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Solving Pell's equation using suborbital graphs

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Abstract. We establish a connection between suborbital graphs and integer solutions of Pell's equation of the form $x^2 - Ny^2 = 1$, where N is a non-square positive integer. We derive new suborbital graphs generated by the action of some specific modular subgroups on extended rational numbers. By using these graphs, we obtain a new combinatorial notation for the integer solutions of Pell's equation and some results on the vertices of the graphs studied here.

1. Introduction

Pell's equation is used to describe an equation of the form

$$x^2 - Ny^2 = 1, \tag{1}$$

with the positive integer N greater than 1. If N is a square integer, then the solutions of (1) are trivial. So, we assume that N is a square-free integer. The integer solutions of the Pell's equation (1) are significant because of the applications in number theory and cryptography.

There are several techniques for finding integer solutions to Pell's equation. The classical solution of this equation is associated with the continuous fractions. In this contribution, we use the suborbital graph theory, which is an efficient method for providing new approaches to various problems in number theory. Sims introduced the concept of suborbital graph theory as relation to permutation groups in [6]. Later, in [7], Jones et al. investigated the suborbital graph, which is a directed graph arising from the transitive group action for the modular group, by studying the notion introduced in [6]. Inspired by [7], extensive research has been done on suborbital graphs for the modular group and related objects, including references to [5, 9, 10, 12], and [11].

Kader et al. analyzed suborbital graphs for the extended modular group in [13]. Köroğlu et al. investigated the suborbital graphs for the Atkin-Lehner group [15]. Değer et al. gave some results on continued fractions in a corresponding suborbital graph [1]. Güler et al. studied the solutions of some congruence equations via suborbital graphs [3, 4]. Recently, some new studies on suborbital graphs are directly related to well-known number sequences such as Fibonacci and Pascal numbers [8, 14, 16, 18].

In this study, we observe a new form for the integer solutions of Pell's equation (1), by using some new suborbital graphs derived from the natural action of a certain modular subgroup on the extended rationals.

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Using a similar notion in [18], we obtain some new sequences of numbers derived from the vertices of suborbital graphs induced by the elements of the modular group;

$$K_N := \begin{pmatrix} a & cN \\ c & a \end{pmatrix},$$

where a and c are integers satisfying $a^2 - Nc^2 = 1$. The corresponding fractional linear transformation of K_N is denoted by

$$K_N(z) = \frac{az + cN}{cz + a}.$$

By using this transformation we obtain new rational sequences converging to the fixed points $\pm\sqrt{N}$. Moreover, it is obtained new paths by considering the orbit of ∞ in suborbital graphs derived from the action of a certain subgroup generated K_N. In general, these paths are as follows:

$$\infty \rightarrow K_N(\infty) = \frac{a}{c} \rightarrow K_N^2(\infty) = \frac{a^2 + Nc^2}{2ac} \rightarrow K_N^3(\infty) = \frac{a^3 + 3aNc^2}{3a^2c + Nc^3} \rightarrow \cdots \tag{2}$$

Moreover, by iterating $K_N(z)$, we obtain new results for Pell's equation in Theorem 3.1. In Section 3.2 we present a new approach to solving Pell's equation.

2. Preliminaries

Let $PSL(2,\mathbb{Z})$ denote the group of fractional linear transformations of the form $Y : z \rightarrow \frac{az+b}{cz+d}$, where a, b, c and d are integers and $ad - bc = 1$. In terms of the matrix representation, the elements of $PSL(2,\mathbb{Z})$ correspond to the matrices $\bar{Y} := \pm\begin{pmatrix} a & b \\ c & d \end{pmatrix}$; with $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$. These matrix representations consist of a particular linear group which is known as the modular group, $\Gamma = SL(2,\mathbb{Z})$. The modular group is generated by the elements

$$\bar{S} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } \bar{T} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

with the relation $\bar{S}^2 = \bar{T}^3 = \bar{I} = \pm I$, where I is the identity matrix.

Many fields of mathematics such as hyperbolic geometry, elliptic curves, modular curves, modular forms and modular functions include the modular group and its subgroups. The modular group acts on the upper half plane $\mathbb{H} := \{z \in \mathbb{C} : Im(z) > 0\}$, such that $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$ and $z \in \mathbb{H}$, and the action is given by:

$$\gamma(z) = \frac{az+b}{cz+d}, \quad \gamma(\infty) = \frac{a}{c} \quad \text{and} \quad \gamma(-\frac{d}{c}) = \infty.$$

These transformations are known as Möbius transformations. The trace of $\bar{Y}$ is denoted by $Tr(\bar{Y}) := |a + d|$ and it can be used for classification of the modular group elements such that $\bar{Y}$ is called elliptic, parabolic or hyperbolic if its trace $Tr(\bar{Y}) < 2$, $Tr(\bar{Y}) = 2$ or $Tr(\bar{Y}) > 2$ respectively (see e.g., [2] and references cited in).

The following classical results about the natural action of modular group on the set of extended rational numbers $\hat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$ will be used for our discussion later.

Any element of $\hat{\mathbb{Q}}$ can be written as a reduced fraction $\frac{x}{y}$, since $\frac{x}{y} = \frac{-x}{-y}$, this representation is not unique. We represent ∞ as $\frac{1}{0} = \frac{-1}{0}$. The action of the modular group on $\hat{\mathbb{Q}}$ becomes,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \frac{x}{y} \rightarrow \frac{ax + by}{cx + dy}.$$

From the determinant $ad - bc = 1$, we obtain equations

$$c(ax + by) - a(cx + dy) = -y$$

and

$$d(ax+by)-b(cx+dy)=x$$

easily. So, we get $(ax+by,cx+dy)=1$ and it follows that $(ax+by)/(cx+dy)$ is a reduced fraction.
In this section, let us give some basic information about Pell's equation of the form (1).

Theorem 2.1. *[17] For every positive integer N that is not a square, the equation $x^2-Ny^2=1$ has infinite number of non trivial solutions in integers x, y.*

Theorem 2.2. *[17] Suppose N is a square-free positive integer. A pair (x,y) of positive integers solves the Pell equation $x^2-Ny^2=1$ if and only if there exists $n\in\mathbb{N}$ such that*

$$x+y\sqrt{N}=(a+b\sqrt{N})^n,$$

where (a,b) is the fundamental solution such that the positive integer b is minimal. Moreover, such solutions may be computed using floors and ceilings:

$$x=\left\lceil\frac{1}{2}(a+b\sqrt{N})^n\right\rceil \quad y=\left\lfloor\frac{1}{2\sqrt{N}}(a+b\sqrt{N})^n\right\rfloor=\left\lfloor\frac{x}{\sqrt{N}}\right\rfloor,$$

where $\lfloor x\rfloor$ is the greatest integer that is less than or equal to x and $\lceil x\rceil$ is the least integer that is greater than or equal to x.

3. Main Calculations and Results

In this section, we introduce a new number sequence derived from the action of a particular subgroup of the modular group on the extended rationals. We use the modular group element;

$$K_N=\begin{pmatrix} a & Nc \\ c & a \end{pmatrix} \quad \text{with} \quad a^2-Nc^2=1, \tag{3}$$

to define a new subgroup. Throughout this paper, we choose a and c as the fundamental solutions of the equation (1), both for compatibility with our problem and to unify the transformation K_N. We obtain a new rational number sequence generated by the forward iterations of K_N. So, r times forward iteration of K_N is denoted by K_N^r and defined as $K_N^r := K_N\circ K_N\circ\cdots\circ K_N$ (r times compositions of K_N). For every r, we define a new rational number sequence:

$$K_N^r(\infty)=\begin{pmatrix} a & Nc \\ c & a \end{pmatrix}^r\begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{4}$$

We represent this sequence with $\left\{K_N^r(\infty)\right\}$, where r is a positive integer. In a similar way, it can be defined backward iterations of K_N by composing the inverse transforms K_N^{-1}. The following theorem generalizes the terms of $\left\{K_N^r(\infty)\right\}$ with respect to the matrix entries a,c and N.

Theorem 3.1. *Let K_N be defined in (3). A^r and C^r are polynomials, defined as,*

$$A^r=\sum_{i=0}^{\lfloor\frac{r}{2}\rfloor}\binom{r}{2i}a^{r-2i}c^{2i}N^i, \tag{5}$$

and

$$C^r=\sum_{i=0}^{\lfloor\frac{r}{2}\rfloor}\binom{r}{2i+1}a^{r-2i-1}c^{2i+1}N^i, \tag{6}$$

where a, c are integers and r, N are positive integers. Then we have,

$$K_N^r(\infty) = \frac{A^r}{C^r}. \tag{7}$$

Proof. We use the mathematical induction for r.
Bases Step : If $r = 1$ then, it is clear that $K_N^1(\infty) = \frac{a}{c} = \frac{A^1}{C^1}$.
Inductive hypothesis: We suppose that the equation (7) is true for any arbitrary positive integer r.
Inductive Step: We must prove that; $K_N^{r+1}(\infty) = \frac{A^{r+1}}{C^{r+1}}$. From the inductive hypothesis, it is easily seen;

$$\frac{A^{r+1}}{C^{r+1}} = K_N^{r+1}(\infty) = K_N(K_N^r)(\infty) = \frac{aA^r + NcC^r}{cA^r + aC^r}. \tag{8}$$

From (8) and determinant of K_N, we can verify the following two equations;

$$A^{r+1} = aA^r + NcC^r, \tag{9}$$

$$C^{r+1} = cA^r + aC^r \tag{10}$$

for any positive integer r.

First, using the basic rules of combinatorics, we show that the equation (9) is true as follows:

$$\begin{aligned} aA^r + NcC^r &= \sum_{i=0}^{\lfloor \frac{r}{2} \rfloor} \binom{r}{2i} a^{r-2i+1} c^{2i} N^i + \sum_{i=0}^{\lfloor \frac{r}{2} \rfloor} \binom{r}{2i+1} a^{r-2i-1} c^{2i+2} N^{i+1} \\ &= \sum_{i=0}^{\lfloor \frac{r}{2} \rfloor} \binom{r}{2i} a^{r-2i+1} c^{2i} N^i + \sum_{i=1}^{\lfloor \frac{r}{2} \rfloor + 1} \binom{r}{2i-1} a^{r-2i+1} c^{2i} N^i \\ &= a^{r+1} + \sum_{i=1}^{\lfloor \frac{r}{2} \rfloor + 1} \left[\binom{r}{2i} + \binom{r}{2i-1} \right] a^{r-2i+1} c^{2i} N^i \\ &= a^{r+1} + \sum_{i=1}^{\lfloor \frac{r+1}{2} \rfloor} \binom{r+1}{2i} a^{r-2i+1} c^{2i} N^i \\ &= \sum_{i=0}^{\lfloor \frac{r+1}{2} \rfloor} \binom{r+1}{2i} a^{r-2i+1} c^{2i} N^i = A^{r+1}. \end{aligned}$$

So, we have the left side of the equation (9). Then, we use the following trivial identity to get (10),

$$\sum_{i=0}^{\lfloor \frac{r}{2} \rfloor} \binom{r+1}{2i+1} = \sum_{i=0}^{\lfloor \frac{r+1}{2} \rfloor} \binom{r+1}{2i+1}. \tag{11}$$

Now, we show the equation (10) is true by using the basic rules of combinatorics and the equation (11), as follows:

$$
\begin{aligned}
cA^r + aC^r &= \sum_{i=0}^{\lfloor \frac{r}{2} \rfloor} \binom{r}{2i} a^{r-2i} c^{2i+1} N^i + \sum_{i=0}^{\lfloor \frac{r}{2} \rfloor} \binom{r}{2i+1} a^{r-2i} c^{2i+1} N^i \\
&= \sum_{i=0}^{\lfloor \frac{r}{2} \rfloor} \left[\binom{r}{2i} + \binom{r}{2i+1} \right] a^{r-2i} c^{2i+1} N^i \\
&= \sum_{i=0}^{\lfloor \frac{r}{2} \rfloor} \binom{r+1}{2i+1} a^{r-2i} c^{2i+1} N^i \\
&= \sum_{i=0}^{\lfloor \frac{r+1}{2} \rfloor} \binom{r+1}{2i+1} a^{r-2i} c^{2i+1} N^i \qquad \text{“by (11)”} \\
&= C^{r+1}.
\end{aligned}
$$

□

Thus, we give the following corollary as a result of Theorem 3.1.

Corollary 3.2. *The matrix K_N with positive and negative integer powers are written as* $K_N^r = \begin{pmatrix} A^r & NC^r \\ C^r & A^r \end{pmatrix}$ *and* $K_N^{-r} = \begin{pmatrix} A^r & -NC^r \\ -C^r & A^r \end{pmatrix}$ *respectively.*

From the Corollary 3.2, we say that the integer powers of K_N generates the infinite cyclic subgroup of the modular group. We represent this group with G_N and it is written as $G_N = \langle \begin{pmatrix} a & Nc \\ c & a \end{pmatrix} \rangle$. The action of G_N on the extended rationals gives new results for the suborbital graphs and the integer solutions of (1). It is clear that this action is not transitive on $\hat{\mathbb{Q}}$, but with the following proposition, we can determine a maximal subset on which the action is transitive.

Proposition 3.3. *The group G_N acts transitively on a subset of $\hat{\mathbb{Q}}$ defined as*

$$D_N := \left\{ \pm K_N^r(\infty) = \pm \frac{A^r}{C^r} : \ r \in \mathbb{N} \cup \{0\} \right\}$$

Furthermore, D_N is the maximal subset on which the action is transitive and ∞ is the element of D_N for each positive square free integer N.

Proof. Transitivity is clear from the definitions of G_N and D_N. Other hand, for $r = 0$, it is obtain that $\infty \in D_N$, for each positive square free integer N. □

Proposition 3.4. *The sets D_N and D_M have no common elements other than infinity if N and M are distinct positive integers that are not squares.*

Proof. Let N and M be distinct positive integers that are not squares. In this case there is no common solution other than $(\pm 1, 0)$ for Pell's equations $x^2 - Ny^2 = 1$ and $x^2 - My^2 = 1$. So we get $G_N \cap G_M = \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ and $D_N \cap D_M = \{\infty\}$. □

3.1. Results on Related Graphs

In this section, we first mention the general theory of suborbital graphs very shortly and then continue for the special case. If (G, Ω) is a transitive permutation group. Then, G as well, acts on $\Omega \times \Omega$ by $g : (\alpha, \beta) \to (g(\alpha), g(\beta))$. The orbits of the action are called suborbitals of the group G. From the suborbital $O(\alpha, \beta)$ containing (α, β) we form the suborbital graph $G(\alpha, \beta)$: Its vertices are the elements of Ω and there is an edge from u to v denoted by $u \to v$ if $(u, v) \in O(\alpha, \beta)$, represented as hyperbolic geodesics in the upper half plane $\mathbb{H}$. For more detailed information, see in [6].

In this paper we put $G = G_N$ and $\Omega = D_N$. Since G_N acts transitively on D_N, there is only one orbit contains a pair $\left(\infty, \frac{a}{c}\right)$ for some $\frac{a}{c}$ in D_N such that a and c are the fundamental solutions of (1). In this case, we denote the related suborbital graphs by $\mathbf{F}_N^+ := \mathbf{F}_N(a, c)$ and $\mathbf{F}_N^- := \mathbf{F}_N(a, -c)$. All rational vertices of the graph $\mathbf{F}_N^+ \cup \mathbf{F}_N^- =: \mathbf{F}_N$ can be considered the terms of the sequence $\{\pm K_N^r(\infty)\}$. Also, some of these subgraphs are considered as infinite paths for each N. We will represent to these paths with $\mathbf{K}_N^+ \subset \mathbf{F}_N^+$ and $\mathbf{K}_N^- \subset \mathbf{F}_N^-$ as follows:

$$\mathbf{K}_N^+ \ : \ \infty \to K_N(\infty) \to K_N^2(\infty) \to \cdots \to K_N^r(\infty) \to \cdots$$

$$\mathbf{K}_N^- \ : \ \infty \to -K_N(\infty) \to -K_N^2(\infty) \to \cdots \to -K_N^r(\infty) \to \cdots$$

Figure 1 illustrates the positioning of these graphs in the upper half plane for consecutive square-free positive integers N and $M := N + 1$.

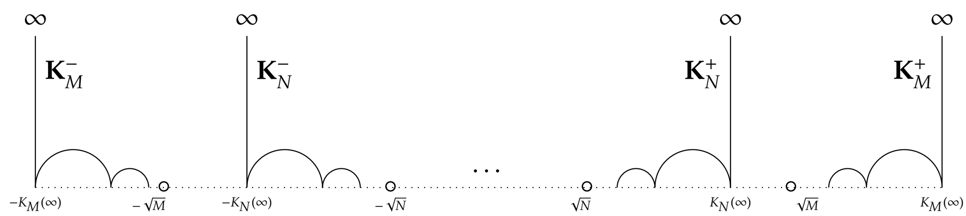

Figure 1: The paths $\mathbf{K}_N \cup \mathbf{K}_M$.

With the following proposition we give the edge conditions for the graph $\mathbf{F}_N$.

Proposition 3.5. *There is an edge from $u = \frac{u_1}{u_2}$ to $v = \frac{v_1}{v_2}$ in the suborbital graph $\mathbf{F}_N = \mathbf{F}_N(a, c)$ iff there exists $m \in \mathbb{Z}$ such that,*

$$v_1 = A^m u_1 + NC^m u_2,$$
$$v_2 = C^m u_1 + A^m u_2.$$

Proof. Since the group G_N acts on the set D_N transitively there exists k, l such that $K_N^k \begin{pmatrix} 1 \\ 0 \end{pmatrix} = u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ and $K_N^l \begin{pmatrix} 1 \\ 0 \end{pmatrix} = v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$. Let $m := l - k$. Then, it is obtained that $K_N^m(u) = v$, which is desired. □

Proposition 3.6. *The suborbital graphs $\mathbf{F}_N^+ = \mathbf{F}_N(a, c)$ and $\mathbf{F}_N^- = \mathbf{F}_N(a, -c)$ are isomorphic.*

Proof. Let g and h group elements of G_N and $g(\infty) \to h(\infty)$ is an edge in $\mathbf{F}_N^+$ then it is clear that $g^{-1}(\infty) \to h^{-1}(\infty)$ is an edge in $\mathbf{F}_N^-$. □

3.2. The Results on The Pell's Equation

In this section, we give some results for the integer solutions of the equation (1) obtained from the sequence $\{K_N^r(\infty)\}$. As is well known, the möbius transformation corresponding to K_N is

$$K_N(z) = \frac{az + Nc}{cz + a},$$

and it has two fixed points. If $ac > 0$, then the points $\sqrt{N}$ and $-\sqrt{N}$ are called "attracting" and "repelling" fixed points respectively, if $ac < 0$, then vice versa. For more information, see [2]. By considering the fixed points $\pm\sqrt{N}$, we get the normal form of $K_N(z)$ as

$$z \longrightarrow \left(\frac{a - c\sqrt{N}}{a + c\sqrt{N}}\right) z,$$

where the positive real number $\dfrac{a - c\sqrt{N}}{a + c\sqrt{N}}$ is called the multiplier of the transformation $K_N(z)$ and is denoted by λ_N. So, it can be seen that the integer powers of $\lambda_N^{-1/2} = a + c\sqrt{N}$ corresponds to the polynomials A^r and C^r. As a direct consequence of Theorem 2.2 and Theorem 3.1, this relationship has the following corollary.

Corollary 3.7. *$(a + c\sqrt{N})^r = A^r + C^r\sqrt{N}$, where a and c are the fundamental solutions to the Pell's equation (1) and r is a positive integer.*

Now we can provide the subsequent corollary, a crucial outcome of this study as it links the suborbital graph vertices with integer solutions of Pell's equation (1).

Corollary 3.8. *Let N is a square-free positive integer and a, c are the fundamental solutions to the Pell's equation (1). Then, the integer pairs (A^r, C^r) which are defined in Theorem 3.1 are the integer solutions to the Pell's equation (1) for each positive integer r.*

Corollary 3.9. *Suppose that $ac > 0$ then the following statements are true for the sequence generated by K_N;*

(i) The sequence $\{K_N^r(\infty)\}$ is decreasing and $\sqrt{N} < K_N^r(\infty) \le \frac{a}{c} = K_N(\infty)$.

(ii) The sequence $\{K_N^{-r}(\infty)\}$ is increasing and $-K_N(\infty) = -\frac{a}{c} \le K_N^{-r}(\infty) < -\sqrt{N}$.

(iii) The sequences $\{K_N^r(\infty)\}$ and $\{K_N^{-r}(\infty)\}$ converge to the attracting fixed point $\sqrt{N}$ and repelling fixed point $-\sqrt{N}$ for the hyperbolic möbius transformation $K_N(z)$ respectively.

Example 3.10. *Let $K_2 = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$ and $K_3 = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$ are in G_2 and G_3 respectively. It is easily seen that $K_i^m(\infty) \to K_i^{m+1}(\infty)$ is an edge in $\mathbf{K}_i^+$ $(i = 2, 3)$ for all non-negative integers m. From this, we get two infinite long paths:*

$$\mathbf{K}_2^+ \ : \infty \to \frac{3}{2} \to \frac{17}{12} \to \frac{99}{70} \to \frac{577}{408} \to \cdots$$

and

$$\mathbf{K}_3^+ \ : \infty \to \frac{2}{1} \to \frac{7}{4} \to \frac{26}{15} \to \frac{97}{56} \to \cdots .$$

These graphs correspond to hyperbolic geodesics in $\mathbb{H}$, as shown in Figure 2. We can see that the vertices of these paths converge to $\sqrt{2}$ and $\sqrt{3}$, respectively. Furthermore, nominators and denominators of fractions in the paths $\mathbf{K}_2^+$ and $\mathbf{K}_3^+$ give all positive solutions of the equation $x^2 - 2y^2 = 1$ and $x^2 - 3y^2 = 1$, respectively. As we can see, the *positive integer pairs* $(3, 2)$ *and* $(2, 1)$ *are the fundamental solutions of the equations* $x^2 - 2y^2 = 1$ *and* $x^2 - 3y^2 = 1$, *respectively. More generally, we can write the path* $\mathbf{K}_N^+$ *as:*

$$\infty \to \frac{a}{c} \to \frac{a^2 + Nc^2}{2ac} \to \frac{a^3 + 3aNc^2}{3a^2c + Nc^3} \to \cdots \to \frac{A^r}{C^r} \to \cdots$$

so the pair (a, c) *is the fundamental solution of the equation* $x^2 - Ny^2 = 1$ *and the all rational vertices of the* $\mathbf{K}_N^+$ *are corresponded to the solutions of* $x^2 - Ny^2 = 1$.

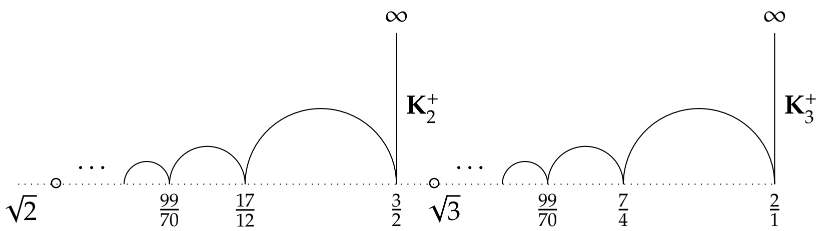

Figure 2: The paths $\mathbf{K}_2^+$ and $\mathbf{K}_3^+$.

4. Conclusion

Many previous studies have examined the relationships between continued fractions and suborbital graphs. It is well known that continued fractions are tools for solving Pell's equation. This study shows that suborbital graphs can also produce solutions of the Pell's equation. We examined the action of a special subgroup of the modular group on $\hat{\mathbb{Q}}$ and by using this action we obtain a new form for the integer solutions of Pell's equation $x^2 - Ny^2 = 1$, via the corollary 3.7. It may be more advantageous to work with suborbital graphs since the arithmetic structures of groups have the potential to give us some extra information in proofs. With this new method presented to the literature, new computer algorithms can be developed to obtain the solutions of the Pell's equation.

References

[1] A.H. Değer, M. Beşenk, B.O. Güler, *On Suborbital Graphs and related continued fractions*, Appl. Math. Comput., **218** (2011), 746–750.
[2] B. P. Palka, *An Introduction to Complex Function Theory*, Springer New York, NY, 1991.
[3] B.Ö. Güler, M. Beşenk, S. Kader, *On congruence equations arising from suborbital graphs*, Turkish J. Math.,**43** (2019), 2396 –2404.
[4] B.Ö. Güler, T. Kör, Z. Şanlı, *Solution to some congruence equations via suborbital graphs*, SpringerPlus, **5** (2016), 1327.
[5] B.Ö. Güler, M. Beşenk, Y. Kesicioğlu, A.H. Değer, *Suborbital graphs for the group* Γ^2, Hacettepe J. Math. and Statist., **44** (2015), 1033–1044.
[6] C.C. Sims, *Graphs and finite permutation groups*, Mathematische Zeitschrift, **95** (1967), 76–86.
[7] G. A. Jones, D. Singerman, K. Wicks, *The Modular Group and Generalized Farey Graphs*, London Math. Soc. Lecture Note Series, CUP, Cambridge,**160** (1991), 316 – 338.
[8] M. Akbaş, T. Kör, Y. Kesicioğlu, *Disconnectedness of the subgraph* F^3 *for the group* Γ^3, J. Inequal. Appl., **283** (2013).
[9] M. Beşenk, *General evaluation of suborbital graphs*, Communication in Mathematical Modeling and Applications, **3** (2018), 42–50.
[10] P. Jaipong, W. Tapanyo, *Generalized classes of suborbital graphs for the congruence subgroups of the modular group*, Algebra Discrete Math., **17**(2019), 20–36.
[11] P. Jaipong, W. Tapanyo, *Connectivity of Suborbital Graphs for the Congruence Subgroups of the Extended Modular Group*, Commun. Math. Appl., **8** (2017), 345–358.
[12] R. Keskin, *Suborbital graphs for the normalizer of* $\Gamma_0(m)$, European J. Combin., **27** (2006), 193–206.
[13] S. Kader, B.O. Güler, *On Suborbital Graphs for Extended Modular Group* $\hat{\Gamma}$, Graphs Combin., **29** (2013), 1813–1825.
[14] S.Öztürk, *A generalization of the suborbital graphs generating Fibonacci numbers for the subgroup* Γ^3, Filomat, **34** (2020), 631–638.
[15] T. Köroğlu, B.Ö. Güler and Z. Şanlı, *Suborbital graphs for the Atkin Lehner group*, Turkish J. Math.,**41** (2017), 235–243.
[16] Ü. Akbaba, A. H. Değer, *Relation between matrices and the suborbital graphs by the special number sequences*, Turkish J. Math., **46** (2022), 753–767.
[17] W. W. Adams, L. J. Goldstein, *Introduction to Number Theory*, Prentice-Hall, Englewood Cliffs, NJ, 1976.
[18] Z. Şanlı, T. Köroğlu, *Some Group Actions and Fibonacci Numbers*, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat., **71** (2022), 273–284.