Filomat 38:18 (2024), 6423–6432
https://doi.org/10.2298/FIL2418423H

Published by Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Structure of rings via pseudo-projective modules

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Abstract. The aim of this paper is to describe pseudo-projective modules over some important classes of rings. It is shown that a ring is semilocal if and only if every finitely generated module with Jacobson radical zero is pseudo-projective. We shall study the structure of artinian principal ideal rings. It is shown that a ring is an artinian principal ideal ring if and only if the class of pseudo-injective modules and the class of pseudo-projective modules coincide.

1. Introduction

Throughout this article all rings are associative rings with unity and all modules are right unital modules over a ring. We denote by $|X|$ the cardinality of a set X. For a submodule N of M, we write $N \leq M$ ($N < M, N \leq^e M$) if N is a submodule of M (respectively, a proper submodule, an essential submodule). We denote by $J(M)$ and $Soc(M)$ the radical of the module M and the socle of M, respectively. We also denote by $E(M)$ the injective envelope of M. For any term not defined here the reader is referred to [3], [8] and [19].

A module M is called *pseudo-injective* (resp., *quasi-injective*) if every monomorphism (resp., homomorphism) from each submodule of M to M is extended to an endomorphism of M ([13], [14]). It is well-known that M is pseudo-injective if M is invariant under all automorphisms of its injective envelope ([26]). These modules are called *automorphism-invariant* ([18]). Some properties and the structure of rings via automorphism invariant modules are studied ([1, 2, 12, 15, 16, 23, 27, 29]). Dualizing the notion of a pseudo-injective module, the authors ([28]) introduced pseudo-projective modules. A module M is called *pseudo-projective* (resp., *quasi-projective*) if every epimorphism (resp., homomorphism) from M to each quotient module of M can be lifted to an endomorphism of M. One can check that every quasi-projective module is pseudo-projective. But the converse is not true in general (see Example 2.1). It is well-known that over a right perfect ring, a module is pseudo-projective if and only if it is coinvariant under all automorphisms of its projective cover ([26]). This is not natural as pseudo-injective modules, because of the existence of injective envelopes and projective covers. In fact, every module has an injective envelope. But, it is not true for projective covers. It means that there are modules with no projective covers. Moreover, every module has a projective cover over perfect rings and every finitely generated module has a projective cover over semiperfect rings.

2020 *Mathematics Subject Classification*. 16D40; 16D50; 16D60; 16D70; 16D80
Keywords. pseudo-projective module, pseudo-injective module, quasi-Frobenius ring, serial ring, principal ideal ring
Received: 28 September 2023; Revised: 20 January 2024; Accepted: 02 February 2024
Communicated by Dijana Mosić
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In this paper, we study pseudo-projective modules over the classical rings. The structure of the paper is as follows: In Section 1, we will give the basic concepts and some known results that are used or cited throughout in this paper. In Section 2, we study the structure of semiperfect and perfect rings via the pseudo-projectivity of modules. It is shown that a ring R is right perfect if and only if every flat right R-module is pseudo-projective, if and only if $M \oplus F$ has a pseudo-projective cover for each semisimple right R-module M and free right R-module F (see Theorem 2.8). In Section 3, we characterize of (semi)hereditary rings via finitely generated submodules of pseudo-projective flat modules. We show that a ring R is a right semihereditary ring if and only if every finitely generated submodule of a projective right R-module is pseudo-projective (see Theorem 3.1). A ring R is a right semihereditary right S-ring if and only if every finitely generated submodule of a flat right R-module is pseudo-projective (see Theorem 3.3). We also prove that R is a semiprimary hereditary ring if and only if every torsionless right (left) R-module is pseudo-projective (see Theorem 3.10). In Section 4, we study some properties of pseudo-projective modules over semilocal rings. We have some characterizations of the relatively projective of direct summands of pseudo-projective modules. From these, we obtain the structure of semilocal rings via the pseudo-projectivity of modules. It is shown that a ring R is semilocal if and only if every finitely generated right R-module with Jacobson radical zero is pseudo-projective (see Theorem 4.4). A "good" generalization of the class of semisimple artinian rings is the class of principal ideal quasi-Frobenius rings. They are called artinian principal ideal rings. These classes are studied by the famous mathematician of Ring and Module theory as Fuller, Faith, Byrd ([4, 8, 9]). Of course, they are interesting in the structure of classical rings. In these rings, one-sided ideals are principal and the class of injective modules and the class of projective coincide. Moreover, every module over these rings is a direct sum of cyclic uniserial submodules which are isomorphic to one-sided ideals. In [4], Byrd proved that a ring is an artinian principal ideal ring if and only if every quasi-projective is quasi-injective. This is equivalent to every quasi-injective module is quasi-projective. In this paper, we continue study for the coincidence of pseudo-injective modules and pseudo-projective modules. It is shown that a ring is an artinian principal ideal ring if and only if the class of pseudo-injective modules and the class of pseudo-projective coincide (see Theorem 4.9).

2. On (semi)perfect rings

As mentioned in the introduction, every quasi-projective module is pseudo-projective. The following example shows that the converse is not true in the general.

Example 2.1 ([12, Example 5.1], [29, Example 2.1]). *Let* $R = \begin{bmatrix} \mathbb{F}_2 & \mathbb{F}_2 & \mathbb{F}_2 \\ 0 & \mathbb{F}_2 & 0 \\ 0 & 0 & \mathbb{F}_2 \end{bmatrix}$ *where* $\mathbb{F}_2$ *is the field of two elements and* $M = e_{11}R$. *Then,* $\mathrm{Hom}_{\mathbb{F}_2}(M, \mathbb{F}_2)$ *is a pseudo-projective left R-module and it is not quasi-projective.*

Next, we give some basic properties of pseudo-projective modules and pseudo-injective modules. Firstly, Osofsky ([20]) proved that a ring is semisimple artinian if and only if every finitely generated module is injective. It is equivalent to every cyclic injective module ([21]). We also get the similar result for pseudo-projective modules and pseudo-injective modules.

The following lemma follows from [14, Theorem 1] and [28, Lemma 1.3].

Lemma 2.2. *Let R be a ring.*

1. *If* $A \oplus B$ *is a pseudo-projective right R-module, then every epimorphism from A to B splits ([29, Corollary 2.18]).*
2. *If* $A \oplus B$ *is a pseudo-injective right R-module, then every monomorphism from A to B splits.*

Proposition 2.3. *Let R be any ring. The following conditions are equivalent:*

1. *R is semisimple artinian.*
2. *Each finitely generated right R-module is a pseudo-projective right R-module ([29, Proposition 2.20]).*
3. *Each finitely generated right R-module is pseudo-injective.*

Proof. (1)⇒ (2), (3) are obvious.

(2)⇒ (1) by [29, Proposition 2.20].

(3)⇒ (1) Let C be a cyclic right R-module. Take $N = R_R \oplus C$. Then, N is a finitely generated right R-module, and so it is pseudo-injective. It follows, from [13, Theorem 2.2], that C is R_R-injective and so it is injective. We deduce that R is semisimple artinian by the Osofsky's result. □

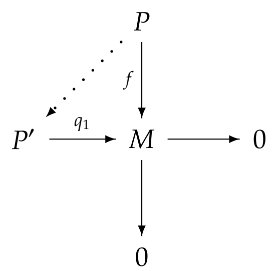
Recall that a module P is a *pseudo-projective cover* (resp., *projective cover*) of a right R-module M if, there exists an epimorphism $p : P \to M$ such that P is pseudo-projective (resp., *projective*) and $\mathrm{Ker}(p)$ is small in P ([29]).

Proposition 2.4. *Let M be a right R-module and let $f : P \to M$ be an epimorphism with P a projective right R-module. The following statements hold:*

1. *M is projective if and only if $P \oplus M$ is pseudo-projective.*
2. *M has a projective cover if and only if $P \oplus M$ has a pseudo-projective cover.*

Proof. (1) is obvious by Lemma 2.2.

(2) If M has a projective cover, then one can check that $P \oplus M$ has a pseudo-projective cover. Assume that $P \oplus M$ has a pseudo-projective cover. We show that M has a projective cover. Let $q : Q \to P \oplus M$ be an epimorphism with small kernel and Q be a pseudo-projective right R-module. Take $\pi : P \oplus M \to P$ the canonical projection. Then, $\pi \circ q : Q \to P$ is an epimorphism, and so it is an splitting epimorphism (since P is projective). There is a monomorphism $\beta : P \to Q$ such that $\pi \circ q \circ \beta = 1_P$, and so $Q = \mathrm{Im}(\beta) \oplus \mathrm{Ker}(\pi \circ q)$. Let $P' = \mathrm{Ker}(\pi \circ q)$ and $q_1 = q|_{P'}$. Then, we have $q_1(P') = q(P') = \mathrm{Ker}(\pi) = M$ which implies that $q_1 : P' \to M$ is an epimorphism. One can check that $\mathrm{Ker}(q_1) = \mathrm{Ker}(q)$, and so $\mathrm{Ker}(q_1)$ is small in P'. Next, we show that P' is projective. We consider the following diagram



Since P is projective, there is a homomorphism $g : P \to P'$ such that the above diagram is commutative. It means that $q_1 \circ g = f$. We have that $\mathrm{Ker}(q_1)$ is small in P' and obtain that g is an epimorphism. Moreover, $Q = \mathrm{Im}(\beta) \oplus P' \cong P \oplus P'$ is pseudo-projective. By Lemma 2.2, g splits and so P' is isomorphic to a direct summand of P. Thus, P' is projective. □

Corollary 2.5 ([28, Theorem 1.16]). *A ring R is right perfect if and only if every right R-module has a pseudo-projective cover.*

Corollary 2.6. *Let R be a ring.*

1. *A ring R is semiperfect if and only if every finitely generated right R-module has a pseudo-projective cover.*
2. *A ring R is semiregular if and only if every finitely presented right R-module has a pseudo-projective cover.*

We denote by $M_n(R)$ the ring of n by n matrices over R. Now applying the same proof of [10, Theorem 3.1], we get the following result.

Proposition 2.7. *The following conditions are equivalent for a ring R:*

1. *R is semiperfect.*
2. *For all $n \geq 1$, every cyclic right $M_n(R)$-module has a pseudo-projective cover.*
3. *There exists an $n > 1$ such that every cyclic right (left) $M_n(R)$-module has a pseudo-projective cover.*

Theorem 2.8. *The following statements are equivalent for a ring R:*

1. *R is right perfect.*
2. *Every flat right R-module is pseudo-projective.*
3. *For any semisimple right R-module M, then $M \oplus F$ has a pseudo-projective cover for each free right R-module F.*

Proof. (1) ⇒ (2) and (1) ⇒ (3) are clear.
(2) ⇒ (1) Let M be a flat right R-module, F a free right R-module and let $f : F \to M$ be an R-epimorphism. We have that $T = M \oplus F$ is flat and obtain that it is a pseudo-projective module by the hypothesis. By Lemma 2.2, f splits, and so M is projective. Hence R is a right perfect ring.
(3) ⇒ (1) Let M be a semisimple right R-module. There exists a free module F and an epimorphism $\psi : F \to M$. By (3), there exists an epimorphism $\phi : X \to F \oplus M$ such that X is a pseudo-projective module and $\mathrm{Ker}(\phi)$ is small in X. Now, by the same proof of (2) of Proposition 2.4, there exists a projective cover of M, as desired. □

A ring R is called a *right S-ring* if every finitely generated flat right R-module is projective ([22]). A ring R is *an S-ring* if it is both a left and right S-ring. Note that every semiperfect ring is an S-ring.

Theorem 2.9. *The following statements are equivalent for a ring R:*

1. *R is a right S-ring.*
2. *Every finitely generated flat right R-module is quasi-projective.*
3. *Every finitely generated flat right R-module is pseudo-projective.*

Proof. This is similar to the proof of Theorem 2.8. □

Recall that R is called a *right PP-ring* if every principal right ideal of R is projective.

Theorem 2.10. *The following statements are equivalent for a ring R:*

1. *R is a right PP-ring.*
2. *Every principal right ideal of $M_2(R)$, generated by a diagonal matrix, is a pseudo-projective right $M_2(R)$-module.*

Proof. (1) ⇒ (2) This follows from [30, Lemma 3].
(2) ⇒ (1) Let $S = M_2(R), a \in R$ and I a principal right ideal of S generated by the diagonal matrix $\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$. As a right S-module, I is a pseudo-projective module. If $e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, then S and R are Morita equivalent via $M \to Me$, where M is a right S-module. Since $Ie \cong aR \oplus R$ as right R-modules and I is a pseudo-projective module, we can obtain that $aR \oplus R$ is pseudo-projective as a right R-module, and so the canonical epimorphism $\eta : R \to aR$ splits. Thus aR is projective and R is a right PP-ring. □

3. On (semi)hereditary rings

A ring R is called a right *(semi)hereditary* if every (finitely generated) right ideal of R is projective, equivalently, if every (finitely generated) submodule of a projective right R-module is projective. A ring R is called *(semi)hereditary* if it is both left and right (semi)hereditary.

Theorem 3.1. *The following conditions are equivalent for a ring R:*

1. *R is a right semihereditary ring.*
2. *Every finitely generated submodule of a projective right R-module is pseudo-projective.*
3. *For any finitely generated free right R-module F, every principal right ideal of $S = \mathrm{End}(F_R)$ is a pseudo-projective right S-module.*

Proof. (1) ⇒ (2) This is obvious.
(2) ⇒ (1) Let K be a finitely generated submodule of a projective right R-module P and let $f : F \to K$ be an R-epimorphism with a finitely generated free right R-module F. Since $F \oplus K$ is a finitely generated submodule of the projective right R-module $F \oplus P$, we can obtain that $F \oplus K$ is a pseudo-projective module by the hypothesis. From Lemma 2.2, it infers that K is projective.
(1) ⇒ (3) This follows from [7, Theorem 2.4].
(3) ⇒ (1) Let F be a finitely generated free right R-module and $S = \text{End}(F_R) \cong M_n(R)$. Then $F \oplus F$ is a free right R-module such that $\text{End}(F \oplus F) = M_2(S)$. By Theorem 2.10, since each principal right ideal of $M_2(S)$ is a pseudo-projective module, S is a right PP-ring. By [7, Theorem 2.4], R is a right semihereditary ring. □

Corollary 3.2. *The following statements are equivalent for a ring R:*

1. *R is a right semihereditary ring.*
2. *Every finitely generated submodule of R^n is pseudo-projective for all integer numbers n.*
3. *Every principal right ideal of $M_n(R)$ is a pseudo-projective module for all integer numbers n.*

Theorem 3.3. *The following conditions are equivalent for a ring R:*

1. *R is a right semihereditary right S-ring.*
2. *R is a left semihereditary left S-ring.*
3. *Every finitely generated submodule of a flat right R-module is pseudo-projective.*
4. *Every finitely generated submodule of a flat left R-module is pseudo-projective.*

Proof. (1) ⇔ (2) This follows from [22, Proposition 4.10].
(1) ⇒ (3) Let K be a finitely generated submodule of a flat right R-module F. Since R is right semihereditary, K is flat by [17, Theorem 4.67] and so it is projective. It follows that K is a pseudo-projective module.
(3) ⇒ (1) This follows from Theorems 2.9 and 3.1.
(2) ⇔ (4) This is similar to (1) ⇔ (3). □

Corollary 3.4. *The following statements are equivalent for a ring R:*

1. *R is a semihereditary S-ring.*
2. *Every finitely generated submodule of a flat right R-module is pseudo-projective.*
3. *Every finitely generated submodule of a flat left R-module is pseudo-projective.*

A right R-module M is called *coherent* if every finitely generated submodule of M is finitely presented. A ring R is called left (right) coherent if R_R (resp., ${}_RR$) is coherent.

Let ΠR_R be an arbitrary product of copies of R_R. According to Camillo [5], a ring R is called right Π-*coherent* if every finitely generated submodule of ΠR_R is finitely presented. A ring R is called Π-*coherent* if it is both left and right Π-coherent.

We say that a right R-module M is called a Π-*pseudo-projective* module if ΠM_R is a pseudo-projective right R-module.

Theorem 3.5. *The following conditions are equivalent for a ring R:*

1. *R is right perfect left coherent.*
2. *The direct product of any family of copies of R is projective as a right R-module.*
3. *All direct products of projective right R-modules are pseudo-projective.*
4. *All direct products of flat right R-modules are pseudo-projective.*
5. *Every projective right R-module is* Π*-pseudo-projective.*

Proof. (1) ⇔ (2) ⇒ (4) ⇒ (3) They follow from [6, Theorem 3.2].
(3) ⇒ (1) Let $M = \prod_{i\in I} M_i$ be a direct product of projective right R-modules and $f : F \to M$ an R-epimorphism with F free. By Proposition 2.4, M is projective since $M \times F \cong M \oplus F$ is a pseudo-projective module. By [6, Theorem 3.3], R is right perfect and left coherent.
(3) ⇒ (5) This is clear.
(5) ⇒ (3) Let $M = \prod\limits_{i\in I} M_i$ be a direct product of projective right R-modules, $f : F \to M$ an R-epimorphism with F free, and $g_i : M \to M_i$ the canonical projection, $i \in I$. Since each M_i is projective, the epimorphism $g_i \circ f : F \to M_i$ splits; i.e., there exist submodules A_i and T_i of F such that $F = A_i \oplus T_i$ and $M_i \cong A_i$ for all $i \in I$. As

$$\prod_{i\in I} F = \prod_{i\in I} (A_i \oplus T_i) \cong (\prod_{i\in I} A_i) \oplus (\prod_{i\in I} T_i)$$

and $\prod\limits_{i\in I} F$ is a pseudo-projective module, it follows that $\prod\limits_{i\in I} A_i$ is a pseudo-projective module. Consequently, $M = \prod\limits_{i\in I} M_i \cong \prod\limits_{i\in I} A_i$ is a pseudo-projective module. □

Corollary 3.6. *The following conditions are equivalent for a ring R:*

1. *R is right perfect left coherent.*
2. *Every free right R-module is* Π*-pseudo-projective.*

A right R-module M is called *torsionless* if it can be embedded in a direct product of copies of R_R.

Theorem 3.7. *The following statements are equivalent for a ring R:*

1. *R is a semihereditary* Π*-coherent ring.*
2. *Every finitely generated torsionless right R-module is projective.*
3. *Every finitely generated torsionless right R-module is pseudo-projective.*
4. *Every finitely generated torsionless left R-module is projective.*
5. *Every finitely generated torsionless left R-module is pseudo-projective.*

Proof. (1) ⇒ (2) This follows from [24, Theorem 3.5].
(2) ⇒ (3) This is obvious.
(3) ⇒ (2) Let M be a finitely generated torsionless right R-module, F a finitely generated free right R-module and $\eta : F \to M$ an R-epimorphism. Since $F \oplus M$ is finitely generated and torsionless, the module M is projective by Lemma 2.2.
(2) ⇒ (1) Since every finitely generated torsionless right R-module is projective, the ring R is right semihereditary right Π-coherent.
(1) ⇔ (4) ⇔ (5) They are symmetric. □

The following observation is a special case of Theorem 3.1 and [7, Theorem 2.3].

Theorem 3.8. *The following conditions are equivalent for a ring R:*

1. *R is right hereditary.*
2. *Every submodule of a projective right R-module is pseudo-projective.*
3. *Every principal right ideal of* $S = End(F_R)$ *is a pseudo-projective module, for any free right R-module F.*

We recall a ring R *semiprimary* if the Jacobson radical $J(R)$ of R is nilpotent and the ring R is semilocal, i.e., $R/J(R)$ has a finite length, equivalently, it is semisimple.

Theorem 3.9. *The following statements are equivalent for a ring R:*

1. *R is a semiprimary hereditary ring.*
2. *R is a right hereditary right perfect ring.*

3. *R is a left hereditary left perfect ring.*
4. *Every submodule of a flat right R-module is pseudo-projective.*
5. *Every submodule of a flat left R-module is pseudo-projective.*

Proof. (2) ⇔ (4) and (3) ⇔ (5) They are consequences of Theorems 2.8 and 3.8.
(1) ⇔ (2) ⇔ (3) They follow from [25, Corollary 2]. □

It is well-known that a ring R is right semihereditary if and only if every torsionless left R-module is flat (see [6, Theorem 4.1]).

Theorem 3.10. *The following conditions are equivalent for a ring R:*

1. *R is a semiprimary hereditary ring.*
2. *Every torsionless right R-module is projective.*
3. *Every torsionless left R-module is projective.*
4. *Every torsionless right R-module is pseudo-projective.*
5. *Every torsionless left R-module is pseudo-projective.*

Proof. (1) ⇒ (2) By Theorem 3.5, the direct product of any family of copies of R is projective as a right R-module. Since R is right hereditary, every submodule of ΠR_R is projective. This means every torsionless right R-module is projective.
(2) ⇒ (4) This is clear.
(4) ⇒ (2) Let M be a torsionless right R-module, F a free right R-module, and $\eta : F \to M$ an R-epimorphism. Since $F \oplus M$ is torsionless, we obtain that M is projective by Lemma 2.2.
(2) ⇒ (1) By the hypothesis and Theorem 3.5, R is a right hereditary right perfect ring. Now, by Theorem 3.9, R is a semiprimary hereditary ring.
(1) ⇔ (3) ⇔ (5) They follow by a symmetrical argument. □

4. On semilocal and artinian principal ideal rings

Recall that a module M is called *semiprimitive* if its Jacobson radical is zero ([11]). Next, we give the structure of rings via semiprimitive finitely generated modules accompanying with the pseudo-projectivity of modules.

Lemma 4.1. *If each semiprimitive finitely generated right R-module is pseudo-projective, then every quotient ring of R has this property.*

Proof. Let S be a quotient ring of R. Assume that M is a semiprimitive finitely generated right S-module. Then M is also a semiprimitive finitely generated right R-module. By the hypothesis, M is a pseudo-projective right R-module. It follows that M is a pseudo-projective right S-module. □

From Proposition 10.15 in [3], we have the following lemma.

Lemma 4.2. *Let M be a right R-module. The following conditions are equivalent:*

1. *M is semiprimitive artinian.*
2. *M is semiprimitive finitely cogenerated.*
3. *M is a semisimple finitely generated module.*

Corollary 4.3. *A semiprimitive artinian module is pseudo-projective and pseudo-injective.*

Theorem 4.4. *The following statements are equivalent for a ring R:*

1. *R is a semilocal ring.*
2. *Each semiprimitive finitely generated right R-module is artinian.*

3. *Each semiprimitive finitely generated right R-module is pseudo-projective.*

It follows from (1) that the conditions (2)-(3) are left right symmetric.

Proof. (1) ⇒ (2). Assume that R is semilocal. We show that every semiprimitive finitely generated right R-module is artinian. In order to conclude the proof we shall show by induction on generated elements of M. Assume that M is generated by n elements. The case $n = 1$, we have M is a cyclic module. This means that $M \cong R/K$ for some right ideal K of R. By assumption, we have $J(R/K) = 0$ or $J(R)$ is contained in K, and so $R/K \cong (R/J(R))/(K/J(R))$. We have that R is a semilocal ring and obtain that $R/J(R)$ is semisimple artinian, and so R/K is semisimple. It follows that R/K is artinian. Suppose now that each semiprimitive right R-module generated by $n = k$ elements is artinian. Call $M = m_1R + m_2R + \cdots + m_{k+1}R$ a semiprimitive finitely generated right R-module. We show that M is artinian. Indeed, we have the following short exact sequence:

$$0 \to m_1R \to M \to M/m_1R \to 0$$

The induction hypothesis can be applied to the modules m_1R and M/m_1R. It follows that m_1R and M/m_1R are artinian modules. Which implies that M is artinian. Thus, it shown that every semiprimitive finitely generated right R-module is artinian. This completes the proof of the theorem.

(2) ⇒ (3) by Corollary 4.3.

(3) ⇒ (1) Let $\bar{R} = R/J(R)$. We show that every simple right $\bar{R}$-module is projective. Indeed, let S be an arbitrary simple right $\bar{R}$-module. Take $M = \bar{R}_{\bar{R}} \oplus S$. Then, M is a semiprimitive finitely generated $\bar{R}$-module. By (3) and Lemma 4.1, we have that M is pseudo-projective. Note that S is an epimorphic image of $\bar{R}_{\bar{R}}$. It follows, from Lemma 2.2, that S is isomorphic to a direct summand of $\bar{R}_{\bar{R}}$, and so S is projective. We deduce that R is a semilocal ring.

□

We do not know that the hypothesis "finitely generated" in Theorem 4.4 is removed or not. We give the following question.

Question 4.5. *If each semiprimitive right R-module is pseudo-projective then R is semilocal or not?*

Next, we study properties of modules over artinian principal ideal rings.

Proposition 4.6 ([4, Proposition 2.1]). *If $A \oplus M$ is quasi-projective whenever A is injective then M is projective. Dually, if $A \oplus M$ is quasi-injective whenever A is projective then M is injective.*

A ring R is called *quasi-Frobenius* if R is one-sided artinan one-sided self-injective (see [8]). We have a characterization of quasi-Frobenius rings via the projectivity and the injectivity of modules:

Theorem 4.7 ([8]). *The following conditions are equivalent for a ring R:*

1. *R is a quasi-Frobenius ring.*
2. *Every injective right R-module is projective.*
3. *Every projective right R-module is injective.*

The authors Faith, Fuller and Byrd demonstrated the following theorem ([4, 8, 9]):

Theorem 4.8. *The following conditions are equivalent for a ring R:*

1. *R is an artinian principal ideal ring.*
2. *Every quasi-injective right R-module is quasi-projective.*
3. *Every quasi-projective right R-module is quasi-injective.*
4. *Every quotient ring of R is a quasi-Frobenius ring.*

For pseudo-projective modules and pseudo-injective modules, we also have the following result.

Theorem 4.9. *The following conditions are equivalent for a ring R:*

1. *R is an artinian principal ideal ring.*
2. *Every pseudo-projective right R-module is quasi-injective.*
3. *Every pseudo-projective right R-module is pseudo-injective.*
4. *Every pseudo-injective right R-module is quasi-projective.*
5. *Every pseudo-injective right R-module is pseudo-projective.*

Proof. (1) ⇒ (2) Assume that R is an artinian principal ideal ring. Let M be a pseudo-projective right R-module. Then, by [8, Proposition 25.4.6B, 25.4.17A] we have $M = \oplus_I M_i$, where each M_i is cyclic uniserial. We show that M is quasi-projective. By [19, Corollary 4.37 and Propositions 4.32, 4.35], it suffices to show that each M_i is M_j-projective for all $i, j \in I$. Note that M is pseudo-projective, M_i is M_j-projective for all $i \neq j$ with $i, j \in I$. We show that M_i is quasi-projective for all $i \in I$. Indeed, we have that R is an artinian principal ideal ring and obtain, from [9, Theorem 5.3], that every indecomposable R-module M_i is quasi-projective. Thus, M is quasi-projective. By Theorem 4.8, M is quasi-injective.

(2) ⇒ (3) and (4) ⇒ (5) are obvious.

(3) ⇒ (1) Let T be a quotient ring of R. Since R has the property of (3), so is T. Firstly, we show that every projective right T-module P is quasi-injective. Indeed, we consider the right T-module $M := P \oplus P$. Clearly, M is a projective right T-module, and so it is pseudo-projective. By assumption, M is pseudo-injective. It follows that P is quasi-injective by [13, Theorem 2.2]. Next, we show that T is a quasi-Frobenius ring. To prove this, we show that every projective right T-module Q is injetive. Indeed, let A be an arbitrary projective right T-module. Then, $A \oplus Q$ is a projective right T-module. Because of the above proof, we have that $A \oplus Q$ is quasi-injective. From Proposition 4.6, it infers that Q is injective. We deduce that R is an artinian principal ideal ring by Theorem 4.8.

(1) ⇒ (4) Assume that R is an artinian principal ideal ring. Let M be a pseudo-injective module. Then, $M = \oplus_I M_i$ where each M_i is cyclic uniserial. We show that M is quasi-injective. By [19, Proposition 1.18], it suffices to show that each M_j is quasi-injective and $\bigoplus_{I\setminus\{j\}} M_i$ is M_j-injective for all $j \in I$. Note that M is pseudo-injective, and so $\bigoplus_{I\setminus\{j\}} M_i$ is M_j-injective by [13, Theorem 2.2]. We show that M_j is quasi-inective for all $j \in I$. Indeed, we have that R is an artinian principal ideal ring and obtain, from [9, Theorem 5.3], that every indecomposable R-module M_i is quasi-injective. Thus, M is quasi-injective, and so it is quasi-projective.

(5) ⇒ (1) is proved as (3) ⇒ (1).

□

Corollary 4.10. *The following conditions are equivalent for a ring R:*

1. *R is an artinian principal ideal ring.*
2. *The class of pseudo-injective right R-modules coincides with the class of pseudo-projective right R-modules.*
3. *The class of pseudo-injective left R-modules coincides with the class of pseudo-projective left R-modules.*

Acknowledgements

Acknowledgments. The authors are grateful to the referee for her/his valuable comments. Nguyen Thi Thu Ha is partially founded by Scientific Research Contract No 150/HD-DHCN, code 23.1CB02. The second author would like to thank the Funds for Science and Technology Development of the Ministry of Education and Training under the Project Number B2023.DNA.14.

References

[1] A. N. Abyzov, T. C. Quynh and D. D. Tai, *Dual automorphism-invariant modules over perfect rings*, Siberian Math. J., **58** (2017), 743-751.

[2] A. N. Abyzov, L. V. Thuyet, T. C. Quynh and A. A. Tuganbaev, *Modules coinvariant under the idempotents endomorphisms of their covers*, Siberian Math. J., **60** (2019), 1191-1208.

[3] F. W. Anderson and K. R. Fuller, *Rings and Categories of Modules*, Springer-Verlag, 1974.

[4] K. A. Byrd, *Some characterizations of uniserial rings*, Math. Ann. **186** (1970), 163-170.

[5] V. Camillo, *Coherence for polynomial rings*, J. Algebra **132** (1990), 72-76.

[6] S. Chase, *Direct products of modules*, Trans. Amer. Math. Soc. **97** (1960), 457-473.

[7] R. R. Colby and E. A. Rutter, *Generalizations of QF-3 algebras*, Trans. Amer. Math. Soc. **153** (1971), 371-386.

[8] C. Faith, *Algebra II: Ring Theory*, Grundl. Math. Wiss. 191, Springer-Verlag, 1976.

[9] K. R. Fuller, *On indecomposable injectives over artinian rings*, Pacific J. Math., **29** (1969), 115-135.

[10] J. Golan, *Characterizations of rings using quasi-projective modules II*, Proc. Amer. Math. Soc., **28** (1971), 337-343.

[11] Y. Hirano, D. V. Huynh and J. K. Park, *Rings characterised by semiprimitive modules*, Bull. Austral. Math. Soc., **52** (1995), 107-116.

[12] P. A. Guil Asensio, T. C. Quynh and A. Srivastava, *Additive unit structure of endomorphism rings and invariance of modules*, Bull. Math. Sci., **7** (2017), 229–246.

[13] D. Q. Hai, *A note on pseudo-injective modules*, Comm. Algebra **33** (2005), 361-369.

[14] S. K. Jain and S. Singh, *Quasi-injective and pseudo-injective modules*, Canadian Math. Bull., **18** (1975), 359-366.

[15] M. T. Koşan, T. C. Quynh and A. K. Srivastava, *Rings with each right ideal automorphism-invariant*, J. Pure Appl. Algebra **220** (2016), 1525-1537.

[16] M. T. Koşan and T. C. Quynh, *Rings whose (proper) cyclic modules have cyclic automorphism-invariant hulls*, Appl. Algebra Engrg. Comm. Comput. **32** (2021), 385-397.

[17] T. Y. Lam, *Lectures on Modules and Rings*. New York: Springer-Verlag, 1999.

[18] T. K. Lee and Y. Zhou, *Modules which are invariant under automorphisms of their injective hulls*, J. Algebra Appl., **12** (2013), 1250159, 9 pp.

[19] S. H. Mohamed and B. J. Müller, *Continous and Discrete Modules*, Cambridge Univ. Press., Cambridge, 1990.

[20] B. Osofsky, *Rings all of whose finitely generated modules are injective*, Pacific J. Math. **14** (1964), 645-650.

[21] B. Osofsky, *Noninjective cyclic modules*, Proc. Amer. Math. Soc. **19** (1968), 1383-1384.

[22] G. Puninski and P. Rothmaler, *When every finitely generated flat is projective*, J. Algebra **277** (2004), 542-558.

[23] T. C. Quynh and M. T. Koşan, *On automorphism-invariant modules*, J. Algebra Appl. **14** (2015), 1550074, 11 pp.

[24] S. T. Rizvi and C. S. Roman, *On direct sums of Baer modules*, J. Algebra **321** (2009), 682-696.

[25] L. Small, *Semihereditary rings*, Bull. Amer. Math. Soc. **73** (1967), 656-658.

[26] A. K. Srivastava, A. A. Tuganbaev and P. A. Guil Asensio, *Invariance of Modules under Automorphisms of their Envelopes and Covers*, Cambridge Univ. Press., Cambridge, 2021.

[27] L. V. Thuyet and T. C. Quynh, *On the automorphism-invariance of finitely generated ideals and formal matrix rings*, Filomat **37**(29), (2023), 9961-9971

[28] A. K. Tiwary and B. M. Pandeya, *Pseudo projective and pseudo injective modules*, Indian J. Pure Appl. Math., **9** (1978), 941-949.

[29] D. T. Trang and B. D. Dung, *Endomorphism rings and formal matrix rings of pseudo-projective modules*, Filomat, **38** (4) (2024), 1389-1400.

[30] W. Xue, *Characterization of rings using direct-projective modules and direct-injective modules*, J. Pure Appl. Algebra **8** (1993),99–104.