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On an inverse problem for a tempered fractional diffusion equation

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Abstract. In this paper, we consider a tempered fractional diffusion equation with an integral condition. We present two main results. The first result concerns the well-posedness of the mild solution and provides estimates for the upper and lower bounds of the solution. We also investigate the continuity of the solution with respect to the fractional order. The second result pertains to the regularization of the inverse problem. The first method is based on the quasi-reversibility method, and we provide an error estimate in L^2 spaces. For the second regularized solution, we employ the Fourier truncation method and obtain error estimates in the higher-order spaces $\mathbb{H}^s$.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N (N \geq 1)$ with smooth boundary $\partial\Omega$. Let T be a positive constant. We are interested in studying the following problem

$$\begin{cases} D_t^{\alpha,k} u(x,t) + \mathcal{A}u(x,t) = 0, & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \end{cases} \tag{1}$$

with the integral condition

$$\int_0^T u(x,s)\mathrm{d}s = f(x), \quad x \in \Omega, \tag{2}$$

where $0 < \alpha < 1$, $\mathcal{A} = -\Delta$ is the Laplacian operator. Here $D_t^{\alpha,k}$ is called the Caputo tempered fractional derivative of order α which is defined by (see [23], p. 430)

$$D_t^{\alpha,k} w(\cdot,t) = \frac{e^{-kt}}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{\mathrm{d}}{\mathrm{d}s}\Big(e^{ks} w(\cdot,s)\Big)\mathrm{d}s, \tag{3}$$

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where Γ is the Gamma function. If $k = 0$ then tempered Caputo derivative becomes Caputo derivative which was studied in [3–8, 22, 28–30].

Fractional differential equations can represent many natural phenomena with long-time behavior such as unexpected dispersion, analytical chemistry, biological science, artificial neural networks, time-frequency analysis, etc. This research direction is very exciting and has attracted many mathematicians to participate. Types of fractional derivatives that attract a lot and have important meanings such as Caputo derivative, Riemann-Liouville, Caputo-Fabrizio. In some models, we need to consider new memory effects related to operators for better real-world applications. The tempered fractional derivative was introduced in [10]. The tempered fractional derivative is one of the generalized forms of the Caputo and Riemann-Liouville fractional derivatives. Multiplying Caputo and Riemann-Liouville fractional derivatives by an exponential factor gives that the tempered fractional derivative. As we know, the tempered fractional calculus has been developed to deal with elasticity [11], geophysical flows,[12] ground water hydrology [13].

Let us collect some results on problems related to differential equations containing tempered fractional derivatives. The authors of [1] focused on discussing the properties of the time tempered fractional derivative, then investigated the well-posedness and the algorithm for the tempered fractional ordinary differential equation. In the interesting paper [23], M.A. Zaky studied the existence, uniqueness, and structural stability of solutions to nonlinear tempered fractional differential equations as follows

$$D_t^{\alpha,k}u(t) = g(z, u(t)), \quad 0 \le t \le T, \tag{4}$$

associated with a general boundary condition

$$au(0) + be^{kT}u(T) = c. \tag{5}$$

Here $g : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, a,b,c are real constants that satisfy $a + b \neq 0$. If $a = 0$ and $b = 1$, the problem (4)-(5) is reduced to the model considered in [2].

In [14], the authors investigated some existence and uniqueness results for a class of problems for nonlinear Caputo tempered implicit fractional differential equations in b-metric spaces.

To the best of our knowledge, there are very few papers refer to Problem (1) when $\mathcal{A}$ is an operator in Hilbert space. Our aim in this paper is described as follows. The first goal is to prove the well-posedness of the solution. We give the upper and lower bound of the mild solution. We also consider the continuity of the solution according to the parameter k. The second result is to investigate the ill-posedness and regularize the solution for our problem. We will provide two regularize method: the quasi-reversibility method and truncation method. Our method is oriented towards research in infinite dimensional space with the use of Fourier series, see [9, 15–21, 31–33].

This paper is organized as follows. In section 2, we introduce some preliminaries which contains some definitions on solutions spaces and some properties on the Mittag-Leffler functions. In section 3, we study the well-posedness of the problem. This section includes many theorems with different contents related to continuity, upper and lower bounds of the solution. Section 4 mentions to the regularized solutions of our problem. Theorem 4.1 provides a regularized solution using the QR method and evaluates the error estimate between the regularized solution and the exact solution in L^2 space.

2. Preliminaries

First of all, we introduce some suitable Sobolev spaces, and fix some notation. Let us recall that the spectral problem

$$\begin{cases} (-\Delta)e_n(x) = \lambda_n e_n(x), & \text{in } \Omega, \\ e_n(x) = 0, & \text{on } \partial\Omega, \end{cases} \tag{6}$$

admits a family of eigenvalues

$$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le ... \le \lambda_n \le ... \nearrow \infty.$$

The notation $\|\cdot\|_B$ stands for the norm in the Banach space B. We denote by $L^q(0,T;B), 1 \le p \le \infty$ for the Banach space of real-valued functions $w : (0;T) \to B$ measurable, provided that

$$\|w\|_{L^q(0,T;B)} = \left(\int_0^T \|w(t)\|_B^q \mathrm{d}t\right)^{\frac{1}{q}}, \quad \text{for } 1 \le q < \infty; \tag{7}$$

while

$$\|w\|_{L^\infty(0,T;B)} = \operatorname*{ess\,sup}_{t\in(0,T)} \|w(t)\|_B, \quad \text{for } q = \infty. \tag{8}$$

For any $p \ge 0$, we define the space

$$\mathbb{H}^p(\Omega) = \Big\{v \in L^2(\Omega); \sum_{n=1}^{\infty} \lambda_n^{2p} |\langle v(x), e_n(x)\rangle|^2 < \infty\Big\},$$

where $\langle\cdot,\cdot\rangle$ is the inner product in $L^2(\Omega)$, then $\mathcal{H}(\Omega)$ is a Hilbert space with the norm

$$\|v\|_{\mathbb{H}^p(\Omega)} = \left(\sum_{n=1}^{\infty} \lambda_n^p |\langle v(x), e_n(x)\rangle|^2\right)^{\frac{1}{2}}.$$

For any $\theta > 0$, we introduce the following space

$$C^\theta\left([0,T];\mathbb{H}^p(\Omega)\right) = \Big\{v \in C\Big([0,T];L^2(\Omega)\Big) : \sup_{0\le t<s\le T} \frac{\|v(\cdot,t) - v(\cdot,s)\|_{\mathbb{H}^p(\Omega)}}{|t-s|^\theta} < \infty\Big\}. \tag{9}$$

and the following norm

$$\|v\|_{C^\theta([0,T];L^2(\Omega))} = \sup_{0\le t<s\le T} \frac{\|v(.,t) - v(.s)\|_{L^2(\Omega)}}{|t-s|^\theta}. \tag{10}$$

Definition 2.1 (Kilbas, [27]). *The Mittag-Leffler function $E_\alpha(z)$ defined by*

$$E_\alpha(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)} \quad (z \in \mathbb{C}, \Re(\alpha) > 0).$$

Lemma 2.2 (Kilbas, [27]). *For $\lambda > 0, \alpha > 0$ and positive integer $m \in \mathbb{N}$, we have*

$$\frac{\mathrm{d}^m}{\mathrm{d}t^m} E_{\alpha,1}(-\lambda t^\alpha) = -\lambda t^{\alpha-m} E_{\alpha,\alpha-m+1}(-\lambda t^\alpha),$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(t E_{\alpha,2}(-\lambda t^\alpha)\Big) = E_{\alpha,1}(-\lambda t^\alpha),$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha)\Big) = -t^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda t^\alpha).$$

It is well-know that, for $0 < \alpha < 1$ and for $z \in \mathbb{C}$

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t} E_{\alpha,1}(z t^\alpha) = z E_{\alpha,1}(z t^\alpha). \tag{11}$$

Lemma 2.3 (Kilbas, [27]). *Let $\lambda > 0$, and $1 < \alpha < 2$. Then the identities*

$$\partial_t E_{\alpha,1}(-\lambda t^\alpha) = -\lambda t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha),$$

and

$$\partial_t (t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha)) = t^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda t^\alpha),$$

hold for all $t > 0$.

Lemma 2.4. *Let $\lambda > 0$, and $0 < \alpha < 1$. Then the identities*

$$\partial_t^\alpha E_{\alpha,1}(-\lambda t^\alpha) = -\lambda E_{\alpha,1}(-\lambda t^\alpha),$$

hold for all $t > 0$.

3. The homogeneous problem with integral condition

This section is devoted to the study of Problem (1) with the nonlocal integral condition (2). Let us now give the explicit formula of the mild solution. Let us assume that Problem (1) has a unique solution u. Let $u(x,t) = \sum_{n=1}^\infty u_n(t)e_n(x)$ be the Fourier series in $L^2(\Omega)$ with $u_n(t) = \langle u(\cdot,t), e_n(\cdot)\rangle_{L^2(\Omega)}$. From the first equation of (1), taking the inner product of both sides of (1) with $e_n(x)$, we obtain

$$D_t^{\alpha,k} + \lambda_n u_n(t) = 0, \quad u_n(0) = \langle u_0, e_n\rangle_{L^2(\Omega)}, \tag{12}$$

where $F_n(t) = \langle F(t), e_n(\cdot)\rangle_{L^2(\Omega)}$. The theory of fractional ordinary differential equations [27] gives a unique function u_n as follows

$$u_n^{\alpha,k}(t) = e^{-kt}E_{\alpha,1}(-\lambda_n t^\alpha)u_{0,n}, \tag{13}$$

where we denote $u_{0,n} = \langle u(0), e_n(\cdot)\rangle_{L^2(\Omega)}$. This implies that

$$\int_0^T u_n^{\alpha,k}(t)\mathrm{d}t = \int_0^T e^{-kt}E_{\alpha,1}(-\lambda_n t^\alpha)u_{0,n}\mathrm{d}t = f_n. \tag{14}$$

Hence, one has

$$u_{0,n} = \frac{f_n}{\int_0^T e^{-kt}E_{\alpha,1}(-\lambda_n t^\alpha)dt}.$$

This equality together with (13) imply that

$$u_n^{\alpha,k}(t) = \frac{e^{-kt}E_{\alpha,1}(-\lambda_n t^\alpha)}{\int_0^T e^{-kt}E_{\alpha,1}(-\lambda_n t^\alpha)\mathrm{d}t}. \tag{15}$$

The mild solution is defined by

$$u^{\alpha,k}(x,t) = \sum_{n=1}^\infty \frac{e^{-kt}E_{\alpha,1}(-\lambda_n t^\alpha)}{\int_0^T e^{-kt}E_{\alpha,1}(-\lambda_n t^\alpha)dt} f_n e_n(x). \tag{16}$$

Theorem 3.1. *Let $f \in \mathbb{H}^{s+1-\theta}(\Omega)$ for any $s \geq 0$ and $0 < \theta < 1$. Then we get*

$$\left\|u^{\alpha,k}(.,t)\right\|_{L^p(0,T;\mathbb{H}^s(\Omega))} \leq C(\varepsilon,\alpha,p,T)e^{kT}k^{-\varepsilon}\left\|f\right\|_{\mathbb{H}^{s+1-\theta}(\Omega)}, \tag{17}$$

for $1 < p < \frac{1}{\varepsilon+\alpha\theta}$ and $1 < \varepsilon < 1-\alpha\theta$. If $f \in \mathbb{H}^{s+\frac{3}{2}}(\Omega)$ then we $u^{\alpha,k} \in C^{\frac{\alpha}{2}}([0,T];\mathbb{H}^s(\Omega))$ and we get

$$\left\|u^{\alpha,k}\right\|_{C^{\frac{\alpha}{2}}([0,T];\mathbb{H}^s(\Omega))} \leq C(\alpha,k,\lambda_1,T)\left\|f\right\|_{\mathbb{H}^{s+\frac{3}{2}}(\Omega)}. \tag{18}$$

Proof. Since the inequality $e^{-kt} \geq e^{-kT}$, we see that

$$\int_0^T e^{-kt}E_{\alpha,1}(-\lambda_n t^\alpha)\mathrm{d}t \geq e^{-kT}\int_0^T E_{\alpha,1}(-\lambda_n t^\alpha)\mathrm{d}t \geq C_\alpha^- e^{-kT}\int_0^T \frac{1}{1+\lambda_n t^\alpha}\mathrm{d}t. \tag{19}$$

Since $1 \le \lambda_n \lambda_1^{-1}$, we know that

$$\int_0^T \frac{1}{1+\lambda_n t^\alpha} \mathrm{d}t \ge \frac{1}{\lambda_n} \int_0^T \frac{dt}{\lambda_1^{-1} + t^\alpha} = \widetilde{M}(\alpha, T) \frac{1}{\lambda_n}.$$

Here we denote by $\widetilde{M}(\alpha, T) = \int_0^T \frac{dt}{\lambda_1^{-1}+t^\alpha}$. This implies that

$$\int_0^T e^{-kt} E_{\alpha,1}(-\lambda_n t^\alpha) \mathrm{d}t \ge \frac{C_\alpha^- \widetilde{M}(\alpha, T)}{e^{kT} \lambda_n}. \tag{20}$$

Using the upper bound of the Mittag-Leffler function $E_{\alpha,1}$ and the inequality $e^{-z} \le C_\varepsilon z^{-\varepsilon}$ for any $\varepsilon > 0$, we know that

$$e^{-kt} E_{\alpha,1}(-\lambda_n t^\alpha) \le C_\varepsilon k^{-\varepsilon} t^{-\varepsilon} \frac{C_\alpha^+}{1+\lambda_n t^\alpha} \le C(\varepsilon, \alpha) k^{-\varepsilon} t^{-\varepsilon} t^{-\alpha\theta} \lambda_n^{-\theta}, \tag{21}$$

for any $0 < \theta < 1$ and $\varepsilon > 0$. Using (20) and (41), we find that

$$\frac{e^{-kt} E_{\alpha,1}(-\lambda_n t^\alpha)}{\int_0^T e^{-kt} E_{\alpha,1}(-\lambda_n t^\alpha) dt} \le C(\varepsilon, \alpha) e^{kT} k^{-\varepsilon} t^{-(\varepsilon+\alpha\theta)} \lambda_n^{1-\theta}. \tag{22}$$

Using Parseval's equality, we find that

$$\begin{aligned} \left\| u^{\alpha,k}(.,t) \right\|^2_{\mathbb{H}^s(\Omega)} &= \sum_{n=1}^\infty \lambda_n^{2s} \left(\frac{e^{-kt} E_{\alpha,1}(-\lambda_n t^\alpha)}{\int_0^T e^{-kt} E_{\alpha,1}(-\lambda_n t^\alpha) \mathrm{d}t} \right)^2 |f_n|^2 \\ &\le |C(\varepsilon, \alpha) e^{kT} k^{-\varepsilon}|^2 t^{-2(\varepsilon+\alpha\theta)} \sum_{n=1}^\infty \lambda_n^{2s} \lambda_n^{2-2\theta} |f_n|^2. \end{aligned} \tag{23}$$

This implies that

$$\left\| u^{\alpha,k}(.,t) \right\|_{\mathbb{H}^s(\Omega)} \le C(\varepsilon, \alpha) e^{kT} k^{-\varepsilon} t^{-(\varepsilon+\alpha\theta)} \left\| f \right\|_{\mathbb{H}^{s+1-\theta}(\Omega)}. \tag{24}$$

Let us choose ε such that $1 < \varepsilon < 1 - \alpha\theta$. Then by taking p such that $1 < p < \frac{1}{\varepsilon+\alpha\theta}$, then we deduce that $u^{\alpha,k} \in L^p(0, T; \mathbb{H}^s(\Omega))$ and

$$\left\| u^{\alpha,k}(.,t) \right\|_{L^p(0,T;\mathbb{H}^s(\Omega))} \le C(\varepsilon, \alpha, p, T) e^{kT} k^{-\varepsilon} \left\| f \right\|_{\mathbb{H}^{s+1-\theta}(\Omega)}. \tag{25}$$

By (16), one has

$$u^{\alpha,k}(x, t+h) - u^{\alpha,k}(x, t) = \sum_{n=1}^\infty \frac{e^{-k(t+h)} E_{\alpha,1}(-\lambda_n (t+h)^\alpha) - e^{-kt} E_{\alpha,1}(-\lambda_n t^\alpha)}{\int_0^T e^{-kt} E_{\alpha,1}(-\lambda_n t^\alpha) \mathrm{d}t} f_n e_n(x). \tag{26}$$

It is obvious to see that

$$\begin{aligned} &\left| e^{-k(t+h)} E_{\alpha,1}(-\lambda_n (t+h)^\alpha) - e^{-kt} E_{\alpha,1}(-\lambda_n t^\alpha) \right| \\ &\qquad \le e^{-k(t+h)} \left| E_{\alpha,1}(-\lambda_n (t+h)^\alpha) - E_{\alpha,1}(-\lambda_n t^\alpha) \right| \\ &\qquad + E_{\alpha,1}(-\lambda_n t^\alpha) \left| e^{-k(t+h)} - e^{-kt} \right|. \end{aligned} \tag{27}$$

Since the formula $\frac{\mathrm{d}}{\mathrm{d}t}E_{\alpha,1}(-\lambda t^\alpha) = -\lambda t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha)$ and the upper bound of the Mittag-Leffler function $E_{\alpha,\alpha}$, we obtain the following estimate

$$\begin{aligned}\Big|E_{\alpha,1}(-\lambda_n(t+h)^\alpha) - E_{\alpha,1}(-\lambda_n t^\alpha)\Big| &= \lambda_n\Big|\int_t^{t+h}\tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n\tau^\alpha)\mathrm{d}\tau\Big|\\ &\le \lambda_n\int_t^{t+h}\tau^{\alpha-1}\frac{M_\alpha}{1+\lambda_n\tau^\alpha}\mathrm{d}\tau. \end{aligned}\tag{28}$$

In view of the inequality $a+b \ge 2\sqrt{ab}$ for any $a,b\ge 0$, we know that

$$\int_t^{t+h}\tau^{\alpha-1}\frac{M_\alpha}{1+\lambda_n\tau^\alpha}\mathrm{d}\tau \le \frac{M_\alpha}{2\sqrt{\lambda_n}}\int_t^{t+h}\tau^{\frac{\alpha}{2}-1}\mathrm{d}\tau = \frac{M_\alpha}{\alpha\sqrt{\lambda_n}}\Big((t+h)^{\frac{\alpha}{2}} - t^{\frac{\alpha}{2}}\Big).\tag{29}$$

Using the inequality $(a+b)^\alpha \le a^\alpha + b^\alpha,\ \ 0<\alpha<1,\ \ a,b\ge 0$, and looking at (28), we infer that

$$\Big|E_{\alpha,1}(-\lambda_n(t+h)^\alpha) - E_{\alpha,1}(-\lambda_n t^\alpha)\Big| \le \frac{M_\alpha}{2}\sqrt{\lambda_n}h^{\frac{\alpha}{2}}.\tag{30}$$

In addition, we also get

$$\begin{aligned}|E_{\alpha,1}(-\lambda_n t^\alpha)|\Big|e^{-k(t+h)} - e^{-kt}\Big| &\le C_\alpha^+ k^{\frac{\alpha}{2}}\Big((t+h)^{\frac{\alpha}{2}} - t^{\frac{\alpha}{2}}\Big)\\ &\le C_\alpha^+ k^{\frac{\alpha}{2}}h^{\frac{\alpha}{2}}.\end{aligned}\tag{31}$$

Combining (27), (30), (31), we find that

$$\begin{aligned}&\Big|e^{-k(t+h)}E_{\alpha,1}(-\lambda_n(t+h)^\alpha) - e^{-kt}E_{\alpha,1}(-\lambda_n t^\alpha)\Big|\\ &\qquad\le \frac{M_\alpha}{2}\sqrt{\lambda_n}h^{\frac{\alpha}{2}} + C_\alpha^+ k^{\frac{\alpha}{2}}h^{\frac{\alpha}{2}} \le C(\alpha,k,\lambda_1)\sqrt{\lambda_n}h^{\frac{\alpha}{2}}.\end{aligned}\tag{32}$$

This inequality together with (20) give that

$$\left|\frac{e^{-k(t+h)}E_{\alpha,1}(-\lambda_n(t+h)^\alpha) - e^{-kt}E_{\alpha,1}(-\lambda_n t^\alpha)}{\int_0^T e^{-kt}E_{\alpha,1}(-\lambda_n t^\alpha)\mathrm{d}t}\right| \le C(\alpha,k,\lambda_1)e^{kT}\lambda_n^{3/2}h^{\frac{\alpha}{2}}.\tag{33}$$

By (26), we obtain the following bound

$$\begin{aligned}&\Big\|u^{\alpha,k}(.,t+h) - u^{\alpha,k}(.,t)\Big\|^2_{\mathbb{H}^s(\Omega)}\\ &= \sum_{n=1}^\infty \lambda_n^{2s}\left|\frac{e^{-k(t+h)}E_{\alpha,1}(-\lambda_n(t+h)^\alpha) - e^{-kt}E_{\alpha,1}(-\lambda_n t^\alpha)}{\int_0^T e^{-kt}E_{\alpha,1}(-\lambda_n t^\alpha)\mathrm{d}t}\right|^2 f_n^2\\ &\le |C(\alpha,k,\lambda_1)|^2 e^{2kT}h^\alpha\sum_{n=1}^\infty \lambda_n^{2s+3}f_n^2.\end{aligned}\tag{34}$$

Hence, we find that

$$\Big\|u^{\alpha,k}(.,t+h) - u^{\alpha,k}(.,t)\Big\|^2_{\mathbb{H}^s(\Omega)} \le C(\alpha,k,\lambda_1)e^{kT}h^{\frac{\alpha}{2}}\Big\|f\Big\|_{\mathbb{H}^{s+\frac{3}{2}}(\Omega)}.\tag{35}$$

This implies that $u^{\alpha,k}\in C^{\frac{\alpha}{2}}([0,T];\mathbb{H}^s(\Omega))$ and we also give the following estimate

$$\Big\|u^{\alpha,k}\Big\|_{C^{\frac{\alpha}{2}}([0,T];\mathbb{H}^s(\Omega))} \le C(\alpha,k,\lambda_1,T)\Big\|f\Big\|_{\mathbb{H}^{s+\frac{3}{2}}(\Omega)}.\tag{36}$$

□

Theorem 3.2. *Let $f \in \mathbb{H}^{s+2-\theta}(\Omega)$ for any $0 < \theta < 1$. Then we get*

$$\left\|\frac{\partial}{\partial t}u^{\alpha,k}(x,t)\right\|_{\mathbb{H}^s(\Omega)} \le \mathbf{C_1} k^{-\varepsilon} t^{\alpha-1-\varepsilon-\alpha\theta}\left\|f\right\|_{\mathbb{H}^{s+2-\theta}(\Omega)}, \tag{37}$$

and

$$\left\|D_t^{\alpha}u^{\alpha,k}(.,t)\right\|_{\mathbb{H}^s(\Omega)} \le \mathbb{C}_3 t^{-\varepsilon-\alpha\theta}\left\|f\right\|_{\mathbb{H}^{s+2-\theta}(\Omega)}, \tag{38}$$

where $\mathbf{C_1}$ depends on $k, \varepsilon, \alpha, \widetilde{M}$. The constant $\mathbb{C}_3$ depends on $\varepsilon, \alpha, k, T, \theta, \widetilde{M}$.

Proof. Since (16) and $\frac{\mathrm{d}}{\mathrm{d}t}E_{\alpha,1}(-\lambda t^{\alpha}) = -\lambda t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^{\alpha})$, we infer that

$$\begin{aligned}\frac{\partial}{\partial t}u^{\alpha,k}(x,t) &= \sum_{n=1}^{\infty}\frac{-ke^{-kt}E_{\alpha,1}(-\lambda_n t^{\alpha}) - \lambda_n e^{-kt}t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n t^{\alpha})}{\int_0^T e^{-kt}E_{\alpha,1}(-\lambda_n t^{\alpha})\mathrm{d}t} f_n e_n(x)\\ &= -ku^{\alpha,k}(x,t) - W(x,t),\end{aligned} \tag{39}$$

where

$$W(x,t) = \sum_{n=1}^{\infty}\frac{\lambda_n e^{-kt}t^{\alpha-1}E_{\alpha,1}(-\lambda_n t^{\alpha})}{\int_0^T e^{-kt}E_{\alpha,\alpha}(-\lambda_n t^{\alpha})\mathrm{d}t} f_n e_n(x). \tag{40}$$

Using the upper bound of the Mittag-Leffler $E_{\alpha,\alpha}$ and the inequality $e^{-z} \le C_{\varepsilon} z^{-\varepsilon}$ for any $\varepsilon > 0$, we know that

$$e^{-kt}E_{\alpha,\alpha}(-\lambda_n t^{\alpha}) \le C_{\varepsilon}k^{-\varepsilon}t^{-\varepsilon}\frac{C_{\alpha}^{++}}{1+\lambda_n t^{\alpha}} \le C_0(\varepsilon,\alpha)k^{-\varepsilon}t^{-\varepsilon}t^{-\alpha\theta}\lambda_n^{-\theta}. \tag{41}$$

Thus, we get that

$$\lambda_n t^{\alpha-1}e^{-kt}E_{\alpha,\alpha}(-\lambda_n t^{\alpha}) \le C_0(\varepsilon,\alpha)k^{-\varepsilon}t^{\alpha-1-\varepsilon-\alpha\theta}\lambda_n^{1-\theta}, \tag{42}$$

for any $\varepsilon > 0$ and $0 < \theta < 1$. Using the bound (20), we derive that

$$\left|\frac{\lambda_n e^{-kt}t^{\alpha-1}E_{\alpha,1}(-\lambda_n t^{\alpha})}{\int_0^T e^{-kt}E_{\alpha,\alpha}(-\lambda_n t^{\alpha})\mathrm{d}t}\right| \le C_1(\varepsilon,\alpha,\widetilde{M})k^{-\varepsilon}t^{\alpha-1-\varepsilon-\alpha\theta}\lambda_n^{2-\theta}. \tag{43}$$

Using Parseval's equality, one has

$$\begin{aligned}\left\|W(.,t)\right\|^2_{\mathbb{H}^s(\Omega)} &= \sum_{n=1}^{\infty}\left(\frac{\lambda_n e^{-kt}t^{\alpha-1}E_{\alpha,1}(-\lambda_n t^{\alpha})}{\int_0^T e^{-kt}E_{\alpha,\alpha}(-\lambda_n t^{\alpha})\mathrm{d}t}\right)^2|f_n|^2\\ &\le \left|C_1(\varepsilon,\alpha,\widetilde{M})\right|^2 k^{-2\varepsilon}t^{2\alpha-2-2\varepsilon-2\alpha\theta}\sum_{n=1}^{\infty}\lambda_n^{2s+4-2\theta}f_n^2.\end{aligned} \tag{44}$$

Hence, we have immediately that

$$\left\|W(.,t)\right\|_{\mathbb{H}^s(\Omega)} \le C_1(\varepsilon,\alpha,\widetilde{M})k^{-\varepsilon}t^{\alpha-1-\varepsilon-\alpha\theta}\left\|f\right\|_{\mathbb{H}^{s+2-\theta}(\Omega)}. \tag{45}$$

Combining (24), (45), we obtain

$$\left\|\frac{\partial}{\partial t}u^{\alpha,k}(x,t)\right\|_{\mathbb{H}^s(\Omega)} \le k\left\|u^{\alpha,k}(x,t)\right\|_{\mathbb{H}^s(\Omega)} + \left\|W(.,t)\right\|_{\mathbb{H}^s(\Omega)}$$

$$\leq kC(\varepsilon,\alpha)e^{kT}k^{-\varepsilon}t^{-(\varepsilon+\alpha\theta)}\left\|f\right\|_{\mathbb{H}^{s+1-\theta}(\Omega)} + C_1(\varepsilon,\alpha,\widetilde{M})k^{-\varepsilon}t^{\alpha-1-\varepsilon-\alpha\theta}\left\|f\right\|_{\mathbb{H}^{s+2-\theta}(\Omega)}. \tag{46}$$

By a simple calculation for the above expression, we get the desired result (37).

The Caputo derivative of the function $u^{\alpha,k}$ is given by

$$\begin{aligned} D_t^\alpha u^{\alpha,k}(x,t) &= \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}\frac{\partial}{\partial\tau}u^{\alpha,k}(x,\tau)\mathrm{d}\tau \\ &= \frac{-k}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}u^{\alpha,k}(x,\tau)\mathrm{d}\tau - \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}W(x,\tau)\mathrm{d}\tau \\ &= \mathbb{J}_1(x,t) - \mathbb{J}_2(x,t), \end{aligned} \tag{47}$$

where

$$\begin{aligned} \mathbb{J}_1(x,t) &= \frac{-k}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}u^{\alpha,k}(x,\tau)\mathrm{d}\tau, \\ \mathbb{J}_2(x,t) &= \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}W(x,\tau)\mathrm{d}\tau. \end{aligned}$$

Using(24), we give the following estimate

$$\begin{aligned} \left\|\mathbb{J}_1(.,t)\right\|_{\mathbb{H}^s(\Omega)} &\leq C(\varepsilon,\alpha)\frac{-k}{\Gamma(1-\alpha)}e^{kT}k^{-\varepsilon}\left\|f\right\|_{\mathbb{H}^{s+1-\theta}(\Omega)}\int_0^t (t-\tau)^{-\alpha}\tau^{-(\varepsilon+\alpha\theta)}\mathrm{d}\tau \\ &= C(\varepsilon,\alpha)\frac{-k}{\Gamma(1-\alpha)}e^{kT}k^{-\varepsilon}\left\|f\right\|_{\mathbb{H}^{s+1-\theta}(\Omega)}t^{1-\alpha-(\varepsilon+\alpha\theta)}B(1-\alpha,1-\varepsilon-\alpha\theta), \end{aligned} \tag{48}$$

where we note that $\varepsilon < 1-\alpha\theta$. This implies that

$$\left\|\mathbb{J}_1(.,t)\right\|_{\mathbb{H}^s(\Omega)} \leq \mathbb{C}_1 t^{1-\alpha-(\varepsilon+\alpha\theta)}\left\|f\right\|_{\mathbb{H}^{s+1-\theta}(\Omega)}, \tag{49}$$

where $\mathbb{C}_1$ depends on $\varepsilon,\alpha,k,T,\theta$. For the second term $\mathbb{J}_2$, we use (45) in order to obtain

$$\begin{aligned} \left\|\mathbb{J}_2(.,t)\right\|_{\mathbb{H}^s(\Omega)} &\leq \frac{1}{\Gamma(1-\alpha)}C_1(\varepsilon,\alpha,\widetilde{M})k^{-\varepsilon}\left\|f\right\|_{\mathbb{H}^{s+2-\theta}(\Omega)}\int_0^t (t-\tau)^{-\alpha}\tau^{\alpha-1-\varepsilon-\alpha\theta}\mathrm{d}\tau \\ &= \frac{1}{\Gamma(1-\alpha)}C_1(\varepsilon,\alpha,\widetilde{M})k^{-\varepsilon}B(1-\alpha,\alpha-\varepsilon-\alpha\theta)t^{-\varepsilon-\alpha\theta}\left\|f\right\|_{\mathbb{H}^{s+2-\theta}(\Omega)}, \end{aligned} \tag{50}$$

where we note that $\varepsilon < \alpha-\alpha\theta$. Thus, we deduce that

$$\left\|\mathbb{J}_2(.,t)\right\|_{\mathbb{H}^s(\Omega)} \leq \mathbb{C}_2 t^{-\varepsilon-\alpha\theta}\left\|f\right\|_{\mathbb{H}^{s+2-\theta}(\Omega)}, \tag{51}$$

where $\mathbb{C}_2$ depends on $\varepsilon,\alpha,k,T,\theta,\widetilde{M}$. Combining (47), (49) and (51), we deduce that

$$\begin{aligned} \left\|D_t^\alpha u^{\alpha,k}(.,t)\right\|_{\mathbb{H}^s(\Omega)} &\leq \left\|\mathbb{J}_1(.,t)\right\|_{\mathbb{H}^s(\Omega)} + \left\|\mathbb{J}_2(.,t)\right\|_{\mathbb{H}^s(\Omega)} \\ &\leq \mathbb{C}_1 t^{1-\alpha-(\varepsilon+\alpha\theta)}\left\|f\right\|_{\mathbb{H}^{s+1-\theta}(\Omega)} + \mathbb{C}_2 t^{-\varepsilon-\alpha\theta}\left\|f\right\|_{\mathbb{H}^{s+2-\theta}(\Omega)} \\ &\leq \mathbb{C}_3 t^{-\varepsilon-\alpha\theta}\left\|f\right\|_{\mathbb{H}^{s+2-\theta}(\Omega)}, \end{aligned} \tag{52}$$

where we note that $t^{1-\alpha-(\varepsilon+\alpha\theta)} \leq T^{1-\alpha}t^{-\varepsilon-\alpha\theta}$ and $\left\|f\right\|_{\mathbb{H}^{s+2-\theta}(\Omega)} \leq C\left\|f\right\|_{\mathbb{H}^{s+1-\theta}(\Omega)}$.

□

Theorem 3.3. *Let $f \in \mathbb{H}^s(\Omega)$ for any $s \geq 0$. Then we get that*

$$\left\| u^{\alpha,k}(.,t) \right\|_{\mathbb{H}^{s+\varepsilon}(\Omega)} \geq \widetilde{C} \left\| f \right\|_{\mathbb{H}^s(\Omega)}, \tag{53}$$

where $\widetilde{C}$ depends on $k, T, \alpha, \lambda_1, \varepsilon$.

Proof. Let the following function

$$\Psi(t) = e^{-kt} t^{1-\alpha}, \quad 0 \leq t \leq T.$$

Its derivative is

$$\Psi'(t) = e^{-kt} t^{1-\alpha} \Big(1 - \alpha - kt \Big), \quad k > 0.$$

The extreme point is $t_0 = \frac{1-\alpha}{k}$. Thus, since $0 \leq t \leq T$, we deduce that

$$\Psi(t) \leq \max \Big(\Psi(T), \Psi(t_0) \Big) = \max \Big(e^{-kT} T^{1-\alpha}, (ke)^{\alpha-1} (1-\alpha)^{1-\alpha} \Big) = \overline{M}_0(\alpha, T). \tag{54}$$

Hence, we get that

$$\begin{aligned} \int_0^T e^{-kt} E_{\alpha,1}(-\lambda_n t^\alpha) \mathrm{d}t &= \int_0^T e^{-kt} t^{1-\alpha} t^{\alpha-1} E_{\alpha,1}(-\lambda_n t^\alpha) \mathrm{d}t \\ &\leq \overline{M}_0(\alpha, T) \int_0^T t^{\alpha-1} E_{\alpha,1}(-\lambda_n t^\alpha) \mathrm{d}t \\ &\leq \overline{M}_0(\alpha, T) \int_0^T t^{\alpha-1} \frac{C_\alpha^+}{1 + \lambda_n t^\alpha} \mathrm{d}t = \overline{M}_0(\alpha, T) \alpha C_\alpha^+ \frac{\log(1 + \lambda_n T^\alpha)}{\lambda_n}. \end{aligned} \tag{55}$$

Using the lower bound of the Mittag-Leffler function $E_{\alpha,1}$, we know that

$$e^{-kt} E_{\alpha,1}(-\lambda_n t^\alpha) \geq e^{-kT} \frac{C_\alpha^-}{1 + \lambda_n t^\alpha} \geq e^{-kT} \frac{C_\alpha^-}{1 + \lambda_n T^\alpha}. \tag{56}$$

In view of (55) and (56), we know that

$$\begin{aligned} \frac{e^{-kt} E_{\alpha,1}(-\lambda_n t^\alpha)}{\int_0^T e^{-kt} E_{\alpha,1}(-\lambda_n t^\alpha) \mathrm{d}t} &\geq \frac{e^{-kT}}{C(\alpha) \overline{M}_0(\alpha, T)} \frac{\lambda_n}{1 + \lambda_n T^\alpha} \frac{1}{\log(1 + \lambda_n T^\alpha)} \\ &\geq \frac{e^{-kT}}{C(\alpha) \overline{M}_0(\alpha, T)} \frac{\lambda_1}{1 + \lambda_1 T^\alpha} \frac{1}{\log(1 + \lambda_n T^\alpha)}. \end{aligned} \tag{57}$$

Thus we get that

$$\begin{aligned} \left\| u^{\alpha,k}(.,t) \right\|^2_{\mathbb{H}^{s+\varepsilon}(\Omega)} &= \sum_{n=1}^\infty \lambda_n^{2s} \lambda_n^{2\varepsilon} \left(\frac{e^{-kt} E_{\alpha,1}(-\lambda_n t^\alpha)}{\int_0^T e^{-kt} E_{\alpha,1}(-\lambda_n t^\alpha) \mathrm{d}t} \right)^2 |f_n|^2 \\ &\geq \left(\frac{e^{-kT}}{C(\alpha) \overline{M}_0(\alpha, T)} \frac{\lambda_1}{1 + \lambda_1 T^\alpha} \right)^2 \sum_{n=1}^\infty \lambda_n^{2s} \frac{\lambda_n^{2\varepsilon}}{\log^2(1 + \lambda_n T^\alpha)} |f_n|^2. \end{aligned} \tag{58}$$

In view of the inequality $\log(1 + z) \leq C_\varepsilon z^\varepsilon$ for any $\varepsilon > 0$, we find that

$$\frac{\lambda_n^{2\varepsilon}}{\log^2(1 + \lambda_n T^\alpha)} \geq \frac{1}{|C_\varepsilon|^2}.$$

Hence, from two latter observations, we derive that

$$\left\|u^{\alpha,k}(.,t)\right\|^2_{\mathbb{H}^{s+\varepsilon}(\Omega)} \geq |\widetilde{C}|^2 \sum_{n=1}^{\infty} \lambda_n^{2s}|f_n|^2 = |\widetilde{C}|^2 \left\|f\right\|^2_{\mathbb{H}^s(\Omega)}, \tag{59}$$

where $\widetilde{C}$ depends on $k, T, \alpha, \lambda_1, \varepsilon$.
□

Theorem 3.4. *Let* $f \in \mathbb{H}^s(\Omega)$ *for any* $s \geq 0$. *Then we get*

$$\left\|u^{\alpha,k}(.,t) - u^{\alpha,k'}(.,t)\right\|_{L^q(0,T;\mathbb{H}^s(\Omega))} \leq C(\varepsilon,\alpha,T,q)T^{\varepsilon}|k-k'|^{\varepsilon}\left\|f\right\|_{\mathbb{H}^s(\Omega)}, \tag{60}$$

for any $1 \leq q \leq \frac{1}{\alpha}$.

Proof. Since (16), we find that

$$u^{\alpha,k}(x,t) - u^{\alpha,k'}(x,t) = \sum_{n=1}^{\infty}\left[\frac{e^{-kt}E_{\alpha,1}(-\lambda_n t^{\alpha})}{\int_0^T e^{-kt}E_{\alpha,1}(-\lambda_n t^{\alpha})\mathrm{d}t} - \frac{e^{-k't}E_{\alpha,1}(-\lambda_n t^{\alpha})}{\int_0^T e^{-k't}E_{\alpha,1}(-\lambda_n t^{\alpha})\mathrm{d}t}\right]f_n e_n(x). \tag{61}$$

It is easy to see that

$$\begin{aligned}&\frac{e^{-kt}E_{\alpha,1}(-\lambda_n t^{\alpha})}{\int_0^T e^{-kt}E_{\alpha,1}(-\lambda_n t^{\alpha})\mathrm{d}t} - \frac{e^{-k't}E_{\alpha,1}(-\lambda_n t^{\alpha})}{\int_0^T e^{-k't}E_{\alpha,1}(-\lambda_n t^{\alpha})\mathrm{d}t}\\ &= \frac{e^{-kt}E_{\alpha,1}(-\lambda_n t^{\alpha})\int_0^T e^{-k't}E_{\alpha,1}(-\lambda_n t^{\alpha})\mathrm{d}t - e^{-k't}E_{\alpha,1}(-\lambda_n t^{\alpha})\int_0^T e^{-kt}E_{\alpha,1}(-\lambda_n t^{\alpha})\mathrm{d}t}{\int_0^T e^{-kt}E_{\alpha,1}(-\lambda_n t^{\alpha})\mathrm{d}t\int_0^T e^{-k't}E_{\alpha,1}(-\lambda_n t^{\alpha})\mathrm{d}t}.\end{aligned} \tag{62}$$

Let us give the following observation

$$\begin{aligned}e^{-kt}\int_0^T e^{-k't}E_{\alpha,1}(-\lambda_n t^{\alpha})\mathrm{d}t - e^{-k't}\int_0^T e^{-kt}E_{\alpha,1}(-\lambda_n t^{\alpha})\mathrm{d}t&\\ = e^{-kt}\int_0^T \left(e^{-k't} - e^{-kt}\right)E_{\alpha,1}(-\lambda_n t^{\alpha})\mathrm{d}t&\\ + \left(e^{-kt} - e^{-k't}\right)\int_0^T e^{-kt}E_{\alpha,1}(-\lambda_n t^{\alpha})\mathrm{d}t.&\end{aligned} \tag{63}$$

Using the inequality $|e^{-a} - e^{-b}| \leq C_{\varepsilon}|a-b|^{\varepsilon}$ for $\varepsilon > 0$, we find that

$$\begin{aligned}&\left|e^{-kt}\int_0^T e^{-k't}E_{\alpha,1}(-\lambda_n t^{\alpha})\mathrm{d}t - e^{-k't}\int_0^T e^{-kt}E_{\alpha,1}(-\lambda_n t^{\alpha})\mathrm{d}t\right|\\ &\quad\leq C_{\varepsilon}T^{\varepsilon}|k-k'|^{\varepsilon}\int_0^T E_{\alpha,1}(-\lambda_n t^{\alpha})\mathrm{d}t + C_{\varepsilon}t^{\varepsilon}|k-k'|^{\varepsilon}\int_0^T E_{\alpha,1}(-\lambda_n t^{\alpha})\mathrm{d}t\\ &\quad\leq 2C_{\varepsilon}T^{\varepsilon}|k-k'|^{\varepsilon}\int_0^T E_{\alpha,1}(-\lambda_n t^{\alpha})\mathrm{d}t.\end{aligned} \tag{64}$$

Combining (62) and (64), we derive that

$$\left|\frac{e^{-kt}E_{\alpha,1}(-\lambda_n t^{\alpha})}{\int_0^T e^{-kt}E_{\alpha,1}(-\lambda_n t^{\alpha})\mathrm{d}t} - \frac{e^{-k't}E_{\alpha,1}(-\lambda_n t^{\alpha})}{\int_0^T e^{-k't}E_{\alpha,1}(-\lambda_n t^{\alpha})\mathrm{d}t}\right|$$

$$\leq \frac{2C_\varepsilon T^\varepsilon |k-k'|^\varepsilon E_{\alpha,1}(-\lambda_n t^\alpha)\int_0^T E_{\alpha,1}(-\lambda_n t^\alpha)\mathrm{d}t}{\int_0^T e^{-kt}E_{\alpha,1}(-\lambda_n t^\alpha)\mathrm{d}t\int_0^T e^{-k't}E_{\alpha,1}(-\lambda_n t^\alpha)\mathrm{d}t}$$
$$\leq \frac{2C_\varepsilon T^\varepsilon |k-k'|^\varepsilon E_{\alpha,1}(-\lambda_n t^\alpha)\int_0^T E_{\alpha,1}(-\lambda_n t^\alpha)\mathrm{d}t}{e^{-kT}\int_0^T E_{\alpha,1}(-\lambda_n t^\alpha)\mathrm{d}t\int_0^T e^{-k't}E_{\alpha,1}(-\lambda_n t^\alpha)\mathrm{d}t}. \tag{65}$$

This follows from (20) that

$$\left|\frac{e^{-kt}E_{\alpha,1}(-\lambda_n t^\alpha)}{\int_0^T e^{-kt}E_{\alpha,1}(-\lambda_n t^\alpha)\mathrm{d}t}-\frac{e^{-k't}E_{\alpha,1}(-\lambda_n t^\alpha)}{\int_0^T e^{-k't}E_{\alpha,1}(-\lambda_n t^\alpha)\mathrm{d}t}\right| \leq 2C_\varepsilon T^\varepsilon|k-k'|^\varepsilon E_{\alpha,1}(-\lambda_n t^\alpha)\frac{\lambda_n}{C_\alpha^- e^{-k'T}\widetilde{M}(\alpha,T)}. \tag{66}$$

Using the inequality $E_{\alpha,1}(-\lambda_n t^\alpha) \leq \frac{C_\alpha^+}{\lambda_n t^\alpha}$, we get the following estimate

$$\begin{aligned}\left|\frac{e^{-kt}E_{\alpha,1}(-\lambda_n t^\alpha)}{\int_0^T e^{-kt}E_{\alpha,1}(-\lambda_n t^\alpha)\mathrm{d}t}-\frac{e^{-k't}E_{\alpha,1}(-\lambda_n t^\alpha)}{\int_0^T e^{-k't}E_{\alpha,1}(-\lambda_n t^\alpha)\mathrm{d}t}\right| &\leq C(\varepsilon,\alpha,T)E_{\alpha,1}(-\lambda_n t^\alpha)\lambda_n|k-k'|^\varepsilon\\ &\leq C(\varepsilon,\alpha,T)T^\varepsilon|k-k'|^\varepsilon t^{-\alpha}.\end{aligned} \tag{67}$$

This inequality together with (61) yields that

$$\begin{aligned}&\left\|u^{\alpha,k}(.,t)-u^{\alpha,k'}(.,t)\right\|_{\mathbb{H}^s(\Omega)}\\ &= \left(\sum_{n=1}^\infty \lambda_n^{2s}\left[\frac{e^{-kt}E_{\alpha,1}(-\lambda_n t^\alpha)}{\int_0^T e^{-kt}E_{\alpha,1}(-\lambda_n t^\alpha)\mathrm{d}t}-\frac{e^{-k't}E_{\alpha,1}(-\lambda_n t^\alpha)}{\int_0^T e^{-k't}E_{\alpha,1}(-\lambda_n t^\alpha)\mathrm{d}t}\right]^2 f_n^2\right)^{1/2}\\ &\leq C(\varepsilon,\alpha,T)T^\varepsilon|k-k'|^\varepsilon t^{-\alpha}\left(\sum_{n=1}^\infty \lambda_n^{2s}f_n^2\right)^{1/2} \leq C(\varepsilon,\alpha,T)T^\varepsilon|k-k'|^\varepsilon t^{-\alpha}\left\|f\right\|_{\mathbb{H}^s(\Omega)}.\end{aligned} \tag{68}$$

Let any q such that $1 \leq q < \frac{1}{\alpha}$. Then since the convergence of the integral $\int_0^T t^{-\alpha q}\mathrm{d}t$, we obtain the following estimate

$$\left\|u^{\alpha,k}(.,t)-u^{\alpha,k'}(.,t)\right\|_{L^q(0,T;\mathbb{H}^s(\Omega))} \leq C(\varepsilon,\alpha,T,q)T^\varepsilon|k-k'|^\varepsilon\left\|f\right\|_{\mathbb{H}^s(\Omega)}. \tag{69}$$

□

4. Ill-posedness and regularization

In practice, the exact data f is noised by the observed data $f_\delta \in L^2(\Omega)$ which satisfies that

$$\left\|f_\delta - f\right\|_{L^2(\Omega)} \leq \delta. \tag{70}$$

In this section, we will provide two methods for regularizing our inverse problem. The first method is quasi-reversibility method (QR method) which was introduced by [24]. This method was developed for other models in the papers [25]. The truncation method proves to be useful in many different problems, see [26]. The reason we add a truncated correction method is because we want to handle errors on the space $\mathbb{H}^s$ for $s > 0$. Note that it is very difficult to solve the problem on $\mathbb{H}^s$ using the QR method.

4.1. Quasi-reversibility method

Let us give the following regularized problem by using quasi-reversibility method

$$\begin{cases} D_t^{\alpha,k} u^\delta(x,t) + \mathcal{A} u^\delta(x,t) + \beta D_t^{\alpha,k} \mathcal{A} u^\delta(x,t) = 0, & (x,t) \in \Omega \times (0,T), \\ u^\delta(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ \displaystyle\int_0^T u^\delta(x,s) \mathrm{d}s = f^\delta(x), & x \in \Omega, \end{cases} \tag{71}$$

Theorem 4.1. *Let the input data $f \in \mathbb{H}^{1+\mu}(\Omega)$ for any $\mu > 0$. Let $\beta = \delta^h$ for $0 < h < 1$ then we get*

$$\left\| u^\delta(.,t) - u(.,t) \right\|_{L^2(\Omega)} \le \mathbf{C}_2 \delta^{1-h} + \mathbf{C}_5 \delta^{h\mu} \left\| f \right\|_{\mathbb{H}^{1+\mu}(\Omega)}, \tag{72}$$

here $\mathbf{C}_2$ depends on $\alpha, k, T, \widetilde{M}$. The constant $\mathbf{C}_5$ depends on $\alpha, k, T, C_\alpha^-, C_\alpha^+, \widetilde{M}(\alpha,T), \mu$.

Proof. Let us now give the explicit formula of the mild solution to Problem (71). Let us assume that Problem (71) has a unique solution u^δ. Let $u^\delta(x,t) = \sum_{n=1}^\infty u_n^\delta(t) e_n(x)$ be the Fourier series in $L^2(\Omega)$. From the first equation of (71), taking the inner product of both sides of (71) with $e_n(x)$, we obtain

$$D_t^{\alpha,k} u_n^\delta(t) + \lambda_n u_n^\delta(t) + \beta \lambda_n D_t^{\alpha,k} u_n^\delta(t) = 0. \tag{73}$$

This implies that

$$u_n^\delta(t) = e^{-kt} E_{\alpha,1}\Big(- \frac{\lambda_n}{1+\beta\lambda_n} t^\alpha \Big) u_n^\delta(0). \tag{74}$$

Since the condition

$$\int_0^T u^\delta(x,t) \mathrm{d}x = f^\delta(x),$$

we know that

$$\Big(\int_0^T e^{-kt} E_{\alpha,1}\Big(- \frac{\lambda_n}{1+\beta\lambda_n} t^\alpha \Big) \mathrm{d}t \Big) u_n^\delta(0) = f_n^\delta. \tag{75}$$

Thus, we have

$$u_n^\delta(0) = \frac{e^{-kt} E_{\alpha,1}(-\frac{\lambda_n}{1+\beta\lambda_n} t^\alpha)}{\int_0^T e^{-kt} E_{\alpha,1}(-\frac{\lambda_n}{1+\beta\lambda_n} t^\alpha) \mathrm{d}t} f_n^\delta. \tag{76}$$

By the definition of Fourier series, one has

$$u^\delta(x,t) = \sum_{n=1}^\infty \frac{e^{-kt} E_{\alpha,1}(-\frac{\lambda_n}{1+\beta\lambda_n} t^\alpha)}{\int_0^T e^{-kt} E_{\alpha,1}(-\frac{\lambda_n}{1+\beta\lambda_n} t^\alpha) \mathrm{d}t} f_n^\delta e_n(x). \tag{77}$$

By a similar techniques as in (20), we know that

$$\int_0^T e^{-kt} E_{\alpha,1}(-\frac{\lambda_n}{1+\beta\lambda_n} t^\alpha) \mathrm{d}t \ge \frac{C_\alpha^- \widetilde{M}(\alpha,T)}{e^{kT} \frac{\lambda_n}{1+\beta\lambda_n}} = \frac{C_\alpha^- \widetilde{M}(\alpha,T)(1+\beta\lambda_n)}{e^{kT} \lambda_n}. \tag{78}$$

Using the upper bound of $E_{\alpha,1}$, we get that

$$E_{\alpha,1}\Big(-\frac{\lambda_n}{1+\beta\lambda_n}t^\alpha\Big) \le \frac{C_\alpha^+}{1+\frac{\lambda_n}{1+\beta\lambda_n}t^\alpha} \le C_\alpha^+. \tag{79}$$

From two latter estimates, we derive that

$$\frac{e^{-kt}E_{\alpha,1}(-\frac{\lambda_n}{1+\beta\lambda_n}t^\alpha)}{\int_0^T e^{-kt}E_{\alpha,1}(-\frac{\lambda_n}{1+\beta\lambda_n}t^\alpha)\mathrm{d}t} \le \frac{C_\alpha^+ e^{kT}\lambda_n}{C_\alpha^-\widetilde{M}(\alpha,T)(1+\beta\lambda_n)} \le \mathbf{C_2}\frac{1}{\beta}. \tag{80}$$

Here $\mathbf{C_2}$ depends on $\alpha, k, T, \widetilde{M}$. This implies that

$$\Big\|u^\delta(.,t)\Big\|^2_{\mathbb{H}^s(\Omega)} = \sum_{n=1}^\infty \lambda_n^{2s}\bigg(\frac{e^{-kt}E_{\alpha,1}(-\frac{\lambda_n}{1+\beta\lambda_n}t^\alpha)}{\int_0^T e^{-kt}E_{\alpha,1}(-\frac{\lambda_n}{1+\beta\lambda_n}t^\alpha)\mathrm{d}t}\bigg)^2|f_n|^2 \le \frac{\mathbf{C_2}^2}{\beta^2}\sum_{n=1}^\infty \lambda_n^{2s}|f_n|^2. \tag{81}$$

This implies that

$$\Big\|u^\delta(.,t)\Big\|_{\mathbb{H}^s(\Omega)} \le \frac{\mathbf{C_2}}{\beta}\Big\|f\Big\|_{\mathbb{H}^s(\Omega)}.$$

Let the following function

$$v^\delta(x,t) = \sum_{n=1}^\infty \frac{e^{-kt}E_{\alpha,1}(-\frac{\lambda_n}{1+\beta\lambda_n}t^\alpha)}{\int_0^T e^{-kt}E_{\alpha,1}(-\frac{\lambda_n}{1+\beta\lambda_n}t^\alpha)\mathrm{d}t}f_n e_n(x). \tag{82}$$

It is obvious to see that

$$\Big\|u^\delta(.,t) - v^\delta(.,t)\Big\|_{L^2(\Omega)} \le \frac{\mathbf{C_2}}{\beta}\Big\|f^\delta - f\Big\|_{L^2(\Omega)} \le \frac{\mathbf{C_2}\delta}{\beta}. \tag{83}$$

Let us now give the bound $\Big\|v^\delta(.,t) - u^*(.,t)\Big\|_{L^2(\Omega)}$. First, we need to find the upper bound for the difference

$$\mathcal{F} = \frac{e^{-kt}E_{\alpha,1}(-\frac{\lambda_n}{1+\beta\lambda_n}t^\alpha)}{\int_0^T e^{-kt}E_{\alpha,1}(-\frac{\lambda_n}{1+\beta\lambda_n}t^\alpha)\mathrm{d}t} - \frac{e^{-kt}E_{\alpha,1}(-\lambda_n t^\alpha)}{\int_0^T e^{-kt}E_{\alpha,1}(-\lambda_n t^\alpha)\mathrm{d}t}. \tag{84}$$

It is easy to verify that

$$\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2, \tag{85}$$

where

$$\mathcal{F}_1 = \frac{e^{-kt}E_{\alpha,1}(-\frac{\lambda_n}{1+\beta\lambda_n}t^\alpha)\int_0^T e^{-kt}\Big(E_{\alpha,1}(-\lambda_n t^\alpha) - E_{\alpha,1}(-\frac{\lambda_n}{1+\beta\lambda_n}t^\alpha)\Big)\mathrm{d}t}{\Big(\int_0^T e^{-kt}E_{\alpha,1}(-\frac{\lambda_n}{1+\beta\lambda_n}t^\alpha)\mathrm{d}t\Big)\Big(\int_0^T e^{-kt}E_{\alpha,1}(-\lambda_n t^\alpha)\mathrm{d}t\Big)} \tag{86}$$

and

$$\mathcal{F}_2 = \frac{e^{-kt}\Big(E_{\alpha,1}(-\frac{\lambda_n}{1+\beta\lambda_n}t^\alpha) - E_{\alpha,1}(-\lambda_n t^\alpha)\Big)\int_0^T e^{-kt}E_{\alpha,1}(-\frac{\lambda_n}{1+\beta\lambda_n}t^\alpha)\mathrm{d}t}{\Big(\int_0^T e^{-kt}E_{\alpha,1}(-\frac{\lambda_n}{1+\beta\lambda_n}t^\alpha)\mathrm{d}t\Big)\Big(\int_0^T e^{-kt}E_{\alpha,1}(-\lambda_n t^\alpha)\mathrm{d}t\Big)}$$

$$= \frac{e^{-kt}\Big(E_{\alpha,1}(-\frac{\lambda_n}{1+\beta\lambda_n}t^\alpha) - E_{\alpha,1}(-\lambda_n t^\alpha)\Big)}{\int_0^T e^{-kt}E_{\alpha,1}(-\lambda_n t^\alpha)\mathrm{d}t}. \tag{87}$$

Since

$$\frac{d}{dz}E_{\alpha,1}(-\omega) = \frac{E_{\alpha,\alpha}(-\omega)}{\alpha}. \tag{88}$$

We know that

$$\begin{aligned}\Big|E_{\alpha,1}(-\frac{\lambda_n}{1+\beta\lambda_n}t^\alpha) - E_{\alpha,1}(-\lambda_n t^\alpha)\Big| &= \frac{1}{\alpha}\Big|\int_{\frac{\lambda_n}{1+\beta\lambda_n}t^\alpha}^{\lambda_n t^\alpha} E_{\alpha,\alpha}(-\tau)\mathrm{d}\tau\Big| \\ &\le \frac{1}{\alpha}\int_{\frac{\lambda_n}{1+\beta\lambda_n}t^\alpha}^{\lambda_n t^\alpha} |E_{\alpha,\alpha}(-\tau)|\mathrm{d}\tau,\end{aligned} \tag{89}$$

where we note that $\frac{\lambda_n}{1+\beta\lambda_n}t^\alpha < \lambda_n t^\alpha$. Since the bound $|E_{\alpha,\alpha}(-\tau)| \le \frac{N_\alpha}{1+\tau}$, we find that

$$\begin{aligned}\int_{\frac{\lambda_n}{1+\beta\lambda_n}t^\alpha}^{\lambda_n t^\alpha} |E_{\alpha,\alpha}(-\tau)|\mathrm{d}\tau &\le N_\alpha \int_{\frac{\lambda_n}{1+\beta\lambda_n}t^\alpha}^{\lambda_n t^\alpha} \frac{\mathrm{d}\tau}{1+\tau} \\ &= N_\alpha \log\Big(1+\lambda_n t^\alpha\Big) - N_\alpha \log\Big(1+\frac{\lambda_n}{1+\beta\lambda_n}t^\alpha\Big) \\ &= N_\alpha \log\Bigg(\frac{1+\lambda_n t^\alpha}{1+\frac{\lambda_n}{1+\beta\lambda_n}t^\alpha}\Bigg).\end{aligned} \tag{90}$$

It is not difficult to check that $\frac{1+\lambda_n t^\alpha}{1+\frac{\lambda_n}{1+\beta\lambda_n}t^\alpha} \le 1+\beta\lambda_n$. This implies that

$$\Big|E_{\alpha,1}(-\frac{\lambda_n}{1+\beta\lambda_n}t^\alpha) - E_{\alpha,1}(-\lambda_n t^\alpha)\Big| \le \frac{1}{\alpha}\log\Big(1+\beta\lambda_n\Big). \tag{91}$$

This follows from (87) and (20) that

$$|\mathcal{F}_2| \le \frac{e^{kT}}{\alpha C_\alpha^- \widetilde{M}(\alpha,T)}\lambda_n \log\Big(1+\beta\lambda_n\Big). \tag{92}$$

Let us observe the term $\mathcal{F}_1$. Using (91), one has

$$\begin{aligned}\Bigg|\int_0^T e^{-kt}\Bigg(E_{\alpha,1}(-\lambda_n t^\alpha) - E_{\alpha,1}(-\frac{\lambda_n}{1+\beta\lambda_n}t^\alpha)\Bigg)\mathrm{d}t\Bigg| &\le \frac{1}{\alpha}\log\Big(1+\beta\lambda_n\Big)\int_0^T e^{-kt}\mathrm{d}t \\ &= \frac{1-e^{-kT}}{k}\frac{1}{\alpha}\log\Big(1+\beta\lambda_n\Big).\end{aligned} \tag{93}$$

This inequality together with (20) yield that

$$\frac{\Bigg|\int_0^T e^{-kt}\Big(E_{\alpha,1}(-\lambda_n t^\alpha) - E_{\alpha,1}(-\frac{\lambda_n}{1+\beta\lambda_n}t^\alpha)\Big)\mathrm{d}t\Bigg|}{\int_0^T e^{-kt}E_{\alpha,1}(-\lambda_n t^\alpha)\mathrm{d}t} \le \frac{e^{kT}-1}{k\alpha C_\alpha^- \widetilde{M}(\alpha,T)}\lambda_n \log\Big(1+\beta\lambda_n\Big). \tag{94}$$

We continute to estimate the term $\frac{e^{-kt}E_{\alpha,1}(-\frac{\lambda_n}{1+\beta\lambda_n}t^\alpha)}{\int_0^T e^{-kt}E_{\alpha,1}(-\frac{\lambda_n}{1+\beta\lambda_n}t^\alpha)\mathrm{d}t}$. Indeed, we get that

$$E_{\alpha,1}(-\frac{\lambda_n}{1+\beta\lambda_n}t^\alpha) \le \frac{C_\alpha^+}{1+\frac{\lambda_n}{1+\beta\lambda_n}t^\alpha} \le C_\alpha^+\frac{1+\beta\lambda_n}{\lambda_n}t^{-\alpha}, \tag{95}$$

and

$$\int_0^T e^{-kt}E_{\alpha,1}(-\frac{\lambda_n}{1+\beta\lambda_n}t^{\alpha})\mathrm{d}t \geq \int_0^T e^{-kt}\frac{C_{\alpha}^{-}}{1+\frac{\lambda_n}{1+\beta\lambda_n}t^{\alpha}}\mathrm{d}t \geq \frac{C_{\alpha}^{-}}{1+\frac{\lambda_n}{1+\beta\lambda_n}T^{\alpha}}\int_0^T e^{-kt}\mathrm{d}t$$
$$= \frac{1-e^{-kT}}{k}C_{\alpha}^{-}\frac{1+\beta\lambda_n}{1+\beta\lambda_n+\lambda_n T^{\alpha}}. \quad (96)$$

Thus, we get that

$$\frac{e^{-kt}E_{\alpha,1}(-\frac{\lambda_n}{1+\beta\lambda_n}t^{\alpha})}{\int_0^T e^{-kt}E_{\alpha,1}(-\frac{\lambda_n}{1+\beta\lambda_n}t^{\alpha})\mathrm{d}t} \leq \frac{C_{\alpha}^{+}k}{(1-e^{-kT})C_{\alpha}^{-}}\frac{1+\beta\lambda_n+\lambda_n T^{\alpha}}{\lambda_n}. \quad (97)$$

Combining (94) and (97), we derive that

$$|\mathcal{F}_1| \leq \mathbf{C_3}\Big(\frac{1}{\lambda_1}+\beta+T^{\alpha}\Big)\lambda_n\log\Big(1+\beta\lambda_n\Big), \quad (98)$$

where $\mathbf{C_3}$ depends on $\alpha, k, T, C_{\alpha}^{-}, C_{\alpha}^{+}$. By collecting (85), (92) and (98), we confirm that

$$|\mathcal{F}| \leq \mathbf{C_4}\lambda_n\log\Big(1+\beta\lambda_n\Big), \quad (99)$$

where $\mathbf{C_4}$ depends on $\alpha, k, T, C_{\alpha}^{-}, C_{\alpha}^{+}, \widetilde{M}(\alpha, T)$. Using Parseval's equality, one gets

$$\Big\|v^{\delta}(.,t)-u^{*}(.,t)\Big\|_{L^2(\Omega)}^2 = \sum_{j=1}^{\infty}\Bigg(\frac{e^{-kt}E_{\alpha,1}(-\frac{\lambda_n}{1+\beta\lambda_n}t^{\alpha})}{\int_0^T e^{-kt}E_{\alpha,1}(-\frac{\lambda_n}{1+\beta\lambda_n}t^{\alpha})\mathrm{d}t} - \frac{e^{-kt}E_{\alpha,1}(-\lambda_n t^{\alpha})}{\int_0^T e^{-kt}E_{\alpha,1}(-\lambda_n t^{\alpha})\mathrm{d}t}\Bigg)^2 f_n^2$$
$$\leq |\mathbf{C_4}|^2\sum_{j=1}^{\infty}\lambda_n^2\log^2\Big(1+\beta\lambda_n\Big)f_n^2. \quad (100)$$

Using the inequality $1-e^{-z} \leq C_{\mu}z^{\mu}$ for any $\mu > 0$, we obtain

$$\Big\|v^{\delta}(.,t)-u^{*}(.,t)\Big\|_{L^2(\Omega)}^2 \leq |\mathbf{C_4}|^2 C_{\mu}^2\beta^{2\mu}\sum_{j=1}^{\infty}\lambda_n^{2+2\mu}f_n^2. \quad (101)$$

Hence, we infer that

$$\Big\|v^{\delta}(.,t)-u^{*}(.,t)\Big\|_{L^2(\Omega)} \leq |\mathbf{C_4}|C_{\mu}\beta^{\mu}\Big\|f\Big\|_{\mathbb{H}^{1+\mu}(\Omega)}. \quad (102)$$

The proof is completed.

□

4.2. Truncation method

In the subsection, we introduce a regularized solution using truncation method. In practice, since k is a positive real number, so it is approximated by a rational number k_{δ} such that

$$|k_{\delta}-k| \leq \delta. \quad (103)$$

Under the perturbation function f^{δ}, we give the following function

$$u_N^{\alpha,\delta}(x,t) = \sum_{n=1}^{\lambda_n\leq N}\frac{e^{-k_{\delta}t}E_{\alpha,1}(-\lambda_n t^{\alpha})}{\int_0^T e^{-k_{\delta}t}E_{\alpha,1}(-\lambda_n t^{\alpha})\mathrm{d}t}f_n^{\delta}e_n(x), \quad (104)$$

which is called a regularized solution. Here N is a positive constant which depends on δ and satisfies that $\lim_{\delta\to 0}N = +\infty$.

Theorem 4.2. *Let $f^\delta \in L^2(\Omega)$, assume that $u^* \in L^\infty(0,T;\mathbb{H}^{s+\beta}(\Omega))$ for any $q \geq 0$ and $\beta > 0$. Then we get*

$$\begin{aligned}\Big\|u_N^{\alpha,\delta}(.,t) - u^*(.,t)\Big\|_{\mathbb{H}^s(\Omega)} &\leq C(\alpha,T)\delta N^{s+1}\Big\|f^\delta\Big\|_{L^2(\Omega)} \\ &\quad + \frac{C_\alpha^+ e^{kT}}{C_\alpha^- \widetilde{M}(\alpha,T)} N^{s+1}\delta + N^{-\beta}\Big\|u^*\Big\|_{L^\infty(0,T;\mathbb{H}^{s+\beta}(\Omega))},\end{aligned} \tag{105}$$

where we choose N such that

$$\lim_{\delta\to 0} N = +\infty, \quad \lim_{\delta \to 0} \delta N^{s+1} = 0. \tag{106}$$

Remark 4.3. *Let us choose $N = \theta^{\frac{\theta-1}{s+1}}$ for $0 < \theta < 1$. Then we deduce that the error $\Big\|u_N^{\alpha,\delta}(.,t) - u^*(.,t)\Big\|_{\mathbb{H}^s(\Omega)}$ is of order*

$$\max\Big(\delta^\theta, \theta^{\frac{\beta(1-\theta)}{s+1}}\Big).$$

Proof. Let us set the following function

$$u_N^{\alpha,\delta}(x,t) = \sum_{n=1}^{\lambda_n \leq N} \frac{e^{-kt}E_{\alpha,1}(-\lambda_n t^\alpha)}{\int_0^T e^{-kt}E_{\alpha,1}(-\lambda_n t^\alpha)\mathrm{d}t} f_n^\delta e_n(x), \tag{107}$$

and

$$v_N^{\alpha,\delta}(x,t) = \sum_{n=1}^{\lambda_n \leq N} \frac{e^{-kt}E_{\alpha,1}(-\lambda_n t^\alpha)}{\int_0^T e^{-kt}E_{\alpha,1}(-\lambda_n t^\alpha)\mathrm{d}t} f_n e_n(x). \tag{108}$$

Using the triangle inequality, we find that

$$\begin{aligned}\Big\|u_N^{\alpha,\delta}(.,t) - u^*(.,t)\Big\|_{\mathbb{H}^s(\Omega)} &\leq \Big\|u_N^{\alpha,\delta}(.,t) - u_N^{\alpha,\delta}(.,t)\Big\|_{\mathbb{H}^s(\Omega)} + \Big\|u_N^{\alpha,\delta}(.,t) - v_N^{\alpha,\delta}(.,t)\Big\|_{\mathbb{H}^s(\Omega)} \\ &\quad + \Big\|v_N^{\alpha,\delta}(.,t) - u^*(.,t)\Big\|_{\mathbb{H}^s(\Omega)}.\end{aligned} \tag{109}$$

In view of (67), one gets

$$\begin{aligned}&\left|\frac{e^{-k_\delta t}E_{\alpha,1}(-\lambda_n t^\alpha)}{\int_0^T e^{-k_\delta t}E_{\alpha,1}(-\lambda_n t^\alpha)\mathrm{d}t} - \frac{e^{-kt}E_{\alpha,1}(-\lambda_n t^\alpha)}{\int_0^T e^{-kt}E_{\alpha,1}(-\lambda_n t^\alpha)\mathrm{d}t}\right| \\ &\qquad \leq C(\gamma,\alpha,T)|k_\delta - k|^\gamma E_{\alpha,1}(-\lambda_n t^\alpha)\lambda_n \\ &\qquad \leq C(\gamma,\alpha,T)|k_\delta - k|^\gamma \lambda_n \leq C(\gamma,\alpha,T)\delta^\gamma \lambda_n,\end{aligned} \tag{110}$$

for any $\gamma > 0$. Using Parseval's equality, we derive that

$$\begin{aligned}&\Big\|u_N^{\alpha,\delta}(.,t) - u_N^{\alpha,\delta}(.,t)\Big\|^2_{\mathbb{H}^s(\Omega)} \\ &\quad = \sum_{n=1}^{\lambda_n\leq N} \lambda_n^{2s}\left|\frac{e^{-k_\delta t}E_{\alpha,1}(-\lambda_n t^\alpha)}{\int_0^T e^{-k_\delta t}E_{\alpha,1}(-\lambda_n t^\alpha)\mathrm{d}t} - \frac{e^{-kt}E_{\alpha,1}(-\lambda_n t^\alpha)}{\int_0^T e^{-kt}E_{\alpha,1}(-\lambda_n t^\alpha)\mathrm{d}t}\right|^2 \\ &\quad \leq \Big|C(\gamma,\alpha,T)\Big|^2 \delta^{2\gamma}\sum_{n=1}^{\lambda_n\leq N}\lambda_n^{2s+2}|f_n^\delta|^2 \leq \Big|C(\gamma,\alpha,T)\Big|^2\delta^{2\gamma}N^{2s+2}\sum_{n=1}^{\lambda_n\leq N}|f_n^\delta|^2.\end{aligned} \tag{111}$$

Hence, we arrive at

$$\left\|u_N^{\alpha,\delta}(.,t) - u_N^{\alpha,\delta}(.,t)\right\|_{\mathbb{H}^s(\Omega)} \leq C(\gamma,\alpha,T)\delta^\gamma N^{s+1}\left\|f^\delta\right\|_{L^2(\Omega)}. \tag{112}$$

In view of (107) and (108), one gets

$$\left\|u_N^{\alpha,\delta}(.,t) - v_N^{\alpha,\delta}(.,t)\right\|^2_{\mathbb{H}^s(\Omega)} = \sum_{n=1}^{\lambda_n \leq N} \left(\frac{e^{-kt}E_{\alpha,1}(-\lambda_n t^\alpha)}{\int_0^T e^{-kt}E_{\alpha,1}(-\lambda_n t^\alpha)\mathrm{d}t}\right)^2 \left(f_n^\delta - f_n\right)^2. \tag{113}$$

Using (20), we get the following bound

$$\frac{e^{-kt}E_{\alpha,1}(-\lambda_n t^\alpha)}{\int_0^T e^{-kt}E_{\alpha,1}(-\lambda_n t^\alpha)\mathrm{d}t} \leq \frac{C_\alpha^+ e^{kT}\lambda_n}{C_\alpha^- \widetilde{M}(\alpha,T)}. \tag{114}$$

It follows from (113) that

$$\begin{aligned}\left\|u_N^{\alpha,\delta}(.,t) - v_N^{\alpha,\delta}(.,t)\right\|^2_{\mathbb{H}^s(\Omega)} &\leq \Big(\frac{C_\alpha^+ e^{kT}}{C_\alpha^- \widetilde{M}(\alpha,T)}\Big)^2 \sum_{n=1}^{\lambda_n \leq N} \lambda_n^{2+2s}\Big(f_n^\delta - f_n\Big)^2 \\ &\leq \Big(\frac{C_\alpha^+ e^{kT}}{C_\alpha^- \widetilde{M}(\alpha,T)}\Big)^2 N^{2+2s}\left\|f^\delta - f\right\|^2_{L^2(\Omega)}.\end{aligned} \tag{115}$$

Under the assumption (70), we obtain that

$$\left\|u_N^{\alpha,\delta}(.,t) - v_N^{\alpha,\delta}(.,t)\right\|_{\mathbb{H}^s(\Omega)} \leq \frac{C_\alpha^+ e^{kT}}{C_\alpha^- \widetilde{M}(\alpha,T)} N^{s+1}\delta. \tag{116}$$

Since (108), we know that

$$v_N^{\alpha,\delta}(x,t) - u^*(x,t) = \sum_{n=1}^{\lambda_n > N} \frac{e^{-kt}E_{\alpha,1}(-\lambda_n t^\alpha)}{\int_0^T e^{-kt}E_{\alpha,1}(-\lambda_n t^\alpha)\mathrm{d}t} f_n e_n(x) = \sum_{n=1}^{\lambda_n > N} u_n^*(t)e_n(x). \tag{117}$$

Thus, using Parseval's equality, one gets

$$\left\|u_N^{\alpha,\delta}(.,t) - u^*(.,t)\right\|^2_{\mathbb{H}^s(\Omega)} = \sum_{n=1}^{\lambda_n > N} \lambda_n^{2s}\lambda_n^{-2\beta}\lambda_n^{2\beta}|u_n^*(t)|^2 \leq N^{-2\beta}\sum_{n=1}^{\lambda_n > N} \lambda_n^{2s+2\beta}|u_n^*(t)|^2. \tag{118}$$

Hence, we infer that

$$\left\|u_N^{\alpha,\delta}(.,t) - u^*(.,t)\right\|_{\mathbb{H}^s(\Omega)} \leq N^{-\beta}\left\|u^*\right\|_{L^\infty(0,T;\mathbb{H}^{s+\beta}(\Omega))}. \tag{119}$$

Combining (109), (112), (116) and (119), we deduce that

$$\begin{aligned}\left\|u_N^{\alpha,\delta}(.,t) - u^*(.,t)\right\|_{\mathbb{H}^s(\Omega)} &\leq \mathbb{C}(\alpha,T)\delta N^{s+1}\left\|f^\delta\right\|_{L^2(\Omega)} \\ &\quad + \frac{C_\alpha^+ e^{kT}}{C_\alpha^- \widetilde{M}(\alpha,T)} N^{s+1}\delta + N^{-\beta}\left\|u^*\right\|_{L^\infty(0,T;\mathbb{H}^{q+\beta}(\Omega))},\end{aligned} \tag{120}$$

where we choose $\gamma = 1$. □

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5. Conclusion

In summary, the paper presents a comprehensive study of a tempered fractional diffusion equation with an integral condition. The significant contribution is the regularization of this problem, employing two regularization methods: the quasi-reversibility method and the Fourier truncation method, both of which enhance the accuracy of solutions in fractional calculus. Moreover, we also investigate the continuity of the solution with respect to the fractional order.

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