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A general form for precise asymptotics for the stochastic wave equation

Jingyu Li[a], Yong Zhang[a,*]

[a] *School of Mathematics, Jilin University, Changchun 130012, P. R. China*

Abstract. Consider the stochastic wave equation driven by a general Gaussian multiplicative noise, which is temporally white and colored in space including the cases of the spatial covariance given by a fractional noise, a Riesz kernel, and an integrable function that satisfies Dalang's condition. In this paper, we present the precise asymptotics for complete convergence and complete moment convergence for the spatial averages of the solution to the equation over a Euclidean ball, as the radius of the ball diverges to infinity. Some general results on precise asymptotics are obtained, which can describe the relations among the boundary function, weighted function, convergence rate and the limit value.

1. Introduction

Consider the following stochastic wave equation (SWE for short):

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(t,x) = \Delta u(t,x) + \sigma(u(t,x))\dot{W}(t,x), \quad t>0, x\in\mathbb{R}^d, \\ u(0,x) = 1, \quad x\in\mathbb{R}^d, \\ \dfrac{\partial u}{\partial t}(0,x) = 0, \quad x\in\mathbb{R}^d, \end{cases} \tag{1.1}$$

where Δ stands for the Laplacian operator in space variables and $\dot{W}$ denotes a centered, generalized Gaussian noise whose covariance is given by

$$\mathbb{E}(\dot{W}(t,x)\dot{W}(s,y)) = \delta_0(t-s)f(x-y), \qquad s,t\geq 0, x,y\in\mathbb{R}^d. \tag{1.2}$$

The coefficient σ is a Lipschitz function. In order to avoid triviality, we assume that $\sigma(1)\neq 0$.

In this paper, we are interested in the following three cases:

Case 1.1. *The Gaussian noise behaves as a fractional noise in space with Hurst parameter* $H\in[1/2,1)$, *that is,* $f(x)=|x|^{2H-2}$ *for* $H\in(1/2,1)$, *and* $f(x)=\delta_0(x)$ *for* $H=1/2$.

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* Corresponding author: Yong Zhang
Email addresses: lijingyu@jlu.edu.cn (Jingyu Li), zyong2661@jlu.edu.cn (Yong Zhang)

Case 1.2. *The Gaussian noise has a spatial covariance described by the Riesz kernel, that is, $d = 2$ and $f(x) = |x|^{-\beta}$, $\beta \in (0, 2)$.*

Case 1.3. *f is a tempered nonnegative and nonnegative definite function, whose Fourier transform μ satisfies Dalang's condition:*

$$\int_{\mathbb{R}^d} \frac{\mu(\mathrm{d}z)}{1 + |z|^2} < \infty \tag{1.3}$$

for $d = 1, 2$. Suppose also that f satisfies $f \in L^1(\mathbb{R})$ if $d = 1$ and $f \in L^1\left(\mathbb{R}^2\right) \cap L^\ell\left(\mathbb{R}^2\right)$ for some $\ell > 1$ if $d = 2$.

Notice that following Dalang [4], under Case 1.1, Case 1.2 or Case 1.3, we can interpret the solution to (1.1) in the following mild form:

$$u(t, x) = 1 + \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x - y)\sigma(u(s, y))W(\mathrm{d}s, \mathrm{d}y), \tag{1.4}$$

where the above stochastic integral is understood in the sense of Dalang-Walsh and $G_{t-s}(x - y)$ denotes the fundamental solution to the corresponding deterministic wave equation, that is,

$$G_t(x) := \begin{cases} \frac{1}{2}\mathbf{1}_{\{|x|<t\}}, & \text{if } d = 1; \\ \frac{1}{2\pi\sqrt{t^2-|x|^2}}\mathbf{1}_{\{|x|<t\}}, & \text{if } d = 2. \end{cases} \tag{1.5}$$

There have been many achievements for limit theorems for spatial averages of the solutions to SWEs. Delgado-Vences et al. [5] obtained a quantitative central limit theorem (CLT for short) and a functional CLT for spatial average of the one-dimensional SWE driven by a Gaussian multiplicative noise, which is white in time and has the covariance of a fractional Brownian motion with Hurst parameter $H \in [1/2, 1)$ in the spatial variable. Bolaños Guerrero et al. [3] considered the case that $d = 2$ and the Gaussian noise is temporally white and spatially colored described by the Riesz kernel. Fix $d \in \{1, 2\}$, Nualart and Zheng [15] studied the d-dimensional SWE driven by a Gaussian noise, which is temporally white and colored in space such that the spatial correlation function is integrable and satisfies Dalang's condition. Nualart and Zheng [14] investigated the spatial ergodicity for a class of SWEs with spatial dimension less than or equal to 3. We refer to Balan et al. [1], Balan et al. [2] and Li and Zhang [9–12] for several other investigations on stochastic partial differential equations.

In this paper, we are interested in the asymptotic behavior of the spatial averages of the solution to (1.1). Specifically, we aim to study the precise asymptotics for complete convergence and complete moment convergence for the stochastic wave equation.

The concept of complete convergence was first introduced by Hsu and Robbins [7], since then there have been extensions in several directions. One important topic of them is to discuss the precise rate and limit value of

$$\sum_{n=n_0}^{\infty} \psi(n)\mathbb{E}[|X_n|^p I\{|X_n| \geq \varepsilon\nu(n)\}]$$

as $\varepsilon \to a, a \geq 0$, where $p \geq 0$, $\psi(x)$ and $\nu(x)$ are the positive functions defined on $[n_0, \infty)$, $X_n = \sum_{i=1}^n \zeta_i$ for $n \geq 1$ and $\{\zeta_n, n \geq 1\}$ is a sequence of independent and identically distributed random variables with $\mathbb{E}\zeta_1 = 0$ and $\mathbb{E}\zeta_1^2 < \infty$. We call $\psi(x)$ and $\nu(x)$ weighted function and boundary function, respectively. A first result in this direction was given by Heyde [6], who proved that

$$\lim_{\varepsilon\to 0} \varepsilon^2 \sum_{n=1}^{\infty} P\{|X_n| \geq \varepsilon n\} = \mathbb{E}\zeta_1^2.$$

The research in this field is called the precise asymptotics. For analogous results in more general case, see Lu et al. [13], Wu and Jiang [16], Wu and Wang [17], Zhang et al. [18], Zhao et al. [19] and the references therein.

The main objective of this paper is to study the general form of precise asymptotics for the stochastic wave equation. The rest of the paper is organized as follows. In Section 2, we establish the precise asymptotics for complete convergence for the stochastic wave equation. And the precise asymptotics for complete moment convergence for the stochastic wave equation are given in Section 3. Finally, we put some technical lemmas into the Appendix. Throughout the paper, C represents a positive constant although its value may change from one appearance to the next, $\mathcal{N}$ denotes the standard normal random variable, and for every $Z \in L^k(\Omega)$, we write $\|Z\|_k$ instead of $\{\mathbb{E}(|Z|^k)\}^{1/k}$.

2. Precise asymptotics for complete convergence

In order to state the main results, let us introduce

$$\mathcal{S}_{R,t} = \int_{B_R} (u(t,x) - 1)\mathrm{d}x, \qquad \text{for all } R > 0, \text{ fixed } t > 0, \tag{2.1}$$

where $B_R = \left\{x \in \mathbb{R}^d : |x| \leq R\right\}$ and $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^d$. Here and in the sequel, n_0 represents a non-negative constant although its value may change as the function g changes in the main results.

The following two results concern the precise asymptotics for complete convergence.

Theorem 2.1. *Let $h(x)$ be a positive and differentiable function defined on $[n_0, \infty)$, which is strictly increasing to ∞. Let f be a kernel of Cases 1.1-1.3, then for any $\theta > 0$ and fixed $t > 0$, we have*

$$\lim_{\varepsilon\to 0} \varepsilon^{1/\theta} \int_{n_0}^{\infty} h'(r) P\left\{\frac{\left|\mathcal{S}_{r,t}\right|}{\sqrt{\mathrm{Var}\left(\mathcal{S}_{r,t}\right)}} \geq \varepsilon h^{\theta}(r)\right\} \mathrm{d}r = \mathbb{E}|\mathcal{N}|^{1/\theta}. \tag{2.2}$$

Theorem 2.2. *Let $h(x)$ be a positive and differentiable function defined on $[n_0, \infty)$, which is strictly increasing to ∞. Let f be a kernel of Cases 1.1-1.3, then for any $\theta > 0$ and fixed $t > 0$, we have*

$$\lim_{\varepsilon\to 0} \frac{1}{-\log\varepsilon} \int_{n_0}^{\infty} \frac{h'(r)}{h(r)} P\left\{\frac{\left|\mathcal{S}_{r,t}\right|}{\sqrt{\mathrm{Var}\left(\mathcal{S}_{r,t}\right)}} \geq \varepsilon h^{\theta}(r)\right\} \mathrm{d}r = \frac{1}{\theta}. \tag{2.3}$$

Remark 1. *In our main results, it is only assumed that $h(x)$ is positive, differentiable and strictly increasing. Actually, it is quite easy to be satisfied. For example, $h(x) = x^{\alpha}, (\log x)^{\beta}, (\log\log x)^{\gamma}$ with some suitable conditions of $\alpha > 0, \beta > 0, \gamma > 0$ and some others all satisfy these conditions. In the following, some typical examples are given.*

In Theorem 2.1, let $h(x) = x^{\frac{\alpha}{p}-1}, n_0 = 1, \theta = \frac{2-p}{2(\alpha-p)}$, where $0 < p < \alpha < 2$, we have

Corollary 2.1. *Let f be a kernel of Cases 1.1-1.3, then for $0 < p < \alpha < 2$ and fixed $t > 0$,*

$$\lim_{\varepsilon\to 0} \varepsilon^{2(\alpha-p)/(2-p)} \int_1^{\infty} r^{\alpha/p-2} P\left\{\frac{\left|\mathcal{S}_{r,t}\right|}{\sqrt{\mathrm{Var}\left(\mathcal{S}_{r,t}\right)}} \geq \varepsilon r^{\frac{1}{p}-\frac{1}{2}}\right\} \mathrm{d}r = \frac{p}{\alpha-p}\mathbb{E}|\mathcal{N}|^{2(\alpha-p)/(2-p)}.$$

In Theorem 2.1, let $h(x) = (\log x)^{b+1}, n_0 = e, \theta = \frac{1}{2(b+1)}$, where $b > -1$, we have

Corollary 2.2. *Let f be a kernel of Cases 1.1-1.3, for any $b > -1$ and fixed $t > 0$,*

$$\lim_{\varepsilon\to 0} \varepsilon^{2(b+1)} \int_e^{\infty} \frac{(\log r)^b}{r} P\left\{\frac{\left|\mathcal{S}_{r,t}\right|}{\sqrt{\mathrm{Var}\left(\mathcal{S}_{r,t}\right)}} \geq \varepsilon\sqrt{\log r}\right\} \mathrm{d}r = \frac{1}{b+1}\mathbb{E}|\mathcal{N}|^{2(b+1)}.$$

In Theorem 2.1, let $h(x) = (\log\log x)^{b+1}, n_0 = e^e, \theta = \frac{1}{2(b+1)}$, where $b > -1$, we have

Corollary 2.3. *Let f be a kernel of Cases 1.1-1.3, for any $b > -1$ and fixed $t > 0$,*

$$\lim_{\varepsilon\to 0} \varepsilon^{2(b+1)} \int_{e^e}^{\infty} \frac{(\log\log r)^b}{r\log r} P\left\{ \frac{\left|\mathcal{S}_{r,t}\right|}{\sqrt{\mathrm{Var}\left(\mathcal{S}_{r,t}\right)}} \geq \varepsilon\sqrt{\log\log r} \right\} \mathrm{d}r = \frac{1}{b+1}\mathbb{E}|\mathcal{N}|^{2(b+1)}.$$

In Theorem 2.2, let $h(x) = x, n_0 = 1, \theta = \frac{2-p}{2p}$, where $0 < p < 2$, we have

Corollary 2.4. *Let f be a kernel of Cases 1.1-1.3, for $0 < p < 2$ and fixed $t > 0$,*

$$\lim_{\varepsilon\to 0} \frac{1}{-\log\varepsilon} \int_{1}^{\infty} \frac{1}{r} P\left\{ \frac{\left|\mathcal{S}_{r,t}\right|}{\sqrt{\mathrm{Var}\left(\mathcal{S}_{r,t}\right)}} \geq \varepsilon r^{\frac{1}{p}-\frac{1}{2}} \right\} \mathrm{d}r = \frac{2p}{2-p}.$$

In Theorem 2.2, let $h(x) = \log x, , n_0 = e, \theta = \frac{1}{2}$, we have

Corollary 2.5. *Let f be a kernel of Cases 1.1-1.3, for fixed $t > 0$,*

$$\lim_{\varepsilon\to 0} \frac{1}{-\log\varepsilon} \int_{e}^{\infty} \frac{1}{r\log r} P\left\{ \frac{\left|\mathcal{S}_{r,t}\right|}{\sqrt{\mathrm{Var}\left(\mathcal{S}_{r,t}\right)}} \geq \varepsilon\sqrt{\log r} \right\} \mathrm{d}r = 2.$$

In Theorem 2.2, let $h(x) = \log\log x, , n_0 = e^e, \theta = \frac{1}{2}$, we have

Corollary 2.6. *Let f be a kernel of Cases 1.1-1.3, for fixed $t > 0$,*

$$\lim_{\varepsilon\to 0} \frac{1}{-\log\varepsilon} \int_{e^e}^{\infty} \frac{1}{r\log r\log\log r} P\left\{ \frac{\left|\mathcal{S}_{r,t}\right|}{\sqrt{\mathrm{Var}\left(\mathcal{S}_{r,t}\right)}} \geq \varepsilon\sqrt{\log\log r} \right\} \mathrm{d}r = 2.$$

2.1. Proof of Theorem 2.1

In the following propositions, we will denote $\vartheta(\varepsilon) = h^{-1}\left(M\varepsilon^{-\frac{1}{\theta}}\right), M > 0$, and $h^{-1}(x)$ is the inverse function of $h(x)$.

Proposition 2.1. *Under the assumptions of Theorem 2.1, for any $\theta > 0$ and fixed $t > 0$, we have*

$$\lim_{\varepsilon\to 0} \varepsilon^{1/\theta} \int_{n_0}^{\infty} h'(r) P\left\{|\mathcal{N}| \geq \varepsilon h^{\theta}(r)\right\} \mathrm{d}r = \mathbb{E}|\mathcal{N}|^{1/\theta}.$$

Proof. Using the change of variable $\tilde{r} = \varepsilon h^{\theta}(r)$, we can get that

$$\lim_{\varepsilon\to 0} \varepsilon^{1/\theta} \int_{n_0}^{\infty} h'(r) P\left\{|\mathcal{N}| \geq \varepsilon h^{\theta}(r)\right\} \mathrm{d}r = \lim_{\varepsilon\to 0} \frac{1}{\theta} \int_{\varepsilon g^{\theta}(n_0)}^{\infty} \tilde{r}^{\frac{1}{\theta}-1} P\{|\mathcal{N}| \geq \tilde{r}\} \mathrm{d}\tilde{r} = \frac{1}{\theta} \int_{0}^{\infty} \tilde{r}^{\frac{1}{\theta}-1} P\{|\mathcal{N}| \geq \tilde{r}\} \mathrm{d}\tilde{r} = \mathbb{E}|\mathcal{N}|^{1/\theta}.$$

□

Proposition 2.2. *Under the assumptions of Theorem 2.1, for any $M > 1$ and fixed $t > 0$, we have*

$$\lim_{\varepsilon\to 0} \varepsilon^{1/\theta} \int_{n_0}^{\vartheta(\varepsilon)} h'(r) \left| P\left\{ \frac{\left|\mathcal{S}_{r,t}\right|}{\sqrt{\mathrm{Var}\left(\mathcal{S}_{r,t}\right)}} \geq \varepsilon h^{\theta}(r) \right\} - P\left\{|\mathcal{N}| \geq \varepsilon h^{\theta}(r)\right\} \right| \mathrm{d}r = 0.$$

Proof. Set

$$\triangle_R = \sup_x \left| P\left(\frac{\left|\mathcal{S}_{R,t}\right|}{\sqrt{\mathrm{Var}\left(\mathcal{S}_{R,t}\right)}} \geq x \right) - P\left(|\mathcal{N}| \geq x\right) \right|.$$

By Lemmas 1-3, we have $\lim_{R\to\infty} \triangle_R = 0$. Note that

$$\int_{n_0}^{\vartheta(\varepsilon)} h'(r)\mathrm{d}r \approx g(\vartheta(\varepsilon)) = M\varepsilon^{-1/\theta},$$

where $A(x) \approx B(x)$ means that there exist positive constants $C_1 < C_2$ such that $C_1A(x) \le B(x) \le C_2A(x)$. Thus, by Stolz's theorem,

$$\lim_{\varepsilon\to 0} \varepsilon^{1/\theta} \int_{n_0}^{\vartheta(\varepsilon)} h'(r) \left| P\left\{ \frac{|\mathcal{S}_{r,t}|}{\sqrt{\mathrm{Var}(\mathcal{S}_{r,t})}} \ge \varepsilon h^{\theta}(r) \right\} - P\left\{ |\mathcal{N}| \ge \varepsilon h^{\theta}(r) \right\} \right| \mathrm{d}r \le \lim_{\varepsilon\to 0} \varepsilon^{1/\theta} \int_{n_0}^{\vartheta(\varepsilon)} h'(r)\triangle_r \mathrm{d}r = 0.$$

□

Proposition 2.3. *Under the assumptions of Theorem 2.1, for any $\theta > 0$ and fixed $t > 0$, uniformly with respect to $0 < \varepsilon < 1$, we have*

$$\lim_{M\to\infty} \varepsilon^{1/\theta} \int_{\vartheta(\varepsilon)}^{\infty} h'(r) P\left\{ |\mathcal{N}| \ge \varepsilon h^{\theta}(r) \right\} \mathrm{d}r = 0.$$

Proof. Using the change of variable $\tilde{r} = \varepsilon h^{\theta}(r)$, we can get that

$$\lim_{M\to\infty} \varepsilon^{1/\theta} \int_{\vartheta(\varepsilon)}^{\infty} h'(r) P\left\{ |\mathcal{N}| \ge \varepsilon h^{\theta}(r) \right\} \mathrm{d}r = \frac{C}{\theta} \lim_{M\to\infty} \int_{M^{\theta}}^{\infty} \tilde{r}^{\frac{1}{\theta}-1} P\{|\mathcal{N}| \ge \tilde{r}\} \mathrm{d}\tilde{r} = 0.$$

□

Proposition 2.4. *Under the assumptions of Theorem 2.1, for any $\theta > 0$ and fixed $t > 0$, uniformly with respect to $0 < \varepsilon < 1$, we have*

$$\lim_{M\to\infty} \varepsilon^{1/\theta} \int_{\vartheta(\varepsilon)}^{\infty} h'(r) P\left\{ \frac{|\mathcal{S}_{r,t}|}{\sqrt{\mathrm{Var}(\mathcal{S}_{r,t})}} \ge \varepsilon h^{\theta}(r) \right\} \mathrm{d}r = 0.$$

Proof. By Markov inequality, the asymptotic variances (see Lemmas 1-3) and moment bounds for spatial averages (see Lemmas 4-6), for some $q > 2$ such that $q\theta > 1$, we conclude that

$$\begin{aligned}
&\lim_{M\to\infty} \varepsilon^{1/\theta} \int_{\vartheta(\varepsilon)}^{\infty} h'(r) P\left\{ \frac{|\mathcal{S}_{r,t}|}{\sqrt{\mathrm{Var}(\mathcal{S}_{r,t})}} \ge \varepsilon h^{\theta}(r) \right\} \mathrm{d}r \\
&\le \lim_{M\to\infty} \varepsilon^{1/\theta} \int_{\vartheta(\varepsilon)}^{\infty} h'(r) \frac{\mathbb{E}|\mathcal{S}_{r,t}|^q}{\left(\varepsilon h^{\theta}(r)\sqrt{\mathrm{Var}(\mathcal{S}_{r,t})}\right)^q} \mathrm{d}r \\
&\le C \lim_{M\to\infty} \varepsilon^{1/\theta - q} \int_{\vartheta(\varepsilon)}^{\infty} \frac{h'(r)}{g^{\theta q}(r)} \mathrm{d}r \\
&\le C \lim_{M\to\infty} \varepsilon^{1/\theta - q} \int_{M\varepsilon^{-\frac{1}{\theta}}}^{\infty} \frac{1}{\tilde{r}^{\theta q}} \mathrm{d}\tilde{r} \\
&\le C \lim_{M\to\infty} \int_{M}^{\infty} \frac{1}{\bar{r}^{\theta q}} \mathrm{d}\bar{r} \\
&= 0.
\end{aligned}$$

□

Proof. [Proof of Theorem 2.1] Theorem 2.1 is proved by Propositions 2.1-2.4 and the triangular inequality. □

2.2. Proof of Theorem 2.2

Proposition 2.5. *Under the assumptions of Theorem 2.2, for any $\theta > 0$ and fixed $t > 0$, we have*

$$\lim_{\varepsilon \to 0} \frac{1}{-\log \varepsilon} \int_{n_0}^{\infty} \frac{h'(r)}{h(r)} P\left\{|\mathcal{N}| \geq \varepsilon h^{\theta}(r)\right\} \mathrm{d}r = \frac{1}{\theta}.$$

Proof. Using the change of variable $\tilde{r} = \varepsilon h^{\theta}(r)$, we can get that

$$\lim_{\varepsilon \to 0} \frac{1}{-\log \varepsilon} \int_{n_0}^{\infty} \frac{h'(r)}{h(r)} P\left\{|\mathcal{N}| \geq \varepsilon h^{\theta}(r)\right\} \mathrm{d}r = \frac{1}{\theta} \lim_{\varepsilon \to 0} \frac{1}{-\log \varepsilon} \int_{\varepsilon h^{\theta}(n_0)}^{\infty} \frac{1}{\tilde{r}} P\{|\mathcal{N}| \geq \tilde{r}\} \mathrm{d}\tilde{r} = \frac{1}{\theta}.$$

□

Proposition 2.6. *Under the assumptions of Theorem 2.2, for any $M > 1$, $\theta > 0$ and fixed $t > 0$, we have*

$$\lim_{\varepsilon \to 0} \frac{1}{-\log \varepsilon} \int_{n_0}^{\vartheta(\varepsilon)} \frac{h'(r)}{h(r)} \left| P\left\{ \frac{|\mathcal{S}_{r,t}|}{\sqrt{\mathrm{Var}(\mathcal{S}_{r,t})}} \geq \varepsilon h^{\theta}(r) \right\} - P\left\{|\mathcal{N}| \geq \varepsilon h^{\theta}(r)\right\} \right| \mathrm{d}r = 0.$$

Proof. Similarly to Proposition 2.2, we have that

$$\int_{n_0}^{\vartheta(\varepsilon)} \frac{h'(r)}{h(r)} \mathrm{d}r \approx -\log \varepsilon.$$

Thus, by Stolz's theorem,

$$\lim_{\varepsilon \to 0} \frac{1}{-\log \varepsilon} \int_{n_0}^{\vartheta(\varepsilon)} \frac{h'(r)}{h(r)} \left| P\left\{ \frac{|\mathcal{S}_{r,t}|}{\sqrt{\mathrm{Var}(\mathcal{S}_{r,t})}} \geq \varepsilon h^{\theta}(r) \right\} - P\left\{|\mathcal{N}| \geq \varepsilon h^{\theta}(r)\right\} \right| \mathrm{d}r \leq \lim_{\varepsilon \to 0} \frac{1}{-\log \varepsilon} \int_{n_0}^{\vartheta(\varepsilon)} \frac{h'(r)}{h(r)} \triangle_r \mathrm{d}r = 0.$$

□

Proposition 2.7. *Under the assumptions of Theorem 2.2, for any $M > 1$, $\theta > 0$ and fixed $t > 0$, we have*

$$\lim_{\varepsilon \to 0} \frac{1}{-\log \varepsilon} \int_{\vartheta(\varepsilon)}^{\infty} \frac{h'(r)}{h(r)} P\left\{|\mathcal{N}| \geq \varepsilon h^{\theta}(r)\right\} \mathrm{d}r = 0.$$

Proof. Using the change of variable $\tilde{r} = \varepsilon h^{\theta}(r)$, we can get that

$$\lim_{\varepsilon \to 0} \frac{1}{-\log \varepsilon} \int_{\vartheta(\varepsilon)}^{\infty} \frac{h'(r)}{h(r)} P\left\{|\mathcal{N}| \geq \varepsilon h^{\theta}(r)\right\} \mathrm{d}r = \lim_{\varepsilon \to 0} \frac{C}{-\log \varepsilon} \frac{1}{\theta} \int_{M^{\theta}}^{\infty} \frac{1}{\tilde{r}} P\{|\mathcal{N}| \geq \tilde{r}\} \mathrm{d}\tilde{r} \leq \lim_{\varepsilon \to 0} \frac{C}{-\log \varepsilon} = 0.$$

□

Proposition 2.8. *Under the assumptions of Theorem 2.2, for any $M > 1$, $\theta > 0$ and fixed $t > 0$, we have*

$$\lim_{\varepsilon \to 0} \frac{1}{-\log \varepsilon} \int_{\vartheta(\varepsilon)}^{\infty} \frac{h'(r)}{h(r)} P\left\{ \frac{|\mathcal{S}_{r,t}|}{\sqrt{\mathrm{Var}(\mathcal{S}_{r,t})}} \geq \varepsilon h^{\theta}(r) \right\} \mathrm{d}r = 0.$$

Proof. By Markov inequality, the asymptotic variances (see Lemmas 1-3) and moment bounds for spatial averages (see Lemmas 4-6), for some $q > 2$, we conclude that

$$\lim_{\varepsilon \to 0} \frac{1}{-\log \varepsilon} \int_{\vartheta(\varepsilon)}^{\infty} \frac{h'(r)}{h(r)} P\left\{ \frac{|\mathcal{S}_{r,t}|}{\sqrt{\mathrm{Var}(\mathcal{S}_{r,t})}} \geq \varepsilon h^{\theta}(r) \right\} \mathrm{d}r$$

$$
\begin{aligned}
&\le \lim_{\varepsilon\to 0} \frac{1}{-\log \varepsilon} \int_{\vartheta(\varepsilon)}^{\infty} \frac{h'(r)}{h(r)} \frac{\mathbb{E}\left|\mathcal{S}_{r,t}\right|^q}{\left(\varepsilon\sqrt{\mathrm{Var}\,(\mathcal{S}_{r,t})}h^{\theta}(r)\right)^q} \mathrm{d}r \\
&\le C \lim_{\varepsilon\to 0} \frac{\varepsilon^{-q}}{-\log \varepsilon} \int_{\vartheta(\varepsilon)}^{\infty} \frac{h'(r)}{h^{1+\theta q}(r)} \mathrm{d}r \\
&\le C \lim_{\varepsilon\to 0} \frac{\varepsilon^{-q}}{-\log \varepsilon} \int_{M\varepsilon^{-1/\theta}}^{\infty} \frac{1}{\tilde{r}^{1+\theta q}} \mathrm{d}\tilde{r} \\
&\le C \lim_{\varepsilon\to 0} \frac{1}{-\log \varepsilon} \int_{M}^{\infty} \frac{1}{\bar{r}^{1+\theta q}} \mathrm{d}\bar{r} \\
&=0.
\end{aligned}
$$

□

Proof. [Proof of Theorem 2.2] Theorem 2.2 is proved by Propositions 2.5-2.8 and the triangular inequality. □

3. Precise asymptotics for complete moment convergence

The following two results concern the precise asymptotics for complete moment convergence.

Theorem 3.1. *Let $h(x)$ be a positive and differentiable function defined on $[n_0, \infty)$, which is strictly increasing to ∞. Set $\frac{1}{\theta} > p \ge 0$. Let f be a kernel of Cases 1.1-1.3, then for fixed $t > 0$, we have*

$$
\lim_{\varepsilon\to 0} \varepsilon^{\frac{1}{\theta}-p} \int_{n_0}^{\infty} \frac{h'(r)}{h^{p\theta}(r)} \mathbb{E}\left[\left|\frac{\mathcal{S}_{r,t}}{\sqrt{\mathrm{Var}\,(\mathcal{S}_{r,t})}}\right|^p I\left\{\frac{\left|\mathcal{S}_{r,t}\right|}{\sqrt{\mathrm{Var}\,(\mathcal{S}_{r,t})}} \ge \varepsilon h^{\theta}(r)\right\}\right] \mathrm{d}r = \frac{1}{1-p\theta} \mathbb{E}|\mathcal{N}|^{1/\theta}. \tag{3.1}
$$

Theorem 3.2. *Let $h(x)$ be a positive and differentiable function defined on $[n_0, \infty)$, which is strictly increasing to ∞. Let f be a kernel of Cases 1.1-1.3, then for any $\theta > 0$ and fixed $t > 0$, we have*

$$
\lim_{\varepsilon\to 0} \frac{1}{-\log \varepsilon} \int_{n_0}^{\infty} \frac{h'(r)}{h(r)} \mathbb{E}\left[\left|\frac{\mathcal{S}_{r,t}}{\sqrt{\mathrm{Var}\,(\mathcal{S}_{r,t})}}\right|^{1/\theta} I\left\{\frac{\left|\mathcal{S}_{r,t}\right|}{\sqrt{\mathrm{Var}\,(\mathcal{S}_{r,t})}} \ge \varepsilon h^{\theta}(r)\right\}\right] \mathrm{d}r = \frac{1}{\theta} \mathbb{E}|\mathcal{N}|^{1/\theta}. \tag{3.2}
$$

In Theorem 3.1, let $h(x) = x, , n_0 = 1, \theta = \frac{1}{2}, 0 \le p < 2$, we have

Corollary 3.1. *Let f be a kernel of Cases 1.1-1.3, for fixed $t > 0$,*

$$
\lim_{\varepsilon\to 0} \varepsilon^{2-p} \int_{1}^{\infty} r^{-\frac{p}{2}} \mathbb{E}\left[\left|\frac{\mathcal{S}_{r,t}}{\sqrt{\mathrm{Var}\,(\mathcal{S}_{r,t})}}\right|^p I\left\{\frac{\left|\mathcal{S}_{r,t}\right|}{\sqrt{\mathrm{Var}\,(\mathcal{S}_{r,t})}} \ge \varepsilon\sqrt{r}\right\}\right] \mathrm{d}r = \frac{2}{2-p}.
$$

In Theorem 3.1, let $h(x) = (\log x)^{b+1}, n_0 = e, \theta = \frac{1}{2(b+1)}, p = 2$, where $b > 0$, we have

Corollary 3.2. *Let f be a kernel of Cases 1.1-1.3, for fixed $t > 0$,*

$$
\lim_{\varepsilon\to 0} \varepsilon^{2b} \int_{e}^{\infty} \frac{(\log r)^{b-1}}{r} \mathbb{E}\left[\left|\frac{\mathcal{S}_{r,t}}{\sqrt{\mathrm{Var}\,(\mathcal{S}_{r,t})}}\right|^2 I\left\{\frac{\left|\mathcal{S}_{r,t}\right|}{\sqrt{\mathrm{Var}\,(\mathcal{S}_{r,t})}} \ge \varepsilon\sqrt{\log r}\right\}\right] \mathrm{d}r = \frac{1}{b} \mathbb{E}|\mathcal{N}|^{2(b+1)}.
$$

In Theorem 3.2, let $h(x) = x, n_0 = 1, \theta = \frac{1}{2}$, we have

Corollary 3.3. *Let f be a kernel of Cases 1.1-1.3, for fixed $t > 0$,*

$$
\lim_{\varepsilon\to 0} \frac{1}{-\log \varepsilon} \int_{1}^{\infty} \frac{1}{r} \mathbb{E}\left[\left|\frac{\mathcal{S}_{r,t}}{\sqrt{\mathrm{Var}\,(\mathcal{S}_{r,t})}}\right|^2 I\left\{\frac{\left|\mathcal{S}_{r,t}\right|}{\sqrt{\mathrm{Var}\,(\mathcal{S}_{r,t})}} \ge \varepsilon\sqrt{r}\right\}\right] \mathrm{d}r = 2.
$$

3.1. Proof of Theorem 3.1

Proposition 3.1. *Under the assumptions of Theorem 3.1, for* $\frac{1}{\theta} > p > 0$ *and fixed* $t > 0$*, we have*

$$\lim_{\varepsilon\to 0} \varepsilon^{\frac{1}{\theta}-p} \int_{n_0}^{\infty} \frac{h'(r)}{h^{p\theta}(r)} \int_{\varepsilon h^{\theta}(r)}^{\infty} px^{p-1} P\{|\mathcal{N}| \geq x\} \mathrm{d}x \mathrm{d}r = \frac{p\theta}{1-p\theta} \mathbb{E}|\mathcal{N}|^{1/\theta}.$$

Proof. Noting that $\frac{1}{\theta} > p$, using change of variables, we have that

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \varepsilon^{\frac{1}{\theta}-p} \int_{n_0}^{\infty} \frac{h'(r)}{h^{p\theta}(r)} \int_{\varepsilon h^{\theta}(r)}^{\infty} px^{p-1} P\{|\mathcal{N}| \geq x\} \mathrm{d}x \mathrm{d}r \\
=&\lim_{\varepsilon\to 0} \varepsilon^{\frac{1}{\theta}-p} \int_{h(n_0)}^{\infty} \frac{1}{\tilde{r}^{p\theta}} \int_{\varepsilon \tilde{r}^{\theta}}^{\infty} px^{p-1} P\{|\mathcal{N}| \geq x\} \mathrm{d}x \mathrm{d}\tilde{r} \\
=&\lim_{\varepsilon\to 0} \frac{p}{\theta} \int_{\varepsilon h^{\theta}(n_0)}^{\infty} \frac{1}{\bar{r}^{p+1-\frac{1}{\theta}}} \int_{\bar{r}}^{\infty} x^{p-1} P\{|\mathcal{N}| \geq x\} \mathrm{d}x \mathrm{d}\bar{r} \\
=&\frac{p}{\theta} \int_{0}^{\infty} \frac{1}{\bar{r}^{p+1-\frac{1}{\theta}}} \int_{\bar{r}}^{\infty} x^{p-1} P\{|\mathcal{N}| \geq x\} \mathrm{d}x \mathrm{d}\bar{r} \\
=&\frac{p}{\theta} \int_{0}^{\infty} x^{p-1} P\{|\mathcal{N}| \geq x\} \int_{0}^{x} \frac{1}{\bar{r}^{p+1-\frac{1}{\theta}}} \mathrm{d}\bar{r} \mathrm{d}x \\
=&\frac{p\theta}{1-p\theta} \int_{0}^{\infty} \frac{1}{\theta} x^{\frac{1}{\theta}-1} P\{|\mathcal{N}| \geq x\} \mathrm{d}x \\
=&\frac{p\theta}{1-p\theta} \mathbb{E}|\mathcal{N}|^{1/\theta}.
\end{aligned}$$

□

Proposition 3.2. *Under the assumptions of Theorem 3.1, for any* $M > 1$, $\frac{1}{\theta} > p > 0$ *and fixed* $t > 0$*, we have*

$$\lim_{\varepsilon\to 0} \varepsilon^{\frac{1}{\theta}-p} \int_{n_0}^{\vartheta(\varepsilon)} \frac{h'(r)}{h^{p\theta}(r)} \int_{\varepsilon h^{\theta}(r)}^{\infty} px^{p-1} \left[P\left\{ \frac{|\mathcal{S}_{r,t}|}{\sqrt{\mathrm{Var}(\mathcal{S}_{r,t})}} \geq x \right\} - P\{|\mathcal{N}| \geq x\} \right] \mathrm{d}x \mathrm{d}r = 0.$$

Proof. Obviously,

$$\begin{aligned}
&\left| \int_{\varepsilon h^{\theta}(r)}^{\infty} px^{p-1} \left[P\left\{ \frac{|\mathcal{S}_{r,t}|}{\sqrt{\mathrm{Var}(\mathcal{S}_{r,t})}} \geq x \right\} - P\{|\mathcal{N}| \geq x\} \right] \mathrm{d}x \right| \\
\leq& \int_{0}^{\infty} p\left(x + \varepsilon h^{\theta}(r)\right)^{p-1} \left| P\left\{ \frac{|\mathcal{S}_{r,t}|}{\sqrt{\mathrm{Var}(\mathcal{S}_{r,t})}} \geq \left(x + \varepsilon h^{\theta}(r)\right) \right\} - P\left\{ |\mathcal{N}| \geq \left(x + \varepsilon h^{\theta}(r)\right) \right\} \right| \mathrm{d}x \\
\leq& \int_{0}^{\triangle_r^{-\frac{1}{2p}}} p\left(x + \varepsilon h^{\theta}(r)\right)^{p-1} \left| P\left\{ \frac{|\mathcal{S}_{r,t}|}{\sqrt{\mathrm{Var}(\mathcal{S}_{r,t})}} \geq \left(x + \varepsilon h^{\theta}(r)\right) \right\} - P\left\{ |\mathcal{N}| \geq \left(x + \varepsilon h^{\theta}(r)\right) \right\} \right| \mathrm{d}x \\
&+ \int_{\triangle_r^{-\frac{1}{2p}}}^{\infty} p\left(x + \varepsilon h^{\theta}(r)\right)^{p-1} P\left\{ \frac{|\mathcal{S}_{r,t}|}{\sqrt{\mathrm{Var}(\mathcal{S}_{r,t})}} \geq \left(x + \varepsilon h^{\theta}(r)\right) \right\} \mathrm{d}x \\
&+ \int_{\triangle_r^{-\frac{1}{2p}}}^{\infty} p\left(x + \varepsilon h^{\theta}(r)\right)^{p-1} P\left\{ |\mathcal{N}| \geq \left(x + \varepsilon h^{\theta}(r)\right) \right\} \mathrm{d}x \\
:=&\Phi_1 + \Phi_2 + \Phi_3.
\end{aligned}$$

Since $r \leq \vartheta(\varepsilon)$ implies $\varepsilon h^{\theta}(r) \leq M^{\theta}$, we have

$$\Phi_1 \leq \int_0^{\triangle_r^{-\frac{1}{2p}}} p\left(x+\varepsilon h^{\theta}(r)\right)^{p-1} \triangle_r \mathrm{d}x \leq \triangle_r\left(\triangle_r^{-\frac{1}{2p}}+\varepsilon h^{\theta}(r)\right)^p \leq \left(\triangle_r^{\frac{1}{2p}}+M^{\theta}\triangle_r^{\frac{1}{p}}\right)^p. \tag{3.3}$$

Now we estimate Φ_2. By Markov inequality, the asymptotic variances (see Lemmas 1-3) and moment bounds for spatial averages (see Lemmas 4-6), we have

$$\Phi_2 \leq C\int_{\triangle_r^{-\frac{1}{2p}}}^{\infty} \frac{\mathbb{E}\left|\mathcal{S}_{r,t}\right|^{2+p}}{\left(\sqrt{\mathrm{Var}\left(\mathcal{S}_{r,t}\right)}\right)^{2+p}(x+\varepsilon h^{\theta}(r))^3}\mathrm{d}x \leq C\int_{\triangle_r^{-\frac{1}{2p}}}^{\infty} \frac{1}{(x+\varepsilon h^{\theta}(r))^3}\mathrm{d}x \leq C\triangle_r^{\frac{1}{p}}. \tag{3.4}$$

For Γ_3, by Markov inequality, we have

$$\Phi_3 \leq C\int_{\triangle_r^{-\frac{1}{2p}}}^{\infty} \frac{1}{(x+\varepsilon h^{\theta}(r))^3}\mathrm{d}x \leq C\triangle_r^{\frac{1}{p}}. \tag{3.5}$$

From (3.3)-(3.5) and the fact $\triangle_r \to 0$ as $r \to \infty$, we can get

$$\Phi_1+\Phi_2+\Phi_3 \to 0. \tag{3.6}$$

Then by (3.6) and Stolz's theorem,

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \varepsilon^{\frac{1}{\theta}-p}\int_{n_0}^{\vartheta(\varepsilon)} \frac{h'(r)}{h^{p\theta}(r)}\int_{\varepsilon h^{\theta}(r)}^{\infty} px^{p-1}\left[P\left\{\frac{\left|\mathcal{S}_{r,t}\right|}{\sqrt{\mathrm{Var}\left(\mathcal{S}_{r,t}\right)}} \geq x\right\} - P\{|\mathcal{N}| \geq x\}\right]\mathrm{d}x\mathrm{d}r \\
\leq &\lim_{\varepsilon\to 0} \varepsilon^{\frac{1}{\theta}-p}\int_{n_0}^{\vartheta(\varepsilon)} \frac{h'(r)}{h^{p\theta}(r)}(\Phi_1+\Phi_2+\Phi_3)\mathrm{d}r \\
=&0.
\end{aligned}$$

□

Proposition 3.3. *Under the assumptions of Theorem 3.1, for any* $\frac{1}{\theta} > p > 0$ *and fixed* $t > 0$*, uniformly with respect to* $0 < \varepsilon < 1$*, we have*

$$\lim_{M\to\infty} \varepsilon^{\frac{1}{\theta}-p}\int_{\vartheta(\varepsilon)}^{\infty} \frac{h'(r)}{h^{p\theta}(r)}\int_{\varepsilon h^{\theta}(r)}^{\infty} px^{p-1}P\left\{\frac{\left|\mathcal{S}_{r,t}\right|}{\sqrt{\mathrm{Var}\left(\mathcal{S}_{r,t}\right)}} \geq x\right\}\mathrm{d}x\mathrm{d}r = 0.$$

Proof. By Markov inequality, the asymptotic variances and moment bounds (see Lemmas 1-3) for spatial averages (see Lemmas 4-6), for some $q > 2+p$ such that $q\theta > 1$, we have that

$$\begin{aligned}
&\lim_{M\to\infty} \varepsilon^{\frac{1}{\theta}-p}\int_{\vartheta(\varepsilon)}^{\infty} \frac{h'(r)}{h^{p\theta}(r)}\int_{\varepsilon h^{\theta}(r)}^{\infty} px^{p-1}P\left\{\frac{\left|\mathcal{S}_{r,t}\right|}{\sqrt{\mathrm{Var}\left(\mathcal{S}_{r,t}\right)}} \geq x\right\}\mathrm{d}x\mathrm{d}r \\
\leq &\lim_{M\to\infty} C\varepsilon^{\frac{1}{\theta}-p}\int_{\vartheta(\varepsilon)}^{\infty} \frac{h'(r)}{h^{p\theta}(r)}\int_{\varepsilon h^{\theta}(r)}^{\infty} px^{p-1}\frac{\mathbb{E}\left|\mathcal{S}_{r,t}\right|^q}{\left(\mathrm{Var}\left(\mathcal{S}_{r,t}\right)\right)^{\frac{q}{2}}x^q}\mathrm{d}x\mathrm{d}r \\
\leq &\lim_{M\to\infty} C\varepsilon^{\frac{1}{\theta}-p}\int_{\vartheta(\varepsilon)}^{\infty} \frac{h'(r)}{h^{p\theta}(r)}\int_{\varepsilon h^{\theta}(r)}^{\infty} \frac{1}{x^{q-p+1}}\mathrm{d}x\mathrm{d}r \\
\leq &\lim_{M\to\infty} C\varepsilon^{\frac{1}{\theta}-q}\int_{\vartheta(\varepsilon)}^{\infty} \frac{h'(r)}{h^{q\theta}(r)}\mathrm{d}r
\end{aligned}$$

$$
\begin{aligned}
&\leq \lim_{M\to\infty} C\varepsilon^{\frac{1}{\theta}-q}\int_{M\varepsilon^{-\frac{1}{\theta}}}^{\infty}\frac{1}{\tilde{r}^{q\theta}}\mathrm{d}\tilde{r}\\
&\leq \lim_{M\to\infty} C\int_{M}^{\infty}\frac{1}{\bar{r}^{q\theta}}\mathrm{d}\bar{r}\\
&=0.
\end{aligned}
$$

□

Proposition 3.4. *Under the assumptions of Theorem 3.1, for any* $\frac{1}{\theta} > p > 0$ *and fixed* $t > 0$*, uniformly with respect to* $0 < \varepsilon < 1$*, we have*

$$
\lim_{M\to\infty}\varepsilon^{\frac{1}{\theta}-p}\int_{\vartheta(\varepsilon)}^{\infty}\frac{h'(r)}{h^{p\theta}(r)}\int_{\varepsilon h^{\theta}(r)}^{\infty}px^{p-1}P\{|\mathcal{N}|\geq x\}\mathrm{d}x\mathrm{d}r=0.
$$

Proof. Similarly to Proposition 3.3, by Markov inequality, for some $q > 2 + p$ such that $q\theta > 1$, we have that

$$
\begin{aligned}
&\lim_{M\to\infty}\varepsilon^{\frac{1}{\theta}-p}\int_{\vartheta(\varepsilon)}^{\infty}\frac{h'(r)}{h^{p\theta}(r)}\int_{\varepsilon h^{\theta}(r)}^{\infty}px^{p-1}P\{|\mathcal{N}|\geq x\}\mathrm{d}x\mathrm{d}r\\
&\leq \lim_{M\to\infty}C\varepsilon^{\frac{1}{\theta}-p}\int_{\vartheta(\varepsilon)}^{\infty}\frac{h'(r)}{h^{p\theta}(r)}\int_{\varepsilon h^{\theta}(r)}^{\infty}px^{p-1}\frac{\mathbb{E}|\mathcal{N}|^{q}}{x^{q}}\mathrm{d}x\mathrm{d}r\\
&\leq \lim_{M\to\infty}C\varepsilon^{\frac{1}{\theta}-p}\int_{\vartheta(\varepsilon)}^{\infty}\frac{h'(r)}{h^{p\theta}(r)}\int_{\varepsilon h^{\theta}(r)}^{\infty}\frac{1}{x^{q-p+1}}\mathrm{d}x\mathrm{d}r\\
&=0.
\end{aligned}
$$

□

Proof. [Proof of Theorem 3.1] Since

$$
\mathbb{E}\left[\left|\frac{\mathcal{S}_{r,t}}{\sqrt{\mathrm{Var}\,(\mathcal{S}_{r,t})}}\right|^{p}I\left\{\frac{|\mathcal{S}_{r,t}|}{\sqrt{\mathrm{Var}\,(\mathcal{S}_{r,t})}}\geq\varepsilon h^{\theta}(r)\right\}\right]=P\left\{\frac{|\mathcal{S}_{r,t}|}{\sqrt{\mathrm{Var}\,(\mathcal{S}_{r,t})}}\geq\varepsilon h^{\theta}(r)\right\},
$$

when $p = 0$, by (2.2) we can get Theorem 3.1. Therefore we just need to discuss the case $\frac{1}{\theta} > p > 0$. Note that

$$
\begin{aligned}
&\int_{n_0}^{\infty}\frac{h'(r)}{h^{p\theta}(r)}\mathbb{E}\left[\left|\frac{\mathcal{S}_{r,t}}{\sqrt{\mathrm{Var}\,(\mathcal{S}_{r,t})}}\right|^{p}I\left\{\frac{|\mathcal{S}_{r,t}|}{\sqrt{\mathrm{Var}\,(\mathcal{S}_{r,t})}}\geq\varepsilon h^{\theta}(r)\right\}\right]\mathrm{d}r\\
&=\varepsilon^{p}\int_{n_0}^{\infty}h'(r)P\left\{\frac{|\mathcal{S}_{r,t}|}{\sqrt{\mathrm{Var}\,(\mathcal{S}_{r,t})}}\geq\varepsilon h^{\theta}(r)\right\}\mathrm{d}r+\int_{n_0}^{\infty}\frac{h'(r)}{h^{p\theta}(r)}\int_{\varepsilon h^{\theta}(r)}^{\infty}px^{p-1}P\left\{\frac{|\mathcal{S}_{r,t}|}{\sqrt{\mathrm{Var}\,(\mathcal{S}_{r,t})}}\geq x\right\}\mathrm{d}x\mathrm{d}r.
\end{aligned}
$$

From Theorem 2.1, it suffices to show

$$
\lim_{\varepsilon\to 0}\varepsilon^{\frac{1}{\theta}-p}\int_{n_0}^{\infty}\frac{h'(r)}{h^{p\theta}(r)}\int_{\varepsilon h^{\theta}(r)}^{\infty}px^{p-1}P\left\{\frac{|\mathcal{S}_{r,t}|}{\sqrt{\mathrm{Var}\,(\mathcal{S}_{r,t})}}\geq x\right\}\mathrm{d}x\mathrm{d}r=\frac{p\theta}{1-p\theta}\mathbb{E}|\mathcal{N}|^{1/\theta}. \tag{3.7}
$$

(3.7) can be proved by Propositions 3.1-3.4 and the triangular inequality. □

3.2. Proof of Theorem 3.2

Proposition 3.5. *Under the assumptions of Theorem 3.2, for any $\theta > 0$ and fixed $t > 0$, we have*

$$\lim_{\varepsilon\to 0}\frac{1}{-\log\varepsilon}\int_{n_0}^{\infty}\frac{h'(r)}{h(r)}\int_{\varepsilon h^{\theta}(r)}^{\infty}\frac{1}{\theta}x^{\frac{1}{\theta}-1}P\{|\mathcal{N}|\ge x\}\mathrm{d}x\mathrm{d}r=\frac{1}{\theta}\mathbb{E}|\mathcal{N}|^{1/\theta}.$$

Proof. Using change of variables, we conclude that

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\frac{1}{-\log\varepsilon}\int_{n_0}^{\infty}\frac{h'(r)}{h(r)}\int_{\varepsilon h^{\theta}(r)}^{\infty}\frac{1}{\theta}x^{\frac{1}{\theta}-1}P\{|\mathcal{N}|\ge x\}\mathrm{d}x\mathrm{d}r\\
=&\lim_{\varepsilon\to 0}\frac{1}{-\log\varepsilon}\int_{h(n_0)}^{\infty}\frac{1}{\tilde{r}}\int_{\varepsilon\tilde{r}^{\theta}}^{\infty}\frac{1}{\theta}x^{\frac{1}{\theta}-1}P\{|\mathcal{N}|\ge x\}\mathrm{d}x\mathrm{d}\tilde{r}\\
=&\lim_{\varepsilon\to 0}\frac{1}{-\log\varepsilon}\frac{1}{\theta}\int_{\varepsilon h^{\theta}(n_0)}^{\infty}\frac{1}{\bar{r}}\int_{\bar{r}}^{\infty}\frac{1}{\theta}x^{\frac{1}{\theta}-1}P\{|\mathcal{N}|\ge x\}\mathrm{d}x\mathrm{d}\bar{r}\\
=&\lim_{\varepsilon\to 0}\frac{1}{\theta}\int_{\varepsilon h^{\theta}(n_0)}^{\infty}\frac{1}{\theta}x^{\frac{1}{\theta}-1}P\{|\mathcal{N}|\ge x\}\mathrm{d}x\\
=&\frac{1}{\theta}\mathbb{E}|\mathcal{N}|^{1/\theta}.
\end{aligned}$$

□

Proposition 3.6. *Under the assumptions of Theorem 3.2, for any $M > 1$, $\theta > 0$ and fixed $t > 0$, we have*

$$\lim_{\varepsilon\to 0}\frac{1}{-\log\varepsilon}\int_{n_0}^{\vartheta(\varepsilon)}\frac{h'(r)}{h(r)}\int_{\varepsilon h^{\theta}(r)}^{\infty}\frac{1}{\theta}x^{\frac{1}{\theta}-1}\left[P\left\{\frac{\left|\mathcal{S}_{r,t}\right|}{\sqrt{\mathrm{Var}\left(\mathcal{S}_{r,t}\right)}}\ge x\right\}-P\{|\mathcal{N}|\ge x\}\right]\mathrm{d}x\mathrm{d}r=0.$$

Proof. Obviously,

$$\begin{aligned}
&\left|\int_{\varepsilon h^{\theta}(r)}^{\infty}\frac{1}{\theta}x^{\frac{1}{\theta}-1}\left[P\left\{\frac{\left|\mathcal{S}_{r,t}\right|}{\sqrt{\mathrm{Var}\left(\mathcal{S}_{r,t}\right)}}\ge x\right\}-P\{|\mathcal{N}|\ge x\}\right]\mathrm{d}x\right|\\
=&\left|\int_{0}^{\infty}\frac{1}{\theta}(x+\varepsilon h^{\theta}(r))^{\frac{1}{\theta}-1}\left[P\left\{\frac{\left|\mathcal{S}_{r,t}\right|}{\sqrt{\mathrm{Var}\left(\mathcal{S}_{r,t}\right)}}\ge (x+\varepsilon h^{\theta}(r))\right\}-P\{|\mathcal{N}|\ge (x+\varepsilon h^{\theta}(r))\}\right]\mathrm{d}x\right|\\
\le&\int_{0}^{\triangle_r^{-\frac{\theta}{2}}}\frac{1}{\theta}(x+\varepsilon h^{\theta}(r))^{\frac{1}{\theta}-1}\left|P\left\{\frac{\left|\mathcal{S}_{r,t}\right|}{\sqrt{\mathrm{Var}\left(\mathcal{S}_{r,t}\right)}}\ge (x+\varepsilon h^{\theta}(r))\right\}-P\{|\mathcal{N}|\ge (x+\varepsilon h^{\theta}(r))\}\right|\mathrm{d}x\\
&+\int_{\triangle_r^{-\frac{\theta}{2}}}^{\infty}\frac{1}{\theta}(x+\varepsilon h^{\theta}(r))^{\frac{1}{\theta}-1}P\left\{\frac{\left|\mathcal{S}_{r,t}\right|}{\sqrt{\mathrm{Var}\left(\mathcal{S}_{r,t}\right)}}\ge (x+\varepsilon h^{\theta}(r))\right\}\mathrm{d}x\\
&+\int_{\triangle_r^{-\frac{\theta}{2}}}^{\infty}\frac{1}{\theta}(x+\varepsilon h^{\theta}(r))^{\frac{1}{\theta}-1}P\{|\mathcal{N}|\ge (x+\varepsilon h^{\theta}(r))\}\mathrm{d}x\\
:=&\Psi_1+\Psi_2+\Psi_3.
\end{aligned}$$

Since $r \le \vartheta(\varepsilon)$ implies $\varepsilon h^{\theta}(r) \le M^{\theta}$, we have

$$\Psi_1\le\int_{0}^{\triangle_r^{-\frac{\theta}{2}}}\frac{1}{\theta}(x+\varepsilon h^{\theta}(r))^{\frac{1}{\theta}-1}\triangle_r\mathrm{d}x\le\triangle_r\left(\triangle_r^{-\frac{\theta}{2}}+\varepsilon h^{\theta}(r)\right)^{\frac{1}{\theta}}\le\left(\triangle_r^{\frac{\theta}{2}}+M^{\theta}\triangle_r^{\theta}\right)^{\frac{1}{\theta}}. \tag{3.8}$$

Now we estimate Ψ_2. By Markov inequality, the asymptotic variances (see Lemmas 1-3) and moment bounds for spatial averages (see Lemmas 4-6), we have

$$\Psi_2 \leq C \int_{\triangle_r^{-\frac{\theta}{2}}}^{\infty} \frac{\mathbb{E}\left|\mathcal{S}_{r,t}\right|^{2+1/\theta}}{\left(\sqrt{\mathrm{Var}\left(\mathcal{S}_{r,t}\right)}\right)^{2+1/\theta}(x+\varepsilon h^{\theta}(r))^3} \mathrm{d}x \leq C\triangle_r^{\theta}. \tag{3.9}$$

For Ψ_3, by Markov inequality, we have

$$\Psi_3 \leq C \int_{\triangle_r^{-\frac{\theta}{2}}}^{\infty} \frac{1}{(x+\varepsilon h^{\theta}(r))^3} \mathrm{d}x \leq C\triangle_r^{\theta}. \tag{3.10}$$

From (3.8)-(3.10) and the fact $\triangle_r \to 0$ as $r \to \infty$, we can get

$$\Psi_1 + \Psi_2 + \Psi_3 \to 0. \tag{3.11}$$

Then by (3.11) and Stolz's theorem,

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \frac{1}{-\log\varepsilon} \int_{n_0}^{\vartheta(\varepsilon)} \frac{h'(r)}{h(r)} \int_{\varepsilon h^{\theta}(r)}^{\infty} \frac{1}{\theta} x^{\frac{1}{\theta}-1} \left[P\left\{ \frac{\left|\mathcal{S}_{r,t}\right|}{\sqrt{\mathrm{Var}\left(\mathcal{S}_{r,t}\right)}} \geq x \right\} - P\{|\mathcal{N}| \geq x\} \right] \mathrm{d}x\mathrm{d}r \\
\leq& \lim_{\varepsilon\to 0} \frac{1}{-\log\varepsilon} \int_{n_0}^{\vartheta(\varepsilon)} \frac{h'(r)}{h(r)} (\Psi_1+\Psi_2+\Psi_3)\mathrm{d}r \\
=&0.
\end{aligned}$$

□

Proposition 3.7. *Under the assumptions of Theorem 3.2, for any $\theta > 0$ and fixed $t > 0$, uniformly with respect to $0 < \varepsilon < 1$, we have*

$$\lim_{M\to\infty} \frac{1}{-\log\varepsilon} \int_{\vartheta(\varepsilon)}^{\infty} \frac{h'(r)}{h(r)} \int_{\varepsilon h^{\theta}(r)}^{\infty} \frac{1}{\theta} x^{\frac{1}{\theta}-1} P\left\{ \frac{\left|\mathcal{S}_{r,t}\right|}{\sqrt{\mathrm{Var}\left(\mathcal{S}_{r,t}\right)}} \geq x \right\} \mathrm{d}x\mathrm{d}r = 0.$$

Proof. By Markov inequality, the asymptotic variances (see Lemmas 1-3) and moment bounds for spatial averages (see Lemmas 4-6), for some $q > 2$ such that $q\theta > 1$, we have that

$$\begin{aligned}
&\lim_{M\to\infty} \frac{1}{-\log\varepsilon} \int_{\vartheta(\varepsilon)}^{\infty} \frac{h'(r)}{h(r)} \int_{\varepsilon h^{\theta}(r)}^{\infty} \frac{1}{\theta} x^{\frac{1}{\theta}-1} P\left\{ \frac{\left|\mathcal{S}_{r,t}\right|}{\sqrt{\mathrm{Var}\left(\mathcal{S}_{r,t}\right)}} \geq x \right\} \mathrm{d}x\mathrm{d}r \\
\leq& \lim_{M\to\infty} \frac{1}{-\log\varepsilon} \int_{\vartheta(\varepsilon)}^{\infty} \frac{h'(r)}{h(r)} \int_{\varepsilon h^{\theta}(r)}^{\infty} \frac{1}{\theta} x^{\frac{1}{\theta}-1} \frac{\mathbb{E}\left|\mathcal{S}_{r,t}\right|^q}{\left(\mathrm{Var}\left(\mathcal{S}_{r,t}\right)\right)^{\frac{q}{2}} x^q} \mathrm{d}x\mathrm{d}r \\
\leq& \lim_{M\to\infty} C \frac{1}{-\log\varepsilon} \int_{\vartheta(\varepsilon)}^{\infty} \frac{h'(r)}{h(r)} \int_{\varepsilon h^{\theta}(r)}^{\infty} \frac{1}{x^{q-\frac{1}{\theta}+1}} \mathrm{d}x\mathrm{d}r \\
\leq& \lim_{M\to\infty} \frac{C\varepsilon^{\frac{1}{\theta}-q}}{-\log\varepsilon} \int_{\vartheta(\varepsilon)}^{\infty} \frac{h'(r)}{h^{q\theta}(r)} \mathrm{d}r \\
\leq& \lim_{M\to\infty} \frac{C\varepsilon^{\frac{1}{\theta}-q}}{-\log\varepsilon} \int_{M\varepsilon^{-\frac{1}{\theta}}}^{\infty} \frac{1}{\tilde{r}^{q\theta}} \mathrm{d}\tilde{r} \\
\leq& \lim_{M\to\infty} C \int_{M}^{\infty} \frac{1}{\bar{r}^{q\theta}} \mathrm{d}\bar{r} \\
=&0.
\end{aligned}$$

□

Proposition 3.8. *Under the assumptions of Theorem 3.2, for any $\theta > 0$ and fixed $t > 0$, uniformly with respect to $0 < \varepsilon < 1$, we have*

$$\lim_{M\to\infty} \frac{1}{-\log\varepsilon} \int_{\vartheta(\varepsilon)}^{\infty} \frac{h'(r)}{h(r)} \int_{\varepsilon h^{\theta}(r)}^{\infty} \frac{1}{\theta} x^{\frac{1}{\theta}-1} P\{|\mathcal{N}| \geq x\} \mathrm{d}x\mathrm{d}r = 0.$$

Proof. Similarly to Proposition 3.7, by Markov inequality, for some $q > 2$ such that $q\theta > 1$, we have that

$$\begin{aligned}
&\lim_{M\to\infty} \frac{1}{-\log\varepsilon} \int_{\vartheta(\varepsilon)}^{\infty} \frac{h'(r)}{h(r)} \int_{\varepsilon h^{\theta}(r)}^{\infty} \frac{1}{\theta} x^{\frac{1}{\theta}-1} P\{|\mathcal{N}| \geq x\} \mathrm{d}x\mathrm{d}r \\
\leq &\lim_{M\to\infty} \frac{1}{-\log\varepsilon} \int_{\vartheta(\varepsilon)}^{\infty} \frac{h'(r)}{h(r)} \int_{\varepsilon h^{\theta}(r)}^{\infty} \frac{1}{\theta} x^{\frac{1}{\theta}-1} \frac{\mathbb{E}|\mathcal{N}|^q}{x^q} \mathrm{d}x\mathrm{d}r \\
\leq &\lim_{M\to\infty} \frac{C}{-\log\varepsilon} \int_{\vartheta(\varepsilon)}^{\infty} \frac{h'(r)}{h(r)} \int_{\varepsilon h^{\theta}(r)}^{\infty} \frac{1}{x^{q-\frac{1}{\theta}+1}} \mathrm{d}x\mathrm{d}r \\
=&0.
\end{aligned}$$

□

Proof. [Proof of Theorem 3.2] Note that

$$\begin{aligned}
&\int_{n_0}^{\infty} \frac{h'(r)}{h(r)} \mathbb{E}\left[\left|\frac{\mathcal{S}_{r,t}}{\sqrt{\mathrm{Var}(\mathcal{S}_{r,t})}}\right|^{1/\theta} I\left\{\frac{|\mathcal{S}_{r,t}|}{\sqrt{\mathrm{Var}(\mathcal{S}_{r,t})}} \geq \varepsilon h^{\theta}(r)\right\}\right] \mathrm{d}r \\
=&\varepsilon^{1/\theta} \int_{n_0}^{\infty} h'(r) P\left\{\frac{|\mathcal{S}_{r,t}|}{\sqrt{\mathrm{Var}(\mathcal{S}_{r,t})}} \geq \varepsilon h^{\theta}(r)\right\} \mathrm{d}r + \int_{n_0}^{\infty} \frac{h'(r)}{h(r)} \int_{\varepsilon h^{\theta}(r)}^{\infty} \frac{1}{\theta} x^{\frac{1}{\theta}-1} P\left\{\frac{|\mathcal{S}_{r,t}|}{\sqrt{\mathrm{Var}(\mathcal{S}_{r,t})}} \geq x\right\} \mathrm{d}x\mathrm{d}r.
\end{aligned}$$

According to Theorem 2.1, it suffices to prove

$$\lim_{\varepsilon\to 0} \frac{1}{-\log\varepsilon} \int_{n_0}^{\infty} \frac{h'(r)}{h(r)} \int_{\varepsilon h^{\theta}(r)}^{\infty} \frac{1}{\theta} x^{\frac{1}{\theta}-1} P\left\{\frac{|\mathcal{S}_{r,t}|}{\sqrt{\mathrm{Var}(\mathcal{S}_{r,t})}} \geq x\right\} \mathrm{d}x\mathrm{d}r = \frac{1}{\theta} \mathbb{E}|\mathcal{N}|^{1/\theta}. \tag{3.12}$$

(3.12) can be proved by Propositions 3.5-3.8 and the triangular inequality. □

Appendix A. Some important lemmas

Appendix A.1. CLTs for SWEs

Recall that the total variance distance between two random variables F and G is defined by

$$d_{\mathrm{TV}}(F, G) := \sup_{B\in\mathcal{B}(\mathbb{R})} |P(F \in B) - P(G \in B)|,$$

where $\mathcal{B}(\mathbb{R})$ is the collection of all Borel sets in $\mathbb{R}$.

Set $\xi(s) = \mathbb{E}\left[\sigma^2(u(s, 0))\right]$, $\eta(s) = \mathbb{E}[\sigma(u(s, 0))]$ and $k_\beta = \int_{B_1^2} |x_1 - x_2|^{-\beta} \mathrm{d}x_1\mathrm{d}x_2$. The following lemmas are useful for the proofs of main results.

Lemma 1. *(Delgado-Vences et al. [5]) Let $u(t, x)$ be the mild solution to (1.1) under Case 1.1, then for all fixed $t > 0$,*

$$d_{\mathrm{TV}}\left(\frac{\mathcal{S}_{R,t}}{\sqrt{\mathrm{Var}\,\mathcal{S}_{R,t}}}, \mathcal{N}\right) \leq CR^{H-1} \quad \textit{for every } R \geq 0.$$

Moreover, if $H = 1/2$,

$$\lim_{R\to\infty} \frac{\mathrm{Var}\,(\mathcal{S}_{R,t})}{R} = \int_0^t (t-s)^2 \xi(s)\mathrm{d}s,$$

if $H \in (1/2, 1)$,

$$\lim_{R\to\infty} \frac{\mathrm{Var}\,(\mathcal{S}_{R,t})}{R^{2H}} = \int_0^t (t-s)^2 \eta^2(s)\mathrm{d}s.$$

Lemma 2. *(Bolaños Guerrero et al. [3]) Let* $u(t,x)$ *be the mild solution to (1.1) under Case 1.2, then for all fixed* $t > 0$,

$$d_{\mathrm{TV}}\left(\frac{\mathcal{S}_{R,t}}{\sqrt{\mathrm{Var}\,\mathcal{S}_{R,t}}}, \mathcal{N}\right) \leq CR^{-\beta/2} \quad \textit{for every } R \geq 0.$$

Moreover,

$$\lim_{R\to\infty} \frac{\mathrm{Var}\,(\mathcal{S}_{R,t})}{R^{4-\beta}} = \kappa_\beta \int_0^t (t-s)^2 \eta^2(s)\mathrm{d}s.$$

Lemma 3. *(Nualart and Zheng [15]) Let* $u(t,x)$ *be the mild solution to (1.1) under Case 1.3, then for all fixed* $t > 0$,

$$d_{\mathrm{TV}}\left(\frac{\mathcal{S}_{R,t}}{\sqrt{\mathrm{Var}\,\mathcal{S}_{R,t}}}, \mathcal{N}\right) \leq CR^{-d/2} \quad \textit{for every } R \geq 0.$$

Moreover,

$$\lim_{R\to\infty} \frac{\mathrm{Var}\,(\mathcal{S}_{R,t})}{R^{d}} = \omega_d \int_{\mathbb{R}^d} \mathrm{Cov}\,(u\,(t,x)\,, u\,(t,0))\,\mathrm{d}x,$$

where ω_d *denotes the volume of the unit ball, that is,* $\omega_d = 2$ *for* $d = 1$ *and* $\omega_d = \pi$ *for* $d = 2$.

Appendix A.2. Moment bounds for spatial averages of SWEs

Set

$$\varphi_{R,t}(s,y) = \int_{B_R} G_{t-s}(x-y)\mathrm{d}x.$$

Lemma 4. *Let* $u(t,x)$ *be the mild solution to (1.1) under Case 1.1. Then for any* $p \geq 2$ *and fixed* $t > 0$,

$$\mathbb{E}\left|\mathcal{S}_{R,t}\right|^p \leq CR^{pH}. \tag{A.1}$$

Proof. Suppose that $H = 1/2$. Recalling the definition of $\mathcal{S}_{R,t}$ and applying Fubini's theorem, we can get

$$\begin{aligned}
\mathcal{S}_{R,t} &= \int_{B_R} (u(t,x) - 1)\mathrm{d}x \\
&= \int_{B_R} \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)\sigma(u(s,y))W(\mathrm{d}s,\mathrm{d}y)\mathrm{d}x \\
&= \int_0^t \int_{\mathbb{R}} \left(\int_{B_R} G_{t-s}(x-y)\sigma(u(s,y))\mathrm{d}x\right) W(\mathrm{d}s,\mathrm{d}y).
\end{aligned}$$

Set $K_p(t) = \sup_{0\leq s\leq t} \sup_{y\in\mathbb{R}} \|\sigma(u(s,y))\|_p$. Using Burkholder-Davis-Gundy inequality and Minkowski's inequality, we can get that

$$\mathbb{E}\left|\mathcal{S}_{R,t}\right|^p \leq C\mathbb{E}\left(\int_0^t \int_{\mathbb{R}} \left(\varphi_{R,t}(s,y)\right)^2 \sigma^2(u(s,y))\mathrm{d}y\mathrm{d}s\right)^{\frac{p}{2}}$$

$$\leq C\left(\int_0^t \int_{\mathbb{R}} (\varphi_{R,t}(s,y))^2 \|\sigma(u(s,y))\|_p^2 \mathrm{d}y\mathrm{d}s\right)^{\frac{p}{2}}$$
$$\leq CK_p^p(t)\left(\int_0^t \int_{\mathbb{R}} (\varphi_{R,t}(s,y))^2 \,\mathrm{d}y\mathrm{d}s\right)^{\frac{p}{2}}.$$

Notice that $2\varphi_{R,t}(s,y)$ is the length of $[-R,R]\cap[y-t+s,y+t-s]$, so

$$\varphi_{R,t}(s,y)=\frac{1}{2}([R\wedge(y+t-s)]-[-R\vee(y-t+s)])_+.$$

As a consequence, we deduce that

$$\varphi_{R,t}(s,y)=0, \text{if } |y|\geq R+t-s \text{ and } \varphi_{R,t}(s,y)\leq R\wedge(t-s). \tag{A.2}$$

Hence, we have that

$$\mathbb{E}\left|\mathcal{S}_{R,t}\right|^p \leq C\left(\int_0^t \int_{|y|\leq R+t-s} (t-s)^2 \mathrm{d}y\mathrm{d}s\right)^{\frac{p}{2}} \leq CR^{\frac{p}{2}}.$$

Suppose that $H\in(1/2,1)$. Using Burkholder-Davis-Gundy inequality and Minkowski's inequality, we can get that

$$\mathbb{E}\left(\left|\mathcal{S}_{R,t}\right|^p\right)$$
$$\leq C\mathbb{E}\left(\int_0^t \int_{\mathbb{R}^{2d}} \varphi_{R,t}(s,y)\varphi_{R,t}(s,y')\sigma(u(s,y))\sigma(u(s,y'))\left|y-y'\right|^{2H-2}\mathrm{d}y\mathrm{d}y'\mathrm{d}s\right)^{\frac{p}{2}}$$
$$\leq C\left(\int_0^t \int_{\mathbb{R}^{2d}} \varphi_{R,t}(s,y)\varphi_{R,t}(s,y')\left\|\sigma(u(s,y))\sigma\left(u\left(s,y'\right)\right)\right\|_{p/2}\left|y-y'\right|^{2H-2}\mathrm{d}y\mathrm{d}y'\mathrm{d}s\right)^{\frac{p}{2}}$$
$$\leq C\left(\int_0^t \int_{\mathbb{R}^{2d}} \varphi_{R,t}(s,y)\varphi_{R,t}(s,y')\left|y-y'\right|^{2H-2}\mathrm{d}y\mathrm{d}y'\mathrm{d}s\right)^{\frac{p}{2}}$$
$$\leq C\left(\int_0^t \int_{|y'|\leq R+t-s}\int_{|y|\leq R+t-s} (t-s)^2\left|y-y'\right|^{2H-2}\mathrm{d}y\mathrm{d}y'\mathrm{d}s\right)^{\frac{p}{2}}$$

where in the penultimate line we have used the fact that $\left\|\sigma(u(s,y))\sigma\left(u\left(s,y'\right)\right)\right\|_{p/2}$ is uniformly bounded. Using the fact that (see (3.10) in Huang et al. [8])

$$\sup_{z\in\mathbb{R}}\int_{B_2}|y+z|^{2H-2}\mathrm{d}y<+\infty, \tag{A.3}$$

we can get

$$\int_{|y|\leq R+t-s}\int_{|y'|\leq R+t-s}|y'-y|^{2H-2}\mathrm{d}y'\mathrm{d}y$$
$$=\left(\frac{R+t-s}{2}\right)^{2H-1}\int_{|y|\leq R+t-s}\int_{B_2}\left|\tilde{y}-\frac{2y}{R+t-s}\right|^{2H-2}\mathrm{d}\tilde{y}\mathrm{d}y$$
$$\leq C(R+t-s)^{2H}\sup_{x\in\mathbb{R}}\int_{B_2}|\tilde{y}-x|^{2H-2}\mathrm{d}\tilde{y}$$
$$\leq CR^{2H}.$$

Therefore the proof is completed. □

Lemma 5. *Let $u(t,x)$ be the mild solution to (1.1) under Case 1.2. Then for any $p \geq 2$ and fixed $t > 0$,*

$$\mathbb{E}\left|\mathcal{S}_{R,t}\right|^p \leq CR^{(4-\beta)p/2}. \tag{A.4}$$

Proof. Notice that by Lemma 2.1 in Bolaños Guerrero et al. [3], for $0 < s < t$, we have

$$\varphi_{R,t}(s,y) \leq (t-s)\mathbf{1}_{\{|y|\leq R+t\}}. \tag{A.5}$$

Similarly to the case $H \in (1/2,1)$ in Lemma 4, we can get that

$$\mathbb{E}\left|\mathcal{S}_{R,t}\right|^p \leq C\left(\int_0^t \int_{\mathbb{R}^4} \varphi_{R,t}(s,y)\varphi_{R,t}(s,y')|y-y'|^{-\beta}\mathrm{d}y\mathrm{d}y'\mathrm{d}s\right)^{p/2} \leq C\left(\int_0^t \int_{|y'|\leq R+t}\int_{|y|\leq R+t} |y-y'|^{-\beta}\mathrm{d}y\mathrm{d}y'\mathrm{d}s\right)^{p/2}$$

Using (3.10) in Huang et al. [8], we can get

$$\begin{aligned}
&\int_{|y'|\leq R+t}\int_{|y|\leq R+t} |y-y'|^{-\beta}\mathrm{d}y\mathrm{d}y' \\
=&\left(\frac{R+t}{2}\right)^{2-\beta}\int_{|y'|\leq R+t}\int_{|\tilde{y}|\leq 2}\left|\tilde{y}-\frac{2y'}{R+t}\right|^{-\beta}\mathrm{d}\tilde{y}\mathrm{d}y' \\
\leq& C(R+t)^{4-\beta}\sup_{x\in\mathbb{R}^2}\int_{B_2}\left|\tilde{y}-x\right|^{-\beta}\mathrm{d}\tilde{y} \\
\leq& C(R+t)^{4-\beta}.
\end{aligned}$$

Therefore the proof is completed. □

Lemma 6. *Let $u(t,x)$ be the mild solution to (1.1) under Case 1.3. Then for any $p \geq 2$ and fixed $t > 0$,*

$$\mathbb{E}\left|\mathcal{S}_{R,t}\right|^p \leq CR^{dp/2}. \tag{A.6}$$

Proof. According to the fact that (see (2.8) and (2.9) in Nualart and Zheng [15])

$$\varphi_{R,t}(s,y) \leq \int_{\mathbb{R}^d} G_{t-s}(x-y)\mathrm{d}x = t-s, \tag{A.7}$$

$$\int_{\mathbb{R}^d}\varphi_{R,t}(s,y)\mathrm{d}y = \int_{B_R}\mathrm{d}x\int_{\mathbb{R}^d}\mathrm{d}yG_{t-s}(x-y) \leq C(t-s)R^d, \tag{A.8}$$

for $0 < s < t$, similarly to the case $H \in (1/2,1)$ in Lemma 4, we can get that

$$\begin{aligned}
\mathbb{E}\left|\mathcal{S}_{R,t}\right|^p \leq& C\left(\int_0^t\int_{\mathbb{R}^{2d}}\varphi_{R,t}(s,y)\varphi_{R,t}(s,y')f\left(y-y'\right)\mathrm{d}y'\mathrm{d}y\mathrm{d}s\right)^{p/2} \\
\leq& C\left(\int_0^t\int_{\mathbb{R}^{2d}}(t-s)\varphi_{R,t}(s,y)f(y-y')\mathrm{d}y'\mathrm{d}y\mathrm{d}s\right)^{p/2} \\
\leq& C\left(\|f\|_{L^1(\mathbb{R}^d)}\int_0^t\int_{\mathbb{R}^d}(t-s)\varphi_{R,t}(s,y)\mathrm{d}y\mathrm{d}s\right)^{p/2} \\
\leq& C\left(R^d\|f\|_{L^1(\mathbb{R}^d)}\int_0^t(t-s)^2\mathrm{d}s\right)^{p/2} \\
\leq& CR^{dp/2}.
\end{aligned}$$

□

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