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On sequential warped product η-Ricci-Bourguignon solitons

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Abstract. We investigate η-Ricci-Bourguignon solitons structure on sequential warped product manifolds and prove that an η-Ricci-Bourguignon soliton sequential warped manifold whose potential vector field is Killing or conformal must be a quasi-Einstein manifold. Finally, we deduce two applications of η-Ricci-Bourguignon solitons sequential warped product namely standard static space-times and generalized Robertson-Walker space-times.

1. Introduction

Ricci-Bourguignons flow introduced by J. P. Bourguignon [6] are defined as an extension of Ricci flow [21]. R. S. Hamilton [22] defined the Ricci solitons as a self-similar solutions of Ricci flow. From there, for generalizing and particularizing gradient Ricci-Bourguignon solitons many examples were given. Then, generalization results of Ricci solitons were given in [16]. In this study, inspiring the work of Ricci almost solitons, he initiated the concept of almost Ricci-Bourguignon solitons. He gave some important results which were qualified as the generalizing results for Ricci almost solitons. Therefore the notion of η-Ricci soliton was introduced in [14] which was developed in [8] on Hopf hypersurfaces in complex space forms.

Besides, several authors studied the almost η-Ricci solitons, (see [3] [28]). The almost Ricci-Bourguignon solitons provided some special potential vector fields and almost η-Ricci-Bourguignon solitons on a doubly warped product were studied in [2]. The almost η-Ricci-Bourguignon solitons on compact and non compact case were investigated by Traore et al. [31].

The notion of warped product Riemannian manifolds introduced in [1] is a generalization of the direct product of Riemannian manifolds and plays a very important role in physics, as well as in differential geometry, especially in the theory of relativity. On the other hand, doubly and multiply warped manifolds generalize the warped product manifolds which were studied in ([17], [25], [26]). Indeed the sequential warped product manifolds, was introduced in [15]. From there, several authors studied the Ricci solitons on warped product manifolds, (see [5], [19], [20], [29]), [23], [24], [13], [10], [11], [27].

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Motivated by the above studies, we give basic background of η-Ricci-Bourguignon solitons, we establish the structure of sequential warped product manifolds and gradient concurrent vector field in section 2. Then, we construct a gradient η-Ricci-Bourguignon soliton on a sequential warped product manifold. We give the necessary conditions for an η-Ricci-Bourguignon soliton sequential warped product to be quasi-Einstein manifold under some conditions on gradient concurrent and its potential vector fields in section 3. In section 4, we investigate warped product manifold sequential standard static space-times and generalized Robertson-Walker space-times.

2. Preliminaries

Let $(M^n, \tilde{g})$ be an n-dimensional Riemannian manifold, then we defined on M^n the *Ricci-Bourguignon solitons* as a self-similar solutions to *Ricci-Bourguignon flow* [7] defined:

$$\frac{\partial}{\partial t}\tilde{g}(t) = -2(\mathrm{Ric} - \tilde{\rho}\tau\tilde{g}), \tag{1}$$

where τ is the scalar curvature of the Riemannian metric $\tilde{g}$, Ric is the Ricci curvature tensor of the metric, and $\tilde{\rho}$ is a real constant. When $\tilde{\rho} = 0$ in (1), then we get a Ricci flow.

Definition 2.1. *Let $(M^n, \tilde{g})$ be a Riemannian manifold of dimension $n \geq 3$. Then it is called Ricci-Bourguignon soliton [16] if*

$$\mathrm{Ric} + \frac{1}{2}\pounds_\xi \tilde{g} = (\tilde{\lambda} + \tilde{\rho}\tau)\tilde{g}, \tag{2}$$

where $\pounds_\xi$ denotes the Lie derivative operator along the vector field ξ which is called soliton or potential, $\tilde{\rho}$ and $\tilde{\lambda}$ are real constants.

Considering η the $\tilde{g}$-dual 1-form of ξ, $(M^n, \tilde{g})$ is called *η-Ricci-Bourguignon soliton* [28] if the following equation holds

$$\mathrm{Ric} + \frac{1}{2}\pounds_\xi \tilde{g} = (\tilde{\lambda} + \tilde{\rho}\tau)\tilde{g} + \mu\eta \otimes \eta, \tag{3}$$

for a vector field ξ, where $\tilde{\lambda}, \mu$ are real constants. Particularly, taking $\tilde{\rho} = 0$ in equation (3), we get the *η-Ricci soliton* [4].

In (3), if ξ is the gradient of a function l on M, then we get a *gradient η-Ricci-Bourguignon soliton*. Then, equation (3) can be written as

$$\mathrm{Ric} + \nabla^2 l = (\tilde{\lambda} + \tilde{\rho}\tau)\tilde{g} + \mu\eta \otimes \eta, \tag{4}$$

where $\nabla^2 l$ is the Hessian of l and it is denoted by $(M, \tilde{g}, \nabla l, \tilde{\lambda}, \mu)$.

Definition 2.2. *Let K_1, K_2 and K_3 be three Riemannian manifolds of dimensions k_1, k_2 and k_3 endowed with the Riemannian metric tensors $\tilde{g}_1$, $\tilde{g}_2$ and $\tilde{g}_3$, respectively and let t and l be two smooth positive functions defined on K_1 and $K_1 \times K_2$. Then the sequential warped product [15] $(K_1 \times_t K_2) \times_l K_3$ of $(K_1, \tilde{g}_1)$, $(K_2, \tilde{g}_2)$ and $(K_3, \tilde{g}_3)$ is the product manifold $K = (K_1 \times K_2) \times K_3$ endowed with the metric $\tilde{g}$ given by*

$$\tilde{g} = (\tilde{g}_1 \oplus t^2 \tilde{g}_2) \oplus l^2 \tilde{g}_3.$$

The functions t and l are called warping functions. The sequential warped product will be denoted by $(K, \tilde{g})$

If ζ is a vector field on K then it shall be written as

$$\zeta = \zeta_1 + \zeta_2 + \zeta_3, \quad where \quad \zeta_i \in \mathfrak{X}(K_i), \quad i = 1, 2, 3. \tag{5}$$

Denoted by ∇ and Ric the Levi–Civita connection and the Ricci tensor of a sequential warped product manifold and let ∇^i and Ric^i the Levi–Civita connection and the Ricci tensor of $(K_i, \tilde{g}_i)$, for $i = 1, 2, 3$.

The covariant derivative formulas of sequential warped product manifold are given by the following.

Lemma 2.3. *[15] Let $(K_1 \times_t K_2) \times_l K_3$ be a sequential warped product manifold. Then for $\zeta_i, U_i \in \mathfrak{X}(K_i)$, $i = 1, 2, 3$, we have*

1. $\nabla_{\zeta_1} U_1 = \nabla^1_{\zeta_1} U_1$,
2. $\nabla_{\zeta_1} \zeta_2 = \nabla_{\zeta_2} \zeta_1 = \zeta_1(\ln t)\zeta_2$,
3. $\nabla_{\zeta_2} U_2 = \nabla^2_{\zeta_2} U_2 - t\tilde{g}_2(\zeta_2, U_2)\nabla^1 t$,
4. $\nabla_{\zeta_3} \zeta_1 = \nabla_{\zeta_1} \zeta_3 = \zeta_1(\ln l)\zeta_3$,
5. $\nabla_{\zeta_2} \zeta_3 = \nabla_{\zeta_3} \zeta_2 = \zeta_2(\ln l)\zeta_3$,
6. $\nabla_{\zeta_3} U_3 = \nabla^3_{\zeta_3} U_3 - l\tilde{g}_3(\zeta_3, U_3)\nabla l$,

where $\nabla^1 t$ and ∇l are the gradient of t on K_1 and l on $K_1 \times_l K_2$, respectively.

Lemma 2.4. *[15] Let $(K_1 \times_t K_2) \times_l K_3$ be a sequential warped product manifold. Then for $\zeta_i, U_i \in \mathfrak{X}(K_i)$, $i = 1, 2, 3$, we have*

1. $\mathrm{Ric}(\zeta_1, U_1) =^1 \mathrm{Ric}(\zeta_1, U_1) - \frac{k_2}{t}\nabla^2_1 t(\zeta_1, U_1) - \frac{k_3}{l}\nabla^2 l(\zeta_1, U_1)$,
2. $\mathrm{Ric}(\zeta_2, U_2) =^2 \mathrm{Ric}(\zeta_2, U_2) - t^\sharp \tilde{g}_2(\zeta_2, U_2) - \frac{k_3}{l}\nabla^2 l(\zeta_2, U_2)$,
3. $\mathrm{Ric}(\zeta_3, U_3) =^3 \mathrm{Ric}(\zeta_3, U_3) - l^\sharp \tilde{g}_3(\zeta_3, U_3)$,
4. $\mathrm{Ric}(\zeta_i, U_j) = 0$, *for $i \neq j$, where $t^\sharp = t\Delta^1 t + (k_2 - 1)\|\nabla^1 t\|^2$ and $l^\sharp = l\Delta l + (k_3 - 1)\|\nabla l\|^2$,*

where $\nabla^2_1 t$, $\Delta^1 t$ and $\nabla^2 l$, Δl are the Hessian and the Laplacian of t on K_1 and l on $K_1 \times_l K_2$, respectively.

A vector field ξ on $(\tilde{M}, \tilde{g})$ is called *concircular* [18]

$$\nabla_\zeta \xi = \psi\zeta, \tag{6}$$

where ψ is a smooth function on $\tilde{M}$. It is called *conccurent* if $\psi = 1$ [12]. Moreover, if the equation

$$\pounds_\xi \tilde{g} = 2\psi\tilde{g}, \tag{7}$$

holds, then ξ is called *conformal vector field*, where ψ is a smooth function on $\tilde{M}$. If $\psi = 0$, then ξ is called a *Killing vector field*.

Lemma 2.5. *[15] Let $(K, \tilde{g})$ be a sequential warped product manifold. Then the vector field $\xi \in \mathfrak{X}(K)$ satisfies*

$$\begin{aligned}(\pounds_\xi \tilde{g})(\zeta, U) &= (\pounds^1_{\xi_1}\tilde{g}_1)(\zeta_1, U_1) + t^2(\pounds^2_{\xi_2}\tilde{g}_2)(\zeta_2, U_2) + l^2(\pounds^3_{\xi_3}\tilde{g}_3)(\zeta_3, U_3) \\ &\quad + 2t\xi_1(t)\tilde{g}_2(\zeta_2, U_2) + 2l(\xi_1 + \xi_2)(l)\tilde{g}_3(\zeta_3, U_3)\end{aligned} \tag{8}$$

for any $\zeta, U \in \mathfrak{X}(K)$.

Recall that a non-flat Riemannian manifold $(\tilde{M}, \tilde{g})$ $(n \geq 3)$ is said to be a quasi-Einstein manifold [9], if Ricci tensor is not identically zero and satisfies

$$\mathrm{Ric} = \alpha_1\tilde{g} + \alpha_2 A \otimes A, \tag{9}$$

for α_1 and α_2 non-zero smooth functions and A a non-zero 1-form. The functions α_1 and α_2 are called associated functions also (see [30]). Now we investigate the properties of η-Ricci-Bourguignon solitons on sequential warped product manifolds.

3. Main results

We start with the following results on a gradient η-Ricci-Bourguignon soliton sequential warped product manifold.

Theorem 3.1. *Let* $(K, \tilde{g})$ *be a sequential warped product manifold. Then* $(K, \tilde{g}, \nabla\varphi, \tilde{\lambda}, \mu)$ *is a gradient* η*-Ricci-Bourguignon soliton with* φ *defined on* K_1 *and* η *the* $\tilde{g}$*-dual* 1*-form of the gradient* $\xi = \nabla\varphi$ *if and only if*

$$^1\operatorname{Ric} = (\tilde{\lambda} + \tilde{\rho}\tau)\tilde{g}_1 - \nabla_1^2\varphi + \frac{k_2}{t}\nabla_1^2 t + \frac{k_3}{l}\nabla^2 l + \mu d\varphi \otimes d\varphi, \tag{10}$$

$$^2\operatorname{Ric} = \Big(t^2(\tilde{\lambda} + \tilde{\rho}\tau) - t\nabla\varphi(t) + t\Delta^1 t + (k_2 - 1)\|\nabla^1 t\|^2\Big)\tilde{g}_2 + \frac{k_3}{l}\nabla^2 l \tag{11}$$

and

$$^3\operatorname{Ric} = \Big(l^2(\tilde{\lambda} + \tilde{\rho}\tau) - l\nabla\varphi(l) + l\Delta l + (k_3 - 1)\|\nabla l\|^2\Big)\tilde{g}_3. \tag{12}$$

Proof. If $(K, \tilde{g}, \nabla\varphi, \tilde{\lambda}, \mu)$ is a gradient η-Ricci-Bourguignon soliton, then we have

$$\operatorname{Ric} + \nabla^2\varphi = (\tilde{\lambda} + \tilde{\rho}\tau)\tilde{g} + \mu d\varphi \otimes d\varphi. \tag{13}$$

For any $\zeta_1, U_1 \in \mathfrak{L}(K_1)$, using Lemma 2.4, we obtain

$$^1\operatorname{Ric}(\zeta_1, U_1) = (\tilde{\lambda} + \tilde{\rho}\tau)\tilde{g}_1(\zeta_1, U_1) - \nabla^2\varphi(\zeta_1, U_1) + \frac{k_2}{t}\nabla_1^2 t(\zeta_1, U_1) + \frac{k_3}{l}\nabla^2 l(\zeta_1, U_1) + \mu d\varphi(\zeta_1)d\varphi(U_1). \tag{14}$$

Therefore from Lemma 2.3, we have $\nabla^2\varphi(\zeta_1, U_1) = \nabla_1^2\varphi(\zeta_1, U_1)$. Hence the equation (10) is proved.

Now for any $\zeta_2, U_2 \in \mathfrak{L}(K_2)$, we have

$$^2\operatorname{Ric}(\zeta_2, U_2) = t^2(\tilde{\lambda} + \tilde{\rho}\tau)\tilde{g}_2(\zeta_2, U_2) - \nabla^2\varphi(\zeta_2, U_2) + \Big(t\Delta^1 t + (k_2 - 1)\|\nabla^1 t\|^2\Big)\tilde{g}_2(\zeta_2, U_2) + \frac{k_3}{l}\nabla^2 l(\zeta_2, U_2). \tag{15}$$

Then using the fact that

$$\begin{aligned} \nabla^2\varphi(\zeta_2, U_2) &= \tilde{g}(\nabla_{\zeta_2}\nabla\varphi, U_2) \\ &= t\nabla\varphi(t)\tilde{g}_2(\zeta_2, U_2) \end{aligned} \tag{16}$$

and putting equation (16) in (15), we obtain

$$^2\operatorname{Ric}(\zeta_2, U_2) = \Big(t^2(\tilde{\lambda} + \tilde{\rho}\tau) - t\nabla\varphi(t) + t\Delta^1 t + (k_2 - 1)\|\nabla^1 t\|^2\Big)\tilde{g}_2(\zeta_2, U_2) + \frac{k_3}{l}\nabla^2 l(\zeta_2, U_2). \tag{17}$$

Hence we get the equation (11).

Moreover for any $\zeta_3, U_3 \in \mathfrak{L}(K_3)$, using the same calculus like the previous result, we get (12)

$$^3\operatorname{Ric}(\zeta_3, U_3) = l^2(\tilde{\lambda} + \tilde{\rho}\tau)\tilde{g}_3(\zeta_3, U_3) - l\nabla\varphi(l) + l\Delta l + (k_3 - 1)\|\nabla l\|^2\tilde{g}(\zeta_3, U_3). \tag{18}$$

The converse is just a verification. □

The following lemma is a necessary and sufficient condition for a gradient vector field on a Riemannian manifold to be concurrent.

Lemma 3.2. *Let k be a smooth function defined on $(\tilde{M}, \tilde{g})$. Then ∇k of k is a concurrent vector field if and only if $\nabla^2 k$ of k satisfies*

$$\nabla^2 k(\zeta, U) = \tilde{g}(\zeta, U), \tag{19}$$

where ∇k and $\nabla^2 k$ are the gradient and Hessian of k respectively and ζ, U are vector fields on M.

Proof. Let k be a smooth function on M and suppose that $\nabla^2 k$ of k satisfies (19). Then for any $\zeta, U \in \mathfrak{X}(M)$ on $\tilde{M}$, we have

$$\begin{aligned} \tilde{g}(\zeta, U) &= \zeta U(k) - \nabla_\zeta U(k) \\ &= \zeta \tilde{g}(\nabla k, U) - \tilde{g}(\nabla_\zeta U, \nabla k) = \tilde{g}(\nabla_\zeta(\nabla k), U). \end{aligned} \tag{20}$$

Thus, we obtain $\nabla_\zeta(\nabla k) = \zeta$. Hence $\xi = \nabla k$ is a concurrent vector field. The converse is a simple verification. □

From Lemma 2.4, we can state the following corollary:

Corollary 3.3. *Let $(K, \tilde{g})$ be a sequential warped product manifold. Then $(K, \tilde{g}, \xi, \tilde{\lambda}, \mu)$ is an η-Ricci-Bourguignon soliton with ξ defined on K_1 and η the $\tilde{g}$-dual 1-form of ξ if and only if*

$$\mathrm{Ric}^1 + \frac{1}{2}\pounds^1_\xi \tilde{g} = (\tilde{\lambda} + \tilde{\rho}\tau)\tilde{g}_1 + \frac{k_2}{t}\nabla^2_1 t + \frac{k_3}{l}\nabla^2 l + \mu\eta \otimes \eta, \tag{21}$$

$$\mathrm{Ric}^2 = \Big(t^2(\tilde{\lambda} + \tilde{\rho}\tau) - t\xi(t) + t\Delta^1 t + (k_2 - 1)\|\nabla^1 t\|^2\Big)\tilde{g}_2 + \frac{k_3}{l}\nabla^2 l \tag{22}$$

and

$$\mathrm{Ric}^3 = \Big(l^2(\tilde{\lambda} + \tilde{\rho}\tau) - l\xi(l) + l\Delta l + (k_3 - 1)\|\nabla l\|^2\Big)\tilde{g}_3. \tag{23}$$

Proof. The proof is similar to the proof of Theorem 3.1. It is sufficient to use Lemma 2.4 and the fact that

$$\begin{aligned} \pounds_\xi \tilde{g}(\zeta_2, U_2) &= \tilde{g}(\nabla_{\zeta_2}\xi, U_2) + \tilde{g}(\zeta_2, \nabla_{U_2}\xi) \\ &= 2\tfrac{\xi(t)}{t}\tilde{g}(\zeta_2, U_2) \\ &= 2t\xi(t)\tilde{g}_2(\zeta_2, U_2), \end{aligned} \tag{24}$$

which allows us to complete the proof. □

We give the following theorem:

Theorem 3.4. *Let $\xi \in \mathfrak{X}(K)$ be a potential vector on K. Assume that the gradient of t and l are concurrent vector fields on K_1 and K, respectively. If $(K, \tilde{g}, \xi, \tilde{\lambda}, \mu)$ is an η-Ricci-Bourguignon soliton then we have the following conditions:*

- ***(a)*** *$(K_1, \tilde{g}_1, \xi_1, \tilde{\lambda}_1, , \mu)$ is an η-Ricci-Bourguignon soliton with $\tilde{\lambda}_1 + \tilde{\rho}_1\tau_1 + \mu = \tilde{\lambda} + \tilde{\rho}\tau + \frac{k_2}{t} + \frac{k_3}{l}$.*
- ***(b)*** *If ξ_2 is a Killing vector field, then K_2 is a quasi-Einstein manifold with factors $t^2\tilde{\lambda} + \tilde{\rho}\tau t^2 + t^\sharp + \frac{k_3}{l}t^2 - t\xi_1(t)$ and μt^4.*
- ***(c)*** *$(K_3, \tilde{g}_3, l^2\xi_3, \tilde{\lambda}_3, \mu l^4)$ is an η-Ricci-Bourguignon soliton with $\tilde{\lambda}_3 + \tilde{\rho}_3, \tau_3 + \mu l^4 = \tilde{\lambda} l^2 + \tilde{\rho}\tau l^2 + l^\sharp - l(\xi_1 + \xi_2)(l)$.*

Proof. The η-Ricci-Bourguignon soliton $(K, \tilde{g}, \xi, \tilde{\lambda}, \mu)$ is given by

$$\mathrm{Ric} + \frac{1}{2}\pounds_{\xi}\tilde{g} = (\tilde{\lambda} + \tilde{\rho}\tau)\tilde{g} + \mu\eta\otimes\eta. \tag{25}$$

Therefore for any $\zeta, U \in \mathfrak{X}(K)$, from Lemma 2.4 and Lemma 2.5, we have

$$\begin{aligned}
&{}^{1}\mathrm{Ric}(\zeta_1, U_1) - \tfrac{k_2}{t}\nabla_1^2 t(\zeta_1, U_1) - \tfrac{k_3}{l}\nabla^2 l(\zeta_1, U_1) +{}^{2}\mathrm{Ric}(\zeta_2, U_2) - t^{\sharp}\tilde{g}_2(\zeta_2, U_2) - \tfrac{k_3}{l}\nabla^2 l(\zeta_2, U_2)\\
&+{}^{3}\mathrm{Ric}(\zeta_3, U_3) - l^{\sharp}\tilde{g}_3(\zeta_3, U_3) + \tfrac{1}{2}(\pounds^1_{\xi_1}\tilde{g}_1)(\zeta_1, U_1) + \tfrac{1}{2}t^2(\pounds^2_{\xi_2}\tilde{g}_2)(\zeta_2, U_2) + \tfrac{1}{2}l^2(\pounds^3_{\xi_3}\tilde{g}_3)(\zeta_3, U_3)\\
&+t\xi_1(t)\tilde{g}_2(\zeta_2, U_2) + l(\xi_1 + \xi_2)(l)\tilde{g}_3(\zeta_3, U_3)\\
&= (\tilde{\lambda} + \tilde{\rho}\tau)\tilde{g}_1(\zeta_1, U_1) + t^2(\tilde{\lambda} + \tilde{\rho}\tau)\tilde{g}_2(\zeta_2, U_2) + l^2(\tilde{\lambda} + \tilde{\rho}\tau)\tilde{g}_3(\zeta_3, U_3)\\
&+\mu\tilde{g}_1(\zeta_1, \xi_1)\tilde{g}_1(U_1, \xi_1) + \mu t^4\tilde{g}_2(\zeta_2, \xi_2)\tilde{g}_2(\zeta_2, \xi_2) + \mu l^4\tilde{g}_3(\zeta_3, \xi_3)\tilde{g}_3(\zeta_3, \xi_3).
\end{aligned} \tag{26}$$

Now let $\zeta = \zeta_1$ $U = U_1$ and taking $\eta_1(\zeta_1)\eta_1(U_1) = \tilde{g}_1(\zeta_1, \xi_1)\tilde{g}_1(U_1, \xi_1)$. Using the fact that $\nabla_1^2 t = \tilde{g}_1$ and $\nabla^2 l = \tilde{g}_1$ from 3.2, then (26) becomes

$$\begin{aligned}
{}^{1}\mathrm{Ric}(\zeta_1, U_1) + \tfrac{1}{2}(\pounds^1_{\xi_1}\tilde{g}_1)(\zeta_1, U_1) &= \tilde{\lambda}_1\tilde{g}_1(\zeta_1, U_1) + \mu\eta_1(\zeta_1)\eta_1(U_1)\\
&+(-\tilde{\lambda}_1 + \tilde{\lambda} + \tilde{\rho}\tau + \tfrac{k_2}{t} + \tfrac{k_3}{l})\tilde{g}_1(\zeta_1, U_1)\\
&= \tilde{\lambda}_1\tilde{g}_1(\zeta_1, U_1) + \tilde{\rho}_1\tau_1\tilde{g}_1(\zeta_1, U_1) + \mu\eta_1(\zeta_1)\eta_1(U_1).
\end{aligned} \tag{27}$$

Thus $(K_1, \tilde{g}_1, \xi_1, \lambda_1, \mu)$ is an η-Ricci-Bourguignon soliton.

Taking now $\zeta = \zeta_2$, $U = U_2$ and $\eta_2(\zeta_2)\eta_2(U_2) = \tilde{g}_2(\zeta_2, \xi_2)\tilde{g}_2(U_2, \xi_2)$, we get

$$\begin{aligned}
&{}^{2}\mathrm{Ric}(\zeta_2, U_2) - t^{\sharp}\tilde{g}_2(\zeta_2, U_2) - \tfrac{k_3}{l}\nabla^2 l(\zeta_2, U_2) + \tfrac{1}{2}t^2(\pounds^2_{\xi_2}\tilde{g}_2)(\zeta_2, U_2) + t\xi_1(t)\tilde{g}_2(\zeta_2, U_2)\\
&= t^2(\tilde{\lambda} + \tilde{\rho}\tau)\tilde{g}_2(\zeta_2, U_2) + \mu t^4\eta_2(\zeta_2)\eta_2(U_2).
\end{aligned} \tag{28}$$

If ξ_2 is a Killing vector field and $\nabla^2 l = \tilde{g}_2$ from 3.2, we get

$${}^{2}\mathrm{Ric}(\zeta_2, U_2) = (\tilde{\lambda}t^2 + \tilde{\rho}\tau t^2 + t^{\sharp} + \tfrac{k_3}{l}t^2 - t\xi_1(t)\tilde{g}_2(\zeta_2, U_2) + \mu t^4\eta_2(\zeta_2)\eta_2(U_2), \tag{29}$$

which implies that K_2 is quasi-Einstein manifold.

Finally, let $\zeta = \zeta_3$, $U = U_3$ and $\eta_3(\zeta_3)\eta_(U_3) = \tilde{g}_3(\zeta_3, \xi_3)\tilde{g}_3(U_3, \xi_3)$, Then

$$\begin{aligned}
{}^{3}\mathrm{Ric}(\zeta_3, U_3) + \tfrac{1}{2}l^2\pounds^3_{\xi_3}\tilde{g}_3(\zeta_3, U_3) &= \tilde{\lambda}_3\tilde{g}_3(\zeta_3, U_3) + \mu l^4\eta_3(\zeta_3)\eta_3(U_3)\\
&+(-\tilde{\lambda}_3 + \tilde{\lambda}l^2 + \tilde{\rho}\tau l^2 + l^{\sharp} - l(\xi_1 + \xi_2)(l)\tilde{g}_3(\zeta_3, U_3)\\
&= \tilde{\lambda}_3\tilde{g}_3(\zeta_3, U_3) + \tilde{\rho}_3\tau_3\tilde{g}_3(\zeta_3, U_3) + \mu l^4\eta_3(\zeta_3)\eta_3(U_3).
\end{aligned} \tag{30}$$

Hence $(K_3, \tilde{g}_3, l^2\xi_3, \tilde{\lambda}_3, \mu l^4)$ is an η-Ricci-Bourguignon soliton. Thus the proof is completed. □

Below, we state some necessary conditions for the sequential warped product manifold to be quasi-Einstein manifold.

Theorem 3.5. *Let $\xi \in \mathfrak{X}(K)$ be a Killing vector field on K. Assume that the gradient of t and l are concurrent vector fields on K_1 and K. If $(K, \tilde{g}, \xi, \tilde{\lambda}, \mu)$ is an η-Ricci-Bourguignon soliton then, we have the following conditions:*

(a) *K_1 is quasi-Einstein manifold with factors* $(\tilde{\lambda} + \tilde{\rho}\tau + \frac{k_2}{t} + \frac{k_3}{l})$ *and* μ.

(b) *K_2 is quasi-Einstein manifold with factors* $\tilde{\lambda}t^2 + \tilde{\rho}\tau t^2 + t^\sharp + \frac{k_3}{l}t^2$ *and* μt^4.

(c) *K_3 is quasi-Einstein manifold with factors* $\tilde{\lambda}l^2 + \tilde{\rho}\tau l^2 + l^\sharp$ *and* μl^4.

Proof. Let $(K, \tilde{g}, \xi, \lambda, \rho, \mu)$ be an η-Ricci-Bourguignon soliton and ξ is a Killing vector field on K. Then we have

$$\mathrm{Ric} = (\tilde{\lambda} + \tilde{\rho}\tau)\tilde{g} + \mu\eta \otimes \eta. \tag{31}$$

For any $\zeta, U \in \mathfrak{X}(K)$, using equation (26), we get

$$^1\,\mathrm{Ric}(\zeta_1, U_1) = (\tilde{\lambda} + \tilde{\rho}\tau + \frac{k_2}{t} + \frac{k_3}{l})\tilde{g}_1(\zeta_1, U_1) + \mu\eta_1(\zeta_1)\eta_1(U_1), \tag{32}$$

$$^2\,\mathrm{Ric}(\zeta_2, U_2) = (\tilde{\lambda}t^2 + \tilde{\rho}\tau t^2 + t^\sharp + \frac{k_3}{l}t^2)\tilde{g}_2(\zeta_2, U_2) + \mu t^4\eta_2(\zeta_2)\eta_2(U_2) \tag{33}$$

and

$$^3\,\mathrm{Ric}(\zeta_3, U_3) = (\tilde{\lambda}l^2 + \tilde{\rho}\tau l^2 + l^\sharp)\tilde{g}_3(\zeta_3, U_3) + \mu l^4\eta_3(\zeta_3)\eta_3(U_3). \tag{34}$$

Then the proof is completed. □

Theorem 3.6. *Let $\xi \in \mathfrak{X}(K)$ be a potential vector field on K. Let $(K, \tilde{g}, \xi, \tilde{\lambda}, \mu)$ be an η-Ricci-Bourguignon soliton and the gradient of t and l are concurrent vector fields on K_1 and K. Then K_1, K_2 and K_3 are quasi-Einstein manifolds if the following conditions hold:*

(a) *$\xi = \xi_1$ and ξ_1 is Killing on K_1.*
(b) *$\xi = \xi_2$ and ξ_2 is Killing on K_2.*
(c) *$\xi = \xi_3$ and ξ_3 is Killing on K_3.*

Proof. If $\xi = \xi_1$ and ξ_1 is Killing on K_1 and using Lemma 2.5, we get

$$\pounds_\xi \tilde{g} = 2t\xi_1(t)\tilde{g}_2. \tag{35}$$

Using the previous equation in (26), we get

$$^1\,\mathrm{Ric}(\zeta_1, U_1) = (\tilde{\lambda} + \tilde{\rho}\tau + \tfrac{k_2}{t} + \tfrac{k_3}{l})\tilde{g}_1(\zeta_1, U_1) + \mu\eta_1(\zeta_1)\eta_1(U_1), \tag{36}$$

$$^2\,\mathrm{Ric}(\zeta_2, U_2) = (\tilde{\lambda}t^2 + \tilde{\rho}\tau t^2 + t^\sharp + \tfrac{k_3}{l}t^2 - t\xi_1(t)\tilde{g}_2(\zeta_2, U_2) + \mu t^4\eta_2(\zeta_2)\eta_2(U_2) \tag{37}$$

and

$$^3\,\mathrm{Ric}(\zeta_3, U_3) = (\tilde{\lambda}l^2 + \tilde{\rho}\tau l^2 + l^\sharp)\tilde{g}_3(\zeta_3, U_3) + \mu l^4\eta_3(\zeta_3)\eta_3(U_3). \tag{38}$$

Hence the manifolds K_1, K_2 and K_3 are quasi-Einstein manifolds. Assertions (2) and (3) should be verified by the same calculus like the assertion (1). □

Theorem 3.7. *Let $\xi \in \mathfrak{X}(K)$ be a conformal vector field on K with factor ψ. Assume that the gradient of t and l are concurrent vector fields on K_1 and K. If $(K, \tilde{g}, \xi, \tilde{\lambda}, \mu)$ is an η-Ricci-Bourguignon soliton then we have the following conditions:*

(a) *K_1 is quasi-Einstein manifold with factors $\tilde{\lambda} + \tilde{\rho}\tau - \psi + \dfrac{k_2}{t} + \dfrac{k_3}{l}$ and μ.*

(b) *K_2 is quasi-Einstein manifold with factors $\tilde{\lambda}t^2 + \tilde{\rho}\tau t^2 - \psi t^2 + t^\sharp + \dfrac{k_3}{l}t^2$ and μt^4.*

(c) *K_3 is quasi-Einstein manifold with factors $\tilde{\lambda}l^2 + \tilde{\rho}\tau l^2 - \psi l^2 + l^\sharp$ and μl^4.*

Proof. The η-Ricci-Bourguignon soliton $(K, \tilde{g}, \xi, \tilde{\lambda}.\mu)$ with conformal factor ψ is given by

$$\mathrm{Ric} = (\tilde{\lambda} + \tilde{\rho}\tau - \psi)\tilde{g} + \mu\eta \otimes \eta. \tag{39}$$

Hence using (26), we get

$$\begin{aligned}
&{}^1\mathrm{Ric}(\zeta_1, U_1) - \tfrac{k_2}{t}\nabla_1^2 t(\zeta_1, U_1) - \tfrac{k_3}{l}\nabla^2 l(\zeta_1, U_1) +{}^2\mathrm{Ric}(\zeta_2, U_2) - t^\sharp \tilde{g}_2(\zeta_2, U_2) - \tfrac{k_3}{l}\nabla^2 l(\zeta_2, U_2) \\
&+{}^3\mathrm{Ric}(\zeta_3, U_3) - l^\sharp \tilde{g}_3(\zeta_3, U_3) \\
&= (\tilde{\lambda} + \tilde{\rho}\tau - \psi)\tilde{g}_1(\zeta_1, U_1) + \mu\eta_1(\zeta_1)\eta_1(U_1) + t^2(\tilde{\lambda} + \tilde{\rho}\tau - \psi)\tilde{g}_2(\zeta_2, U_2) + \mu t^4\eta_2(\zeta_2)\eta_2(U_2) \\
&+l^2(\tilde{\lambda} + \tilde{\rho}\tau - \psi)\tilde{g}_3(\zeta_3, U_3) + \mu l^4\eta_3(\zeta_3)\eta_3(U_3).
\end{aligned} \tag{40}$$

Using the fact that $\nabla^2 t_1 = \tilde{g}_1$ and $\nabla^2 l = \tilde{g}_1$, we obtain

$${}^1\mathrm{Ric}(\zeta_1, U_1) = (\tilde{\lambda} + \tilde{\rho}\tau - \psi + \tfrac{k_2}{t} + \tfrac{k_3}{l})\tilde{g}_1(\zeta_1, U_1) + \mu\eta_1(\zeta_1)\eta_1(U_1), \tag{41}$$

$${}^2\mathrm{Ric}(\zeta_2, U_2) = (\tilde{\lambda}t^2 + \tilde{\rho}\tau t^2 - \psi t^2 + t^\sharp + \tfrac{k_3}{l}t^2)\tilde{g}_2(\zeta_2, U_2) + \mu t^4\eta_2(\zeta_2)\eta_2(U_2) \tag{42}$$

and

$${}^3\mathrm{Ric}(\zeta_3, U_3) = (\tilde{\lambda}l^2 + \tilde{\rho}\tau l^2 - \psi l^2 + l^\sharp)\tilde{g}_3(\zeta_3, U_3) + \mu l^4\eta_3(\zeta_3)\eta_3(U_3). \tag{43}$$

Then, K_1 K_2 and K_3 are quasi-Einstein manifolds. □

The next corollary is deduced from Lemma 2.5:

Corollary 3.8. *Let $(K, \tilde{g}, \xi, \tilde{\lambda}, \mu)$ be a sequential warped product manifold η-Ricci-Bourguignon soliton. Then it is a quasi-Einstein manifold if the following statements hold:*

(a) *$\xi = \xi_3$ and ξ_3 is Killing vector field on K_3.*

(b) *ξ_1 is a Killing vector field on K_1, ξ_2 and ξ_3 are conformal vector fields on K_2 and K_3 with factors $-2\xi_1(\ln t)$ and $-2(\xi_1 + \xi_2)(\ln l)$, respectively.*

(c) *$\xi = \xi_2 + \xi_3$, ξ_2 and ξ_3 are Killing vector fields on K_2 and K_3, respectively and $\xi_2(l) = 0$.*

4. Application

Now, we would like to characterize η-Ricci-Bourguignon solitons on a standard static space-times and on generalized Robertson-Walker space-times within the framework of sequential warped products.

4.1. η-Ricci-Bourguignon Solitons on Sequential Warped Product Space-Times

Let $(K_1, \tilde{g}_1)$ and $(K_2, \tilde{g}_2)$ be two Riemannian manifolds of dimensions k_1 and k_2 and I an open, connected subinterval of $\mathbb{R}$ and dt^2 the Euclidean metric tensor on I. Then the product manifold $K = (K_1 \times K_2) \times I$ of dimension $(k_1 + k_2 + 1)$ equipped with the metric $\tilde{g} = (\tilde{g}_1 \oplus t^2\tilde{g}_2) \oplus l^2(-dt^2)$ is a *sequential standard static space-time* [15] and denoted by $K = (K_1 \times_t K_2) \times_l I$.

Lemma 4.1. *[15] Let K be a sequential standard space-time. Then for $\zeta_i, U_i \in \mathfrak{X}(K_i)$, $i = 1, 2$, we have*

1. $\nabla_{\zeta_1} U_1 = \nabla^1_{\zeta_1} U_1$,
2. $\nabla_{\zeta_1}\zeta_2 = \nabla_{\zeta_2}\zeta_1 = \zeta_1(\ln t)\zeta_2$,
3. $\nabla_{\zeta_2} U_2 = \nabla^2_{\zeta_2} U_2 - t\tilde{g}_2(\zeta_2, U_2)\nabla^1 t$,
4. $\nabla_{\zeta_i}\partial_t = \nabla_{\partial_t}\zeta_i = \zeta_i(\ln l)\partial_t$,
5. $\nabla_{\partial_t}\partial_t = l\nabla l$.

Lemma 4.2. *[15] Let K be a sequential standard space-time. Then for $\zeta_i, U_i \in \mathfrak{X}(K_i)$, $i = 1, 2$, we have*

1. $\mathrm{Ric}(\zeta_1, Y_1) =^1 \mathrm{Ric}(\zeta_1, U_1) - \frac{k_2}{t}\nabla^2 t(\zeta_1, U_1) - \frac{1}{l}\nabla^2 l(\zeta_1, U_1)$,
2. $\mathrm{Ric}(\zeta_2, U_2) =^2 \mathrm{Ric}(\zeta_2, U_2) - t^\sharp\tilde{g}_2(\zeta_2, U_2) - \frac{1}{l}\nabla^2 l(\zeta_2, U_2)$,
3. $\mathrm{Ric}(\partial_t, \partial_t) = l\Delta l$,
3. $\mathrm{Ric}(\zeta_i, U_j) = 0$, *for* $i \neq j$, *where* $t^\sharp = t\Delta^1 t + (k_2 - 1)\|\nabla^1 t\|^2$.

The following corollary is deduced from Lemma 2.5.

Corollary 4.3. *If ξ is a vector field on a sequential warped product standard space-time. Then we have*

$$(\pounds_\xi\tilde{g})(\zeta, U) = (\pounds^1_{\xi_1}\tilde{g}_1)(\zeta_1, U_1) + t^2(\pounds^2_{\xi_2}\tilde{g}_2)(\zeta_2, U_2) - 2l^2\frac{\partial v}{\partial t} + 2t\xi_1(t)\tilde{g}_2(\zeta_2, U_2) - 2l(\xi_1 + \xi_2)(l), \tag{44}$$

where $\xi = \xi_1 + \xi_2 + v\partial_t$, $\zeta = \zeta_1 + \zeta_2 + \partial_t$ *and* $U = U_1 + U_2 + \partial_t \in \mathfrak{X}(K)$.

For any $\zeta, U \in \mathfrak{X}(K)$ a sequential standard space-times is η-Ricci-Bourguignon soliton if

$$\mathrm{Ric}(\zeta, U) + \pounds_\xi\tilde{g}(\zeta, U) = (\tilde{\lambda} + \tilde{\rho}\tau)\tilde{g}(\zeta, U) + \mu\eta(\zeta)\eta(U). \tag{45}$$

We know that if ξ is Killing vector field then K is quasi-Einstein manifold of the form

$$\mathrm{Ric}(\zeta, U) = (\tilde{\lambda} + \tilde{\rho}\tau)\tilde{g}(\zeta, U) + \mu\eta(\zeta)\eta(U), \tag{46}$$

Then we have the following situations. Firstly

$$\mathrm{Ric}(\partial_t, \partial_t) = -l^2(\tilde{\lambda} + \tilde{\rho}\tau) + \mu v^2 l^4, \tag{47}$$

which give us

$$\tilde{\lambda} + \tilde{\rho}\tau = \mu v^2 l^2 - \frac{\Delta l}{l}. \tag{48}$$

Now taking the trace of (46), we get

$$\tau = (k_1 + k_2 + 1)[\mu v^2 l^2 - \frac{\Delta l}{l}] + \mu|\xi|^2.$$

Secondly,

$${}^1\mathrm{Ric}(\zeta_1, U_1) = (\tilde{\lambda} + \tilde{\rho}\tau)\tilde{g}_1(\zeta_1, U_1) - \tfrac{k_2}{t}\nabla^2 t(\zeta_1, U_1) - \tfrac{1}{l}\nabla^2 l(\zeta_1, U_1) + \mu\eta_1(\zeta_1)\eta_1(U_1)$$

and finally

$${}^2\mathrm{Ric}(\zeta_2, U_2) = \left(t^2\tilde{\lambda} + t^2\tilde{\rho}\tau + t^\sharp\right)\tilde{g}_2(\zeta_2, U_2) + \tfrac{1}{l}\nabla^2 l(\zeta_2, U_2) + \mu t^4\eta_2(\zeta_2)\eta_2(U_2).$$

Theorem 4.4. *Let $\xi \in \mathfrak{X}(K)$ be a Killing vector field on a sequential standard space-time. Then the scalar curvature r of K is given by*

$$\tau = (k_1 + k_2 + 1)\Big(\mu v^2 l^2 - \frac{\Delta l}{l}\Big) + \mu|\xi|^2. \tag{49}$$

Corollary 4.5. *Let $\xi \in \mathfrak{X}(K)$ be a Killing vector field on a sequential standard space-time. Then*

(a) *K_1 is quasi-Einstein with factors $\tilde{\lambda} + \tilde{\rho}\tau$ and μ if $k_2 l \nabla^2 t(\zeta_1, U_1) = -t\nabla^2 l(\zeta_1, U_1)$.*

(b) *K_2 is quasi-Einstein with factors $\tilde{\lambda}t^2 + \tilde{\rho}\tau t^2 + t^\sharp$ and μt^4 if $\nabla^2 l(\zeta_2, U_2) = 0$.*

We give the following theorem which comes from to Theorem 3.4:

Theorem 4.6. *Let $\xi \in \mathfrak{X}(K)$ be a potential vector field on a sequential standard static space-time. Assume that the gradient of t and l are concurrent vector fields on K_1 and K, respectively. If $(K, g, \xi, \lambda, \rho, \mu)$ is an η-Ricci-Bourguignon then we have the following conditions:*

(a) *$(K_1, \tilde{g}_1, \xi_1, \tilde{\lambda}_1, \mu)$ is an η-Ricci-Bourguignon soliton and $\tilde{\lambda}_1 + \tilde{\rho}_1\tau_1 + \mu = \tilde{\lambda} + \tilde{\rho}\tau + \frac{k_2}{t} + \frac{1}{l}$.*

(b) *K_2 is quasi-Einstein manifold if ξ_2 is a Killing vector field.*

(c) $-\frac{\Delta l}{l} + \frac{\partial v}{\partial t} + \frac{1}{l}(\xi_1 + \xi_2)(l) = \lambda + \rho r + \mu l^4 v^2.$

Proof. The η-Ricci-Bourguignon soliton $(K, \tilde{g}, \xi, \tilde{\lambda}, \mu)$ is given by

$$\mathrm{Ric} + \frac{1}{2}\pounds_\xi \tilde{g} = (\tilde{\lambda} + \tilde{\rho}\tau)\tilde{g} + \mu\eta \otimes \eta. \tag{50}$$

Using Lemma 4.2 and Corollary 4.3 for any vector fields ζ, U such that $\zeta = \zeta_1 + \zeta_2 + \partial_t$ and $U = U_1 + U_2 + \partial_t$, we have

$$\begin{aligned} &{}^1\mathrm{Ric}(\zeta_1, U_1) - \tfrac{k_2}{t}\nabla^2 t_1(\zeta_1, U_1) - \tfrac{k_3}{l}\nabla^2 l(\zeta_1, U_1) + {}^2\mathrm{Ric}(\zeta_2, U_2) - t^\sharp \tilde{g}_2(\zeta_2, U_2) - \tfrac{1}{l}\nabla^2 l(\zeta_2, U_2) + l\Delta l \\ &+ \tfrac{1}{2}(\pounds^1_{\xi_1}\tilde{g}_1)(\zeta_1, U_1) + \tfrac{t^2}{2}(\pounds^2_{\xi_2}\tilde{g}_2)(\zeta_2, U_2) - l^2\tfrac{\partial v}{\partial t} + t\xi_1(t)\tilde{g}_2(\zeta_2, U_2) - l(\xi_1 + \xi_2)(l) \\ &= (\tilde{\lambda} + \tilde{\rho}\tau)\tilde{g}_1(\zeta_1, U_1) + \mu\eta_1(\zeta_1)\eta_1(U_1) + t^4(\tilde{\lambda} + \tilde{\rho}\tau)\tilde{g}_2(\zeta_2, U_2) + \mu t^2\eta_2(\zeta_2)\eta_2(U_2) - (\tilde{\lambda} + \tilde{\rho}\tau)l^2 + \mu l^4 v^2. \end{aligned} \tag{51}$$

Then separately, we obtain

$${}^1\mathrm{Ric}(\zeta_1, U_1) - \tfrac{k_2}{t}\tilde{g}_1(\zeta_1, U_1) - \tfrac{1}{l}\tilde{g}_1(\zeta_1, U_1) + \tfrac{1}{2}(\pounds^1_{\xi_1}\tilde{g}_1)(\zeta_1, U_1) = (\tilde{\lambda} + \tilde{\rho}\tau)\tilde{g}_1(\zeta_1, U_1) + \mu\eta(\zeta_1)\eta(U_1). \tag{52}$$

Therefore following the same methods as in the Theorem 3.4, we conclude that $(K_1, \tilde{g}_1, \xi_1, \tilde{\lambda}_1, \mu)$ is an η-Ricci-Bourguignon soliton. We have

$$\begin{aligned} &{}^2\mathrm{Ric}(\zeta_2, U_2) - t^\sharp \tilde{g}_2(\zeta_2, U_2) - \tfrac{1}{l}\nabla^2 l(\zeta_2, U_2) + \tfrac{1}{2}t^2(\pounds^2_{\xi_2}\tilde{g}_2)(\zeta_2, U_2) + t\xi_1(t)\tilde{g}_2(\zeta_2, U_2) \\ &= t^2(\tilde{\lambda} + \tilde{\rho}\tau)\tilde{g}_2(\zeta_2, U_2) + \mu t^2\eta_2(\zeta_2)\eta_2(U_2). \end{aligned} \tag{53}$$

If ξ_2 is a Killing vector field, then K_2 is quasi-Einstein manifold. Therefore from (51), we obtain

$$l\Delta l - l^2\frac{\partial v}{\partial t} - l(\xi_1 + \xi_2)(l) = -(\tilde{\lambda} + \tilde{\rho}\tau)l^2 + \mu l^4 v^2, \tag{54}$$

which allows us to conclude the proof. □

The next theorem is deduced from Theorem 3.7, Corollary 3.8.

Theorem 4.7. *Let $\xi \in \mathfrak{X}(K)$ be a conformal vector field on a sequential standard space-time with factor ψ and $(K, \tilde{g}, \xi, \tilde{\lambda}, \mu)$ is an η-Ricci-Bourguignon soliton. Assume that the gradient of t and l are concurrent vector fields on K_1 and K. Then K_1 and K_2 are quasi-Einstein manifolds with factors $\alpha_1 = -\frac{\Delta l}{l} + \mu l^2 v^2 + \frac{k_2}{t} + \frac{1}{l}$, $\alpha_2 = \mu$ and $\beta_1 = -\frac{\Delta l}{l} t^2 + \mu l^2 v^2 t^4 + t^\sharp + \frac{1}{l} t^2$, $\beta_2 = \mu t^4$, respectively.*

Proof. Assume that $(K, \tilde{g}, \xi, \lambda, \rho, \mu)$ be an η-Ricci-Bourguignon soliton and ξ is a conformal vector field with factor ψ. Then we have

$$\mathrm{Ric} = (\tilde{\lambda} + \tilde{\rho}\tau - \psi)\tilde{g} + \mu\eta \otimes \eta. \tag{55}$$

Then for any $X, Y \in \mathfrak{X}(K)$, we get

$$\begin{aligned}&\mathrm{Ric}^1(\zeta_1, U_1) - \tfrac{k_2}{t}\tilde{g}_1(\zeta_1, U_1) - \tfrac{1}{l}\tilde{g}_1(\zeta_1, U_1) +^2 \mathrm{Ric}(\zeta_2, U_2) - t^\sharp\tilde{g}_2(\zeta_2, U_2) - \tfrac{1}{l}t^2 g_2(X_2, Y_2) + l\Delta l \\ &= (\tilde{\lambda} + \tilde{\rho}\tau - \psi)\tilde{g}_1(\zeta_1, U_1) + \mu\eta(\zeta_1)\eta(U_1) + t^2(\tilde{\lambda} + \tilde{\rho}\tau - \psi)\tilde{g}_2(\zeta_2, U_2) \\ &+\mu t^4\eta_2(\zeta_2)\eta(U_2) - (\tilde{\lambda} + \tilde{\rho}\tau - \psi)l^2 + \mu l^4 v^2.\end{aligned} \tag{56}$$

Hence we find

$$^1\mathrm{Ric}(\zeta_1, U_1) = (\tilde{\lambda} + \tilde{\rho}\tau - \psi + \tfrac{k_2}{t} + \tfrac{1}{l})\tilde{g}_1(\zeta_1, U_1) + \mu\eta(\zeta_1)\eta(U_1). \tag{57}$$

$$^2\mathrm{Ric}(\zeta_2, U_2) = (\tilde{\lambda}t^2 + \tilde{\rho}\tau t^2 - \psi t^2 + t^\sharp + \tfrac{1}{l}t^2)\tilde{g}_2(\zeta_2, U_2) + \mu t^4\eta_2(\zeta_2)\eta(U_2) \tag{58}$$

and $= \tilde{\lambda} + \tilde{\rho}\tau - \psi = -\frac{\Delta l}{l} + \mu l^2 v^2$. Hence the proof is completed. □

4.2. η-Ricci-Bourguignon Solitons on Sequential Warped Product generalized Robertson-Walker space-times.

Let $(K_2, \tilde{g}_2)$ and $(K_3, \tilde{g}_3)$ be two Riemannian manifolds of dimension k_2 and k_3, respectively and t and l are positive smooth functions on K_2 and $I \times K_2$. The sequential generalized Robertson-Walker space-time is a product manifold $K = (I \times_t K_2) \times_l K_3$, endowed with the metric tensor $\tilde{g} = (-dt^2 \oplus t^2\tilde{g}_2) \oplus l^2\tilde{g}_3$. [15].

Lemma 4.8. *[15] Let K be a sequential generalized Robertson-Walker space-time. Then for $\zeta_i, U_i \in \mathfrak{X}(K_i)$, $i = 2, 3$, we have*

1. $\nabla_{\partial_t}\partial_t = 0$,
2. $\nabla_{\partial_t}\zeta_i = \nabla_{\zeta_i}\partial_t = \frac{\dot{t}}{t}\zeta_i$,
3. $\nabla_{\zeta_2}U_2 = \nabla^2_{\zeta_2}U_2 - t\dot{t}\tilde{g}_2(\zeta_2, U_2)\partial_t$,
4. $\nabla_{\zeta_2}\zeta_3 = \nabla_{\zeta_3}\zeta_2 = \zeta_2(\ln l)\zeta_3$,
5. $\nabla_{\zeta_3}U_3 = \nabla^2_{\zeta_3}U_3 - l\tilde{g}_3(\zeta_3, U_3)\nabla l$.

Lemma 4.9. *[15] Let K be a sequential generalized Robertson-Walker space-time. Then for $\zeta_i, U_i \in \mathfrak{X}(K_i)$, $i = 2, 3$, we have*

1. $\mathrm{Ric}(\partial_t, \partial_t) = \frac{k_2}{t}\ddot{t} + \frac{k_3}{l}\frac{\partial^2 l}{\partial t^2}$,
2. $\mathrm{Ric}(\zeta_2, U_2) =^2 \mathrm{Ric}(\zeta_2, U_2) - t^\sharp\tilde{g}_2(\zeta_2, U_2) - \frac{k_3}{l}\nabla^2 l(\zeta_2, U_2)$,
3. $\mathrm{Ric}(\zeta_3, U_3) =^3 \mathrm{Ric}(\zeta_3, U_3) - l^\sharp\tilde{g}_3(\zeta_3, U_3)$,
4. $\mathrm{Ric}(\zeta_i, U_j) = 0$, *for $i \neq j$, where $t^\sharp = -t\ddot{t} - (k_2 - 1)\dot{t}^2$ and $l^\sharp = l\Delta l + (k_3 - 1)||\nabla l||^2$.*

From Lemma 2.5, we deduced the following corollary:

Corollary 4.10. *If $(K, \tilde{g})$ is a sequential generalized Robertson-Walker space-time. Then we have*

$$(\pounds_\xi \tilde{g})(\zeta, U) = -2\frac{\partial v}{\partial t} + t^2(\pounds^2_{\xi_2}\tilde{g}_2)(\zeta_2, U_2) + l^2(\pounds^3_{\xi_3}\tilde{g}_3)(\zeta_3, U_3) + 2vt\frac{\partial t}{\partial t}\tilde{g}_2(\zeta_2, U_2) + 2vl(\frac{\partial l}{\partial t} + \xi_2(l))\tilde{g}_3(\zeta_3, U_3), \quad (59)$$

where $\xi = v\partial_t + \xi_2 + \xi_3$, $\zeta = \partial_t + \zeta_2 + \zeta_3$ and $U = \partial_t + U_2 + U_3 \in \mathfrak{X}(K)$.

K is η-Ricci-Bourguignon soliton if

$$\mathrm{Ric}(\zeta, U) + \pounds_\xi \tilde{g}(\zeta, U) = (\tilde{\lambda} + \tilde{\rho}\tau)\tilde{g}(\zeta, U) + \mu\eta(\zeta)\eta(U). \quad (60)$$

We know that if ξ is Killing vector field then K is quasi-Einstein manifold of the form

$$\mathrm{Ric}(\zeta, U) = (\tilde{\lambda} + \tilde{\rho}\tau)\tilde{g}(\zeta, U) + \mu\eta(\zeta)\eta(U). \quad (61)$$

Then we get the following situations. Firstly

$$\mathrm{Ric}(\partial_t, \partial_t) = -(\tilde{\lambda} + \tilde{\rho}\tau) + \mu v^2, \quad (62)$$

which imply from Lemma 4.9

$$\frac{k_2}{t}\ddot{t} + \frac{k_3}{l}\frac{\partial^2 l}{\partial t^2} = -(\tilde{\lambda} + \tilde{\rho}\tau) + \mu v^2.$$

Secondly

$$t^2(\lambda + \rho r)\tilde{g}_2(\zeta_2, U_2) + \mu t^4\eta_2(\zeta_2)\eta_2(U_2) =^2 \mathrm{Ric}(\zeta_2, U_2) - t^\sharp\tilde{g}_2(\zeta_2, U_2) - \tfrac{k_3}{l}\nabla^2 l(\zeta_2, U_2).$$

Then

$$^2\,\mathrm{Ric}(\zeta_2, U_2) = \left(t^2\tilde{\lambda} + t^2\tilde{\rho}\tau + t^\sharp\right)\tilde{g}_2(\zeta_2, U_2) + \tfrac{k_3}{l}\nabla^2 l(\zeta_2, U_2) + \mu t^4\eta_2(\zeta_2)\eta_2(U_2)$$

and finally

$$^3\,\mathrm{Ric}(\zeta_3, U_3) = \left(l^2\tilde{\lambda} + l^2\tilde{\rho}\tau + l^\sharp\right)\tilde{g}_3(\zeta_3, U_3) + \mu l^4\eta_3(\zeta_3)\eta_3(U_3).$$

Theorem 4.11. *Let ξ be a Killing vector field on a sequential generalized Robertson-Walker space-time. Then we have the following situations:*

(a) $\frac{k_2}{t}\ddot{t} + \frac{k_3}{l}\frac{\partial^2 l}{\partial t^2} = -(\tilde{\lambda} + \tilde{\rho}\tau) + \mu v^2$,

(b) *$(K_2, \tilde{g}_2)$ is quasi-Einstein manifold with factors $t^2\tilde{\lambda} + t^2\tilde{\rho}\tau + t^\sharp$ and μt^4 if $\nabla^2 l(\zeta_2, U_2) = 0$ for any $\zeta_2, U_2 \in \mathfrak{X}(K_2)$ and*

(c) *$(K_3, \tilde{g}_3)$ is quasi-Einstein manifold with factors $l^2\tilde{\lambda} + l^2\tilde{\rho}\tau + l^\sharp$ and μl^4.*

The following theorem is an application of Theorem 3.4

Theorem 4.12. *Let ξ be a potential vector field on a sequential generalized Robertson-Walker space-time and $(K, \tilde{g}, \xi, \lambda, \mu)$ is an η-Ricci-Bourguignon soliton. Assume that the gradient of l is concurrent vector field on K. Then we have the following situations:*

(a) *$(K_2, \tilde{g}_2, t^2\xi_2, \tilde{\lambda}_2, \mu)$ is an η-Ricci-Bourguignon soliton with $\tilde{\lambda}_2 + \rho_2\tau_2 + \mu t^4 = \tilde{\lambda}t^2 + \tilde{\rho}\tau t^2 + t^\sharp - vt\dot{t} + \frac{k_3}{l}$.*

(b) *$(K_3, \tilde{g}_3, l^2\xi_3, \tilde{\lambda}_3, \mu)$ is an η-Ricci-Bourguignon soliton with $\tilde{\lambda}_3 + \tilde{\rho}_3\tau_3 + \mu l^4 = \tilde{\lambda}l^2 + \tilde{\rho}\tau l^2 + l^\sharp - vl\frac{\partial l}{\partial t} - vl\xi_2(l)$.*

Proof. For proving, It enough to use lemma 4.2 and Corollary 4.10. □

The following theorem is a consequence of Theorem 4.7

Theorem 4.13. *Let ξ be a conformal vector field on a sequential generalized Roberstson space-time and $(K, g, \xi, \lambda, \mu, \rho)$ is an η-Ricci-Bourguignon soliton. Assume that the gradient of l is concurrent vector field on K. Then K_2 and K_3 are quasi-Einstein manifolds with factors $\alpha_1 = (-\frac{k_2}{t}\ddot{t} - \frac{k_3}{l}\frac{\partial^2 l}{\partial^2 t})t^2 + \mu v^2 + t^\sharp + \frac{k_2}{l}$, $\alpha_2 = \mu t^4$ and $\beta_1 = (-\frac{k_2}{t}\ddot{t} - \frac{k_3}{l}\frac{\partial^2 l}{\partial^2 t})l^2 + \mu v^2 + l^\sharp$, $\beta_2 = \mu l^4$, respectively.*

Proof. The proof is similar to the proof of Theorem 4.7. □

Declarations

Conflicts of interests.
The authors declare that there is no conflict of interests.

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