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Lacunary harmonic summability in neutrosophic normed spaces

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Abstract. In this study, we introduce the concept of strongly lacunary $(H,1)$ convergence in the neutrosophic normed spaces. We investigate a few fundamental properties of this new concept.

1. Introduction

The neutrosophic set (NS) was investigated by Smarandache [22] who defined the degree of indeterminacy (i) as independent component. In [23], neutrosophic logic was firstly examined. It is a logic where each proposition is determined to have a degree of truth (T), falsity (F), and indeterminacy (I). Neutrosophic set and neutrosophic logic has used by applied sciences and theoretical science such as decision making, robotics and summability theory.

The new concept of a neutrosophic metric space (NMS) was defined by Kirişçi and Şimşek [3]. Also, they investigated neutrosophic normed space (NNS) and statistical convergence in NNS [4]. The various convergence properties of the sequences on this space were investigated after NNS was defined. Lacunary statistical convergence and lacunary ideal convergence of sequences in NNS were presented by Kişi ([5, 6]). Some works related to this concept can be found ([2, 7–10, 14–16]).

The harmonic means of the sequence $x = (x_k)$ is defined by

$$\tau_n = \frac{1}{\ell_n}\sum_{k=1}^{n}\frac{x_k}{k}, \text{ where } \ell_n = \sum_{i=1}^{n}\frac{1}{i} \approx \log n \text{ for } n = 1, 2, \ldots.$$

A sequence $x = (x_k)$ is named $(H,1)-$summable to ℓ if

$$\lim_{n\longrightarrow\infty}\frac{1}{\ell_n}\sum_{k=1}^{n}\frac{x_k}{k} = \ell.$$

By a lacunary sequence we mean an increasing integer sequence $\theta = (k_r)$ of non-negative integers such that $k_0 = 0$ and $h_r = (k_r - k_{r-1}) \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$

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and the ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated by q_r, and $q_1 = k_1$ for convenience. In recent years, lacunary sequences have been studied in ([1, 17–20]).

The set of all $(H,1)-$summable sequences is denoted by H and the set of all real sequences which is $(H,1)-$summable to 0 by H^0. It is well known that ordinary convergence does imply harmonic summability. However, the converse implication holds only under additional conditions. Recently, harmonically summability was studied in ([12, 13, 21]).

The lacunary harmonic means of the sequence $x = (x_i)$ is defined by

$$T_r = \frac{1}{L_r}\sum_{k\in I_r}\frac{x_k}{k}, \text{ where } L_r = \sum_{k\in I_r}\frac{1}{k} \text{ for } r = 1, 2, ...$$

2. Preliminaries

Now, we give definition of triangular norms (TN) and its dual operations known as triangular conorms (TC) which are important for fuzzy operations.

Definition 2.1. ([11]) Let $* : [0,1]\times[0,1] \to [0,1]$ be an operation. If the conditions are satisfied;
(i) $p * 1 = p$,
(ii) If $p \le r$ and $q \le s$, then $p * q \le r * s$, for all $p, q, r, s \in [0,1]$,
(iii) $*$ is continuous,
(iv) $*$ associative and commutative,
then the operation $*$ is called continuous TN.

Definition 2.2. ([11]) Let $\diamond : [0,1]\times[0,1] \to [0,1]$ be an operation. If the conditions are satisfied;
(i) $p\diamond 0 = p$,
(ii) If $p \le r$ and $q \le s$, then $p\diamond q \le r\diamond s$ for all $p, q, r, s \in [0,1]$,
(iii) $\diamond$ is continuous,
(iv) $\diamond$ associative and commutative,
then the operation $\diamond$ is said to be continuous TC (Triangular conorms (t-conorms)).

The concepts of the neutrosophic norm and the neutrosophic normed space were defined as the following:

Definition 2.3. ([4]) Let F be a vector space and $N : F\times\mathbb{R}^+ \to [0,1]$ such that $N = \{\langle e, G(e), B(e), Y(e)\rangle : e \in F\}$ be a normed space (NS). Let $*$ and $\diamond$ be the continuous TN and continuous TC, respectively. Provided the following conditions are satisfied, $V = (F, N, *, \diamond)$ is said to be NNS. For each $e, f \in F$ and $\lambda, \mu > 0$ and for all $\sigma \neq 0$,
(i) $0 \le G(e,\lambda) \le 1$, $0 \le B(e,\lambda) \le 1$, $0 \le Y(e,\lambda) \le 1$, $\forall\lambda \in \mathbb{R}^+$,
(ii) $G(e,\lambda) + B(e,\lambda) + Y(e,\lambda) \le 3$, $\forall\lambda \in \mathbb{R}^+$,
(iii) $G(e,\lambda) = 1$ (for $\lambda > 0$) if and only if $e = 0$,
(iv) $G(\sigma e,\lambda) = G\left(e, \frac{\lambda}{|\sigma|}\right)$,
(v) $G(e,\mu) * G(f,\lambda) \le G(e+f, \mu+\lambda)$,
(vi) $G(e,.)$ is non-decreasing continuous function,
(vii) $\lim_{\lambda\to\infty} G(e,\lambda) = 1$,
($viii$) $B(e,\lambda) = 0$ (for $\lambda > 0$) if and only if $e = 0$,
(ix) $B(\sigma e,\lambda) = B\left(e, \frac{\lambda}{|\sigma|}\right)$,
(x) $B(e,\mu)\diamond B(f,\lambda) \ge B(e+f, \mu+\lambda)$,
(xi) $B(e,.)$ is non-increasing continuous function,
(xii) $\lim_{\lambda\to\infty} B(e,\lambda) = 0$,
($xiii$) $Y(e,\lambda) = 0$ (for $\lambda > 0$) if and only if $e = 0$,
(xiv) $Y(\sigma e,\lambda) = Y\left(e, \frac{\lambda}{|\sigma|}\right)$,

(xv) $Y(e,\mu)\Diamond Y(f,\lambda) \geq Y(e+f,\mu+\lambda)$,
(xvi) $Y(e,.)$ is non-increasing continuous function,
$(xvii)$ $\lim_{\lambda\to\infty} Y(e,\lambda) = 0$,
$(xviii)$ If $\lambda \leq 0$, then $G(e,\lambda) = 0, B(e,\lambda) = 1$ and $Y(e,\lambda) = 1$.
Then $N = (G, B, Y)$ is called neutrosophic norm (NN).

Example 2.4. ([4]) Let $(F, \|.\|)$ be a NS. Give the operations $*$ and $\Diamond$ as TN $e * f = ef$; TC $e\Diamond f = e + f - ef$. For $\lambda > \|e\|$,

$$G(e,\lambda) = \frac{\lambda}{\lambda + \|e\|},\ B(e,\lambda) = \frac{\|e\|}{\lambda + \|e\|},\ Y(e,\lambda) = \frac{\|e\|}{\lambda},$$

$\forall e, f \in F$ and $\lambda > 0$. If we take $\lambda \leqslant \|e\|$, then $G(e,\lambda) = 0$, $B(e,\lambda) = 1$ and $Y(e,\lambda) = 1$. Then, $(F, N, *, \Diamond)$ is NNS such that $N : F \times \mathbb{R}^+ \to [0, 1]$.

Definition 2.5. ([4]) Let V be an NNS, the sequence (x_k) in V, $\varepsilon \in (0,1)$ and $\lambda > 0$. Then, the sequence (x_k) is converges to ζ if and only if there is $N \in \mathbb{N}$ such that $G(x_k - \zeta, \lambda) > 1 - \varepsilon, B(x_k - \zeta, \lambda) < \varepsilon, Y(x_k - \zeta, \lambda) < \varepsilon$. That is, $\lim_{k\to\infty} G(x_k - \zeta, \lambda) = 1, \lim_{k\to\infty} B(x_k - \zeta, \lambda) = 0$ and $\lim_{k\to\infty} Y(x_k - \zeta, \lambda) = 0$ as $\lambda > 0$. In this case, the sequence (x_k) is named a convergent sequence in V. A convergent sequence in NNS is indicated by $N - \lim x_k = \zeta$.

Definition 2.6. ([4]) Let V be an NNS. For $\lambda > 0$, $w \in F$ and $\varepsilon \in (0,1)$,

$$OB(w,\varepsilon,\lambda) = \{u \in F : G(w-u,\lambda) > 1-\varepsilon, B(w-u,\lambda) < \varepsilon, Y(w-u,\lambda) < \varepsilon\}$$

is called open ball with center w, radius ε.

Definition 2.7. ([4]) The set $A \subset F$ is called neutrosophic-bounded (NB) in NNS V, if there exist $\lambda > 0$, and $\varepsilon \in (0,1)$ such that $G(u,\lambda) > 1-\varepsilon$, $B(u,\lambda) < \varepsilon$ and $Y(u,\lambda) < \varepsilon$ for each $u \in A$.

3. Main results

In this section we give the main results of this article.

Definition 3.1. Take an NNS V. Let θ be a lacunary sequence. The sequence $x = (x_k)$ is said to be strongly lacunary $(H,1)$−convergent (or $(G,B,Y)_\theta$-convergent) to $\zeta \in F$ with regard to (or w.r.t., briefly) NN $(LC{-}NN)$, provided that for every $\lambda > 0$ and $\varepsilon \in (0,1)$, there is $r_0 \in \mathbb{N}$ such that

$$\frac{1}{L_r}\sum_{k\in I_r} G\left(\frac{x_k}{k} - \zeta, \lambda\right) > 1 - \varepsilon \quad \text{and} \quad \frac{1}{L_r}\sum_{k\in I_r} B\left(\frac{x_k}{k} - \zeta, \lambda\right) < \varepsilon,$$

$$\frac{1}{L_r}\sum_{k\in I_r} Y\left(\frac{x_k}{k} - \zeta, \lambda\right) < \varepsilon$$

for all $r \geq r_0$. In this case, we write $(G,B,Y)_\theta - \lim x = \zeta$. In case of $\theta = (2^r)$, $(G,B,Y) - \lim x = \zeta$ is obtained.

Theorem 3.2. *Let V be an NNS and θ be a lacunary sequence. If a sequence x is $(G,B,Y)_\theta$-convergent to ζ w.r.t. the NN, then $(G,B,Y)_\theta - \lim x$ is unique.*

Proof. Let $(G,B,Y)_\theta - \lim x = \zeta_1$, $(G,B,Y)_\theta - \lim x = \zeta_2$ and $\zeta_1 \neq \zeta_2$. For a given $\varepsilon > 0$, select $\rho \in (0,1)$ such that $(1-\rho) * (1-\rho) > 1-\varepsilon$ and $\rho\Diamond\rho < \varepsilon$. Then, for each $\lambda > 0$, there is $r_1 \in \mathbb{N}$ such that

$$\frac{1}{L_r}\sum_{k\in I_r} G\left(\frac{x_k}{k} - \zeta_1, \lambda\right) > 1 - \rho \quad \text{and} \quad \frac{1}{L_r}\sum_{k\in I_r} B\left(\frac{x_k}{k} - \zeta_1, \lambda\right) < \rho,$$

$$\frac{1}{L_r}\sum_{k\in I_r} Y\left(\frac{x_k}{k}-\zeta_1,\lambda\right)<\rho$$

for all $r \geq r_1$. Also, there is $r_2 \in \mathbb{N}$ such that

$$\frac{1}{L_r}\sum_{k\in I_r} G\left(\frac{x_k}{k}-\zeta_2,\lambda\right)>1-\rho \quad \text{and} \quad \frac{1}{L_r}\sum_{k\in I_r} B\left(\frac{x_k}{k}-\zeta_2,\lambda\right)<\rho,$$

$$\frac{1}{L_r}\sum_{k\in I_r} Y\left(\frac{x_k}{k}-\zeta_2,\lambda\right)<\rho$$

for all $r \geq r_2$. Assume that $r_0 = \max\{r_1, r_2\}$. Then, for $r \geq r_0$, we can find a positive integer $m \in \mathbb{N}$ such that

$$\begin{aligned} G(\zeta_1-\zeta_2,\lambda) &= G\left(\zeta_1+\frac{x_m}{m}-\frac{x_m}{m}-\zeta_2,\lambda\right) \geq G\left(\frac{x_m}{m}-\zeta_1,\frac{\lambda}{2}\right) * G\left(\frac{x_m}{m}-\zeta_2,\frac{\lambda}{2}\right) \\ &> (1-\rho)*(1-\rho) > 1-\varepsilon, \\ B(\zeta_1-\zeta_2,\lambda) &\leq B\left(\frac{x_m}{m}-\zeta_1,\frac{\lambda}{2}\right)\Diamond B\left(\frac{x_m}{m}-\zeta_2,\frac{\lambda}{2}\right) < \rho\Diamond\rho < \varepsilon, \end{aligned}$$

and

$$Y(\zeta_1-\zeta_2,\lambda) \leq Y\left(\frac{x_m}{m}-\zeta_1,\frac{\lambda}{2}\right)\Diamond Y\left(\frac{x_m}{m}-\zeta_2,\frac{\lambda}{2}\right) < \rho\Diamond\rho < \varepsilon.$$

Since $\varepsilon > 0$ is abritrary, we have $G(\zeta_1-\zeta_2,\lambda) = 1$, $B(\zeta_1-\zeta_2,\lambda) = 0$ and $Y(\zeta_1-\zeta_2,\lambda) = 0$ for all $\lambda > 0$, which yields $\zeta_1 = \zeta_2$. □

We show the sequence $x = (x_k)$ is strongly lacunary $(H,1)-$convergence in an *NNS* with an example.

Let $(F, \|.\|)$ be a NS. For $e, f \in [0,1]$ and $e * f = ef$; *TC* $e\Diamond f = \min\{e+f, 1\}$. For all $x \in F$ and every $\lambda > 0$ we take

$$G(x,\lambda)=\frac{\lambda}{\lambda+\|x\|},\ B(x,\lambda)=\frac{\|x\|}{\lambda+\|x\|},\ Y(x,\lambda)=\frac{\|x\|}{\lambda}.$$

Then, V is an *NNS*. Let us take a sequence defined by

$$\frac{x_k}{k}=\begin{cases} 1, & \text{if } \ k=t^2 (t\in\mathbb{N}) \\ 0, & \text{otherwise} \end{cases},$$

and consider

$$A=\left\{k\in I_r : G\left(\frac{x_k}{k},\lambda\right)>1-\varepsilon \text{ and } B\left(\frac{x_k}{k},\lambda\right)<\varepsilon, Y\left(\frac{x_k}{k},\lambda\right)<\varepsilon\right\}.$$

So, the following set for any $\lambda > 0$ and for all $\varepsilon \in (0,1)$,

$$\begin{aligned} A &= \left\{k\in I_r : \frac{\lambda}{\lambda+\left\|\frac{x_k}{k}\right\|}>1-\varepsilon, \text{ and } \frac{\left\|\frac{x_k}{k}\right\|}{\lambda+\left\|\frac{x_k}{k}\right\|}<\varepsilon, \frac{\left\|\frac{x_k}{k}\right\|}{\lambda}<\varepsilon\right\} \\ &= \left\{k\in I_r : \left\|\frac{x_k}{k}\right\| \leq \frac{\lambda\varepsilon}{1-\varepsilon}, \text{ and } \left\|\frac{x_k}{k}\right\|<\lambda\varepsilon\right\} \\ &\subset \left\{k\in I_r : \left\|\frac{x_k}{k}\right\|=1\right\} = \left\{k\in I_r : k=t^2\right\} \end{aligned}$$

i.e.,

$$A_r(\varepsilon,\lambda)=\left\{r\in\mathbb{N} : \frac{1}{L_r}\sum_{k\in I_r} G\left(\frac{x_k}{k},\lambda\right)>1-\varepsilon \text{ and } \frac{1}{L_r}\sum_{k\in I_r} B\left(\frac{x_k}{k},\lambda\right)<\varepsilon, \frac{1}{L_r}\sum_{k\in I_r} Y\left(\frac{x_k}{k},\lambda\right)<\varepsilon\right\}$$

will be a finite set.

Theorem 3.3. *Let V be an NNS and θ be a lacunary sequence. If $(G,B,Y)_\theta - \lim x = \zeta_1$ and $(G,B,Y)_\theta - \lim y = \zeta_2$, then $(G,B,Y)_\theta - \lim(x+y) = \zeta_1 + \zeta_2$ and $c \in F$, $(G,B,Y)_\theta - \lim cx = c\zeta$.*

Proof. For every $\lambda > 0$ and $\varepsilon \in (0,1)$, there is $r_0 \in \mathbb{N}$ such that

$$\frac{1}{L_r}\sum_{k\in I_r} G\left(\frac{x_k}{k} - \zeta_1, \lambda\right) > 1-\rho \quad \text{and} \quad \frac{1}{L_r}\sum_{k\in I_r} B\left(\frac{x_k}{k} - \zeta_1, \lambda\right) < \rho,$$

$$\frac{1}{L_r}\sum_{k\in I_r} Y\left(\frac{x_k}{k} - \zeta_1, \lambda\right) < \rho$$

for all $r \geq r_1$. Also, there is $r_2 \in \mathbb{N}$ such that

$$\frac{1}{L_r}\sum_{k\in I_r} G\left(\frac{y_k}{k} - \zeta_2, \lambda\right) > 1-\rho \quad \text{and} \quad \frac{1}{L_r}\sum_{k\in I_r} B\left(\frac{y_k}{k} - \zeta_2, \lambda\right) < \rho,$$

$$\frac{1}{L_r}\sum_{k\in I_r} Y\left(\frac{y_k}{k} - \zeta_2, \lambda\right) < \rho$$

for all $r \geq r_2$. Assume that $r_0 = \max\{r_1, r_2\}$. Now, for $r \geq r_0$ we get

$$\begin{aligned}\frac{1}{L_r}\sum_{k\in I_r} G\left(\frac{(x_k+y_k)}{k} - (\zeta_1+\zeta_2), \lambda\right) &= \frac{1}{L_r}\sum_{k\in I_r} G\left(\frac{x_k}{k} - \zeta_1 + \frac{y_k}{k} - \zeta_2, \lambda\right) \\ &\geqslant \frac{1}{L_r}\sum_{k\in I_r} G\left(\frac{x_k}{k} - \zeta_1, \frac{\lambda}{2}\right) * G\left(\frac{y_k}{k} - \zeta_2, \frac{\lambda}{2}\right) \\ &> (1-\rho) * (1-\rho) > 1-\varepsilon\end{aligned}$$

and

$$\begin{aligned}\frac{1}{L_r}\sum_{k\in I_r} B\left(\frac{(x_k+y_k)}{k} - (\zeta_1+\zeta_2), \lambda\right) &= \frac{1}{L_r}\sum_{k\in I_r} B\left(\frac{x_k}{k} - \zeta_1 + \frac{y_k}{k} - \zeta_2, \lambda\right) \\ &\leqslant \frac{1}{L_r}\sum_{k\in I_r} B\left(\left(\frac{x_k}{k} - \zeta_1\right), \frac{\lambda}{2}\right) \diamond B\left(\left(\frac{y_k}{k} - \zeta_2\right), \frac{\lambda}{2}\right) < \rho \diamond \rho < \varepsilon.\end{aligned}$$

Further,

$$\begin{aligned}\frac{1}{L_r}\sum_{k\in I_r} Y\left(\left(\frac{x_k+y_k}{k}\right) - (\zeta_1+\zeta_2), \lambda\right) &= \frac{1}{L_r}\sum_{k\in I_r} Y\left(\frac{x_k}{k} - \zeta_1 + \frac{y_k}{k} - \zeta_2, \lambda\right) \\ &\leqslant \frac{1}{L_r}\sum_{k\in I_r} Y\left(\left(\frac{x_k}{k} - \zeta_1\right), \frac{\lambda}{2}\right) \diamond Y\left(\left(\frac{y_k}{k} - \zeta_2\right), \frac{\lambda}{2}\right) < \rho \diamond \rho < \varepsilon.\end{aligned}$$

Similarly we can show that $(G,B,Y)_\theta - \lim cx = c\zeta$. □

Theorem 3.4. *Let V be an NNS and θ be a lacunary sequence. Then there is a subsequence (x_{ρ_k}) of x such that $(G,B,Y)_\theta - \lim x_{\rho_k} = \zeta$, if $(G,B,Y)_\theta - \lim x = \zeta$.*

Proof. Assume that $(G,B,Y)_\theta - \lim x = \zeta$, for every $\lambda > 0$ and $\varepsilon \in (0,1)$, there is $r_0 \in \mathbb{N}$ such that

$$\frac{1}{L_r}\sum_{k\in I_r} G\left(\frac{x_k}{k} - \zeta, \lambda\right) > 1-\varepsilon \quad \text{and} \quad \frac{1}{L_r}\sum_{k\in I_r} B\left(\frac{x_k}{k} - \zeta, \lambda\right) < \varepsilon,$$

$$\frac{1}{L_r}\sum_{k\in I_r} Y\left(\frac{x_k}{k} - \zeta, \lambda\right) < \varepsilon$$

for all $r \geq r_0$. Clearly, we choose $\rho_k \in I_r$ such that,

$$G\left(\frac{x_{\rho_k}}{k} - \zeta, \lambda\right) > \frac{1}{L_r} \sum_{k \in I_r} G\left(\frac{x_k}{k} - \zeta, \lambda\right) > 1 - \varepsilon$$

$$B\left(\frac{x_{\rho_k}}{k} - \zeta, \lambda\right) < \frac{1}{L_r} \sum_{k \in I_r} B\left(\frac{x_k}{k} - \zeta, \lambda\right) < \varepsilon$$

$$Y\left(\frac{x_{\rho_k}}{k} - \zeta, \lambda\right) < \frac{1}{L_r} \sum_{k \in I_r} Y\left(\frac{x_k}{k} - \zeta, \lambda\right) < \varepsilon$$

for each $r \geq r_0$. Therefore $(G, B, Y)_\theta - \lim x_{\rho_k} = \zeta$. □

Theorem 3.5. *Let V be an NNS and θ be a lacunary sequence. Then $(G, B, Y) \subseteq (G, B, Y)_\theta$, if $\liminf_r \frac{\ell_{k_r}}{\ell_{k_{r-1}}} > 1$.*

Proof. Assume that $(G, B, Y) - \lim x = \zeta$. Since $\frac{\ell_{k_r}}{\ell_{k_{r-1}}} > 1$, then there exists $\delta > 0$ such that $1 + \delta \leqslant \frac{\ell_{k_r}}{\ell_{k_{r-1}}}$ for all $r \geq 1$, we have

$$\frac{1}{L_r} \sum_{k \in I_r} G\left(\frac{x_k}{k} - \zeta, \lambda\right) = \frac{1}{L_r} \sum_{k=1}^{k_r} G\left(\frac{x_k}{k} - \zeta, \lambda\right) - \frac{1}{L_r} \sum_{k=1}^{k_{r-1}} G\left(\frac{x_k}{k} - \zeta, \lambda\right)$$

$$= \frac{\ell_{k_r}}{L_r}\left(\frac{1}{\ell_{k_r}} \sum_{k=1}^{k_r} G\left(\frac{x_k}{k} - \zeta, \lambda\right)\right) - \frac{\ell_{k_{r-1}}}{L_r}\left(\frac{1}{\ell_{k_{r-1}}} \sum_{k=1}^{k_{r-1}} G\left(\frac{x_k}{k} - \zeta, \lambda\right)\right)$$

$$> \frac{\ell_{k_r}}{L_r}\left(\frac{1}{\ell_{k_r}} \sum_{k=1}^{k_r} G\left(\frac{x_k}{k} - \zeta, \lambda\right)\right) > \left(\frac{1}{\ell_{k_r}} \sum_{k=1}^{k_r} G\left(\frac{x_k}{k} - \zeta, \lambda\right)\right) > 1 - \varepsilon.$$

Since $L_r = \ell_{k_r} - \ell_{k_{r-1}}$, we can write

$$\frac{\ell_{k_r}}{L_r} \leq \frac{(1+\delta)}{\delta} \text{ and } \frac{\ell_{k_{r-1}}}{L_r} \leq \frac{1}{\delta}.$$

It gives that $(G, B, Y)_\theta - \lim x = \zeta$. From here, $\frac{1}{L_r} \sum_{k \in I_r} B\left(\frac{x_k}{k} - \zeta, \lambda\right) < \varepsilon$ and $\frac{1}{L_r} \sum_{k \in I_r} Y\left(\frac{x_k}{k} - \zeta, \lambda\right) < \varepsilon$ are obtained. □

Theorem 3.6. *Let V be an NNS and θ be a lacunary sequence. Then $(G, B, Y)_\theta \subseteq (G, B, Y)$, if $\liminf_r \frac{\ell_{k_r}}{\ell_{k_{r-1}}} = 1$.*

Proof. Assume that $(G, B, Y)_\theta - \lim x = \zeta$. For $\lambda > 0$, we have

$$G_r = \frac{1}{L_r} \sum_{k \in I_r} G\left(\frac{x_k}{k} - \zeta, \lambda\right) \to 1$$

as $r \to \infty$. Then for $\varepsilon > 0$, there exists $r_0 \in \mathbb{N}$ such that $G_r < 1 + \varepsilon$ for all $r \geqslant r_0$. Also, we can find $P > 0$ such that $G_r < P$, $r = 1, 2, \ldots$. Let n be an integer with $\ell_{k_{r-1}} < \ell_n \leqslant \ell_{k_r}$. Then

$$\frac{1}{\ell_n} \sum_{k=1}^{n} G\left(\frac{x_k}{k} - \zeta, \lambda\right) \leqslant \frac{1}{\ell_{k_{r-1}}} \sum_{k=1}^{k_r} G\left(\frac{x_k}{k} - \zeta, \lambda\right)$$

$$= \frac{1}{\ell_{k_{r-1}}}\left[\sum_{k \in I_1} G\left(\frac{x_k}{k} - \zeta, \lambda\right) + \cdots + \sum_{k \in I_r} G\left(\frac{x_k}{k} - \zeta, \lambda\right)\right]$$

$$\leqslant \sup_{1 \leqslant r \leqslant r_0} G_r \frac{\ell_{k_{r_0}}}{\ell_{k_{r-1}}} + \frac{L_{r_0+1}}{\ell_{k_{r-1}}} G_{r_0+1} + \cdots + \frac{L_r}{\ell_{k_{r-1}}} G_r$$

$$< P\frac{\ell_{k_{r_0}}}{\ell_{k_{r-1}}} + (1+\varepsilon)\frac{\ell_{k_r} - \ell_{k_{r_0}}}{\ell_{k_{r-1}}}.$$

Since $\ell_{k_{r-1}} \to \infty$ as $n \to \infty$, it follows that $\frac{1}{\ell_n}\sum_{k=1}^{n} G\left(\frac{x_k}{k} - \zeta, \lambda\right) \to 1$. Similarly we can show that $\frac{1}{\ell_n}\sum_{k=1}^{n} B\left(\frac{x_k}{k} - \zeta, \lambda\right) \to 0$ and $\frac{1}{\ell_n}\sum_{k=1}^{n} Y\left(\frac{x_k}{k} - \zeta, \lambda\right) \to 0$. □

Theorem 3.7. *Let V be an NNS and $\theta = (k_r)$ and $\theta' = (s_r)$ be two lacunary sequences such that $I_r \subseteq J_r$ for all $r \in \mathbb{N}$. If*

$$\lim_{r\to\infty} \frac{L_r'}{(L_r)} = 1 \tag{1}$$

holds and $A \subset F$ is neutrosophic-bounded (NB) in NNS V then $(G, B, Y)_\theta \subset (G, B, Y)_{\theta'}$, where $I_r = (k_{r-1}, k_r]$, $J_r = (s_{r-1}, s_r]$, $L_r = \sum_{k\in I_r} \frac{1}{k}$, $L_r' = \sum_{k\in J_r} \frac{1}{k}$.

Proof. Assume that $x \in (G, B, Y)_\theta$ and (1) holds. Because of $A \subset F$ is neutrosophic-bounded (NB) in $NNSV$, then there exists some $\lambda > 0$ such that $\frac{1}{L_r}\sum_{k\in I_r} G\left(\frac{x_k}{k} - \zeta, \lambda\right) > 1-\varepsilon$ and $\frac{1}{L_r}\sum_{k\in I_r} B\left(\frac{x_k}{k} - \zeta, \lambda\right) < \varepsilon$, $\frac{1}{L_r}\sum_{k\in I_r} Y\left(\frac{x_k}{k} - \zeta, \lambda\right) < \varepsilon$ for each $\left(\frac{x_k}{k} - \zeta\right) \in A$. Now, since $I_r \subseteq J_r$ and $L_r \le L_r'$ for all $r \in \mathbb{N}$, we may write

$$\begin{aligned}\frac{1}{L_r}\sum_{k\in I_r} G\left(\frac{x_k}{k} - \zeta, \lambda\right) &\le \frac{1}{L_r}\sum_{k\in J_r} G\left(\frac{x_k}{k} - \zeta, \lambda\right)\\ &= \frac{L_r'}{L_r}\frac{1}{L_r'}\sum_{k\in J_r} G\left(\frac{x_k}{k} - \zeta, \lambda\right)\end{aligned}$$

for all $r \in \mathbb{N}$. Therefore,

$$\begin{aligned}\frac{1}{L_r'}\sum_{k\in J_r} B\left(\frac{x_k}{k} - \zeta, \lambda\right) &= \frac{1}{L_r'}\sum_{k\in J_r - I_r} B\left(\frac{x_k}{k} - \zeta, \lambda\right) + \frac{1}{L_r'}\sum_{k\in I_r} B\left(\frac{x_k}{k} - \zeta, \lambda\right)\\ &\le \frac{L_r' - L_r}{L_r'}\varepsilon + \frac{1}{L_r'}\sum_{k\in I_r} B\left(\frac{x_k}{k} - \zeta, \lambda\right)\\ &\le \frac{L_r' - L_r}{L_r}\varepsilon + \frac{1}{L_r}\sum_{k\in I_r} B\left(\frac{x_k}{k} - \zeta, \lambda\right)\\ &\le \left(\frac{L_r'}{L_r} - 1\right)\varepsilon + \frac{1}{L_r}\sum_{k\in I_r} B\left(\frac{x_k}{k} - \zeta, \lambda\right).\end{aligned}$$

Therefore $\frac{1}{L_r'}\sum_{k\in J_r} G\left(\frac{x_k}{k} - \zeta, \lambda\right) > 1-\varepsilon$ and $\frac{1}{L_r'}\sum_{k\in J_r} B\left(\frac{x_k}{k} - \zeta, \lambda\right) < \varepsilon$. It is demonstrable to be $\frac{1}{L_r'}\sum_{k\in J_r} Y\left(\frac{x_k}{k} - \zeta, \lambda\right) < \varepsilon$ by similar operations. Hence $(G, B, Y)_\theta \subset (G, B, Y)_{\theta'}$. □

Definition 3.8. Let V be a NNS and θ be a lacunary sequence. A sequence $x = (x_k)$ is said to be strongly lacunary $(H, 1)$ Cauchy (or $(G, B, Y)_\theta$-Cauchy) w.r.t. the NN N $(LCa - NN)$ if, for every $\varepsilon \in (0, 1)$ and $\lambda > 0$, there is $p = p(\varepsilon) \in \mathbb{N}$ satisfying

$$\frac{1}{L_r}\sum_{k\in I_r} G\left(\frac{x_k}{k} - \frac{x_p}{p}, \lambda\right) > 1 - \varepsilon \quad \text{and} \quad \frac{1}{L_r}\sum_{k\in I_r} B\left(\frac{x_k}{k} - \frac{x_p}{p}, \lambda\right) < \varepsilon,$$

$$\frac{1}{L_r}\sum_{k\in I_r} Y\left(\frac{x_k}{k} - \frac{x_p}{p}, \lambda\right) < \varepsilon.$$

Theorem 3.9. *If a sequence x is $(G,B,Y)_\theta$-convergent to ζ w.r.t. the NN N , then it is strongly lacunary $(H,1)$ Cauchy w.r.t. the NN N .*

Proof. Assume that $(G,B,Y)_\theta - \lim x = \zeta$. Choose $\varepsilon > 0$ then for a given $\rho \in (0,1), (1-\rho) * (1-\rho) > 1-\varepsilon$ and $\rho \diamond \rho < \varepsilon$. Then, we have

$$\frac{1}{L_r}\sum_{k\in I_r} G\left(\frac{x_k}{k} - \zeta, \frac{\lambda}{2}\right) > 1-\rho \quad \text{and} \quad \frac{1}{L_r}\sum_{k\in I_r} B\left(\frac{x_k}{k} - \zeta, \frac{\lambda}{2}\right) < \rho,$$

$$\frac{1}{L_r}\sum_{k\in I_r} Y\left(\frac{x_k}{k} - \zeta, \frac{\lambda}{2}\right) < \rho.$$

We have to show that

$$\frac{1}{L_r}\sum_{k\in I_r} G\left(\frac{x_k}{k} - \frac{x_m}{m}, \lambda\right) > 1-\varepsilon \quad \text{and} \quad \frac{1}{L_r}\sum_{k\in I_r} B\left(\frac{x_k}{k} - \frac{x_m}{m}, \lambda\right) < \varepsilon,$$

$$\frac{1}{L_r}\sum_{k\in I_r} Y\left(\frac{x_k}{k} - \frac{x_m}{m}, \lambda\right) < \varepsilon.$$

There are three possible situations.

Case (*i*) we have for $\lambda > 0$

$$G\left(\frac{x_k}{k} - \frac{x_m}{m}, \lambda\right) \geq G\left(\frac{x_k}{k} - \zeta, \frac{\lambda}{2}\right) * G\left(\frac{x_m}{m} - \zeta, \frac{\lambda}{2}\right) > (1-\rho) * (1-\rho) > 1-\varepsilon.$$

Case (*ii*) we obtain

$$B\left(\frac{x_k}{k} - \frac{x_m}{m}, \lambda\right) \leq B\left(\frac{x_k}{k} - \zeta, \frac{\lambda}{2}\right) \diamond B\left(\frac{x_m}{m} - \zeta, \frac{\lambda}{2}\right) < \rho \diamond \rho < \varepsilon.$$

Case (*iii*) we have

$$Y\left(\frac{x_k}{k} - \frac{x_m}{m}, \lambda\right) \leq Y\left(\frac{x_k}{k} - \zeta, \frac{\lambda}{2}\right) \diamond Y\left(\frac{x_m}{m} - \zeta, \frac{\lambda}{2}\right) < \rho \diamond \rho < \varepsilon.$$

This shows that (x_k) is strongly Cauchy with regards to *NN N*. □

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