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A new classification for almost $C(\alpha)$-manifolds

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Abstract. In this paper, D-conformal curvature tensor on an almost $C(\alpha)$-manifold has been considered. At first, a few fundamental geometric properties of D-conformal curvature tensor on an almost $C(\alpha)$-manifold have been obtained with some interesting outcomes. Moreover, the necessary and sufficient conditions have been addressed for almost $C(\alpha)$-manifold to be D-conformally flat, D-conformally semi-symmetric, $B(\xi, X_1)\widetilde{Z} = 0$, $B(\xi, X_1)P = 0$ and $B(\xi, X_1)S = 0$. It is shown that a D-Conformal flat $C(\alpha)$ manifold reduces to the Kenmotsu manifold and a D-Conformal Semi Symmetric $C(\alpha)$ manifold reduces to the co-Keahler manifold. At the end, an example of an almost $C(\alpha)$ manifold has been presented.

1. Introduction

Almost $C(\alpha)$-manifolds are given as sub-class of almost contact metric manifolds or almost co-Hermitian manifolds. In addition, almost $C(\alpha)$-manifolds are general case of co-Keahler, Kenmotsu and Sasakian manifolds. That is, if $\alpha = 0$, an almost $C(\alpha)$-manifold corresponds to co-Keahler, for $\alpha = 1$, it is Sasakian and it corresponds to Kenmotsu manifold for $\alpha = -1$ [12]. For characterization of special manifold types, generally the covariant derivative of C^∞ $(1, 1)$-type tensor field ϕ is utilized. However, this derivative could have not been defined yet for almost $C(\alpha)$-manifolds. Therefore, investigation of under what conditions an almost $C(\alpha)$-manifold reduces to a co-Keahler, Sasakian, or Kenmotsu manifold is currently a hot topic in the literature.

To reduce an almost $C(\alpha)$ manifold to a more specific type of manifold, i.e. co-Keahler, Sasakian, Kenmotsu, or Einstein, certain curvature conditions should be satisfied. Although there have been several works which address the curvature properties of Keahler, Sasakian and Kenmotsu manifolds, studies on almost $C(\alpha)$-manifold, which is the more general case of these special manifold types, are quite limited.

One of the pioneering studies on almost contact structures and curvature tensors was presented by D. Janssens and L. Vanhecke in 1981 [12]. The paper by M. Atçeken and Ü. Yıldırım can be given as one of the recent studies on curvature tensors on almost C(α) manifolds [1–3]. In addition, T. Mert has studied pseudo-symmetry conditions for the classification of an almost C(α) manifold [13–16].

Let (M, g, ϕ, ξ, η) be an almost co-Hermitian manifold with Riemann connection ∇. Then

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(i)-M is co-Keahlerian if and only if $\nabla_{\phi} = 0$.

(ii)-M is Sasakian if and only if

$\forall X_1, X_2 \in \chi(M) : (\nabla_{X_1}\phi)X_2 = g(X_1, X_2)\xi - \eta(X_2)X_1.$

(iii)-M is Kenmotsu manifold if and only if

$\forall X_1, X_2 \in \chi(M) : (\nabla_{X_1}\phi)X_2 = g(\phi X_1, X_2)\xi - \eta(X_2)\phi X_1.$

Let M be a $(2n+1)$-dimensional C^∞-manifold, $\chi(M)$ be the space of the vector fields on M and $X_1, X_2, X_3, X_5 \in \chi(M)$. If M admits a tensor field ϕ of type $(1,1)$, a vector field ξ and a 1-form η satisfying

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{1}$$

where I denotes the identity transformation, then it is said that M has an almost contact structure. A manifold M with an almost contact structure admits a Riemannian metric g such that

$$g(\phi X_1, \phi X_2) = g(X_1, X_2) - \eta(X_1)\eta(X_2), \tag{2}$$

and M is said to be an almost contact metric manifold.

An almost contact metric manifold is said to be normal if ,

$$[\phi, \phi](X_1, X_2) + 2d\eta(X_1, X_2)\xi = 0, \tag{3}$$

where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ.

Almost contact metric structures (M, ϕ, ξ, η, g) are an almost $C(\alpha)$-manifold if the Riemannian curvature tensor R satisfies the following equality

$$\begin{aligned} R(X_1, X_2, X_3, X_5) &= R(X_1, X_2, \phi X_3, \phi X_5) + \alpha\Big[- g(X_1, X_3)g(X_2, X_5) \\ &+ g(X_1, X_5)g(X_2, X_3) + g(X_1, \phi X_3)g(X_2, \phi X_5) \\ &- g(X_1, \phi X_5)g(X_2, \phi X_3)\Big] \end{aligned} \tag{4}$$

for all $X_1, X_2, X_3, X_5 \in \chi(M)$. Moreover, if such a manifold has constant ϕ-sectional curvature equal to c, then its curvature tensor is given by

$$\begin{aligned} R(X_1, X_2)X_3 &= \Big(\frac{c+3\alpha}{4}\Big)\Big[g(X_2, X_3)X_1 - g(X_1, X_3)X_2\Big] \\ &+ \Big(\frac{c-\alpha}{4}\Big)\Big[g(X_1, \phi X_3)\phi X_2 - g(X_2, \phi X_3)\phi X_1 + 2g(X_1, \phi X_2)\phi X_3\Big] \\ &+ \Big(\frac{c-\alpha}{4}\Big)\Big[\eta(X_1)\eta(X_3)X_2 - \eta(X_2)\eta(X_3)X_1 \\ &+ g(X_1, X_3)\eta(X_2)\xi - g(X_2, X_3)\eta(X_1)\xi\Big]. \end{aligned} \tag{5}$$

Also, a normal almost $C(\alpha)$-manifold is called $C(\alpha)$-manifold.

2. Preliminaries

In 1983, Chuman defined a tensor field B on a n-dimensional Riemannian manifold, (M^n, g), $(n > 4)$ as

$$\begin{aligned} B(X_1, X_2)X_3 &= R(X_1, X_2)X_3 + \frac{1}{n-3}\Big[S(X_1, X_3)X_2 - S(X_2, X_3)X_1 + g(X_1, X_3)QX_2 - g(X_2, X_3)QX_1 \\ &+ S(X_2, X_3)\eta(X_1)\xi - S(X_1, X_3)\eta(X_2)\xi + \eta(X_2)\eta(X_3)QX_1 - \eta(X_1)\eta(X_3)QX_2\Big] \\ &- \frac{K-2}{n-3}\Big[g(X_1, X_3)X_2 - g(X_2, X_3)X_1\Big] \\ &+ \frac{K}{n-3}\Big[g(X_1, X_3)\eta(X_2)\xi - g(X_2, X_3)\eta(X_1)\xi + \eta(X_1)\eta(X_3)X_2 - \eta(X_2)\eta(X_3)X_1\Big]. \end{aligned} \tag{6}$$

Such a tensor field B is known as D-conformal curvature tensor, where $K = \frac{r+2(n-1)}{n-2}$, R is Riemannian curvature tensor, Q is the Ricci operator, S is the Ricci tensor and r is the scalar curvature of M.

Let (M, g) be $(2n+1)$-dimensional Riemannian manifold. Then the concircular curvature tensor $\widetilde{Z}$ and the projective curvature tensor P are defined by

$$\widetilde{Z} = R(X_1, X_2)X_3 - \frac{r}{2n(2n+1)}\Big[g(X_2, X_3)X_1 - g(X_1, X_3)X_2\Big], \tag{7}$$

and

$$P(X_1, X_2)X_3 = R(X_1, X_2)X_3 - \frac{1}{n-1}\Big[S(X_2, X_3)X_1 - S(X_1, X_3)X_2\Big], \tag{8}$$

where S is the Ricci tensor and r is the scalar curvature of M.

In a $(2n+1)$-dimensional almost $C(\alpha)$-manifold the following relations are satisfied:

$$R(\xi, X_2)X_3 = \alpha\Big[g(X_2, X_3)\xi - \eta(X_3)X_2\Big], \tag{9}$$

$$R(X_1, X_2)\xi = \alpha\Big[\eta(X_2)X_1 - \eta(X_1)X_2\Big], \tag{10}$$

$$R(X_1, \xi)\xi = \alpha\Big[X_1 - \eta(X_1)\xi\Big], \tag{11}$$

$$\eta(R(X_1, X_2)X_3) = \alpha\Big[g(X_2, X_3)\eta(X_1) - g(X_1, X_3)\eta(X_2)\Big], \tag{12}$$

$$\widetilde{Z}(\xi, X_2)X_3 = \Big[\alpha - \frac{r}{2n(2n+1)}\Big]\{g(X_2, X_3)\xi - \eta(X_3)X_2\Big], \tag{13}$$

$$P(\xi, X_2)X_3 = \alpha g(X_2, X_3)\xi - \frac{1}{2n}S(X_2, X_3)\xi, \tag{14}$$

$$B(X_2, X_3)\xi = \Big[\frac{\alpha+1}{1-n}\Big]\Big[\eta(X_3)X_2 - \eta(X_2)X_3\Big], \tag{15}$$

and

$$B(\xi, X_2)X_3 = \Big[\frac{\alpha+1}{1-n}\Big]\Big[g(X_2, X_3)\xi - \eta(X_3)X_2\Big]. \tag{16}$$

Also, from (5), we can state

$$\begin{aligned} R(X_1, e_i)e_i + R(X_1, \phi e_i)\phi e_i + R(X_1, \xi)\xi &= \sum_{i=1}^{n}\{(\frac{3\alpha + c}{4})\{nX_1 - g(X_1, e_i)e_i + nX_1 \\ &- g(X_1, \phi e_i)\phi e_i + X_1 - g(X_1, \xi)\xi\} \\ &+ (\frac{c-\alpha}{4})\{3g(X_1, \phi e_i)\phi e_i - 2n\eta(X_1)\xi \\ &+ 3g(X_1, \phi^2 e_i)\phi^2 e_i \eta(X_1)\xi - X_1\}\} \end{aligned} \tag{17}$$

for $\{e_1, e_2, ..., e_n, \phi e_1, \phi e_2, ..., \phi e_n, \xi\}$ orthonormal basis of M. From (17), for $X_2 = \xi \in \chi(M)$, we obtain

$$S(X_1, X_2) = \left(\frac{\alpha(3n-1) + c(n+1)}{2}\right)g(X_1, X_2) + \left(\frac{(\alpha - c)(n+1)}{2}\right)\eta(X_1)\eta(X_2) \tag{18}$$

which is equivalent to

$$QX_1 = \left(\frac{\alpha(3n-1) + c(n+1)}{2}\right)X_1 + \left(\frac{(\alpha - c)(n+1)}{2}\right)\eta(X_1)\xi. \tag{19}$$

Corollary 2.1. *An almost $C(\alpha)$-manifold is always an η-Einstein manifold.*

Also, from (18), we can easily see

$$r = n[\alpha(3n+1) + c(n+1)] \tag{20}$$

and

$$S(X_1, \xi) = 2n\alpha\eta(X_1). \tag{21}$$

This yields to

$$Q\xi = 2n\alpha\xi. \tag{22}$$

3. D-Conformally Flat $C(\alpha)$-manifolds

Theorem 3.1. *Let M be $(2n+1)$-dimensional an almost $C(\alpha)$-manifold. Then M is D-conformally flat if and only if M reduce a Kenmotsu manifold.*

Proof. Let an almost $C(\alpha)$-manifold M be a D-conformally flat, for $X_1, X_2, X_3 \in \chi(M)$, then we have □

$$B(X_1, X_2)X_3 = 0. \tag{23}$$

In (23), choosing $Z = \xi$ we obtain

$$\begin{aligned}
0 &= R(X_1, X_2)\xi + \frac{1}{2n-2}\Big[S(X_1,\xi)X_2 - S(X_2,\xi)X_1 + g(X_1,\xi)QX_2 - g(X_2,\xi)QX_1 \\
&+ S(X_2,\xi)\eta(X_1)\xi - S(X_1,\xi)\eta(X_2)\xi + \eta(X_2)\eta(\xi)QX_1 - \eta(X_1)\eta(\xi)QX_2\Big] \\
&- \frac{K-2}{2n-2}\Big[g(X_1,\xi)X_2 - g(X_2,\xi)X_1\Big] \\
&+ \frac{K}{2n-2}\Big[g(X_1,\xi)\eta(X_2)\xi - g(X_2,\xi)\eta(X_1)\xi + \eta(X_1)\eta(\xi)X_2 - \eta(X_2)\eta(\xi)X_1\Big].
\end{aligned} \tag{24}$$

Using (1) and (21) in (24), we obtain

$$\begin{aligned}
0 &= \alpha\Big[\eta(X_2)X_1 - \eta(X_1)X_2\Big] + \frac{1}{2n-2}\Big[2n\alpha\eta(X_1)X_2 - 2n\alpha\eta(X_2)X_1\Big] \\
&- \frac{K-2}{2n-2}\Big[\eta(X_1)X_2 - \eta(X_2)X_1\Big] + \frac{K}{2n-2}\Big[\eta(X_1)X_2 - \eta(X_2)X_1\Big].
\end{aligned} \tag{25}$$

By direct calculations, we conclude that

$$\frac{\alpha+1}{n-1} = 0. \tag{26}$$

So, for $\alpha = -1$, M is an almost $C(-1)$ (Kenmotsu) manifold.

4. D-Conformally Semi-Symmetric Almost $C(\alpha)$-Manifolds

Theorem 4.1. *Let M be $(2n+1)$-dimensional an almost $C(\alpha)$-manifold. Then M is D-conformally semi-symmetric if and only if M either reduce a Kenmotsu manifold or it is a Keahler manifold.*
Proof. □

We suppose that $(2n+1)$-dimensional almost $C(\alpha)$-manifold M is D-conformally semi-symmetric. Then we have

$$(R(X_1, X_2)B)(X_4, X_5)X_3 = 0. \tag{27}$$

From (27) we obtain

$$\begin{aligned}(R(X_1, X_2)B)(X_4, X_5)X_3 &= R(X_1, X_2)B(X_4, X_5)X_3 - B(R(X_1, X_2)X_4, X_5)X_3 \\ &- B(X_4, R(X_1, X_2)X_5)X_3 - B(X_4, X_5)R(X_1, X_2)X_3,\end{aligned} \tag{28}$$

for all $X_1, X_2, X_3, X_4, X_5 \in \chi(M)$. In (28), choosing $X_1 = \xi$ and using the equation (9) we obtain

$$\begin{aligned}0 &= \alpha\Big[g(X_2, B(X_4, X_5)X_3)\xi - \eta\Big(B(X_4, X_5)X_3\Big)X_2 \\ &- B\Big(\alpha\Big[g(X_2, X_4)\xi - \eta(X_4)X_2\Big], X_5\Big)X_3 \\ &- B\Big(X_4, \alpha\Big[g(X_2, X_5)\xi - \eta(X_5)X_2\Big]\Big)X_3 \\ &- B(X_4, X_5)\Big(\alpha\Big[g(X_2, X_3)\xi - \eta(X_3)X_2\Big]\Big)\Big].\end{aligned} \tag{29}$$

Now, putting $X_4 = \xi$ in (29) and using the equations (15) and (16) we obtain

$$\begin{aligned}0 &= \alpha B(X_2, X_5)X_3 \\ &+ \alpha\Big(\frac{\alpha+1}{1-n}\Big)\Big[g(X_5, X_3)\eta(X_2)\xi - g(X_5, X_3)X_2 + g(X_2, X_3)X_5 - \eta(X_3)\eta(X_2)X_5\Big].\end{aligned} \tag{30}$$

In the same way, choosing $X_3 = \xi$ in (30) and using (15) we conclude

$$\alpha\Big(\frac{\alpha+1}{n-1}\Big) = 0. \tag{31}$$

So, M either reduces a Kenmotsu manifold ($\alpha = -1$) or it is a Keahler manifold ($\alpha = 0$). The converse is trivial. The proof is complete.

5. Curvature Conditions $B(\xi, X_2)\widetilde{Z} = 0$, $B(\xi, X_2)P = 0$ and $B(\xi, X_2)S = 0$

Theorem 5.1. *Let M be $(2n+1)$-dimensional an almost $C(\alpha)$-manifold. Then $B(\xi, X_2)\widetilde{Z} = 0$ if and only if M either reduce a Kenmotsu manifold or real space form with constant sectional curvature $c = \Big(2\alpha - \frac{r}{n(2n+1)}\Big)$.*

Proof. Let $\Big(B(\xi, X_2)\widetilde{Z}\Big)(X_4, X_5)X_3 = 0$ be on $(2n+1)$-dimensional almost $C(\alpha)$-manifold M, for any $X_2, X_3, X_4, X_5 \in \chi(M)$, then we have □

$$\begin{aligned}0 &= B(\xi, X_2)\widetilde{Z}(X_4, X_5)X_3 - \widetilde{Z}(B(\xi, X_2)X_4, X_5)X_3 \\ &- \widetilde{Z}(X_4, B(\xi, X_2)X_5)X_3 - \widetilde{Z}(X_4, X_5)B(\xi, X_2)X_3.\end{aligned} \tag{32}$$

In (32), using (16) we obtain

$$\begin{aligned}0 &= \Big(\frac{\alpha+1}{1-n}\Big)\Big(g(X_2, \widetilde{Z}(X_4, X_5)X_3)\xi - \eta(\widetilde{Z}(X_4, X_5)X_3)X_2\Big) \\ &- \Big(\frac{\alpha+1}{1-n}\Big)\widetilde{Z}\Big(g(X_2, X_4)\xi - \eta(X_4)X_2, X_5\Big)X_3 \\ &- \Big(\frac{\alpha+1}{1-n}\Big)\widetilde{Z}\Big(X_4, \Big(g(X_2, X_5)\xi - \eta(X_5)X_2\Big)\Big) \\ &- \Big(\frac{\alpha+1}{1-n}\Big)\widetilde{Z}(X_4, X_5)\Big(g(X_2, X_3)\xi - \eta(X_3)X_2\Big).\end{aligned} \tag{33}$$

In (33) using (13), we have

$$\begin{aligned} 0 &= \Big(\frac{\alpha+1}{1-n}\Big)\Big[\widetilde{Z}(X_2,X_5)X_3 \\ &+ \Big(\alpha - \frac{r}{2n(2n+1)}\Big)\Big[g(X_5,X_3)X_2 - g(X_2,X_3)X_5\Big]\Big]. \end{aligned} \tag{34}$$

With help to (7), we get

$$0 = \Big(\frac{\alpha+1}{1-n}\Big)\Big[R(X_2,X_5)X_3 - \Big(\alpha - \frac{r}{2n(2n+1)}\Big)\Big[g(X_5,X_3)X_2 - g(X_2,X_3)X_5\Big]\Big]. \tag{35}$$

This shows that the manifold reduces a Kenmotsu manifold (for $\frac{\alpha+1}{1-n} = 0$). On the other hand, from (35) we have $R(X_2,X_5)X_3 = \Big(\alpha - \frac{r}{2n(2n+1)}\Big)\Big[g(X_5,X_3)X_2 - g(X_2,X_3)X_5\Big]$. So, M is a real space form with constant sectional curvature $c = \Big(2\alpha - \frac{r}{n(2n+1)}\Big)$.

Theorem 5.2. *Let M be $(2n+1)$-dimensional an almost $C(\alpha)$-manifold. Then $B(\xi, X_2)P = 0$ if and only if M either reduce a Kenmotsu manifold or an Einstein manifold.*

Proof. Assume that $(B(\xi,X_2)P)(X_4,X_5)X_3 = 0$ for all $X_2, X_3, X_4, X_5 \in \chi(M)$, then we have □

$$\begin{aligned} 0 &= B(\xi,X_2)P(X_4,X_5)X_3 - P(B(\xi,X_2)X_4,X_5)X_3 \\ &- P(X_4,B(\xi,X_2)X_5)X_3 - P(X_4,X_5)B(\xi,X_2)X_3. \end{aligned} \tag{36}$$

In (36), using (16) we obtain

$$\begin{aligned} 0 &= \Big(\frac{\alpha+1}{1-n}\Big)\Big(g(X_2,P(X_4,X_5)X_3)\xi - \eta(P(X_4,X_5)X_3)X_2\Big) \\ &- \Big(\frac{\alpha+1}{1-n}\Big)P\Big(g(X_2,X_4)\xi - \eta(X_4)X_2, X_5\Big)X_3 \\ &- \Big(\frac{\alpha+1}{1-n}\Big)P\Big(X_4, \Big(g(X_2,X_5)\xi - \eta(X_5)X_2\Big)\Big) \\ &- \Big(\frac{\alpha+1}{1-n}\Big)P(X_4,X_5)\Big(g(X_2,X_3)\xi - \eta(X_3)X_2\Big). \end{aligned} \tag{37}$$

Putting $X_4 = \xi$ in (37), we get

$$\begin{aligned} 0 &= \Big(\frac{\alpha+1}{1-n}\Big)\Big[\Big[g(X_2,P(\xi,X_5)X_3)\xi - \eta(P(\xi,X_5)X_3)X_2\Big] \\ &- \eta(X_2)P(\xi,X_5)X_3 + \eta(X_5)P(\xi,X_2)X_3 + \eta(X_3)P(\xi,X_5)X_2 + P(X_2,X_5)X_3\Big]. \end{aligned} \tag{38}$$

With the help of (14), we have

$$\begin{aligned} 0 &= \Big(\frac{\alpha+1}{1-n}\Big)\Big[-\alpha g(X_5,X_3)X_2 + \frac{1}{2n}S(X_5,X_3)X_2 \\ &+ P(X_2,X_5)X_3 + \alpha g(X_2,X_3)\eta(X_5)\xi - \frac{1}{2n}S(X_2,X_3)\eta(X_5)\xi \\ &+ \alpha g(X_5,X_2)\eta(X_3)\xi - \frac{1}{2n}S(X_5,X_2)\eta(X_3)\xi\Big]. \end{aligned} \tag{39}$$

In (39), choosing $X_3 = \xi$ and using the equations (1), (21) we conclude

$$0 = \Big(\frac{\alpha+1}{1-n}\Big)\Big[S(X_2,X_5)\xi - 2n\alpha g(X_2,X_5)\xi\Big] = 0. \tag{40}$$

This tell us M either reduce a Kenmotsu manifold or it is an Einstein manifold. The converse is obvious.

Theorem 5.3. *Let M be $(2n+1)$-dimensional an almost $C(\alpha)$-manifold. Then $B(\xi, X_2)S = 0$ if and only if M either reduces a Kenmotsu manifold or an Einstein manifold.*

Proof. Suppose that $(B(\xi, X_2)S)(X_4, X_5) = 0$, for all $X_2, X_4, X_5 \in \chi(M)$, then we have □

$$S(B(\xi, X_2)X_4, X_5) + S(X_4, B(\xi, X_2)X_5) = 0. \tag{41}$$

In (41), using (16) obtain

$$0 = \Big(\frac{\alpha+1}{1-n}\Big)\Big[S(g(X_2, X_4)\xi - \eta(X_4)X_2, X_5) + S(X_4, g(X_2, X_5)\xi - \eta(X_5)X_2)\Big]. \tag{42}$$

In (42), choosing $X_4 = \xi$, we conclude

$$\Big(\frac{\alpha+1}{1-n}\Big)\Big[S(X_2, X_5) - 2n\alpha g(X_2, X_5)\Big] = 0. \tag{43}$$

So, the almost $C(\alpha)$-manifold is a Kenmotsu manifold or an Einstein manifold. The converse is obvious. This proves our assertion.

6. *D*-Conformal Pseudosymmetric and D-Conformal Ricci Pseudosymmetric Almost $C(\alpha)$-Manifolds

Theorem 6.1. *Let M be a $(2n+1)$-dimensional almost $C(\alpha)$-manifold. M is a D-conformal pseudosymmetric manifold if and only if M is either a Kenmotsu manifold or a D-conformal semisymmetric manifold.*

Proof. Let us suppose that the M is a D-conformal pseudosymmetric manifold. Then we have;

$$(R(X_1, X_2)B)(X_3, X_4, X_5) = \lambda Q(g, B)(X_3, X_4, X_5; X_1, X_2) \tag{44}$$

for any $X_1, X_2, X_3, X_4, X_5 \in \chi(M)$. In this case we get

$$\begin{aligned} & R(X_1, X_2)B(X_3, X_4)X_5 - B(R(X_1, X_2)X_3, X_4)X_5 - B(X_3, R(X_1, X_2)X_4)X_5 - B(X_3, X_4)R(X_1, X_2)X_5 \\ = \; & -\lambda\{B((X_1\Lambda_g X_2)X_3, X_4)X_5 + B(X_3, (X_1\Lambda_g X_2)X_4)X_5 + B(X_3, X_4)(X_1\Lambda_g X_2)X_5\}. \end{aligned}$$

Putting $X_1 = \xi$ in (44), we obtain

$$\begin{aligned} & R(\xi, X_2)B(X_3, X_4)X_5 - B(R(\xi, X_2)X_3, X_4)X_5 - B(X_3, R(\xi, X_2)X_4)X_5 - B(X_3, X_4)R(\xi, X_2)X_5 \\ = \; & -\lambda\{g(X_2, X_3)B(\xi, X_4)X_5 - \eta(X_3)B(X_2, X_4)X_5 + g(X_2, X_4)B(X_3, \xi)X_5 \\ & -\eta(X_4)B(X_3, X_2)X_5 + g(X_2, X_5)B(X_3, X_4)\xi - \eta(X_5)B(X_3, X_4)X_2\}. \end{aligned} \tag{45}$$

When we use equations (9), (15), (16) in (45) then make a direct calculation by choosing $X_3 = X_5 = \xi$, we obtain

$$\lambda(\frac{\alpha+1}{1-n})[g(X_2, X_4) - \eta(X_2)\eta(X_4)] = 0. \tag{46}$$

From (2), we conclude that

$$\lambda(\frac{\alpha+1}{1-n})g(\phi X_2, \phi X_4) = 0. \tag{47}$$

So, M either reduces a Kenmotsu manifold ($\alpha = -1$) or it is D-conformal semisymmetric manifold ($\lambda = 0$). The converse is obvious. □

Theorem 6.2. *Let M be a $(2n+1)$-dimensional almost $C(\alpha)$-manifold. M is a D-conformal Ricci pseudosymmetric manifold, if and only if M is either an Kenmotsu manifold or a D-conformal semisymmetric manifold or an co-Keahler manifold.*

Proof. We us assume that the M is a D-conformal Ricci pseudosymmetric manifold. Then we have

$$(R(X_1, X_2)B)(X_3, X_4, X_5) = \lambda Q(S, B)(X_3, X_4, X_5; X_1, X_2) \tag{48}$$

for any $X_1, X_2, X_3, X_4, X_5 \in \chi(M)$. Then we have

$$\begin{aligned} & R(X_1, X_2)B(X_3, X_4)X_5 - B(R(X_1, X_2)X_3, X_4)X_5 - B(X_3, R(X_1, X_2)X_4)X_5 - B(X_3, X_4)R(X_1, X_2)X_5 \\ = \ & -\lambda\{B((X_1 \Lambda_S X_2)X_3, X_4)X_5 + B(X_3, (X_1 \Lambda_S X_2)X_4)X_5 + B(X_3, X_4)(X_1 \Lambda_S X_4)X_5\}. \end{aligned} \tag{49}$$

In (49), choosing $X_1 = \xi$, we obtain

$$\begin{aligned} & R(\xi, X_2)B(X_3, X_4)X_5 - B(R(\xi, X_2)X_3, X_4)X_5 - B(X_3, R(\xi, X_2)X_4)X_5 - B(X_3, X_4)R(\xi, X_2)X_5 \\ = \ & -\lambda\{S(X_2, X_3)B(\xi, X_4)X_5 - S(\xi, X_3)B(X_2, X_4)X_5 + S(X_2, X_4)B(X_3, \xi)X_5 \\ - \ & S(\xi, X_4)B(X_3, X_2)X_5 + S(X_2, X_5)B(X_3, X_4)\xi - S(\xi, X_5)B(X_3, X_4)X_2\}. \end{aligned} \tag{50}$$

Using (9), (15), (16), (21) in (50) and then direct calculation by choosing $X_3 = X_5 = \xi$, we obtain

$$2n\alpha\lambda(\frac{\alpha + 1}{1 - n})[g(X_2, X_4) - \eta(X_2)\eta(X_4)] = 0. \tag{51}$$

From (2), we conclude

$$2n\alpha\lambda(\frac{\alpha + 1}{1 - n})g(\phi X_2, \phi X_4) = 0. \tag{52}$$

So, for $\alpha = -1$ M is an almost $C(-1)$ (Kenmotsu manifold), for $\alpha = 0$, M is a co-Keahler manifold, for $\lambda = 0$, M is D-conformal semisymmetric manifold. The converse is obvious. □

Example 6.3. *We consider the* 5*-dimensional Riemannian manifold* $M = \{(x_1, x_2, x_3, x_4, x_5) \in R^5 : x_2 \neq 0, x_3 \neq 0\}$, *where* $(x_1, x_2, x_3, x_4, x_5)$ *are standard coordinates in* R^5. *We chose the following vector fields*

$$e_1 = x_2\Big(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3}\Big), \qquad e_2 = x_2\Big(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_3}\Big), \qquad e_3 = x_3\Big(\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_4}\Big),$$

$$e_4 = x_3\Big(\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_4}\Big), \qquad e_5 = \frac{\partial}{\partial x_5}.$$

Let η *be the* 1*-form defined by* $\eta(e_5) = g(X, e_5)$ *for any vector field* X *on* M. *We define the* $(1, 1)$*- tensor field* ϕ *as*

$$\phi e_1 = e_2, \quad \phi e_2 = -e_1, \quad \phi e_3 = e_4, \quad \phi e_4 = -e_3, \quad \phi e_5 = 0.$$

Let g *be the Riemannian metric defined by*

$$g(e_i, e_j) = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

The linearity properties of ϕ *and* g *yield to*

$$\eta(e_5) = 1 \qquad \phi^2 X = -X + \eta(X)e_5,$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for any vector fields $X, Y \in \chi(M)$. *Thus, for* $e_5 = \xi$, $M(\phi, \xi, \eta, g)$ *is almost contact metric manifold. Let* ∇ *be Levi-Civita Connection with respect to* g. *Then we have*

$$[e_1, e_2] = [e_1, e_5] = [e_2, e_5] = [e_3, e_4] = [e_3, e_5] = [e_4, e_5] = 0,$$

$$[e_1, e_3] = \frac{x_2}{x_3}e_1 - \frac{x_3}{x_2}e_1, \quad [e_1, e_4] = -\frac{x_3}{x_2}e_1 + \frac{x_2}{x_3}e_4, \quad [e_2, e_3] = -\frac{x_3}{x_2}e_2 - \frac{x_2}{x_3}e_3, \quad [e_2, e_4] = -\frac{x_3}{x_2}e_1 - \frac{x_2}{x_3}e_4.$$

Takin $e_5 = \xi$ and using Kozsul formula, we obtain

$$\nabla_{e_1} e_1 = \frac{x_3}{x_2}(e_3 + e_4), \quad \nabla_{e_1} e_3 = -\frac{x_3}{x_1} e_1, \quad \nabla_{e_1} e_4 = -\frac{x_3}{x_2} e_1 \quad \nabla_{e_1} e_2 = \nabla_{e_1} e_5 = 0,$$

$$\nabla_{e_2} e_2 = \frac{x_3}{x_2}(e_3 + e_4), \quad \nabla_{e_2} e_3 = -\frac{x_3}{x_2} e_2, \quad \nabla_{e_2} e_4 = -\frac{x_3}{x_2} e_2 \quad \nabla_{e_2} e_1 = \nabla_{e_2} e_5 = 0,$$

$$\nabla_{e_3} e_3 = \frac{x_2}{x_3}(e_1 - e_2), \quad \nabla_{e_3} e_1 = -\frac{x_2}{x_3} e_3, \quad \nabla_{e_3} e_2 = -\frac{x_2}{x_3} e_3 \quad \nabla_{e_3} e_4 = \nabla_{e_3} e_5 = 0,$$

$$\nabla_{e_4} e_4 = \frac{x_2}{x_3}(e_1 - e_2), \quad \nabla_{e_4} e_1 = -\frac{x_2}{x_3} e_4, \quad \nabla_{e_4} e_2 = \frac{x_2}{x_3} e_4 \quad \nabla_{e_4} e_3 = \nabla_{e_4} e_5 = 0,$$

$$\nabla_{e_5} e_1 = \nabla_{e_5} e_2 = \nabla_{e_5} e_3 = \nabla_{e_5} e_4 = \nabla_{e_5} e_5 = 0.$$

By using above results, we can easily obtain the following

$$R(e_1, e_2)e_1 = \frac{x_3^2}{x_2^2} e_2 + e_3 + e_4, \quad R(e_1, e_2)e_2 = (-\frac{x_3^2}{x_1 x_2} - \frac{x_3^2}{x_2^2})e_1 + e_3 + e_4,$$

$$R(e_1, e_2)e_3 = (-\frac{x_2 x_3}{x_1^2} - \frac{x_2}{x_1})e_1 - e_2, \quad R(e_1, e_2)e_4 = -e_1 - e_2,$$

$$R(e_1, e_3)e_1 = (\frac{x_2 - x_1}{x_1})e_1 + e_2 + (\frac{x_2^2}{x_3^2} + \frac{x_3^2}{x_2^2})e_3 + \frac{2x_3^2}{x_2^2} e_4, \quad R(e_1, e_3)e_2 = -\frac{x_2}{x_1} e_1 - \frac{2x_2^2}{x_3^2} e_3,$$

$$R(e_1, e_3)e_3 = \Big(-\frac{x_2 x_3 - x_2}{x_1} - \frac{x_2^2}{x_3^2} - \frac{x_3^2}{x_1 x_2}\Big)e_1 + 2\frac{x_2^2}{x_3^2} e_2 + \Big(\frac{x_1 - x_2}{x_1}\Big)e_3 + e_4,$$

$$R(e_1, e_3)e_4 = -2\frac{x_3^2}{x_2^2} e_1 - e_3, \quad R(e_1, e_4)e_1 = e_2 + \frac{2x_3^2}{x_2^2} e_3 + \Big(-\frac{x_2}{x_3} + \frac{2x_2^2}{x_3^2} + \frac{2x_3^2}{x_2^2}\Big)e_4,$$

$$R(e_1, e_4)e_2 = -e_1 - \frac{2x_2^2}{x_3^2} e_4 \quad R(e_1, e_4)e_3 = -\frac{x_3^2}{x_1 x_2} e_1 - \frac{x_2}{x_1} e_4,$$

$$R(e_1, e_4)e_4 = \Big(\frac{x_2^2 - 2x_3^2}{x_3^2} - \frac{x_2^2}{x_3^2}\Big)e_1 + \frac{2x_2^2}{x_3^2} e_2 + e_3$$

$$R(e_2, e_3)e_1 = e_2, \quad R(e_2, e_3)e_2 = -e_1 + \Big(\frac{2x_2^2}{x_3^2} + \frac{2x_3^2}{x_2^2}\Big)e_3 + \frac{2x_3^2}{x_2^2} e_4,$$

$$R(e_2, e_3)e_3 = \frac{2x_2^2}{x_3^2} e_1 - \frac{2x_3^2}{x_2^2} e_2 - e_4 \quad R(e_2, e_3)e_4 = -\frac{2x_3^2}{x_2^2} e_2 + e_3,$$

$$R(e_2, e_4)e_1 = e_2 - \frac{2x_2^2}{x_3^2} e_4 \quad R(e_2, e_4)e_2 = -e_1 + \frac{2x_3^2}{x_2^2} e_3 + \frac{2x_2^2}{x_3^2} e_4,$$

$$R(e_2,e_4)e_3=-\frac{2x_3^2}{x_2^2}e_2+e_4 \quad R(e_2,e_4)e_4=\frac{2x_2^2}{x_3^2}e_1+\Big(-\frac{2x_2^2}{x_3^2}-\frac{2x_3^2}{x_2^2}\Big)e_2-e_3,$$

$$R(e_i,e_j)\xi=R(e_i,\xi)e_j=0$$

for $e_5=\xi$ and $1\le i,j\le 5$. The definition of Ricci tensor is given as 3-dimensional manifold :

$$S(X,Y)=\sum_{i=1}^{5}g(R(e_i,X)Y,e_i). \tag{53}$$

Using the components of the curvature tensor in (53), we get the following results:

$$S(e_1,e_1)=-\frac{3x_3^2}{x_1x_2}-\frac{x_3x_2}{x_1^2}-\frac{5x_3^2}{x_2^2}-\frac{2x_2^2}{x_3^2}+\frac{2x_2}{x_1}-\frac{x_2x_3}{x_1}-4$$

$$S(e_2,e_2)=-\frac{10x_3^2}{x_2^2}-\frac{2x_2^2}{x_3^2}+4,\quad S(e_3,e_3)=-\frac{2x_3^2}{x_2^2}-\frac{2x_2^2}{x_3^2}+\frac{x_2}{x_1}-1,$$

$$S(e_4,e_4)=-\frac{2x_3^2}{x_2^2}-\frac{x_2^2}{x_3^2}+\frac{x_2}{x_1}+\frac{x_2}{x_3}-1,\quad S(e_1,e_4)=\frac{4x_3^2}{x_2^2}-\frac{x_2}{x_1}-\frac{x_2}{x_3}+3,$$

$$S(e_2,e_3)=\frac{2x_2^2}{x_3^2}+\frac{4x_3^2}{x_2^2}-2,\quad S(e_2,e_4)=\frac{2x_3^2}{x_2^2}-2,\quad S(e_3,e_4)=-\frac{2x_2^2}{x_3^2}+\frac{x_2^2}{x_3^2},$$

and

$$S(e_i,\xi)=0.$$

So, the scalar curvature function r of almost $C(\alpha)$-manifold M is calculated as

$$r=-\frac{19x_3^2}{x_2^2}-\frac{7x_2^2}{x_3^2}-\frac{3x_3^2}{x_1x_2}-\frac{x_3x_2}{x_1^2}-\frac{x_2x_3-4x_2}{x_1}+\frac{x_2}{x_3}-2.$$

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