Filomat 38:19 (2024), 6871–6879
https://doi.org/10.2298/FIL2419871G

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Beta-expansion in Pisot and Salem unit bases in $\mathbb{F}_q((x^{-1}))$

R. Ghorbel[a]

[a] *Faculté des Sciences de Sfax, Departement de Mathematiques, BP 1171, Sfax 3000, Tunisia.*

Abstract. In [11] K. Schmidt studied the lengths of periods occurring in the β-expansion of a rational number r noted by $Per_\beta(r)$ for the Pisot numbers of a very special form satisfying $\beta^2 = n\beta + 1$ for some $n \geq 1$. He showed the curious property "$Per_\beta(\frac{p}{q}) = Per_\beta(\frac{1}{q})$" for all positive integers p, q with $p \wedge q = 1$ and $p < q$. The aim of this paper is to prove that in the case of a formal power series over finite fields this property is true for special cubic Pisot unit basis.

1. Introduction

β-expansions of real numbers were introduced by A. Rényi [9]. Since then, their arithmetic, diophantine and ergodic properties have been extensively studied by several researchers. In this paper, we consider an analogue of this concept in algebraic function fields over finite fields. There are striking analogies between these digit systems and the classical β-expansions of real numbers. In order to pursue this analogy, we recall the definition of real β-expansions and survey the problems corresponding to our results.
Let β be a fixed real number greater than 1 and let x be a positive real number. A convergent series $\sum_{k\leq n} x_k\beta^k$ is called a β-representation of x if

$$x = \sum_{k\leq n} x_k\beta^k$$

and for all k the coefficient x_k is a non-negative integer. If moreover for every $-\infty < N < n$ we have

$$\sum_{k\leq N} x_k\beta^k < \beta^{N+1}$$

the series $\sum_{k\leq n} x_k\beta^k$ is called the β-expansion of x. The β-expansion is an analogue of the decimal or binary expansion of reals and we sometimes use the natural notation $d_\beta(x) = x_n x_{n-1} \ldots x_0.x_{-1} \ldots$ Every $x \geq 0$ has a unique β-expansion which is found by the greedy algorithm [9]. We can introduce lexicographic ordering on β-representations in the following way. The β-representation $x_n\beta^n + x_{n-1}\beta^{n-1} + \ldots$ is lexicographically greater than $x_k\beta^k + x_{k-1}\beta^{k-1} + \cdots$, if $k < n$ and for the corresponding infinite words we have $x_n x_{n-1} \ldots \succ \underbrace{0\ldots00}_{(n-k)times} x_k x_{k-1} \ldots$, where the symbol $\prec$ means the common lexicographic ordering on words in an ordered alphabet.

2020 *Mathematics Subject Classification*. Primary 11R06; Secondary 37B50.
Keywords. Formal power series, β-expansion, Pisot series, Salem series.
Received: 16 October 2023; Revised: 20 December 2023; Accepted: 08 January 2024
Communicated by Paola Bonacini
Email address: rimghorbel87@gmail.com (R. Ghorbel)

We say that $d_\beta(x)$ is finite when $x_i = 0$ for all sufficiently large i. A β-expansion is periodic if there exists $p \geq 1$ and $m \geq 1$ such that $x_k = x_{k+p}$ holds for all $k \geq m$; if $x_k = x_{k+p}$ holds for all $k \geq 1$, then it is purely periodic. We note by $Per(\beta)$ the sets of real numbers in [0,1) with periodic expansions.

Let $\mathbb{Q}(\beta)$ be the smallest field containing $\mathbb{Q}$ and β. An easy argument shows that $Per(\beta) \subseteq \mathbb{Q}(\beta) \cap [0,1)$ for every real number $\beta > 1$. K. Schmidt [11] showed that if β is a Pisot number (an algebraic integer whose conjugates have modulus <1), then $Per(\beta)$=$\mathbb{Q}(\beta) \cap [0,1)$. It establishes a remarkable analogy as in the case of the expansion in a rational integer base.
In the real case and with a quadratic base β satisfied $\beta^2 = n\beta + 1$ for some integer $n \geq 1$, K. Schmidt [11] has given this theorem:

Theorem 1.1. *Suppose that β satisfies $\beta^2 = n\beta + 1$ for some $n \geq 1$. Then every $r \in \mathbb{Q} \cap [0,1)$ has strictly periodic β-expansion. If $r = \frac{p}{q}$ be written in reduced form with $0 < p < q$, then $Per_\beta(\frac{p}{q}) = Per_\beta(\frac{1}{q})$.*

In [2] and [3], D.W. Boyd investigates the length of the period for some Salem numbers of degree 4 and 6.

The aim of this paper is to studies analogue results in the field of power series over finite fields.
This paper is organized as follows: In section 2, we will define the field of formal power series over finite fields as well as the analogues to Pisot and Salem numbers. We will also define the β-expansion algorithm for formal power series. In section 3, we prove that the length of the period of the expansion can be computed algorithmically in a finite group by mimicking an Euclidean algorithm, avoiding difficult computations of power series.

2. β-expansions in $\mathbb{F}_q((x^{-1}))$

Let $\mathbb{F}_q$ be a finite field of q elements, $\mathbb{F}_q[x]$ the ring of polynomials with coefficient in $\mathbb{F}_q$, $\mathbb{F}_q(x)$ the field of rational functions, $\mathbb{F}_q(x,\beta)$ the field of rational functions in base β and $\mathbb{F}_q[x,\beta]$ the ring of polynomials in base β. Let $\mathbb{F}_q((x^{-1}))$ be the field of formal power series of the form :

$$f = \sum_{k=-\infty}^{l} f_k x^k, \quad f_k \in \mathbb{F}_q$$

whereby

$$l = \deg f := \begin{cases} \max\{k : f_k \neq 0\} & \text{for } f \neq 0; \\ -\infty & \text{for } f = 0. \end{cases}$$

Define the absolute value

$$|f| = \begin{cases} q^{\deg f} & \text{for } f \neq 0; \\ 0 & \text{for } f = 0. \end{cases}$$

Note that the set of possible values of $|.|$ is a discrete set. Then $\mathbb{F}_q((x^{-1}))$ is the completion of $\mathbb{F}_q(x)$ with respect to $|.|$. Since $|.|$ is not archimedean, it fulfills the inequality of the strict triangle

$$\begin{aligned} |f+g| &\leq \max(|f|,|g|) && \textit{and} \\ |f+g| &= \max(|f|,|g|) && \textit{if } |f| \neq |g|. \end{aligned}$$

Let $f \in \mathbb{F}_q((x^{-1}))$ define the integer (polynomial) part $[f] = \sum_{k=0}^{l} f_k x^k$ where the empty sum, as usual, is defined to be zero. Therefore $[f] \in \mathbb{F}_q[x]$ and $(f - [f]) \in M_0$ where $M_0 = \{f \in \mathbb{F}_q((x^{-1})) : |f| < 1\}$. Now let

$$\mathbb{F}_q^{\beta}[x] = \{P \in \mathbb{F}_q[x] : |P| < |\beta|\}.$$

Proposition 2.1. *[8] Let K be complete field with respect to (a non archimedean absolute value* $|.|$ *) and* L/K $(K \subset L)$ *be an algebraic extension of degree m. Then* $|.|$ *has a unique extension to L defined by :* $|a| = \sqrt[m]{|N_{L/K}(a)|}$ *and L is complete with respect to this extension.*

We apply Proposition 2.1 to algebraic extensions of $\mathbb{F}_q((x^{-1}))$. Since $\mathbb{F}_q[x] \subset \mathbb{F}_q((x^{-1}))$, every algebraic element over $\mathbb{F}_q[x]$ can be valued. However, since $\mathbb{F}_q((x^{-1}))$ is not algebraically closed, such an element does necessarily need a power series. For a full characterization of the algebraic closure of $\mathbb{F}_q[x]$, we refer to Kedlaya [5].

An element $\beta \in \mathbb{F}_q((x^{-1}))$ is called a Pisot (resp Salem) series if it is an algebraic integer over $\mathbb{F}_q[x]$ such that $|\beta| > 1$ and $|\beta_j| < 1$ for all conjugates β_j (resp $|\beta_j| \leq 1$ and there exists at least one conjugate β_k such that $|\beta_k| = 1$). P. Bateman and A. L. Duquette [1] characterized the Pisot and Salem elements in $\mathbb{F}_q((x^{-1}))$:

Theorem 2.2. *Let* $\beta \in \mathbb{F}_q((x^{-1}))$ *be an algebraic integer over* $\mathbb{F}_q[x]$ *and*

$$P(y) = y^n - A_1 y^{n-1} - \cdots - A_n, \;\; A_i \in \mathbb{F}_q[x],$$

be its minimal polynomial. Then

(*i*) *β is a Pisot series if and only if* $|A_1| > \max_{2 \leq i \leq n} |A_i|$.

(*ii*) *β is a Salem series if and only if* $|A_1| = \max_{2 \leq i \leq n} |A_i|$.

Let $\beta, f \in \mathbb{F}_q((x^{-1}))$ with $|\beta| > 1$. A representation in base β (or β-representation) of f is an infinite sequence $(a_i)_{i\geq 1}$, $a_i \in \mathbb{F}_q[x]$ with

$$f \;\; = \;\; \sum_{i \geq 1} \frac{a_i}{\beta^i}.$$

A particular β-representation of f is called the β-expansion of f in base β, denoted $d_\beta(f)$. This is obtained by using the β-transformation T_β in the unit disk which is given by $T_\beta(f) = \beta f - [\beta f]$. Then $d_\beta(f) = (a_i)_{i\geq 1}$ where $a_i = [\beta T_\beta^{i-1}(f)]$.

An equivalent definition of the β-expansion can be obtained by a greedy algorithm. This algorithm works as follows: $r_0 = f$, $a_i = [\beta r_{i-1}]$ and $r_i = \beta r_{i-1} - a_i$ for all $i \geq 1$. The β-expansion of f will be noted as $d_\beta(f) = (a_i)_{i\geq 1}$.

Notice that $d_\beta(f)$ is finite if and only if there is a $k \geq 0$ with $T_\beta^k(f) = 0$, $d_\beta(f)$ is ultimately periodic if and only if there is some smallest $p \geq 0$ (the pre-period length) and $s \geq 1$ (the period length) when $T_\beta^{p+s}(f) = T_\beta^p(f)$, namely the period length will be noted by $Per_\beta(f)$.
Now, let $f \in \mathbb{F}_q((x^{-1}))$ be an element, with $|f| \geq 1$. Then there is a unique $k \in \mathbb{N}$ having $|\beta|^k \leq |f| < |\beta|^{k+1}$. Hence, $|\frac{f}{\beta^{k+1}}| < 1$. We can represent f by shifting $d_\beta(\frac{f}{\beta^{k+1}})$ by k digits to the left. Therefore, if $d_\beta(f) = 0.d_1d_2d_3\ldots$, then $d_\beta(\beta f) = d_1.d_2d_3\ldots$.
If we have $d_\beta(x) = d_l d_{l-1} \ldots d_0 \cdot d_{-1} \ldots d_{-m}$, then we put $\mathrm{ord}_\beta(x) = -m$.

Afterwards, we will use the following notation:

$$Per(\beta) = \{f \in \mathbb{F}_q((x^{-1})) : \; d_\beta(f) \text{ is eventually periodic } \}$$

Remark 2.3. *In contrast to the real case, there is no carry occurring, when we add two digits. So, if* $z, w \in \mathbb{F}_q((x^{-1}))$*, then we have* $d_\beta(z + w) = d_\beta(z) + d_\beta(w)$ *digitwise and if* $c \in \mathbb{F}_q^*$*, then* $d_\beta(cz) = cd_\beta(z)$.

Theorem 2.4. *[4] A β-representation* $(d_j)_{j\geq 1}$ *of f in the unit disk is its β-expansion if and only if* $|d_j| < |\beta|$ *for* $j \geq 1$.

In the case of the field of formal series, Hbaib - Mkaouar and Scheicher were proved the following theorems independently;

Theorem 2.5. *[10] β is a Pisot or Salem series if and only if $Per(\beta) = \mathbb{F}_q(x, \beta)$.*

Theorem 2.6. *[4] β is a Pisot or Salem series if and only if $d_\beta(1)$ is periodic.*

In the papers [6] and [7], metric results are established and the relation to continued fractions is studied.

3. The length of the period

Now, we will study specifically the length of the period and its extension as well as some of its specific properties. We will prove that the β-expansion of $f \in \mathbb{F}_q(x)$ presents some surprising regularities. Let us start with this proposition.

Proposition 3.1. *Let β be a Pisot or Salem series of algebraic degree n and f a rational series in the unit disk. If there exists a conjugate $\beta^{(k)}$ $(2 \leq k \leq n)$ such that $|\beta^{(k)}| = \frac{1}{|\beta|}$ and $|f| < \frac{1}{|\beta|^s}$ $(s \geq 0)$, then $d_\beta(f) = 0.\overline{a_1 \dots a_n}$ with $n \geq s+1$ and $a_n = \dots = a_{n-s} = 0$.*

Proof:
Assume that β is a Pisot or Salem series, by Theorem 2.5 we can deduce that $d_\beta(f)$ is periodic. Suppose that f does not have a purely periodic β-expansion, so $d_\beta(f) = 0.a_1...a_p\overline{a_{p+1}...a_{p+m}}$ and $a_p \neq a_{p+m}$. Hence

$$f = \frac{a_1}{\beta} + \dots + \frac{a_p}{\beta^p} + \frac{a_{p+1}}{\beta^{p+1}} + \dots + \frac{a_{p+m}}{\beta^{p+m}} + \frac{1}{\beta^m}(f - \frac{a_1}{\beta} - \dots - \frac{a_p}{\beta^p}).$$

Since $a_1, ..., a_{p+m} \in \mathbb{F}_q[x]$ *and* $f \in \mathbb{F}_q(x)$, we have

$$f = \frac{a_1}{\beta^{(k)}} + \dots + \frac{a_p}{(\beta^{(k)})^p} + \frac{a_{p+1}}{(\beta^{(k)})^{p+1}} + \dots + \frac{a_{p+m}}{(\beta^{(k)})^{p+m}} + \frac{1}{(\beta^{(k)})^m}(f - \frac{a_1}{\beta^{(k)}} - \dots - \frac{a_p}{(\beta^{(k)})^p}).$$

So

$$f(1 - \frac{1}{(\beta^{(k)})^m}) = \frac{a_1}{\beta^{(k)}} + \dots + \frac{a_p}{(\beta^{(k)})^p} + \frac{a_{p+1}}{(\beta^{(k)})^{p+1}} + \dots + \frac{a_{p+m}}{(\beta^{(k)})^{p+m}} + \frac{1}{(\beta^{(k)})^m}(-\frac{a_1}{\beta^{(k)}} - \dots - \frac{a_p}{(\beta^{(k)})^p}).$$

Therefore

$$f(-\beta^{(k)})^{m+p} - (\beta^{(k)})^p) = a_1(\beta^{(k)})^{m+p-1} + \dots + a_{p+m} - a_1(\beta^{(k)})^{p-1} - \dots - a_p.$$

Since $|\beta^{(k)}| = \frac{1}{|\beta|}$, then we get

$$|f||(\beta^{(k)})^p| = |a_{p+m} - a_p|.$$

So

$$\frac{|f|}{|\beta|^p} \geq |a_{p+m} - a_p|.$$

Since $a_{p+m} - a_p \neq 0$, $|f| \geq |\beta|^p$. which is absurd because f is in the unit disk.

Now, suppose that $d_\beta(f) = 0.\overline{a_1 \dots a_n}$, so

$$f = \frac{a_1}{\beta} + \dots + \frac{a_n}{\beta^n} + \frac{f}{\beta^n},$$

this gives

$$(1 - \frac{1}{\beta^n})f = \frac{a_1}{\beta} + \cdots + \frac{a_n}{\beta^n}.$$

Since $a_1, \ldots, a_n \in \mathbb{F}_q[x]$, we get

$$(1 - \frac{1}{(\beta^{(k)})^n})f = \frac{a_1}{\beta^{(k)}} + \cdots + \frac{a_n}{(\beta^{(k)})^n}.$$

As we have $|\beta^{(k)}| = \frac{1}{|\beta|}$. Clearly, we get $|\beta^n f| = |a_1\beta + \cdots + a_n\beta^n| = |a_n\beta^n|$.
For $s = 0$, $|f| = |a_n| < 1$, therefore $a_n = 0$. So $|\beta^n f| = |a_1\beta + \cdots + a_{n-1}\beta^{n-1}|$. For $s = 1$, $|f| = |\frac{a_{n-1}}{\beta}| < \frac{1}{|\beta|}$, therefore $a_{n-1} = 0$. By iteration, we get $a_{n-s} = 0$. □

Corollary 3.2. *Let β be a Pisot or Salem series of algebraic degree n. If there exists a conjugate $\beta^{(k)}$ $(2 \le k \le n)$ such that $|\beta^{(k)}| = \frac{1}{|\beta|}$, then every rational r in the unit disk satisfies $d_\beta(r) = 0.\overline{a_1 \ldots a_n}$ with $a_n = 0$.*

Remark 3.3. *If β is a Salem or Pisot series of algebraic degree n which has a conjugate $\beta^{(k)}$ $(2 \le k \le n)$ satisfying $|\beta^{(k)}| = \frac{1}{|\beta|}$, then such $\beta^{(k)}$ is unique and β is unit. This means that all other conjugates (if they exists) are of absolute value one. So β is a quadratic Pisot unit or a special Salem element whose conjugates have absolute value one except β and $\beta^{(k)}$.*

Now, we giving a sufficient condition for β for which the curious property P " every rational in the unit disk has purely periodic β-expansion" is satisfied.

Let β be a Pisot or Salem unit series of minimal polynomial $H(y) = y^n + A_{n-1}y^{n-1} + \cdots + A_0$ where $A_i \in \mathbb{F}_q[x]$ for $i \in \{1, \ldots, n-1\}$ and $A_0 \in \mathbb{F}_q^*$. Let $\beta^{(2)}, \ldots, \beta^{(n)}$ be the conjugates of β. For $f = r_0 + r_1\beta + r_2\beta^2 + \cdots + r_{n-1}\beta^{n-1}$ with $r_i \in \mathbb{F}_q(x)$, the s-th conjugate of f is defined by $f^{(s)} = r_0 + r_1\beta^{(s)} + r_2(\beta^{(s)})^2 + \cdots + r_{n-1}(\beta^{(s)})^{n-1}$.

Proposition 3.4. *Let β be Pisot or Salem unit series of algebraic degree n which has at least one conjugate $\beta^{(s)} \in M_0$ $(2 \le s \le n)$. Then every rational in the unit disk has a purely periodic β-expansion.*

Proof:
Let β be Pisot or Salem unit series. Since $f \in \mathbb{F}_q(x)$, so for all $2 \le i \le n$, we have $f^{(i)} = f$. To complete the proof, we need the following lemma.

Lemma 3.5. *Let β be a Pisot or Salem unit series of algebraic degree n which has at least one conjugate $\beta^{(s)} \in M_0$ $(2 \le s \le n)$ and let $f \in M_0 \cap \mathbb{F}_q(x, \beta)$. If $|f^{(s)}| < \frac{1}{|\beta^{(s)}|}$, then f has a purely periodic β-expansion.*

Proof:
Let β be a Pisot or Salem unit series. Since $|\beta\beta^{(2)} \ldots \beta^{(n)}| = 1$, we must have for all $i \in \{2, \ldots, n\}$, $|\beta^{(i)}| \ge \frac{1}{|\beta|}$ and hence $1 \le \frac{1}{|\beta^{(i)}|} \le |\beta|$. In particular, $1 < \frac{1}{|\beta^{(s)}|} \le |\beta|$. As $|f^{(s)}| < \frac{1}{|\beta^{(s)}|}$ and $f^{(s)} \in \mathbb{F}_q((x^{-1}))$ (because $\beta^{(s)} \in \mathbb{F}_q((x^{-1}))$), then the $(\frac{1}{\beta^{(s)}})$-expansion of $f^{(s)}$ is given by $d_{(\frac{1}{\beta^{(s)}})}(f^{(s)}) = d_0.d_1d_2\ldots$. Therefore

$$f^{(s)} = \sum_{i \ge 0} \frac{d_i}{\frac{1}{(\beta^{(s)})^i}} = \sum_{i \ge 0} d_i(\beta^{(s)})^i; \quad ||d_i| < \frac{1}{|\beta^{(s)}|} \le |\beta|.$$

Now, let P be the polynomial with the smallest degree such that $Pf \in \mathbb{F}_q[x,\beta]$ and put

$$R_P = \{g : |g| < 1,\ Pg \in \mathbb{F}_q[x,\beta] \text{ and } g^{(s)} = \sum_{i\geq 0} a_i(\beta^{(s)})^i \text{ with } a_i \in \mathbb{F}_q^{\beta}[x]\}.$$

Then R_P is a non empty finite set since it is bounded and $\mathbb{F}_q[x,\beta]$ is discrete.
First, we assert that $T_\beta(R_P) \subset R_P$. For $f \in R_P$, we have

$$\begin{aligned} T_\beta(f) &= \beta f - [\beta f]. \\ PT_\beta(f) &= (P\beta f - P[\beta f]) \in \mathbb{F}_q[x,\beta]. \end{aligned}$$

Since

$$\begin{aligned} (T_\beta(f))^{(s)} &= \beta^{(s)} f^{(s)} - [\beta f]. \\ &= \sum_{i\geq 1} a_{i-1}(\beta^{(s)})^i - [\beta f]. \\ &= \sum_{i\geq 0} b_i(\beta^{(s)})^i, \end{aligned}$$

and $|T_\beta(f)| = |\{\beta f\}| < 1$, this implies that $T_\beta(R_P) \subset R_P$.
Second, we claim that T_β is surjective on R_P. Pick $f \in R_P$ and prove that there exists $h \in R_P$ such that $T_\beta(h) = f$. Let $h = \beta^{-1}(f - a_0) = \frac{f-a_0}{\beta}$. Hence $T_\beta(h) = \{f - a_0\} = \{f\} = f$. We argue that $h \in R_P$. As $h = \frac{f-a_0}{\beta}$, $|h| < 1$. We have

$$\begin{aligned} h^{(s)} &= (\beta^{(s)})^{-1}(f^{(s)} - a_0). \\ &= (\beta^{(s)})^{-1}(a_1(\beta^{(s)}) + a_2(\beta^{(s)})^2 + \cdots). \\ &= a_1 + a_2(\beta^{(s)}) + a_3(\beta^{(s)})^2 + \cdots . \\ &= \sum_{j\geq 0} d_j(\beta^{(s)})^j \text{ with } (d_i)_{i\geq 0} = (a_i)_{i\geq 1},\ (d_i)_{i\geq 0} \in (\mathbb{F}_q^{\beta}[x])^{\mathbb{N}}. \end{aligned}$$

On the other hand, $Ph = \frac{P(f-a_0)}{\beta} \in \mathbb{F}_q[x,\beta]$ because β is an algebraic unit series. This proves that T_β is surjective on R_P. Hence T_β/R_P is one-to-one mapping and thus there exists an integer m such that $f = T_\beta^m(f)$, namely, the β-expansion of f is purely periodic. □

Now, let us return to the proof of Proposition 3.4:
In particular, since $|f^{(s)}| = |f| < 1 < \frac{1}{|\beta^{(s)}|}$, then f has a purely periodic β-expansion. □

Corollary 3.6. *Let β be quadratic Pisot unit series. Then every rational in the unit disk has a purely periodic β-expansion.*

Corollary 3.7. *Let β be cubic Salem unit series. Then every rational in the unit disk has a purely periodic β-expansion.*

Theorem 3.8. *Let β be a Pisot cubic unit series of minimal polynomial $P(y) = y^3 + A_2y^2 + A_1y + A_0$ where* $\deg(A_1) = \deg(\beta) - 1$. *Then every rational r in the unit disk has a purely periodic β-expansion.*

Proof:
By using the Newton-Puiseux Theorem, we can easily describe all roots of this polynomial. The Newton polygon is the upper convex function of the zigzag broken line connecting $(0,0)$, $(1,\deg(A_1))$, $(2,\deg(\beta))$

and $(3,0)$ where $\deg(\beta)-1=\deg(A_1)$ and $\deg(\beta)=\deg(A_2)$. The Newton polygon is subdivided into line segments L_i $(i=0,1,2)$, where L_i connects $(0,0)$, $(1,\deg(\beta)-1)$, $(2,\deg(\beta))$ and $(3,0)$ and each L_0,L_1,L_2 has different slope $\alpha_0=(\deg(\beta)-1)/1)$; $\alpha_1=(\deg(\beta)-\deg(A_1)/1)$; $\alpha_2=(-\deg(\beta)/1)$; with $\alpha_0>\alpha_1>\alpha_2$. Denote by b_i the denominator of α_i. Then the set of all roots of P is given in this following manner: Each L_i gives one root of the form:

$$\beta^{(i)}=\sum_{k=0}^{\infty}c_k x^{-\alpha_i-k/b_i}\in\mathbb{F}_q((x^{-1/b_i}));\ (0\le i\le 2).$$

So, we conclude that β has two conjugates in $\mathbb{F}_q((x^{-1}))$. Then, by Proposition 3.4, every rational in the unit disk has a purely periodic β-expansion.

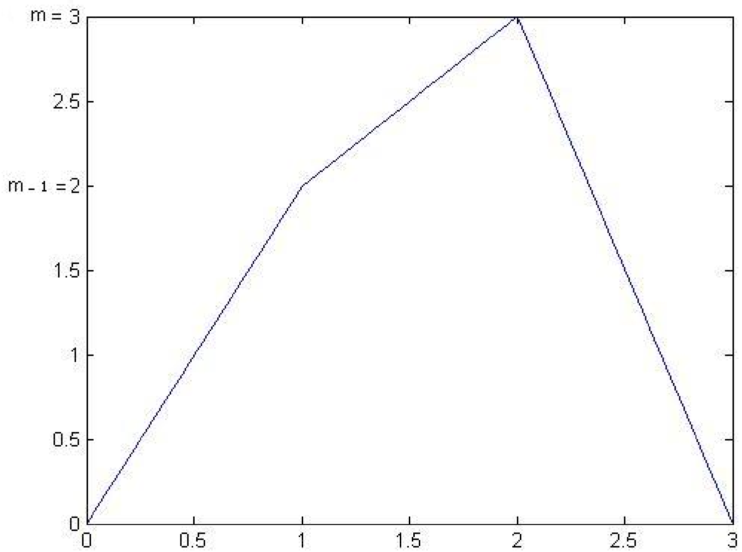

Figure 1: Newton polygon

□

Theorem 3.9. *Let β be a Pisot cubic unit series of minimal polynomial $P(y)=y^3+A_2y^2+A_1y+A_0$ where $\deg(A_1)=\deg(\beta)-1$. Then every rational r in the unit disk satisfies*

$$Per_\beta(xr)=Per_\beta(r).$$

Proof:
Let β be a Pisot cubic unit series of minimal polynomial $P(y)=y^3+A_2y^2+A_1y+A_0$ where $\deg(A_1)=\deg(\beta)-1=m-1$. Then from Theorem 3.8, r has a purely periodic β-expansion and $d_\beta(r)=0.\overline{a_1\dots a_s}$. Therefore

$$r=\frac{a_1}{\beta}+\dots+\frac{a_s}{\beta^s}+\frac{r}{\beta^s}, \tag{1}$$

this gives

$$r(\beta^s-1)=a_1\beta^{s-1}+\dots+a_{s-2}\beta^2+a_{s-1}\beta+a_s.$$

Since $a_1, \ldots, a_s \in \mathbb{F}_q[x]$, we get

$$|r((\beta^{(3)})^s - 1)| = |a_1(\beta^{(3)})^{s-1} + \cdots + a_{s-2}(\beta^{(3)})^2 + a_{s-1}\beta^{(3)} + a_s|.$$

As $|r((\beta^{(3)})^s - 1)| < 1$ and $\deg(a_i(\beta^{(3)})^{s-i}) < m - 1$ for all $1 \leq i \leq s - 1$, so $\deg(a_s) < m - 1$. To complete this proof we need the following lemma:

Lemma 3.10. *Let β be a Pisot cubic unit series of minimal polynomial $P(y) = y^3 + A_2y^2 + A_1y + A_0$ where $\deg(A_1) = \deg(\beta) - 1 = m - 1$. Then $\deg(\beta^{(2)}) = -1$ and $\deg(\beta^{(3)}) = -m + 1$ where $\beta^{(2)}; \beta^{(3)}$ are the conjugates of β.*

Proof:
We have

$$\begin{aligned} |A_1| = q^{m-1} &= |\beta\beta^{(2)} + \beta\beta^{(3)} + \beta^{(3)}\beta^{(2)}| \\ &= |\beta\beta^{(2)} + \beta\beta^{(3)}| \\ &\leq \sup(|\beta\beta^{(2)}|,\ |\beta\beta^{(3)}|). \end{aligned}$$

Moreover, $\sup(|\beta\beta^{(2)}|,\ |\beta\beta^{(3)}|) \leq q^{m-1}$. So $\sup(|\beta\beta^{(2)}|,\ |\beta\beta^{(3)}|) = q^{m-1}$. In this case, we suppose that $\deg(\beta\beta^{(2)}) = m - 1$, hence $\deg(\beta^{(2)}) = -1$. On the other hand, we have

$$\deg(A_0) = \deg(\beta) + \deg(\beta^{(2)}) + \deg(\beta^{(3)}) = 0.$$

Therefore $\deg(\beta^{(3)}) = -m + 1$. □

According to Lemma 3.10, we have $\deg(\beta^{(3)}) = -m + 1$ which implies $\deg(a_i(\beta^{(3)})^{s-i}) < -m + 1$ for all $1 \leq i \leq s-2$. Suppose now $\deg(a_{s-1}) = m-1$, then $\deg(a_{s-1}\beta^{(3)}) = 0$ which is absurd because $|r((\beta^{(3)})^s - 1)| < 1$. We return to the equation (1) and multiplying by x, we get

$$xr = \frac{xa_1}{\beta} + \cdots + \frac{xa_{s-2}}{\beta^{s-2}} + \frac{xa_{s-1}}{\beta^{s-1}} + \frac{xa_s}{\beta^s} + \frac{xr}{\beta^s}, \tag{2}$$

Set $H = \{1 \leq i \leq s : \deg(xa^k_{-i}) = m\}$. To continues the proof we must distinguish two cases:
Case 1: If $H = \emptyset$. So $Per_\beta(xr) = Per_\beta(r)$.
Case 2: If $H \neq \emptyset$. Since $\deg(a_s) < m - 1$ and $\deg(a_{s-1}) < m - 1$, it is clear that $h = \sup H \leq s - 2$.
On the other hand, β is a Pisot cubic unit series, then $\deg A_2 = m$. If c is the dominant coefficient of A_2, we have

$$cx^m\beta^2 = \beta^3 - (A_2 - cx^m)\beta^2 - A_1\beta^1 - A_0.$$

Therefore

$$x^m = c^{-1}\beta - c^{-1}(A_2 - cx^m) - \frac{c^{-1}A_1}{\beta} - \frac{c^{-1}A_0}{\beta^2}.$$

According to Theorem 2.4, the last equality is the β-expansion of x^m. For that $ord_\beta(\frac{xa_h}{\beta^h}) = -h-2 \geq -s$. Finally, we conclude that $Per_\beta(xr) = Per_\beta(r)$. □

Corollary 3.11. *Let β be a Pisot cubic unit series of minimal polynomial $P(y) = y^3 + A_2y^2 + A_1y + A_0$ where $\deg(A_1) = \deg(\beta) - 1$. Then every rational r in the unit disk satisfies*

$$Per_\beta(\alpha x^n r) = Per_\beta(r) \quad \forall\, n \in \mathbb{N} \text{ and } \alpha \in \mathbb{F}_q^*.$$

References

[1] P. Bateman and A. L. Duquette,*The analogue of Pisot-Vijayaraghvan numbers in fields of power series*, III. J. Math., (1962), 594-606.
[2] D.W. Boyd, *Salem numbers of degree four have periodic expansions*, Number Theory (J.M. de Koninck and C. Levesque, eds.), (1989), 57-64.
[3] D.W. Boyd, *On the beta expansion for Salem numbers of degree 6, mathematics of computation*, 65, (1996), 861-875.
[4] M. Hbaib, M. Mkaouar, *Sur le bêta-développement de 1 dans le corps des séries formelles*, Int. J. Number Theory, 2, (2006), 365-377.
[5] k. S. Kedlaya, *The algebraic closure of the power series field in positive characteristic*, Proc. Amer. Math. Soc, 12, (2001), 3461-3470.
[6] B. Li and J. Wu,*Beta-expansions and cotinued fraction expansion over formal Laurent series*, Finite Fields Appl, 14, (2008), 635-647.
[7] B. Li, J. Wu and J. Xu,*Metric properties and exceptional sets of β-expansions over formal Laurent series*, Monatsh. Math, 155, (2008), 145-160.
[8] J. Neukirch. *Algebraic Number Theory, the Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences].* Springer-Verlag, Berlin, 322, (1999).
[9] A. Rènyi, *Representations for real numbers and their ergodic properties, Acta Math. Acad. Sci. Hung*, 8, (1975), 477-493.
[10] K. Scheicher, *Beta-expansions in algebraic function fields over finite fields, Finite Fields Appl*, 13, (2007), 394-410.
[11] K. Schmidt, *On periodic expansions of Pisot numbers and Salem numbers*, Bull. London Math. Soc, 12, (1980), 269-278.