\left(\frac{\varphi\left(A^2\right)}{2\varphi(A)} - \varphi(A)\right)^2 \leq \left(\frac{\varphi\left(A^3\right)}{2\varphi(A^2)} - \varphi(A)\right)^2. \tag{30}$$

To prove (30), we use (2.20) of [15] in the following equivalent form:

$$m\varphi\left(A^2\right) + \frac{m\varphi(A) - \left(\varphi\left(A^2\right)\right)^2}{\varphi(A) - m} \leq \varphi\left(A^3\right),$$

or equivalently

$$\varphi\left(A^3\right) \geq \frac{\left(\varphi\left(A^2\right)\right)^2}{\varphi(A)} + \frac{m\left(\varphi\left(A^2\right) - (\varphi(A))^2\right)\left(\varphi\left(A^2\right) - m\varphi(A)\right)}{\varphi(A)\left(\varphi(A) - m\right)}. \tag{31}$$

In the right-hand side expression of (31), the second quantity is nonnegative, and hence

$$\varphi\left(A^3\right) \geq \frac{\left(\varphi\left(A^2\right)\right)^2}{\varphi(A)},$$

which gives

$$\frac{\varphi\left(A^3\right)}{2\varphi(A^2)} - \varphi(A) \geq \frac{\varphi\left(A^2\right)}{2\varphi(A)} - \varphi(A) \geq 0, \tag{32}$$

because $\varphi\left(A^2\right) \geq 2\left(\varphi(A)\right)^2$. Thus (30) holds.
To prove the second inequality in (29). Applying (2) for a unital positive linear functional φ and write its left-hand side inequality in the following equivalent form:

$$\varphi\left(A^2\right) - \varphi(A)^2 + \left(\varphi(A) - \tfrac{M+m}{2}\right)^2 \leq \frac{(M-m)^2}{4}. \tag{33}$$

Applying Theorem 2.1 for a strictly unital positive linear functional φ and taking $r = 2$, we deduce that

$$M + m \geq \frac{\varphi\left(A^3\right)}{\varphi(A^2)},$$

and hence

$$\frac{M+m}{2} - \varphi(A) \geq \frac{\varphi\left(A^3\right)}{2\varphi\left(A^2\right)} - \varphi(A). \tag{34}$$

By Lemma 2.2, the left-hand side quantity in (34) is non-negative. Therefore, by using (32) we state that the right-hand side quantity in (34) is also non-negative. Hence, the right-hand side inequality of (29) follows by combining (33) and (34). □

Remark 2.12. *By using* (29) *for different choices of unital positive linear functionals φ, one can obtain several lower bounds for spd(A).*

Our next result present an inequality involving Kantorovich ratio and unital positive linear map.

Theorem 2.13. *Let $A \in \mathbb{M}_k$ be a positive definite matrix with $mI_k \leq A \leq MI_k$, and let $\Phi : \mathbb{M}_k \to \mathbb{M}_n$ be a strictly unital positive linear map. If $\Phi\left(A^2\right)$ commutes with $(\Phi(A))^2$, then*

$$\left(\Phi\left(A^2\right) - (\Phi(A))^2\right)\left(\Phi\left(A^2\right)\right)^{-1} \leq (k(A))^2 I_n, \tag{35}$$

where $k(A)$ is the Kantorovich ratio.

Proof. The matrices $MI_k - A$ and $A - mI_k$ are positive semidefinite and commute with each other. This gives

$$A^2 \leq (M+m)A - MmI_k,$$

and hence

$$\Phi\left(A^2\right) \leq (M+m)\Phi(A) - MmI_n. \tag{36}$$

From (36), we have

$$\Phi\left(A^2\right)(\Phi(A))^{-2} \leq (M+m)(\Phi(A))^{-1} - Mm(\Phi(A))^{-2}, \tag{37}$$

because by using the assumption $\Phi\left(A^2\right)$ commutes with $(\Phi(A))^2$.
Let $f(x) = \frac{M+m}{x} - \frac{Mm}{x^2}$. Then $f(x)$ has maxima at $x = \frac{2Mm}{M+m}$ and where its value is $\frac{(M+m)^2}{4Mm}$. So, from (37) we find that

$$\Phi\left(A^2\right)(\Phi(A))^{-2} \leq \frac{(M+m)^2}{4Mm} I_n. \tag{38}$$

A little calculation in (38) leads to (35). □

Corollary 2.14. *Let $A = \left(a_{ij}\right) \in \mathbb{M}_n$ be a positive definite matrix with real entries such that $mI_n \leq A \leq MI_n$. Then for any indices i and j with $i \neq j$,*

$$(k(A))^2 \geq 1 - \frac{d\left(a_{ij}^2 + \left(\frac{a_{ii}+a_{jj}}{2}\right)^2\right) - ca_{ij}\left(a_{ii}+a_{jj}\right)}{d^2 - c^2} + \frac{\left|c\left(a_{ij}^2 + \left(\frac{a_{ii}+a_{jj}}{2}\right)^2\right) - d\left(a_{ii}+a_{jj}\right)\right|}{d^2 - c^2}, \tag{39}$$

where $d = \frac{\sum_{k=1}^n a_{ik}^2 + a_{jk}^2}{2}$ and $c = \sum_{k=1}^n a_{ik}a_{jk}$.

Proof. For $i \neq j$, we define a unital positive linear map $\Phi : \mathbb{M}_n \to \mathbb{M}_2$ $\Phi(A) = \begin{bmatrix} \frac{a_{ii}+a_{jj}}{2} & a_{ij} \\ a_{ji} & \frac{a_{ii}+a_{jj}}{2} \end{bmatrix}$. This map satisfies the condition of Theorem 2.13, that is, $\Phi\left(A^2\right)$ commutes with $(\Phi(A))^2$.
A little calculation leads to

$$\Phi\left(A^2\right) - (\Phi(A))^2 = \begin{bmatrix} d - \left(a_{ij}^2 + \left(\frac{a_{ii}+a_{jj}}{2}\right)^2\right) & c - a_{ij}\left(a_{ii} + a_{jj}\right) \\ c - a_{ij}\left(a_{ii} + a_{jj}\right) & d - \left(a_{ij}^2 + \left(\frac{a_{ii}+a_{jj}}{2}\right)^2\right) \end{bmatrix}$$

and

$$\left(\Phi\left(A^2\right)\right)^{-1} = \frac{1}{d^2-c^2}\begin{bmatrix} d & -c \\ -c & d \end{bmatrix}.$$

The matrix $\left(\Phi\left(A^2\right) - (\Phi(A))^2\right)\left(\Phi\left(A^2\right)\right)^{-1}$ is of the form $\begin{bmatrix} x & y \\ y & x \end{bmatrix}$ which is positive semidefinite and its norm is $x + |y|$. Therefore, by using Theorem 2.13, we have

$$\left\|\left(\Phi\left(A^2\right) - (\Phi(A))^2\right)\left(\Phi\left(A^2\right)\right)^{-1}\right\| \leq (k(A))^2, \tag{40}$$

where $\|A\|$ denotes the operator norm of A. Inequality (39) now follows immediately from (40). □

Example 2.15. *Let*

$$B = \begin{bmatrix} 4 & 0 & 2 & 3 \\ 0 & 5 & 0 & 1 \\ 2 & 0 & 6 & 0 \\ 3 & 1 & 0 & 7 \end{bmatrix}.$$

From (7), *we get* $(k(A))^2 \geq 0.3125$ *while our bound* (39) *provides a better estimate* $(k(A))^2 \geq 0.4318$.

3. Inequalities involving traces of matrices

Let $A \in \mathbb{M}_n$ have nonnegative eigenvalues $m = \lambda_1(A) \leq \lambda_2(A) \ldots \leq \lambda_n(A) = M$. We respectively define the arithmetic mean $\overline{\lambda}$ and the variance S_λ^2 of the eigenvalues of A by

$$\overline{\lambda} = \frac{\operatorname{tr}(A)}{n} \quad \text{and} \quad S_\lambda^2 = \frac{\operatorname{tr}\left(A^2\right)}{n} - \left(\frac{\operatorname{tr}(A)}{n}\right)^2.$$

In [19], Sharma et al. showed that if the eigenvalues of $A \in \mathbb{M}_n$ are nonnegative and $0 < \overline{\lambda} \leq S_\lambda$, then

$$S_\lambda^2 + \left(\frac{S_\lambda^2 - \left(\overline{\lambda}\right)^2}{2\overline{\lambda}}\right)^2 \leq \frac{(M-m)^2}{4}, \tag{41}$$

and with $n \geq 3$,

$$S_\lambda^2 - \frac{2}{n-2}\left(\frac{S_\lambda^2 - \left(\overline{\lambda}\right)^2}{2\overline{\lambda}}\right)^2 \geq \frac{(M-m)^2}{2n}. \tag{42}$$

We now present the following refinements of (41) and (42).

Theorem 3.1. *Let $A \in \mathbb{M}_n$ have nonnegative eigenvalues $m = \lambda_1(A) \leq \lambda_2(A) ... \leq \lambda_n(A) = M$. If $0 < \overline{\lambda} \leq S_\lambda$, then*

$$S_\lambda^2 + \left(\frac{S_\lambda^2 - \left(\overline{\lambda}\right)^2}{2\overline{\lambda}}\right)^2 \leq S_\lambda^2 + \left(\frac{tr\left(A^3\right)}{2tr\left(A^2\right)} - \overline{\lambda}\right)^2 \leq \frac{(M-m)^2}{4} \tag{43}$$

and with $n \geq 3$,

$$S_\lambda^2 - \frac{2}{n-2}\left(\frac{S_\lambda^2 - \left(\overline{\lambda}\right)^2}{2\overline{\lambda}}\right)^2 \geq S_\lambda^2 - \frac{2}{n-2}\left(\frac{tr\left(A^3\right)}{2tr\left(A^2\right)} - \overline{\lambda}\right)^2 \geq \frac{(M-m)^2}{2n}. \tag{44}$$

Proof. By Sharma et al. [14], we have

$$S_\lambda^2 \geq \frac{(M-m)^2}{2n} + \frac{2}{n-2}\left(\left(\frac{M+m}{2}\right) - \overline{\lambda}\right)^2. \tag{45}$$

Applying (2) for a unital positive linear functional $\varphi(A) = \frac{\operatorname{tr}(A)}{n}$, we get $S_\lambda^2 \leq \frac{(M-m)^2}{4}$ and then resulting inequality combining with $0 < \overline{\lambda} \leq S_\lambda$ we find that

$$\frac{M+m}{2} - \overline{\lambda} \geq 0.$$

Also, since all the eigenvalues $\lambda_i(A)$ of A are nonnegative and $M + m - \lambda_i(A) \geq 0$, therefore we have for $i = 1, 2, ..., n$

$$\lambda_i^2(A)\left(M + m - \lambda_i(A)\right) \geq 0. \tag{46}$$

Summing over i from 1 to n in (46), we get that

$$M + m \geq \frac{\operatorname{tr}\left(A^3\right)}{\operatorname{tr}\left(A^2\right)},$$

and hence

$$\frac{M+m}{2} - \overline{\lambda} \geq \frac{\operatorname{tr}\left(A^3\right)}{2\operatorname{tr}\left(A^2\right)} - \overline{\lambda}. \tag{47}$$

We know that the Cauchy-Schwarz inequality for real numbers is given by

$$\textstyle\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 \geq \left(\sum_{i=1}^n a_i b_i\right)^2 \tag{48}$$

for $a = (a_1, ..., a_n)$, $b = (b_1, ..., b_n) \in \mathbb{R}^n$. Applying (48) for positive numbers $\lambda_i(A)$; $i = 1, 2, ..., n$ by taking $a_i = (\lambda_i(A))^{\frac{3}{2}}$ and $b_i = (\lambda_i(A))^{\frac{1}{2}}$, we find that $\operatorname{tr}\left(A^3\right)\operatorname{tr}(A) \geq \left(\operatorname{tr}\left(A^2\right)\right)^2$, and hence

$$\frac{\operatorname{tr}\left(A^3\right)}{2\operatorname{tr}\left(A^2\right)} - \overline{\lambda} \geq \frac{\operatorname{tr}\left(A^2\right)}{2\operatorname{tr}(A)} - \overline{\lambda} = \frac{S_\lambda^2 - \overline{\lambda}^2}{2\overline{\lambda}} \geq 0, \tag{49}$$

holds for $0 < \overline{\lambda} \leq S_\lambda$. It follows from (47) and (49) that

$$\frac{M+m}{2} - \overline{\lambda} \geq \frac{\operatorname{tr}\left(A^3\right)}{2\operatorname{tr}\left(A^2\right)} - \overline{\lambda} \geq \frac{S_\lambda^2 - \overline{\lambda}^2}{2\overline{\lambda}} \geq 0, \tag{50}$$

holds for $0 < \overline{\lambda} \leq S_\lambda$. Combining (45) with (50) we deduce the desired inequalities in (44).
To prove (43). We use Theorem 1 of [2], that is $S_\lambda^2 \leq \left(M - \overline{\lambda}\right)\left(\overline{\lambda} - m\right)$, and therefore we can write from this inequality that

$$S_\lambda^2 + \left(\overline{\lambda} - \frac{M+m}{2}\right)^2 \leq \frac{(M-m)^2}{4}. \tag{51}$$

Inequality (43) now follows by combining (50) and (51). □

Remark 3.2. *It is notice here that for $A \in \mathbb{M}_n$ with all nonnegative eigenvalues, the inequalities* (43) *and* (44) *respectively provided us a lower bound and an upper bound for $spd(A)$.*

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