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Laplacian eigenvalue distribution based on some graph parameters

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Abstract. Let G be a connected graph on n vertices. For an interval I, denote by $m_G I$ the number of Laplacian eigenvalues of G which lie in I. In this paper, we obtain several bounds on $m_G I$ in terms of various structural parameters of the graph G, including chromatic number, pendant vertices, and the number of vertices with degree $n-1$.

1. Introduction

Throughout this paper, we only consider connected, finite and simple graphs. Let $G = \big(V(G), E(G)\big)$ be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$, where $|V(G)| = n$ and $|E(G)| = m$. For $v \in V(G)$, the *neighbour set* of vertex v is defined as $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$ and the number $d(v) = d_G(v) = |N_G(v)|$ is the *degree* of vertex v. A vertex $v \in V(G)$ is called a *pendant vertex* if $d_G(v) = 1$ and a *quasi-pendant vertex* of G is a vertex adjacent to a pendant vertex. The *complement* of G, denoted by $\overline{G}$, is the simple graph whose vertex set is $V(G)$ and whose edges are the pairs of nonadjacent vertices of G.

The *adjacency matrix* of G is defined as the matrix $A(G) = (a_{ij})_{n\times n}$ with $a_{ij} = 1$ if v_i, v_j are adjacent in G, and $a_{ij} = 0$ otherwise. Moreover, let $D(G)$ be the *diagonal matrix* $diag\big(d(v_1), d(v_2), \ldots, d(v_n)\big)$ with $d(v_i)$ is the degree of vertex v_i, for $i = 1, \ldots, n$. Then $L(G) = D(G) - A(G)$ is called the *Laplacian matrix* of G. We denote by $\Theta(G, x) = \det(xI - L(G))$ the characteristic polynomial of $L(G)$. It is known that $L(G)$ is a singular, positive semidefinite symmetric matrix. The eigenvalues of $L(G)$ are called the *Laplacian eigenvalues* of G, which are regarded as

$$\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_n(G) = 0$$

in non-increasing order. The collection of eigenvalues of $L(G)$ together with their multiplicities is called the *Laplacian spectrum* of G, denoted by $Spec(L(G))$. If G has k distinct Laplacian eigenvalues $\mu_1, \mu_2, \ldots, \mu_k$ with multiplicities of $m_1, m_2, \ldots, m_k$ respectively, then we shall write $\mu_1^{(m_1)}, \mu_2^{(m_2)}, \ldots, \mu_k^{(m_k)}$ for the Laplacian spectrum of G. Sometimes, we show it in the matrix form as follows

$$\begin{pmatrix} \mu_1 & \mu_2 & \cdots & \mu_k \\ m_1 & m_2 & \cdots & m_k \end{pmatrix},$$

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where $\mu_1 > \mu_2 > \cdots > \mu_k$, and m_i is the multiplicity of μ_i. The multiplicity of a Laplacian eigenvalue μ_i in a graph G is denoted by $m_G(\mu_i)$ for $i = 1, 2, \dots, k$. Given a real interval I, let $m_G I$ denote the number of Laplacian eigenvalues, multiplicities included, of G in I.

As usual, K_n and $K_{q_1,q_2,\dots,q_t}$ denote respectively the *complete graph* of order n and the *complete multipartite graph* with part sizes $q_1, q_2, \dots, q_t$. In particular, the *complete bipartite graph* with part sizes p and q is denoted by $K_{p,q}$ and the *star* of order n is denoted by S_n. For two graphs G and H, $G \cup H$ denotes their *disjoint union*, and kG stands for the disjoint union of k copies of G. Let y be a real number. Then $\lceil y \rceil$ denotes the smallest integer no less than y.

For a graph G, a subset $H \subseteq V(G)$ is *dominating* if every $v \in V(G) - H$ is adjacent to some member in H. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set. An *independent set* M of G is a subset of vertices of G if no two of its vertices are adjacent, the *independence number* of G, denoted by $\alpha(G)$, is the cardinality of the largest independent sets in G. A proper vertex coloring of a graph G is an assignment of colors to the vertices of G so that no two adjacent vertices are assigned the same color. Alternatively, a *proper vertex coloring* of a graph may be viewed as a partition of the vertex set. The *chromatic number* of G, written as $\chi(G)$, is the minimum number of colors of a proper vertex coloring of G. Generally, the set of all vertices with the same color is called a color class. A *clique* is a complete subgraph of a given graph G. The cardinality of the maximum clique is called the *clique number* of G and is denoted by $\omega(G)$.

It is well known that all Laplacian eigenvalues of any graph G lie in $[0, n]$. But it is unclear how the Laplacian eigenvalues are distributed in the interval $[0, n]$. Therefore, many researchers have focused on the bound of $m_G I$ for some subinterval I of $[0, n]$ since 1990. Grone et al. [5] proved that $m_G[0, 1) \geq \nu(G)$, where $\nu(G)$ is the number of quasi-pendant vertices in G. Merris [10] obtained that $m_G(2, n] \geq \nu(G)$ for a graph G with $n > 2\nu(G)$. Guo et al. [6] studied that if $n > 2\beta(G)$, then $m_G(2, n] > \beta(G)$, where $\beta(G)$ is the matching number of G. Recently, Hedetniemi et al. [7] investigated the relationship between Laplacian eigenvalues distribution and domination number of a graph, and proved that $m_G[0, 1) \leq \gamma(G)$. Cardoso et al. [3] showed that an isolate-free graph G satisfies $\gamma(G) \leq m_G[2, n]$. Wang et al. [12] considered the bounds of $m_G(I)$ for $I = (n-1, n]$. Meanwhile, they proved that $m_G(n-1, n] \leq \kappa(G)$ and $m_G(n-1, n] \leq \chi(G) - 1$, where $\kappa(G)$ is the vertex-connectivity of G. Moreover, Ahanjideh et al. [1] showed that $m_G(n - \alpha(G), n] \leq n - \alpha(G)$ and $m_G(n - d(G) + 3, n] \leq n - d(G) - 1$, where $d(G)$ is the diameter of G.

It is easy to see that researchers characterized the bounds of $m_G I$ with different parameters of graphs. Be inspired by the above works, we consider whether it is possible to determine the bounds of $m_G I$ with other parameters, or change the length of interval I. Therefore, in this paper, for a given interval I, we present several bounds on $m_G I$ in terms of various structural parameters of the graph G, including chromatic number, pendant vertices, and the number of vertices with degree $n - 1$.

The structure of the paper is as follows. In the next section, we give some necessary lemmas which will be used to prove our main results. In Section 3, we research the relations between the chromatic number and the distribution of Laplacian eigenvalues of a graph. Particularly, we obtain the bound on $m_G(n-2, n]$ in terms of chromatic number, which improves the result of Theorem 3.1. in [12]. Meanwhile, we acquire some other conclusions about Laplacian eigenvalue distribution based on the chromatic number. In Section 4, according to the relations between the number of pendant vertices of a graph and Laplacian eigenvalue distribution, we prove that the number of Laplacian eigenvalues in the interval $(n - p(G), n]$ is at most $n - p(G)$, where $p(G)$ is the number of pendant vertices in G. Finally, we give an upper bound on the distribution of Laplacian eigenvalues in the interval $(0, n)$ in terms of the number of vertices having degree $n - 1$ in a graph.

2. Preliminaries

In this section, we shall list some known results which are needed in the following sections.

Lemma 2.1. *[5] The star S_n on n vertices has Laplacian spectrum $0, 1^{(n-2)}, n$.*

Lemma 2.2. *[3] Let $G = (V, E)$ and $H = (V, F)$ be graphs with $F \subseteq E$. Then*

(*i*) *for all i, $\mu_i(H) \leq \mu_i(G)$;*

(*ii*) *for any a*, $m_H[0,a] \geq m_G[0,a]$;
(*iii*) *for any a*, $m_H(a,n] \leq m_G(a,n]$.

Lemma 2.3. *[2] If a graph G has n vertices, and μ is an eigenvalue of L(G), then* $0 \leq \mu \leq n$. *The multiplicity of* 0 *equals the number of components of G, the multiplicity of n is one less than the number of components of* $\overline{G}$.

Lemma 2.4. *[11] Let G be the disjoint union of graphs* $G_1, G_2, \ldots, G_s$. *Then*

$$\Theta(G,x) = \Pi_{i=1}^{s}\Theta(G_i,x).$$

Lemma 2.5. *[8], [9] If G is a graph with n vertices, then*

$$\Theta(\overline{G},x) = (-1)^{n-1}\frac{x}{n-x}\Theta(G,n-x).$$

Lemma 2.6. *[4] Let* $G=(V,E)$ *be a graph with vertex subset* $V' = \{v_1, v_2, \ldots, v_k\}$ *having the same set of neighbors* $\{v_{k+1}, v_{k+2}, \ldots, v_s\}$, *where* $V = \{v_1, v_2, \ldots, v_k, \ldots, v_s, \ldots, v_n\}$. *Then G has at least* $k-1$ *same Laplacian eigenvalues and they are all equal to the cardinality of the neighbor set. Also the corresponding* $(k-1)$ *eigenvectors are*

$$(\underbrace{1,-1}_{2},0,\ldots,0)^T, (\underbrace{1,0,-1}_{3},0,\ldots,0)^T, \ldots, (\underbrace{1,0,0,\ldots,-1}_{k},0,\ldots,0)^T.$$

The Laplacian spectrum of a complete multipartite graph will play an important role in this paper. Thus using a different method, we can also prove the following result on the Laplacian spectrum of a complete multipartite graph in [1].

Lemma 2.7. *Let* $q_1, q_2, \ldots, q_t$ *and n be positive integers such that* $q_1 + q_2 + \cdots + q_t = n$. *Suppose* $S = \{i : q_i \geq 2\}$. *Then the Laplacian spectrum of a complete t-partite graph* $K_{q_1,\ldots,q_t}$ *consists of:*

(*i*) *n with multiplicity* $t-1$;
(*ii*) $n - q_i$ *with multiplicity* $q_i - 1$, *for each* $i \in S$;
(*iii*) *0.*

Proof. Let $K_{q_1,\ldots,q_t}$ be a complete t-partite graph with $q_1 \geq q_2 \geq \cdots \geq q_t$. It is easy to see that $q_1 \geq 2$. Otherwise, the graph is K_n. Suppose $S = \{i : q_i \geq 2\}$. If h is the largest integer in S, then $V(K_{q_1,\ldots,q_t}) = V_1 \cup V_2 \cup \cdots \cup V_h \cup \cdots \cup V_t$ such that $|V_i| = q_i$ for $i = 1, 2, \ldots, t$. By the definition of $K_{q_1,\ldots,q_t}$, we obtain that each vertex of V_i has the same neighbors set, which contains $n - q_i$ vertices. Thus, using Lemma 2.6, we see that $n - q_i$ is a Laplacian eigenvalue of $K_{q_1,\ldots,q_t}$ with multiplicity at least $q_i - 1$. It is obvious that $i \in S$. Therefore, we just determine $\sum_{i\in S}(q_i - 1) = n - t$ Laplacian eigenvalues of $K_{q_1,\ldots,q_t}$. Since the complement of $K_{q_1,\ldots,q_t}$ contains t connected components, according to Lemma 2.3, n is a Laplacian eigenvalue of $K_{q_1,\ldots,q_t}$ with multiplicity $t-1$. Therefore, the Laplacian spectrum of $K_{q_1,\ldots,q_t}$ is $\{n^{(t-1)}, (n-q_i)^{(q_i-1)}, 0\}$, where $i \in S$. □

Lemma 2.8. *Let G be a graph on n vertices. If* $V' = \{v_1, v_2, \ldots, v_k\}$ *is a clique of G such that* $N(v_i) - V' = N(v_j) - V'$ *for all* $i, j \in \{1, 2, \ldots, k\}$, *then* $d(v_i) = d(v_j)$ *and* $d(v_i) + 1$ *is an eigenvalue of L(G) with multiplicity at least* $k-1$, *for all* $i, j \in \{1, 2, \ldots, k\}$.

Proof. Since the vertices in V' share the same neighborhood, all vertices in V' have the same degree, denoted by s. Let $X = (x_1, x_2, \ldots, x_n)^T$ be an eigenvector of $L(G)$ corresponding to the eigenvalue $\mu(G)$. Then

$$\mu(G)x_i = d_i x_i - \sum_{v_iv_j\in E(G)} x_j, \tag{1}$$

where $i = 1, 2, \ldots, n$.

According to (1), we can easily find that $s+1$ is an eigenvalue of $L(G)$ with corresponding eigenvectors

$$l_1 = (\underbrace{1,-1}_{2},0,\ldots,0)^T, l_2 = (\underbrace{1,0,-1}_{3},0,\ldots,0)^T, \ldots, l_{k-1} = (\underbrace{1,0,0,\ldots,-1}_{k},0,\ldots,0)^T.$$

Since $l_1, l_2, \ldots, l_{k-1}$ are $k-1$ linearly independent eigenvectors, $s+1$ is an eigenvalue of $L(G)$ with multiplicity at least $k-1$. □

3. Laplacian eigenvalue distribution and chromatic number

In this section, we give relationships between Laplacian eigenvalue distribution and chromatic number $\chi(G)$ of a graph G. More specifically, we obtain bounds for $m_G I$ in terms of $\chi(G)$ for some interval I. For a given integer t such that $1 \leq t \leq n-2$, $\mathcal{H}(n,t)$ is the graph obtained by making the disjoint union of a star S_{n-t} with t isolated vertices, and then taking its complement, that is $\mathcal{H}(n,t) = \overline{S_{n-t} \cup tK_1}$. See Figure 1 for example, where $n = 6$ and $t \in \{1, 2, 3, 4\}$.

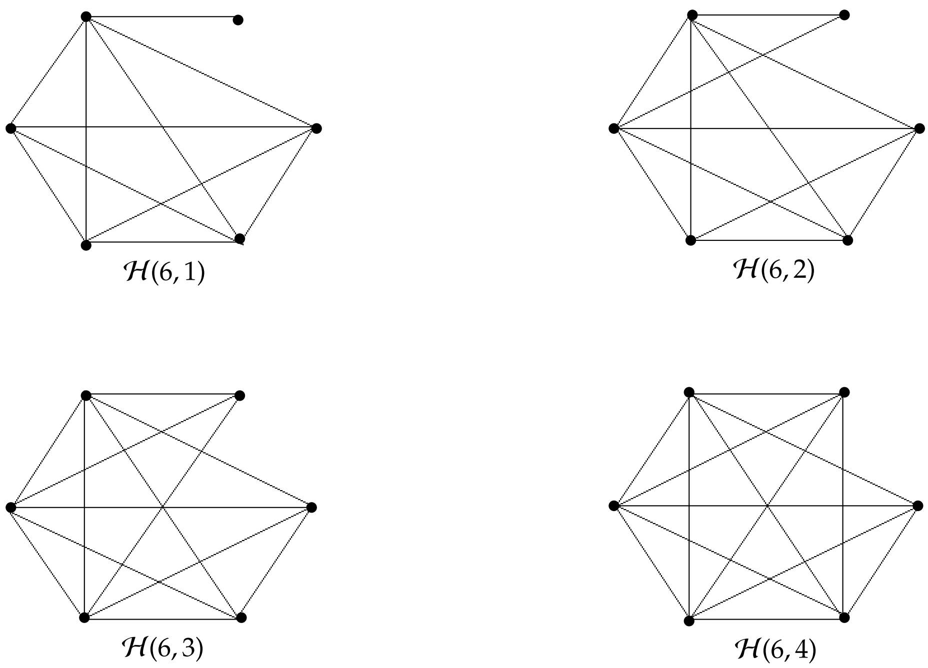

Figure 1: All $\mathcal{H}(6,t)(1 \leq t \leq 4)$ graphs of six vertices.

Theorem 3.1. *Let G be a connected graph of order n and with chromatic number $\chi(G)$. Then*

$$m_G(n-2, n] \leq \chi(G) - 1. \tag{2}$$

The bound is the best possible as shown by all complete multipartite graphs or $G \cong \mathcal{H}(n,t)$, where $1 \leq t \leq n-2$.

Proof. Since the chromatic number of G is $\chi(G)$, the vertex set $V(G)$ of G can be partitioned into $\chi(G)$ disjoint sets as follows

$$V = V_1 \cup V_2 \cup \cdots \cup V_{\chi(G)},$$

where each V_i is an independent set of G, i.e., no edge between vertices of V_i. Let $n_i = |V_i|$. Then it is easy to get that $n_1 + n_2 + \cdots + n_{\chi(G)} = n$. Without loss of generality, we assume that $n_1 \geq n_2 \geq \cdots \geq n_{\chi(G)}$. Let G' be the graph obtained from G by adding edges between V_i and V_j such that the vertices of V_i and V_j are all adjacent, for any pair V_i, V_j of G with $i \neq j$. Apparently, G can be considered as a spanning subgraph of G' and $G' = K_{n_1, n_2, \ldots, n_{\chi(G)}}$ with $n_1 \geq n_2 \geq \cdots \geq n_{\chi(G)}$.

By Lemma 2.2(iii), we can obtain $m_G(n-2, n] \leq m_{G'}(n-2, n]$. It is worth noticing that the chromatic number of G' is also $\chi(G)$. Hence, in order to complete the proof of (2), we only need to prove that $m_{G'}(n-2, n] \leq \chi(G) - 1$.

Using Lemma 2.7, if $n_1 \geq n_2 \geq \cdots \geq n_{\chi(G)}$, then the Laplacian spectrum of G' is

$$\begin{pmatrix} 0 & n-n_1 & n-n_2 & \cdots & \cdots & n-n_{\chi(G)} & n \\ 1 & n_1-1 & n_2-1 & \cdots & \cdots & n_{\chi(G)}-1 & \chi(G)-1 \end{pmatrix}.$$

By observing the Laplacian spectrum of G', we see that $m_{G'}(n-2,n] = \chi(G)-1$. Therefore, $m_G(n-2,n] \leq m_{G'}(n-2,n] = \chi(G)-1$.

By the above analysis, we can know that all complete multipartite graphs satisfy the equality. Next, we show that if $G \cong \mathcal{H}(n,t)$ (t fixed and satisfying $1 \leq t \leq n-2$), then equality holds. By Lemma 2.4 and Lemma 2.5, we can obtain that

$$\Theta(\mathcal{H}(n,t),x) = x(x-t)(x-n)^t\Big(x-(n-1)\Big)^{n-t-2},$$

which implies that the Laplacian spectrum of $\mathcal{H}(n,t)$ is

$$\begin{pmatrix} 0 & t & n-1 & n \\ 1 & 1 & n-t-2 & t \end{pmatrix},$$

where $1 \leq t \leq n-2$. It is obvious that $\chi(\mathcal{H}(n,t)) = n-1$ and $m_{\mathcal{H}(n,t)}(n-2,n] = n-2$. Therefore, we have $m_{\mathcal{H}(n,t)}(n-2,n] = \chi(\mathcal{H}(n,t))-1$.

It completes the proof of Theorem 3.1. □

Remark 3.2. *Wang et al. [12] considered the distribution of Laplacian eigenvalues in subinterval $(n-1,n]$ of length 1 and they proved that $m_G(n-1,n] \leq \chi(G)-1$. It is easy to verify that Theorem 3.1 improves this bound. On the other hand, it is well known that for a graph G, $\chi(G)-1 \leq \Delta(G)$. Therefore, we show that $m_G(n-2,n] \leq \Delta(G)$ by Theorem 3.1.*

Corollary 3.3. *Let G be a connected graph of order n having chromatic number $\chi(G)$. Then*

$$m_G[0,n-2] \geq n-\chi(G)+1,$$

with the equality holding when G is a complete multipartite graph or $G \cong \mathcal{H}(n,t)$, where $1 \leq t \leq n-2$.

Corollary 3.4. *Let G be a connected graph of order n with the chromatic number $\chi(G)$. If each color class of G contains the same number of vertices, then*

$$m_G(n-\frac{n}{\chi(G)},n] \leq \chi(G)-1.$$

Equality holds when $G \cong K_{\frac{n}{\chi(G)},\dots,\frac{n}{\chi(G)}}$.

Proof. Since each color class of G contains the same number of vertices, the graph G satisfies $n = p\chi(G)$ for some positive integer p, which implies that $p = \frac{n}{\chi(G)}$. By using a similar proof as Theorem 3.1, we see that G can be considered as a spanning subgraph of the complete multipartite graph $G' = K_{\frac{n}{\chi(G)},\dots,\frac{n}{\chi(G)}}$. Since G' is a complete $\chi(G)$-partite graph and using Lemma 2.7, we see that the Laplacian spectrum of G' is listed below:

$$\begin{pmatrix} 0 & n-\frac{n}{\chi(G)} & n \\ 1 & n-\chi(G) & \chi(G)-1 \end{pmatrix}.$$

It is not difficult to find that $m_{G'}(n-\frac{n}{\chi(G)},n] = \chi(G)-1$. According to Lemma 2.2(iii), we have $m_G(n-\frac{n}{\chi(G)},n] \leq m_{G'}(n-\frac{n}{\chi(G)},n] = \chi(G)-1$, which implies that $m_G(n-\frac{n}{\chi(G)},n] \leq \chi(G)-1$.

In view of the above analysis, we can obtain that the graph $K_{\frac{n}{\chi(G)},\dots,\frac{n}{\chi(G)}}$ satisfies the equality. Therefore, we complete the proof. □

Corollary 3.5. *Let G be a connected graph of order n with the chromatic number $\chi(G)$. If each color class of G contains the same number of vertices, then*

$$m_G[0,n-\frac{n}{\chi(G)}] \geq n-\chi(G)+1.$$

And the bound is the best possible as shown by the $K_{\frac{n}{\chi(G)},\dots,\frac{n}{\chi(G)}}$.

Now, we consider an upper bound for the number of Laplacian eigenvalues which are contained in the interval $(n - \lceil \frac{n}{\chi(G)} \rceil, n)$ of an arbitrary non-complete graph.

Theorem 3.6. *Suppose that $G \ncong K_n$ is a connected graph on n vertices with chromatic number $\chi(G)$. Then*

$$m_G(n - \lceil \frac{n}{\chi(G)} \rceil, n) \le n - \lceil \frac{n}{\chi(G)} \rceil - c(\overline{G}) + 1, \tag{3}$$

where $c(\overline{G})$ is the number of connected components of $\overline{G}$. Further, for $\chi(G) = 2$ and n is odd, equality in (3) is attained if $G \cong K_{t+1,t}$. For $\chi(G) = n - 1$, equality in (3) is attained if $G \cong K_{2,\underbrace{1,1,\ldots,1}_{n-2}}$.

Proof. Let G be a graph with chromatic number $\chi(G)$. Then the vertex set $V(G)$ of G can be partitioned into $\chi(G)$ independent sets. Suppose that $n_1, n_2, \ldots, n_{\chi(G)}$ are the cardinalities of those independent sets. Without loss of generality, we assume $n_1 \ge n_2 \ge \cdots \ge n_{\chi(G)}$. Then G is a spanning subgraph of the complete $\chi(G)$-partite graph $K_{n_1,n_2,\ldots,n_{\chi(G)}}$. As $G \ncong K_n$, $\chi(G) \le n - 1$ and $n_1 \ge 2$, using Lemma 2.2(i), we have

$$\mu_i(G) \le \mu_i(K_{n_1,n_2,\ldots,n_{\chi(G)}}) = n - n_1, \tag{4}$$

for all $n - n_1 + 1 \le i \le n - 1$.

Since $n_1 \ge n_2 \ge \cdots \ge n_{\chi(G)}$ and $n_1 + n_2 + \cdots + n_{\chi(G)} = n$, $n_1 \ge \frac{n}{\chi(G)}$, it means that $n_1 \ge \lceil \frac{n}{\chi(G)} \rceil$. Hence, from (4), we easily obtain

$$\mu_i(G) \le n - \lceil \frac{n}{\chi(G)} \rceil,$$

for all $n - n_1 + 1 \le i \le n - 1$. It implies that there exist at least $n_1 - 1$ Laplacian eigenvalues of G which are no greater than $n - \lceil \frac{n}{\chi(G)} \rceil$. Further, since 0 is always a Laplacian eigenvalue of a connected graph G, we have

$$m_G[0, n - \lceil \frac{n}{\chi(G)} \rceil] \ge n_1.$$

On the other hand, from Lemma 2.3, we see that n is the Laplacian eigenvalue of G with multiplicity exactly $c(\overline{G}) - 1$. Therefore,

$$\begin{aligned} m_G(n - \lceil \frac{n}{\chi(G)} \rceil, n) &= n - m_G[0, n - \lceil \frac{n}{\chi(G)} \rceil] - (c(\overline{G}) - 1) \\ &\le n - n_1 - (c(\overline{G}) - 1) \\ &\le n - \lceil \frac{n}{\chi(G)} \rceil - c(\overline{G}) + 1, \end{aligned}$$

which completes the proof of (3).

• Let $G \cong K_{t+1,t}$, where $t \ge 1$ and $n = 2t + 1$. Then we easily have $\chi(G) = 2$ and $\lceil \frac{n}{\chi(G)} \rceil = \lceil \frac{n}{2} \rceil = t + 1 = \frac{n+1}{2}$. It is noted that the complement of $K_{t+1,t}$ has just 2 connected components. Therefore, by Lemma 2.7, we obtain that the Laplacian spectrum of $K_{t+1,t}$ is given below:

$$\begin{pmatrix} 0 & \frac{n-1}{2} & \frac{n+1}{2} & n \\ 1 & \frac{n-1}{2} & \frac{n-3}{2} & 1 \end{pmatrix}.$$

Obviously, the equality of (3) holds for $G \cong K_{t+1,t}$, which explains that the bound is best possible when $\chi(G) = 2$ and n is odd.

• Let $G \cong K_{2,\underbrace{1,1,\ldots,1}_{n-2}}$. Then $\chi(G) = n - 1$. So we observe that $\lceil \frac{n}{\chi(G)} \rceil = \lceil \frac{n}{n-1} \rceil = 2$ and the complement of $K_{2,\underbrace{1,1,\ldots,1}_{n-2}}$ has exactly $n - 1$ components. By a simple calculation, $n - \lceil \frac{n}{\chi(G)} \rceil - c(\overline{G}) + 1 = 0$. On the other hand, by Lemma 2.7, the Laplacian spectrum of $K_{2,\underbrace{1,1,\ldots,1}_{n-2}}$ is

$$\begin{pmatrix} 0 & n-2 & n \\ 1 & 1 & n-2 \end{pmatrix}.$$

It implies that $m_G(n - \lceil \frac{n}{\chi(G)} \rceil, n) = 0$. Hence, the equality of (3) holds for $K_{2,\underbrace{1,1,\ldots,1}_{n-2}}$, which shows that the bound is the best possible when $\chi(G) = n - 1$. □

The following conclusions are derived from Theorem 3.6.

Corollary 3.7. *Let $G \ncong K_n$ be a connected graph on n vertices with the chromatic number $\chi(G)$. Then*

$$m_G[0, n - \lceil \frac{n}{\chi(G)} \rceil] \geq \lceil \frac{n}{\chi(G)} \rceil. \tag{5}$$

Moreover, for $\chi(G) = 2$ and n is odd, equality in (5) is attained if $G \cong K_{t+1,t}$. For $\chi(G) = n - 1$, equality in (5) is attained if $G \cong K_{2,\underbrace{1,1,\ldots,1}_{n-2}}$.

Corollary 3.8. *Let $G \ncong K_n$ be a connected graph on n vertices with the chromatic number $\chi(G)$. If $\overline{G}$ is connected, then*

$$m_G(n - \lceil \frac{n}{\chi(G)} \rceil, n) \leq n - \lceil \frac{n}{\chi(G)} \rceil.$$

Proof. Since $\overline{G}$ is connected, $c(\overline{G}) = 1$. Therefore, we can obtain the required conclusion by substituting $c(\overline{G}) = 1$ into the inequality of Theorem 3.6. □

4. Laplacian eigenvalue distribution, pendant vertices and the number of vertices with degree $n - 1$

First, we establish relations between the pendant vertices of a graph and how the Laplacian eigenvalues are distributed. Next, we characterize the distribution of Laplacian eigenvalues by the number of vertices having degree $n - 1$.

Theorem 4.1. *Let $G \ncong K_n$ be a connected graph on n vertices having $p(G) \geq 1$ pendant vertices. Then*

$$m_G(n - p(G), n] \leq n - p(G). \tag{6}$$

Moreover, the equality holds if and only if $G \cong S_n$ for $p(G) = n - 1$.

Proof. Let S be the set of all pendant vertices of G such that $| S |= p(G)$. Then it is easy to know that S is an independent set of G and the induced subgraph of $T = V(G)\backslash S$ is connected, we denote it by H. Let $\chi(H)$ be the chromatic number of H and $n_1 \geq n_2 \geq \cdots \geq n_{\chi(H)}$ be the cardinalities of these chromatic classes, where $1 \leq \chi(H) \leq n - p(G)$ and $n_1 + n_2 + \cdots + n_{\chi(H)} = n - p(G)$. Suppose that $n_k \geq p(G) \geq n_{k+1}$, where $0 \leq k \leq \chi(H)$. Specifically, $n_0 = p(G)$ if $k = 0$ and $n_{\chi(H)+1} = p(G)$ if $k = \chi(H)$. Hence, the vertex set $V(G)$ is partitioned into $\chi(H) + 1$ independent sets. Then we easily see that G can be considered as a spanning subgraph of complete $(\chi(H) + 1)$-partite graph $G' = K_{n_1,n_2,\ldots,n_k,p(G),n_{k+1},\ldots,n_{\chi(H)}}$. If $n_1 = p(G) = 1$, then $G' \cong K_n$. It is worth noting that the Laplacian spectrum of K_n is

$$\begin{pmatrix} 0 & n \\ 1 & n-1 \end{pmatrix}.$$

Hence, we have $m_{G'}(n - p(G), n] = m_{G'}(n - 1, n] = n - 1$. By Lemma 2.2(iii), we see that $m_G(n - p(G), n] \leq m_{G'}(n - p(G), n] = n - 1$, which implies that (6) is established. If $n_1 \geq 2$, we consider the following two cases.

Case 1. $n_1 \geq p(G)$.

By Lemma 2.2(i) and Lemma 2.7, we obtain

$$\mu_i(G) \leq \mu_i(G') = n - n_1 \leq n - p(G),$$

for all $n - n_1 + 1 \leq i \leq n - 1$.

Case 2. $n_1 < p(G)$. Again, using Lemma 2.2(i) and Lemma 2.7, we have

$$\mu_i(G) \leq \mu_i(G^{'}) = n - p(G),$$

for all $n - p(G) + 1 \leq i \leq n - 1$.

By the above analysis, we see that there are at least $p(G) - 1$ Laplacian eigenvalues of G which are no greater than $n - p(G)$. Meanwhile, we notice that 0 is a Laplacian eigenvalue of a connected graph G. Therefore, we have

$$m_G[0, n - p(G)] \geq p(G).$$

Since $m_G[0, n - p(G)] + m_G(n - p(G), n] = n$,

$$m_G(n - p(G), n] \leq n - p(G).$$

It obtains the required inequality (6).

Assume now that the equality holds in (6) for $p(G) = n - 1$. Then it is easy to see that $G \cong S_n$. On the other side, by Lemma 2.1, we know that the Laplacian spectrum of S_n is given as follows

$$\begin{pmatrix} 0 & 1 & n \\ 1 & n-2 & 1 \end{pmatrix}.$$

So the equality holds and the proof is completed. □

Now we have the following corollary which can be derived from Theorem 4.1.

Corollary 4.2. *If $G \ncong K_n$ be a connected graph on n vertices having $p(G) \geq 1$ pendant vertices, then*

$$m_G[0, n - p(G)] \geq p(G).$$

The equality holds if and only if $G \cong S_n$ for $p(G) = n - 1$.

Theorem 4.3. *Let G be a connected graph on $n \geq 4$ vertices with chromatic number $\chi(G)$. For $s \geq \frac{n}{2}$, if $M = \{v_1, v_2, \ldots, v_s\} \subseteq V(G)$ is the set of pendant vertices such that every vertex in M has the same neighbour in $V(G)\backslash M$, then*

$$m_G(1, n] \leq n - \chi(G).$$

Proof. Since $M = \{v_1, v_2, \ldots, v_s\} \subseteq V(G)$ is the set of pendant vertices and every vertex in M has the same neighbour in $V(G)\backslash M$, say $\{v_{s+1}\}$, by Lemma 2.6, there are at least $s - 1$ Laplacian eigenvalues of G which are equal to 1. Note that 0 is a Laplacian eigenvalue of a connected graph G. Thus, there are at least s Laplacian eigenvalues of G which are no greater than 1, that is,

$$m_G[0, 1] \geq s.$$

As $m_G[0, 1] + m_G(1, n] = n$, we have

$$m_G(1, n] \leq n - s.$$

In order to proof $m_G(1, n] \leq n - \chi(G)$, it remains to show $\chi(G) \leq s$.

By contradiction, assume that $\chi(G) > s$. If $s = n - 1$, then G is the connected graph having $n - 1$ pendant vertices, that is $G \cong S_n$. As $\chi(S_n) = 2 \leq \frac{n}{2}$ for $n \geq 4$ and $s \geq \frac{n}{2}$, $\chi(G) \leq s$, contrary to the assumption. If $\frac{n}{2} \leq s \leq n - 2$, then there is at least one vertex, say v, which is not adjacent to any vertex in M. Thus in the minimal coloring of G, at least $s+1$ vertices, say, $v, v_1, \ldots, v_s$ can be colored using only one color. The remaining $n-s-1$ vertices can be colored with at most $n-s-1$ colors. Thus, from $s \geq \frac{n}{2}$, we know that $\chi(G) \leq 1+n-s-1 = n-s \leq n-\frac{n}{2} = \frac{n}{2}$, which implies that $\chi \leq s$. It contradicts the assumption.

In a word, we have $\chi(G) \leq \frac{n}{2} \leq s$. Thus we must have

$$m_G(1, n] \leq n - \chi(G),$$

which completes the proof. □

In order to get a bound only in terms of order n and the number of pendant vertices s, we can relax the conditions $s \geq \frac{n}{2}$ and $n \geq 4$ in Theorem 4.3. This will be given in the following corollary.

Corollary 4.4. *Let G be a connected graph on n vertices. If $M = \{v_1, v_2, \ldots, v_s\} \subseteq V(G)$ is the set of pendant vertices such that every vertex in M has the same neighbour in $V(G)\backslash M$, then*

$$m_G(1, n] \leq n - s.$$

Next, based on the order n and the number of vertices having degree $n-1$ of a graph, we obtain a bound for the number of Laplacian eigenvalues which are in the interval $(0, n)$.

Theorem 4.5. *Let G be a connected graph with n vertices. If $n_d(G) = |\{v \in V(G) : d_G(v) = n-1\}|$, where $1 \leq n_d(G) \leq n$, then*

$$m_G(0, n) \leq n - n_d(G). \tag{7}$$

Further equality holds when $n_d(G) = n$, that is, $G \cong K_n$.

Proof. We consider the following two situations.

• Let $n_d(G) = n$. Then $G \cong K_n$. By Lemma 2.7, we see that $m_{K_n}(0, n) = 0$, the equality holds.

• Let $1 \leq n_d(G) \leq n-1$. Since G contains $n_d(G)$ vertices of degree $n-1$, G contains a clique, say M. Obviously, the size of this clique is $n_d(G)$. Let $V(M) = \{v_1, v_2, \ldots, v_{n_d(G)}\}$. Then we observe that

$$N(v_i) - V(M) = N(v_j) - V(M) = \{v_{n_d(G)+1}, \ldots, v_n\},$$

for $i, j \in \{1, 2, \ldots, n_d(G)\}$. According to Lemma 2.8, we obtain that n is a Laplacian eigenvalue of G with multiplicity at least $n_d(G) - 1$. Moreover, we know that 0 is a Laplacian eigenvalue of a connected graph. Therefore, we have

$$m_G(0, n) \leq n - n_d(G),$$

which completes the proof. □

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Conflicts of Interest

The authors declare no conflict of interest.

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