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Nonlinear skew Lie triple centralizers (derivations) on $*$-algebras

Changjing Li[a,*], Xiaoyi Li[a], Jingxuan Wang[a]

[a] *School of Mathematics and Statistics, Shandong Normal University, Jinan 250014, P. R. China*

Abstract. Let $\mathcal{A}$ be a unital $*$-algebra over the complex field $\mathbb{C}$. In this paper, we prove that every nonlinear skew Lie (triple) centralizer on $\mathcal{A}$ is a linear $*$-centralizer. Under some mild conditions on $\mathcal{A}$, we also prove that a map Φ on $\mathcal{A}$ is a nonlinear skew Lie triple derivation if and only if Φ is an additive $*$-derivation. As applications, nonlinear skew Lie triple derivations on prime $*$-algebras, von Neumann algebras with no central summands of type I_1, factor von Neumann algebras and standard operator algebras are characterized.

1. Introduction

Let $\mathcal{A}$ be a $*$-algebra over the complex field $\mathbb{C}$. For $A, B \in \mathcal{A}$, denote by $[A, B]_* = AB - BA^*$ the skew Lie product of A and B. The skew Lie product is found playing a more and more important role in some research topics, and its study has recently attracted many authors' attention (for example, see[3, 7, 11–15, 19]). The product was extensively studied because it naturally arises in the problem of representing quadratic functionals with sesquilinear functionals (see, for example, [12–14]) and in the problem of characterizing ideals (see, for example, [3, 11]).

Recall that an additive map $\Phi : \mathcal{A} \to \mathcal{A}$ is said to be an additive derivation if $\Phi(AB) = \Phi(A)B + A\Phi(B)$ for all $A, B \in \mathcal{A}$. We say that Φ is an additive $*$-derivation if it is an additive derivation and satisfies $\Phi(A^*) = \Phi(A)^*$ for all $A \in \mathcal{A}$. A map $\Phi : \mathcal{A} \to \mathcal{A}$ is said to be a nonlinear skew Lie derivation if

$$\Phi([A, B]_*) = [\Phi(A), B]_* + [A, \Phi(B)]_*$$

for all $A, B \in \mathcal{A}$. More generally, we say that a map $\Phi : \mathcal{A} \to \mathcal{A}$ is a nonlinear skew Lie triple derivation if

$$\Phi([[A, B]_*, C]_*) = [[\Phi(A), B]_*, C]_* + [[A, \Phi(B)]_*, C]_* + [[A, B]_*, \Phi(C)]_*$$

for all $A, B, C \in \mathcal{A}$. Many authors have paid more attentions on the problem about nonlinear skew Lie derivations and nonlinear skew Lie triple derivations (see [4–6, 9, 10, 17–19, 22]). Yu and Zhang in [19] proved that every nonlinear skew Lie derivation between factor von Neumann algebras is an additive $*$-derivation. In [9], Li et al. showed that if $\mathcal{A} \subseteq B(H)$ is a von Neumann algebra without central abelian projections, then $\Phi : \mathcal{A} \to B(H)$ is a nonlinear skew Lie derivation if and only if Φ is an additive $*$-derivation.

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* Corresponding author: Changjing Li
Email addresses: lcjbxh@163.com (Changjing Li), 1372480897@qq.com (Xiaoyi Li), 190375476@qq.com (Jingxuan Wang)

Li et al. [10] proved that every nonlinear skew Lie triple derivation between factor von Neumann algebras is an additive $*$-derivation. Fu and An [6] proved every nonlinear skew Lie triple derivation between von Neumann algebras without central abelian projections is an additive $*$-derivation. In this paper, we characterize nonlinear skew Lie triple derivations on general $*$-algebras. Our main conclusion generalizes all known results above.

A linear map $\Phi : \mathcal{A} \to \mathcal{A}$ is called a centralizer if $\Phi(AB) = \Phi(A)B = A\Phi(B)$ holds for all $A, B \in \mathcal{A}$. We say that Φ is a $*$-centralizer if it is a centralizer and satisfies $\Phi(A^*) = \Phi(A)^*$ for all $A \in \mathcal{A}$. Centralizers are very important both in theory and applications and have been investigated intensively by many mathematicians (see [2, 8, 20, 21] and references therein). In this paper, we introduced the definition of nonlinear skew Lie (triple) centralizers. Let $\mathcal{A}$ be a $*$-algebra. A map $\Phi : \mathcal{A} \to \mathcal{A}$ is said to be a nonlinear skew Lie centralizer if

$$\Phi([A, B]_*) = [\Phi(A), B]_*$$

for all $A, B \in \mathcal{A}$. More generally, we say that a map $\Phi : \mathcal{A} \to \mathcal{A}$ is a nonlinear skew Lie triple centralizer if

$$\Phi([[A, B]_*, C]_*) = [[\Phi(A), B]_*, C]_*$$

for all $A, B, C \in \mathcal{A}$. Obviously, every $*$-centralizer is a nonlinear Lie (triple) centralizer. In this paper, we prove that every nonlinear skew Lie (triple) centralizer on general unital $*$-algebras is a $*$-centralizer.

2. Nonlinear skew Lie (triple) centralizers

In this section, we will give the characterization of nonlinear skew Lie (triple) centralizers on unital $*$-algebras. The following is our main result in this section.

Theorem 2.1. *Let $\mathcal{A}$ be a unital $*$-algebra with the unit I having the center $\mathcal{Z}(\mathcal{A})$. If a map $\Phi : \mathcal{A} \to \mathcal{A}$ satisfies*

$$\Phi([[A, B]_*, C]_*) = [[\Phi(A), B]_*, C]_*$$

for all $A, B, C \in \mathcal{A}$, then Φ is a $*$-centralizer. Moreover, there exists an element $T = T^* \in \mathcal{Z}(\mathcal{A})$ such that $\Phi(A) = AT = TA$ for all $A \in \mathcal{A}$.

Proof. Since $-\frac{1}{2}iI = [[-\frac{1}{2}iI, -\frac{1}{2}I]_*, -\frac{1}{2}I]_*$, where i is the imaginary unit, we have

$$\begin{aligned}
\Phi(-\frac{1}{2}iI) &= \Phi([[-\frac{1}{2}iI, -\frac{1}{2}I]_*, -\frac{1}{2}I]_*) \\
&= [[\Phi(-\frac{1}{2}iI), -\frac{1}{2}I]_*, -\frac{1}{2}I]_* \\
&= [-\frac{1}{2}\Phi(-\frac{1}{2}iI) + \frac{1}{2}\Phi(-\frac{1}{2}iI)^*, -\frac{1}{2}I]_* \\
&= \frac{1}{2}\Phi(-\frac{1}{2}iI) - \frac{1}{2}\Phi(-\frac{1}{2}iI)^*,
\end{aligned}$$

which implies

$$\Phi(-\frac{1}{2}iI)^* = -\Phi(-\frac{1}{2}iI). \tag{1}$$

Noticing that $-\frac{1}{2}I = [[-\frac{1}{2}iI, -\frac{1}{2}I]_*, \frac{1}{2}iI]_*$, we have

$$\begin{aligned}
\Phi(-\frac{1}{2}I) &= \Phi([[-\frac{1}{2}iI, -\frac{1}{2}I]_*, \frac{1}{2}iI]_*) \\
&= [[\Phi(-\frac{1}{2}iI), -\frac{1}{2}I]_*, \frac{1}{2}iI]_* \\
&= [-\frac{1}{2}\Phi(-\frac{1}{2}iI) + \frac{1}{2}\Phi(-\frac{1}{2}iI)^*, \frac{1}{2}iI]_* \\
&= \frac{1}{2}i\Phi(-\frac{1}{2}iI)^* - \frac{1}{2}i\Phi(-\frac{1}{2}iI).
\end{aligned}$$

By (1), we get

$$\Phi(-\frac{1}{2}I) = -i\Phi(-\frac{1}{2}iI). \tag{2}$$

Let $\lambda \in \mathbb{R}$ be arbitrary, where $\mathbb{R}$ is the real field. Note that

$$\Phi(0) = \Phi([[0,0]_*,0]_*) = [[\Phi(0),0]_*,0]_* = 0.$$

Hence

$$\begin{aligned} 0 &= \Phi([[\lambda I, A]_*, I]_*) \\ &= [[\Phi(\lambda I), A]_*, I]_* \\ &= \Phi(\lambda I)(A + A^*) - (A + A^*)\Phi(\lambda I)^* \end{aligned}$$

holds true for all $A \in \mathcal{A}$. That is,

$$\Phi(\lambda I)(A + A^*) = (A + A^*)\Phi(\lambda I)^*$$

holds true for all $A \in \mathcal{A}$. So

$$\Phi(\lambda I)A = A\Phi(\lambda I)^*$$

holds true for all $A = A^* \in \mathcal{A}$. Since for every $B \in \mathcal{A}$, $B = B_1 + iB_2$ with $B_1 = \frac{B+B^*}{2}$ and $B_2 = \frac{B-B^*}{2i}$, it follows that

$$\Phi(\lambda I)B = B\Phi(\lambda I)^*$$

holds true for all $B \in \mathcal{A}$. Letting $B = I$, we see that

$$\Phi(\lambda I)^* = \Phi(\lambda I). \tag{3}$$

Now we get

$$\Phi(\lambda I)B = B\Phi(\lambda I)$$

holds true for all $B \in \mathcal{A}$. Hence

$$\Phi(\lambda I) \in \mathcal{Z}(\mathcal{A}) \tag{4}$$

for all $\lambda \in \mathbb{R}$. By equation (2), we have

$$\Phi(-\frac{1}{2}iI) \in \mathcal{Z}(\mathcal{A}). \tag{5}$$

For every $A \in \mathcal{A}$, since $iA = [[-\frac{1}{2}iI, -\frac{1}{2}I]_*, A]_*$, by equations (5) and (1), we see that

$$\begin{aligned} \Phi(iA) &= \Phi([[-\frac{1}{2}iI, -\frac{1}{2}I]_*, A]_*) \\ &= [[\Phi(-\frac{1}{2}iI), -\frac{1}{2}I]_*, A]_* \\ &= -\Phi(-\frac{1}{2}iI)A + \Phi(-\frac{1}{2}iI)^*A \\ &= -2\Phi(-\frac{1}{2}iI)A. \end{aligned}$$

Now by equations (2) and (4), we get

$$\Phi(iA) = -2\Phi(-\frac{1}{2}iI)A = -2i\Phi(-\frac{1}{2}I)A = -2iA\Phi(-\frac{1}{2}I). \tag{6}$$

On the other hand, by equation (6), we also have

$$\Phi(I) = \Phi(i(-iI)) = -2i(-iI)\Phi(-\frac{1}{2}I) = -2\Phi(-\frac{1}{2}I). \tag{7}$$

So by equations (6) and (7), we see that

$$\Phi(iA) = i\Phi(I)A = iA\Phi(I). \tag{8}$$

Replacing A by $-iA$ in the above equation, we have

$$\Phi(A) = \Phi(i(-iA)) = \Phi(I)A = A\Phi(I). \tag{9}$$

Furthermore, by by equation (3), we get that

$$\Phi(A)^* = (\Phi(I)A)^* = A^*\Phi(I)^* = A^*\Phi(I) = \Phi(A^*) \tag{10}$$

for all $A \in \mathcal{A}$. Let $T = \Phi(I)$. Then $T = T^* \in \mathcal{Z}(\mathcal{A})$ and $\Phi(A) = AT = TA$ for all $A \in \mathcal{A}$. Hence, Φ is a linear $*$-centralizer. □

Clearly, every nonlinear skew Lie centralizer is a nonlinear skew Lie triple centralizer. Then we have the following corollary.

Corollary 2.2. *Let $\mathcal{A}$ be a unital $*$-algebra with the unit I. If a map $\Phi : \mathcal{A} \to \mathcal{A}$ satisfies*

$$\Phi([A, B]_*) = [\Phi(A), B]_*$$

for all $A, B \in \mathcal{A}$, then Φ is a $*$-centralizer. Moreover, there exists an element $T = T^* \in \mathcal{Z}(\mathcal{A})$ such that $\Phi(A) = AT = TA$ for all $A \in \mathcal{A}$.

3. Nonlinear skew Lie triple derivations

The aim of this section is to characterize nonlinear skew Lie triple derivations on unital $*$-algebras. The following theorem is the main result of this section.

Theorem 3.1. *Let $\mathcal{A}$ be a unital $*$-algebra with the unit I and P be a nontrivial projection in $\mathcal{A}$. Assume that $\mathcal{A}$ satisfies*

$$(\spadesuit) \quad X\mathcal{A}P = 0 \quad \textit{implies} \quad X = 0$$

and

$$(\clubsuit) \quad X\mathcal{A}(I-P) = 0 \quad \textit{implies} \quad X = 0.$$

Then a map $\Phi : \mathcal{A} \to \mathcal{A}$ satisfies

$$\Phi([[A, B]_*, C]_*) = [[\Phi(A), B]_*, C]_* + [[A, \Phi(B)]_*, C]_* + [[A, B]_*, \Phi(C)]_*$$

for all $A, B, C \in \mathcal{A}$ if and only if Φ is an additive $*$-derivation.

In the following, let $P_1 = P$ and $P_2 = I - P$. Denote $\mathcal{A}_{ij} = P_i\mathcal{A}P_j, i, j = 1, 2$. Then $\mathcal{A} = \sum_{i,j=1}^{2} \mathcal{A}_{ij}$. For every $A \in \mathcal{A}$, we may write $A = \sum_{i,j=1}^{2} A_{ij}$. In all that follows, when we write A_{ij}, it indicates that $A_{ij} \in \mathcal{A}_{ij}$. Clearly, we only need to prove the necessity. We will complete the proof by several lemmas.

Lemma 3.2. $\Phi(0) = 0$.

Proof. Indeed,

$$\Phi(0) = \Phi([[0, 0]_*, 0]_*) = [[\Phi(0), 0]_*, 0]_* + [[0, \delta(0)]_*, 0]_* + [[0, 0]_*, \delta(0)]_* = 0.$$

□

Lemma 3.3. $\Phi(I)^* = \Phi(I) \in \mathcal{Z}(\mathcal{A})$.

Proof. It follows from Lemma 3.2 that

$$\begin{aligned}0 &= \Phi([[I, A]_*, I]_*)\\ &= [[\Phi(I), A]_*, I]_* + [[I, \delta(A)]_*, I]_* + [[I, A]_*, \delta(I)]_*\\ &= [[\Phi(I), A]_*, I]_*\\ &= \Phi(I)(A + A^*) - (A + A^*)\Phi(I)^*\end{aligned}$$

holds true for all $A \in \mathcal{A}$. That is,

$$\Phi(I)(A + A^*) = (A + A^*)\Phi(I)^*$$

holds true for all $A \in \mathcal{A}$. So

$$\Phi(I)B = B\Phi(I)^*$$

holds true for all $B = B^* \in \mathcal{A}$. Since for every $C \in \mathcal{A}$, $C = C_1 + iC_2$ with $C_1 = \frac{C+C^*}{2}$ and $C_2 = \frac{C-C^*}{2i}$, it follows that

$$\Phi(I)C = C\Phi(I)^*$$

holds true for all $C \in \mathcal{A}$. Letting $C = I$, we have

$$\Phi(I)^* = \Phi(I).$$

Now we get

$$\Phi(I)C = C\Phi(I)$$

holds true for all $C \in \mathcal{A}$. Hence $\Phi(I) \in \mathcal{Z}(\mathcal{A})$. □

Lemma 3.4. *For all $A = A^* \in \mathcal{A}$, we have $\Phi(A) = \Phi(A)^*$.*

Proof. Using Lemma 3.3, we have that

$$\begin{aligned}0 &= \Phi([[A, I]_*, I]_*)\\ &= [[\Phi(A), I]_*, I]_* + [[A, \Phi(I)]_*, I]_* + [[A, I]_*, \Phi(I)]_*\\ &= [[\Phi(A), I]_*, I]_*\\ &= 2\Phi(A) - 2\Phi(A)^*.\end{aligned}$$

Hence $\Phi(A) = \Phi(A)^*$. □

Lemma 3.5. *For any $A_{ij} \in \mathcal{A}_{ij}, 1 \le i \ne j \le 2$, we have*

$$P_j\Phi(A_{ij})P_i = 0.$$

Proof. Let $A_{ij} \in \mathcal{A}_{ij}, 1 \le i \ne j \le 2$. For any $X_{ij} \in \mathcal{A}_{ij}$, since $0 = [[A_{ij}, X_{ij}]_*, P_j]_*$, we have

$$\begin{aligned}0 &= \Phi([[A_{ij}, X_{ij}]_*, P_j]_*)\\ &= [[\Phi(A_{ij}), X_{ij}]_*, P_j]_* + [[A_{ij}, \Phi(X_{ij})]_*, P_j]_* + [[A_{ij}, X_{ij}]_*, \Phi(P_j)]_*\\ &= \Phi(A_{ij})X_{ij} - X_{ij}\Phi(A_{ij})^*P_j - X_{ij}^*\Phi(A_{ij})^* + P_j\Phi(A_{ij})X_{ij}^*\\ &\quad + A_{ij}\Phi(X_{ij})P_j - P_j\Phi(X_{ij})^*A_{ij}^* - X_{ij}A_{ij}^*\Phi(P_j) + \Phi(P_j)A_{ij}X_{ij}^*.\end{aligned}$$

Multiplying the above equation by P_j from both sides, we get that

$$0 = P_j\Phi(A_{ij})X_{ij} - X_{ij}^*\Phi(A_{ij})^*P_j. \tag{11}$$

Replacing X_{ij} with iX_{ij} in the above equation yields that

$$0 = P_j\Phi(A_{ij})X_{ij} + X_{ij}^*\Phi(A_{ij})^*P_j. \tag{12}$$

Combing (11) and (12), we see that

$$P_j\Phi(A_{ij})X_{ij} = 0$$

for any $X_{ij} \in \mathcal{A}_{ij}$. It follows from (♣) and (♠) that $P_j\Phi(A_{ij})P_i = 0$.
□

Similar to the proof method of Claims 4-8 in [10], we can prove the following lemma.

Lemma 3.6. Φ *is additive.*

Lemma 3.7. $\Phi(I) = 0$.

Proof. For $1 \leq k \neq j \leq 2$, by Lemma 3.4 , we have

$$\begin{aligned} 0 &= \Phi([[iP_k, P_j]_*, P_k]_*) \\ &= [[\Phi(iP_k), P_j]_*, P_k]_* + [iP_k, \Phi(P_j)]_*, P_k]_* + [iP_k, P_j]_*, \Phi(P_k)]_* \\ &= -P_j\Phi(iP_k)^*P_k + P_k\Phi(iP_k)P_j + 2iP_k\Phi(P_j)P_k + i\Phi(P_j)P_k + iP_k\Phi(P_j). \end{aligned}$$

Multiplying both sides of the above equation by P_k, we obtain that

$$0 = P_k\Phi(P_j)P_k, 1 \leq k \neq j \leq 2. \tag{13}$$

For any $A_{jk} \in \mathcal{A}_{jk}, 1 \leq k \neq j \leq 2$, it follows from Lemma 3.6 that

$$\begin{aligned} \Phi(A_{jk}) - \Phi(A_{jk}^*) &= \Phi([[A_{jk}, P_k]_*, P_k]_*) \\ &= [[\Phi(A_{jk}), P_k]_*, P_k]_* + [[A_{jk}, \Phi(P_k)]_*, P_k]_* + [[A_{jk}, P_k]_*, \Phi(P_k)]_* \\ &= \Phi(A_{jk})P_k - P_k\Phi(A_{jk})^*P_k - P_k\Phi(A_{jk})^* + P_k\Phi(A_{jk})P_k \\ &+ A_{jk}\Phi(P_k)P_k - P_k\Phi(P_k)A_{jk}^* + A_{jk}\Phi(P_k) \\ &- A_{jk}^*\Phi(P_k) - \Phi(P_k)A_{jk}^* + \Phi(P_k)A_{jk}. \end{aligned}$$

Multiplying both sides of the above equation by P_j and P_k from the left and right respectively, we obtain that

$$-P_j\Phi(A_{jk}^*)P_k = 2A_{jk}\Phi(P_k)P_k + P_j\Phi(P_k)A_{jk}. \tag{14}$$

It follows from (13) that

$$-P_j\Phi(A_{jk}^*)P_k = 2A_{jk}\Phi(P_k)P_k. \tag{15}$$

By Lemma 3.5, we arrive at

$$0 = A_{jk}\Phi(P_k)P_k \tag{16}$$

for any $A_{jk} \in \mathcal{A}_{jk}$. It follows from (♣) and (♠) that

$$0 = P_k\Phi(P_k)P_k, k = 1, 2. \tag{17}$$

Adding (13) and (17), we get

$$0 = P_k\Phi(I)P_k, k = 1, 2. \tag{18}$$

By Lemma 3.3, we have $0 = P_k\Phi(I), k = 1, 2$. Hence $\Phi(I) = 0$.
□

Lemma 3.8. *For all $A \in \mathcal{A}$, we have $\Phi(A^*) = \Phi(A)^*$.*

Proof. Using Lemma 3.6 and Lemma 3.7 , we have that

$$\begin{aligned} 2\Phi(A) - 2\Phi(A^*) &= \Phi(2A - 2A^*) \\ &= \Phi([[A, I]_*, I]_*) \\ &= [[\Phi(A), I]_*, I]_* \\ &= 2\Phi(A) - 2\Phi(A)^*. \end{aligned}$$

Hence $\Phi(A^*) = \Phi(A)^*$. □

Lemma 3.9. $\Phi(iI) = 0$.

Proof. By Lemma 3.7 and Lemma 3.8 , we see that

$$\begin{aligned} 0 &= -4\Phi(I) = \Phi(-4I) = \Phi([[iI, I]_*, iI]_*) \\ &= [[\Phi(iI), I]_*, iI]_* + [[iI, I]_*, \Phi(iI)]_* \\ &= 8i\Phi(iI). \end{aligned}$$

So $\Phi(iI) = 0$. □

Lemma 3.10. *For all $A \in \mathcal{A}$, we have $\Phi(iA) = i\Phi(A)$.*

Proof. By Lemma 3.7 and Lemma 3.9 , we obtain

$$\begin{aligned} 4\Phi(iA) &= \Phi(4iA) = \Phi([[iI, I]_*, A]_*) \\ &= [[iI, I]_*, \Phi(A)]_* \\ &= 4i\Phi(A). \end{aligned}$$

So $\Phi(iA) = i\Phi(A)$. □

Lemma 3.11. Φ *is a derivation.*

Proof. For all $A, B \in \mathcal{A}$, on one hand, by Lemma 3.9 and Lemma 3.10, we have

$$\begin{aligned} 2i\Phi(AB) + 2i\Phi(BA^*) &= \Phi(2i(AB + BA^*)) \\ &= \Phi([[iI, A]_*, B]_*) \\ &= [[iI, \Phi(A)]_*, B]_* + [[iI, A]_*, \Phi(B)]_* \\ &= 2i(\Phi(A)B + A\Phi(B) + \Phi(B)A^* + B\Phi(A)^*). \end{aligned}$$

From this, we get

$$\Phi(AB) + \Phi(BA^*) = \Phi(A)B + A\Phi(B) + \Phi(B)A^* + B\Phi(A)^*. \tag{19}$$

On the other hand, by Lemma 3.10 and equation (19), we also have

$$\begin{aligned} \Phi(AB) - \Phi(BA^*) &= \Phi((iA)(-iB)) + \Phi((-iB)(iA)^*) \\ &= \Phi(iA)(-iB) + (iA)\Phi(-iB) + \Phi(-iB)(iA)^* + (-iB)\Phi(iA)^* \\ &= \Phi(A)B + A\Phi(B) - \Phi(B)A^* - B\Phi(A)^*. \end{aligned}$$

From this, we get

$$\Phi(AB) - \Phi(BA^*) = \Phi(A)B + A\Phi(B) - \Phi(B)A^* - B\Phi(A)^*. \tag{20}$$

Summing (19) with (20), we get $\Phi(AB) = \Phi(A)B + A\Phi(B)$. □

Recall that an algebra $\mathcal{A}$ is prime if $A\mathcal{A}B = \{0\}$ for $A, B \in \mathcal{A}$ implies either $A = 0$ or $B = 0$. It is easy to see that prime $*$-algebras satisfy (♠) and (♣). Applying Theorem 3.1 to prime $*$-algebras, we have the following corollary.

Corollary 3.12. *Let $\mathcal{A}$ be a prime $*$-algebra with unit I and P be a nontrivial projection in $\mathcal{A}$. Then a map $\Phi : \mathcal{A} \to \mathcal{A}$ satisfies*

$$\Phi([[A, B]_*, C]_*) = [[\Phi(A), B]_*, C]_* + [[A, \Phi(B)]_*, C]_* + [[A, B]_*, \Phi(C)]_*$$

for all $A, B, C \in \mathcal{A}$ if and only if Φ is an additive $*$-derivation.

Let $B(\mathcal{H})$ be the algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$ and $\mathcal{F}(H) \subseteq B(\mathcal{H})$ be the subalgebra of all bounded finite rank operators. A subalgebra $\mathcal{A} \subseteq B(\mathcal{H})$ is called a standard operator algebra if it contains $\mathcal{F}(\mathcal{H})$. Now we have the following corollary.

Corollary 3.13. *Let $\mathcal{A}$ be a standard operator algebra on an infinite dimensional complex Hilbert space $\mathcal{H}$ containing the identity operator I. Suppose that $\mathcal{A}$ is closed under the adjoint operation. Then $\Phi : \mathcal{A} \to \mathcal{A}$ satisfies*

$$\Phi([[A, B]_*, C]_*) = [[\Phi(A), B]_*, C]_* + [[A, \Phi(B)]_*, C]_* + [[A, B]_*, \Phi(C)]_*$$

for all $A, B, C \in \mathcal{A}$ if and only if Φ is a linear $*$-derivation. Moreover, there exists an operator $T \in B(\mathcal{H})$ satisfying $T + T^* = 0$ such that $\Phi(A) = AT - TA$ for all $A \in A$, i.e., Φ is inner.

Proof. Since $\mathcal{A}$ is prime, we have that Φ is an additive $*$-derivation. It follows from [16] that Φ is a linear inner derivation, i.e., there exists an operator $S \in B(\mathcal{H})$ such that $\Phi(A) = AS - SA$. Since $\Phi(A^*) = \Phi(A)^*$, we have

$$A^*S - SA^* = \Phi(A^*) = \Phi(A)^* = -A^*S^* + S^*A^*$$

for all $A \in A$. Hence $A^*(S + S^*) = (S + S^*)A^*$, and then $S + S^* = \lambda I$ for some $\lambda \in \mathbb{R}$. Let $T = S - \frac{1}{2}\lambda I$. It is easy to see that $T + T^* = 0$ such that $\Phi(A) = AT - TA$. □

A von Neumann algebra $\mathcal{M}$ is a weakly closed, self-adjoint algebra of operators on a Hilbert space $\mathcal{H}$ containing the identity operator I. $\mathcal{M}$ is a factor von Neumann algebra if its center only contains the scalar operators. It is well known that a factor von Neumann algebra is prime. Now we have the following corollary.

Corollary 3.14. *[10] Let $\mathcal{M}$ be a factor von Neumann algebra with dim$(\mathcal{M}) \geq 2$. Then a map $\Phi : \mathcal{M} \to \mathcal{M}$ satisfies*

$$\Phi([[A, B]_*, C]_*) = [[\Phi(A), B]_*, C]_* + [[A, \Phi(B)]_*, C]_* + [[A, B]_*, \Phi(C)]_*$$

if and only if Φ is an additive $*$-derivation.

It is shown in [9] that if a von Neumann algebra has no central summands of type I_1, then $\mathcal{M}$ satifies (♠) and (♣). Now we have the following corollary.

Corollary 3.15. *[6] Let $\mathcal{M}$ be a von Neumann algebra with no central summands of type I_1. Then a map $\Phi : \mathcal{M} \to \mathcal{M}$ satisfies*

$$\Phi([[A, B]_*, C]_*) = [[\Phi(A), B]_*, C]_* + [[A, \Phi(B)]_*, C]_* + [[A, B]_*, \Phi(C)]_*$$

if and only if Φ is an additive $*$-derivation.

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