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A note on nonlinear skew Jordan-type derivations on $*$-rings

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Abstract. Let $\mathcal{A}$ be a 2-torsion free unital $*$-ring containing non-trivial symmetric idempotent. For $\mu_1, \mu_2 \in \mathcal{A}$, the product $\mu_1 \circ \mu_2 = \mu_1\mu_2 + \mu_2\mu_1^*$ is called the skew Jordan product of elements μ_1 and μ_2. In this article, it is shown that if a map $\varphi : \mathcal{A} \to \mathcal{A}$ (not necessarily additive) fulfills $\varphi(P_n(\nu_1, \nu_2, \ldots, \nu_n)) = \sum_{i=1}^{n} P_n(\nu_1, \ldots, \nu_{i-1}, \varphi(\nu_i), \nu_{i+1}, \ldots, \nu_n)$ for all $\nu_1, \nu_2, \ldots, \nu_n \in \mathcal{A}$, then φ is additive. Moreover, if $\varphi(I)$ is self-adjoint then φ is a $*$-derivation. As applications, our main result is applied to several special classes of unital $*$-rings and unital $*$-algebras such as prime $*$-ring, prime $*$-algebra, factor von Neumann algebra.

1. Introduction

Throughout the paper, we are implicitly assuming that $\mathcal{A}$ is an associative ring with centre $Z(\mathcal{A})$. An involution '$*$' on $\mathcal{A}$ is an anti automorphism of order 1 or 2. Ring $\mathcal{A}$ together with an involution '$*$' is called a $*$-ring. An element $P \in \mathcal{A}$ is said to be a symmetric idempotent if $P^2 = P = P^*$. Moreover, a symmetric idempotent P of $\mathcal{A}$ is called a nontrivial symmetric idempotent if $P \neq 0$ and $P \neq I$. A mapping $\varphi : \mathcal{A} \to \mathcal{A}$ is called an additive derivation if $\varphi(\mu_1 + \mu_2) = \varphi(\mu_1) + \varphi(\mu_2)$ and $\varphi(\mu_1\mu_2) = \varphi(\mu_1)\mu_2 + \mu_1\varphi(\mu_2)$ holds for all $\mu_1, \mu_2 \in \mathcal{A}$. If $\mathcal{A}$ admits an involution '$*$', then an additive mapping $\varphi : \mathcal{A} \to \mathcal{A}$ is called an additive $*$-derivation, if $\varphi(\mu_1\mu_2) = \varphi(\mu_1)\mu_2 + \mu_1\varphi(\mu_2)$ and $\varphi(\mu_1^*) = \varphi(\mu_1)^*$ holds for all $\mu_1, \mu_2 \in \mathcal{A}$. For $\mu_1, \mu_2 \in \mathcal{A}$, describe the skew Jordan product and bi-skew Jordan product of μ_1 and μ_2 by $\mu_1 \circ \mu_2 = \mu_1\mu_2 + \mu_2\mu_1^*$ and $\mu_1 \bullet \mu_2 = \mu_1\mu_2^* + \mu_2\mu_1^*$, respectively. A map $\varphi : \mathcal{A} \to \mathcal{A}$ (not necessarily additive) is said to be nonlinear skew Jordan derivation (resp. nonlinear skew Jordan triple derivation) if

$$\begin{aligned} \varphi(\mu_1 \circ \mu_2) &= \varphi(\mu_1) \circ \mu_2 + \mu_1 \circ \varphi(\mu_2) \\ (resp.\ \varphi((\mu_1 \circ \mu_2) \circ \mu_3) &= (\varphi(\mu_1) \circ \mu_2) \circ \mu_3 + (\mu_1 \circ \varphi(\mu_2)) \circ \mu_3 + (\mu_1 \circ \mu_2) \circ \varphi(\mu_3)) \end{aligned}$$

holds for all $\mu_1, \mu_2, \mu_3 \in \mathcal{A}$. Analogously, a map $\mathcal{A} \to \mathcal{A}$ (not necessarily additive) is called nonlinear bi-skew Jordan derivation (resp. nonlinear bi-skew Jordan triple derivation) if

$$\begin{aligned} \varphi(\mu_1 \bullet \mu_2) &= \varphi(\mu_1) \bullet \mu_2 + \mu_1 \bullet \varphi(\mu_2) \\ (resp.\ \varphi((\mu_1 \bullet \mu_2) \bullet \mu_3) &= (\varphi(\mu_1) \bullet \mu_2) \bullet \mu_3 + (\mu_1 \bullet \varphi(\mu_2)) \bullet \mu_3 + (\mu_1 \bullet \mu_2) \bullet \varphi(\mu_3)) \end{aligned}$$

holds for all $\mu_1, \mu_2, \mu_3 \in \mathcal{A}$. Over the last few years, numerous mathematicians focused their attention on mappings involving new products on various types of rings and algebras. These newly maps are discovered

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to be playing an ever-increasingly significant role in several research fields, and their study has caught the attention of numerous authors. (see [1, 2, 4, 5, 7–9, 15, 17–21]).
For any $\tau_1, \tau_2, \dots, \tau_n \in \mathcal{A}$ and integer $n \geq 2$, define a sequence of polynomials as follows: $P_1(\tau_1) = \tau_1$, $P_2(\tau_1, \tau_2) = \tau_1 \circ \tau_2 = \tau_1\tau_2 + \tau_2\tau_1^*$ and $P_n(\tau_1, \tau_2, \dots, \tau_n) = P_{n-1}(\tau_1, \tau_2, \dots, \tau_{n-1}) \circ \tau_n$. The polynomial $P_n(\tau_1, \tau_2, \dots, \tau_n)$ is called the skew Jordan n-product of elements $\tau_1, \tau_2, \dots, \tau_n \in \mathcal{A}$. A map $\varphi : \mathcal{A} \to \mathcal{A}$ (not necessarily additive) is said to be nonlinear skew Jordan n-derivation if

$$\varphi(P_n(\nu_1, \nu_2, \dots, \nu_n)) = P_n(\varphi(\nu_1), \nu_2, \dots, \nu_n) + P_n(\nu_1, \varphi(\nu_2), \dots, \nu_n) + \cdots + P_n(\nu_1, \nu_2, \dots, \varphi(\nu_n))$$

holds for all $\nu_1, \nu_2, \dots, \nu_n \in \mathcal{A}$.
Obviously, a nonlinear skew Jordan 2-derivation is a nonlinear skew Jordan derivation, and a nonlinear skew Jordan 3-derivation is a nonlinear skew Jordan triple derivation. Nonlinear skew Jordan 2-derivations, nonlinear skew Jordan 3-derivations and nonlinear skew Jordan n-derivations are collectively known as nonlinear skew Jordan-type derivations. Zhang in [20], proved that every nonlinear skew Jordan derivation on factor von Neumann algebra is an additive $*$-derivation. Zhao et al. [21], proved that if $\mathcal{A}$ is a von Neumann algebra with no central summands of type I_1, then a map $\varphi : \mathcal{A} \to \mathcal{A}$ is a $*$-Jordan triple derivation if and only if φ is an additive $*$-derivation. This result has been extended to the case of nonlinear $*$-Jordan type derivations on arbitrary $*$-algebra by Li in [9]. Khan [5], proved that every multiplicative bi-skew Jordan triple derivation on a prime $*$-algebra is an additive $*$-derivation. This result has been extended to the case of nonlinear bi-skew Jordan-type derivation by Ashraf et al. in [2]. Yu et al. [19], proved that if $\mathcal{A}$ is a factor von Neumann algebra acting on a complex Hilbert space $\mathcal{H}$ with $\dim(\mathcal{H}) \geq 2$ and $\varphi : \mathcal{A} \to \mathcal{A}$ is a skew Lie derivation, then φ is an additive $*$-derivation. Recently, Kong and Zhang [6], uplifted this result to prime $*$-rings and proved that if $\mathcal{A}$ is a 2-torsion free unital prime $*$-ring containing a nontrivial symmetric idempotent, then a map $\varphi : \mathcal{A} \to \mathcal{A}$ is a skew Lie derivation if and only if φ is an additive $*$-derivation. In the entire paper, we assume that $\frac{1}{2} \in \mathcal{A}$.

Motivated by the above mentioned work, in this article, we find the relationship between nonlinear skew Jordan-type derivations and additive $*$-derivation on arbitrary unital $*$-rings. Exactly, we show that, under mild assumptions, every nonlinear skew Jordan-type derivation on a unital $*$-ring is an additive $*$-derivation.

2. The Main Results

The main result of the article states as follows:

Theorem 2.1. *Let $\mathcal{A}$ be a 2-torsion free $*$-ring with the unity I containing a non-trivial symmetric idempotent P_1. Write $P_2 = I - P_1$ and assume that $\mathcal{A}$ satisfies*

$$X\mathcal{A}P_k = 0 \Longrightarrow X = 0 \ \ (k = 1, 2). \tag{$\spadesuit$}$$

If a map $\varphi : \mathcal{A} \to \mathcal{A}$ (not necessarily additive) satisfies

$$\varphi(P_n(\nu_1, \nu_2, \dots, \nu_n)) = \sum_{i=1}^{n} P_n(\nu_1, \dots, \nu_{i-1}, \varphi(\nu_i), \nu_{i+1}, \dots, \nu_n) \tag{1}$$

for all $\nu_1, \nu_2, \dots, \nu_n \in \mathcal{A}$, then φ is additive. Moreover, if $\varphi(I)$ is symmetric, then φ is a $*$-derivation.

Assume $\mathcal{A}_{ij} = P_i\mathcal{A}P_j$ for $i, j = 1, 2$, then by the Peirce decomposition, we have $\mathcal{A} = \mathcal{A}_{11} \oplus \mathcal{A}_{12} \oplus \mathcal{A}_{21} \oplus \mathcal{A}_{22}$. Clearly any $H \in \mathcal{A}$ can be written as $H = H_{11} + H_{12} + H_{21} + H_{22}$, where $H_{ij} \in \mathcal{A}_{ij}$ for $i, j = 1, 2$. To finalize the proof of the theorem stated above, several lemmas are needed. These lemmas are presented as follows:

Lemma 2.2. $\varphi(0) = 0$.

Proof. It is obvious that

$$\begin{aligned}\varphi(0) &= \varphi(P_n(0,0,\ldots,0))\\ &= P_n(\varphi(0),0,\ldots,0)+P_n(0,\varphi(0),\ldots,0)+\cdots+P_n(0,0,\ldots,\varphi(0))=0.\end{aligned}$$

□

Lemma 2.3. *For any $S_{11}\in\mathcal{A}_{11}, S_{12}\in\mathcal{A}_{12}, S_{21}\in\mathcal{A}_{21}$, and $S_{22}\in\mathcal{A}_{22}$, we have*

$$\varphi(S_{11}+S_{12}+S_{21})=\varphi(S_{11})+\varphi(S_{12})+\varphi(S_{21})$$

and

$$\varphi(S_{12}+S_{21}+S_{22})=\varphi(S_{12})+\varphi(S_{21})+\varphi(S_{22}).$$

Proof. Let $T=\varphi(S_{11}+S_{12}+S_{21})-\varphi(S_{11})-\varphi(S_{12})-\varphi(S_{21})$. We need to show that $T=0$. Using the fact $P_n(S_{11},P_2,P_1,\ldots,P_1)=P_n(S_{21},P_2,P_1,\ldots,P_1)=0$ and Lemmas 2.2, we have

$$\begin{aligned}&\varphi(P_n(S_{11}+S_{12}+S_{21},P_2,P_1,\ldots,P_1))\\ &= \varphi(P_n(S_{11},P_2,P_1,\ldots,P_1))+\varphi(P_n(S_{12},P_2,P_1,\ldots,P_1))\\ &\quad+\varphi(P_n(S_{21},P_2,P_1,\ldots,P_1)).\\ &= P_n(\varphi(S_{11}),P_2,P_1,\ldots,P_1)+P_n(S_{11},\varphi(P_2),P_1,\ldots,P_1)+P_n(S_{11},P_2,\varphi(P_1),\ldots,P_1)\\ &\quad+\cdots+P_n(S_{11},P_2,P_1,\ldots,P_1,\varphi(P_1))+P_n(\varphi(S_{12}),P_2,P_1,\ldots,P_1)+P_n(S_{12},\varphi(P_2),P_1,\ldots,P_1)\\ &\quad P_n(S_{12},P_2,\varphi(P_1),\ldots,P_1)+\cdots+P_n(S_{12},P_2,P_1,\ldots,\varphi(P_1))+P_n(\varphi(S_{21}),P_2,P_1,\ldots,P_1)\\ &\quad+P_n(S_{21},\varphi(P_2),P_1,\ldots,P_1)+P_n(S_{21},P_2,\varphi(P_1),\ldots,P_1)+\cdots+P_n(S_{21},P_2,P_1,\ldots,\varphi(P_1))\\ &= P_n((\varphi(S_{11})+\varphi(S_{12})+\varphi(S_{21})),P_2,P_1,\ldots,P_1)+P_n((S_{11}+S_{12}+S_{21}),\varphi(P_2),P_1,\ldots,P_1)\\ &\quad+P_n((S_{11}+S_{12}+S_{21}),P_2,\varphi(P_1),\ldots,P_1)+\cdots+P_n((S_{11}+S_{12}+S_{21}),P_2,P_1,\ldots,\varphi(P_1)).\end{aligned}$$

Alternatively, we obtain

$$\begin{aligned}&\varphi(P_n(S_{11}+S_{12}+S_{21},P_2,P_1,\ldots,P_1))\\ &= P_n((\varphi(S_{11}+S_{12}+S_{21})),P_2,P_1,\ldots,P_1)+P_n((S_{11}+S_{12}+S_{21}),\varphi(P_2),P_1,\ldots,P_1)\\ &\quad+P_n((S_{11}+S_{12}+S_{21}),P_2,\varphi(P_1),\ldots,P_1)+\cdots+P_n((S_{11}+S_{12}+S_{21}),P_2,P_1,\ldots,\varphi(P_1)).\end{aligned}$$

Comparing the above two expressions for $\varphi(P_n(S_{11}+S_{12}+S_{21},P_2,P_1,\ldots,P_1))$, we find that $P_n(T,P_2,P_1,\ldots,P_1)=0$. This leads us to $T_{12}=0$. Invoking the fact $P_n(S_{11},P_1,P_2,\ldots,P_2)=P_n(S_{12},P_1,P_2,\ldots,P_2)=0$ and using Lemma 2.2, we find that

$$\begin{aligned}&\varphi(P_n(S_{11}+S_{12}+S_{21},P_1,P_2,\ldots,P_2))\\ &= \varphi(P_n(S_{11},P_1,P_2,\ldots,P_2))+\varphi(P_n(S_{12},P_1,P_2,\ldots,P_2))+\varphi(P_n(S_{21},P_1,P_2,\ldots,P_2))\\ &= P_n(\varphi(S_{11}),P_1,P_2,\ldots,P_2)+P_n(S_{11},\varphi(P_1),P_2,\ldots,P_2)+P_n(S_{11},P_1,\varphi(P_2),\ldots,P_2)\\ &\quad+\cdots+P_n(S_{11},P_1,P_2,\ldots,\varphi(P_2))+P_n(\varphi(S_{12}),P_1,P_2,\ldots,P_2)\\ &\quad+P_n(S_{12},\varphi(P_1),P_2,\ldots,P_2)+P_n(S_{12},P_1,\varphi(P_2),\ldots,P_2)\\ &\quad+\cdots+P_n(S_{12},P_1,P_2,\ldots,\varphi(P_2))+P_n(\varphi(S_{21}),P_1,P_2,\ldots,P_2)\\ &\quad+P_n(S_{21},\varphi(P_1),P_2,\ldots,P_2)+P_n(S_{21},P_1,\varphi(P_2),\ldots,P_2)\\ &\quad+\cdots+P_n(S_{21},P_1,P_2,\ldots,\varphi(P_2))\\ &= P_n((\varphi(S_{11})+\varphi(S_{12})+\varphi(S_{21})),P_1,P_2,\ldots,P_2)+P_n(S_{11}+S_{12}+S_{21},\varphi(P_1),P_2,\ldots,P_2)\\ &\quad+P_n(S_{11}+S_{12}+S_{21},P_1,\varphi(P_2),\ldots,P_2)+\cdots+P_n(S_{11}+S_{12}+S_{21},P_1,P_2,\ldots,\varphi(P_2)).\end{aligned}$$

On a different way, we have

$$\begin{aligned}
&\varphi(P_n(S_{11}+S_{12}+S_{21},P_1,P_2,\ldots,P_2))\\
&= P_n(\varphi(S_{11}+S_{12}+S_{21}),P_1,P_2,\ldots,P_2)+P_n(S_{11}+S_{12}+S_{21},\varphi(P_1),P_2,\ldots,P_2)\\
&\quad +P_n(S_{11}+S_{12}+S_{21},P_1,\varphi(P_2),\ldots,P_2)+\cdots+P_n(S_{11}+S_{12}+S_{21},P_1,P_2,\ldots,\varphi(P_2)).
\end{aligned}$$

Comparing the above two expressions for $\varphi(P_n(S_{11}+S_{12}+S_{21},P_1,P_2,\ldots,P_2))$, we obtain that $P_n(T,P_1,P_2,\ldots,P_2)=0$. This futher implies that $T_{21}=0$. Referring to the fact $P_n(I,I\ldots,(P_1-P_2),S_{12})=P_n(I,I,\ldots(P_1-P_2),S_{21})=0$ and using Lemma 2.2, we find that

$$\begin{aligned}
&\varphi(P_n(I,I,\ldots,(P_1-P_2),(S_{11}+S_{12}+S_{21})))\\
&= \varphi(P_n(I,I,\ldots,(P_1-P_2),S_{11}))+\varphi(P_n(I,I,\ldots,(P_1-P_2),S_{12})\\
&\quad +\varphi(P_n(I,I,\ldots,(P_1-P_2),S_{21}))\\
&= P_n(\varphi(I),I,\ldots,(P_1-P_2),S_{11})+P_n(I,\varphi(I),\ldots,(P_1-P_2),S_{11})\\
&\quad +\cdots+P_n(I,I,\ldots,\varphi(P_1-P_2),S_{11})+P_n(I,I,\ldots,(P_1-P_2),\varphi(S_{11}))\\
&\quad +P_n(\varphi(I),I,\ldots,(P_1-P_2),S_{12})+P_n(I,\varphi(I),\ldots,(P_1-P_2),S_{12})\\
&\quad +\cdots+P_n(I,I,\ldots,\varphi(P_1-P_2),S_{12})+P_n(I,I,\ldots,(P_1-P_2),\varphi(S_{12}))\\
&\quad +P_n(\varphi(I),I\ldots,(P_1-P_2),S_{21})+P_n(I,\varphi(I),\ldots,(P_1-P_2),S_{21})\\
&\quad +\cdots+P_n(I,I,\ldots,\varphi(P_1-P_2),S_{21})+P_n(I,I,\ldots,(P_1-P_2),\varphi(S_{21}))\\
&= P_n(\varphi(I),I,\ldots,(P_1-P_2),(S_{11}+S_{12}+S_{21}))+P_n(I,\varphi(I),\ldots,(P_1-P_2),(S_{11}+S_{12}+S_{21}))\\
&\quad +\cdots+P_n(I,I,\ldots,\varphi(P_1-P_2),(S_{11}+S_{12}+S_{21}))\\
&\quad +P_n(I,I,\ldots,(P_1-P_2),\varphi(S_{11})+\varphi(S_{12})+\varphi(S_{21})).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\varphi(P_n(I,I,\ldots,(P_1-P_2),\ldots,S_{11}+S_{12}+S_{21}))\\
&= P_n(\varphi(I),I,\ldots,(P_1-P_2),(S_{11}+S_{12}+S_{21}))+P_n(I,\varphi(I),\ldots,(P_1-P_2),(S_{11}+S_{12}+S_{21}))\\
&\quad +\cdots+P_n(I,I,\ldots,\varphi(P_1-P_2),(S_{11}+S_{12}+S_{21}))\\
&\quad +P_n(I,I,\ldots,(P_1-P_2),\varphi(S_{11}+S_{12}+S_{21})).
\end{aligned}$$

Let's compare the two expressions for $\varphi(P_n(I,I,\ldots,(P_1-P_2),S_{11}+S_{12}+S_{21}))$, we obtain that $P_n(I,I,\ldots,(P_1-P_2),T)=0$ which in turn implies that $T_{11}=0$ and $T_{22}=0$. Hence $T=0$, that is,

$$\varphi(S_{11}+S_{12}+S_{21})=\varphi(S_{11})+\varphi(S_{12})+\varphi(S_{21}).$$

Similarly, we can show that $\varphi(S_{12}+S_{21}+S_{22})=\varphi(S_{12})+\varphi(S_{21})+\varphi(S_{22})$. □

Lemma 2.4. *For any $S_{11}\in\mathcal{A}_{11},S_{12}\in\mathcal{A}_{12},S_{21}\in\mathcal{A}_{21}$, and $S_{22}\in\mathcal{A}_{22}$, we have*

$$\varphi(S_{11}+S_{12}+S_{21}+S_{22})=\varphi(S_{11})+\varphi(S_{12})+\varphi(S_{21})+\varphi(S_{22}).$$

Proof. Let $T=\varphi(S_{11}+S_{12}+S_{21}+S_{22})-\varphi(S_{11})-\varphi(S_{12})-\varphi(S_{21})-\varphi(S_{22})$. We show that $T=0$. Using the fact

$P_n(I, I, \ldots, P_1, S_{22}) = 0$ and Lemmas 2.2 and 2.3, we find that

$$\begin{aligned}
&\varphi(P_n(I, I, \ldots, P_1, (S_{11} + S_{12} + S_{21} + S_{22}))) \\
&= \varphi(P_n(I, I, \ldots, P_1, (S_{11} + S_{12} + S_{21}))) + \varphi(P_n(I, I, \ldots, P_1, S_{22})). \\
&= P_n(\varphi(I), I, \ldots, P_1, (S_{11} + S_{12} + S_{21})) + P_n(I, \varphi(I), \ldots, P_1, (S_{11} + S_{12} + S_{21})) \\
&\quad + \cdots + P_n(I, I, \ldots, \varphi(P_1), (S_{11} + S_{12} + S_{21})) + P_n(I, I, \ldots, P_1, (\varphi(S_{11}) + \varphi(S_{12}) + \varphi(S_{21}))) \\
&\quad + P_n(\varphi(I), I, \ldots, P_1, S_{22}) + P_n(I, \varphi(I), \ldots, P_1, S_{22}) \\
&\quad + \cdots + P_n(I, I, \ldots, \varphi(P_1), S_{22}) + P_n(I, I, \ldots, P_1, \varphi(S_{22})). \\
&= P_n(\varphi(I), I, \ldots, P_1, (S_{11} + S_{12} + S_{21} + S_{22})) + P_n(I, \varphi(I), \ldots, P_1, (S_{11} + S_{12} + S_{21} + S_{22})) \\
&\quad + \cdots + P_n(I, I, \ldots, \varphi(P_1), (S_{11} + S_{12} + S_{21} + S_{22})) \\
&\quad + P_n(I, I, \ldots, P_1, (\varphi(S_{11}) + \varphi(S_{12}) + \varphi(S_{21}) + \varphi(S_{22}))).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\varphi(P_n(I, I, \ldots, P_1, (S_{11} + S_{12} + S_{21} + S_{22}))) \\
&= P_n(\varphi(I), I, \ldots, P_1, (S_{11} + S_{12} + S_{21} + S_{22})) + P_n(I, \varphi(I), \ldots, P_1, (S_{11} + S_{12} + S_{21} + S_{22})) \\
&\quad + \cdots + P_n(I, I, \ldots, \varphi(P_1), (S_{11} + S_{12} + S_{21} + S_{22})) \\
&\quad + P_n(I, I, \ldots, P_1, (\varphi(S_{11} + S_{12} + S_{21} + S_{22}))).
\end{aligned}$$

Comparing the above two expressions for $\varphi(P_n(I, I, \ldots, P_1, (S_{11} + S_{12} + S_{21} + S_{22})))$, we obtain that $P_n(I, I, \ldots, P_1, T) = 0$, which futher implies that $T_{11} = T_{12} = T_{21} = 0$. Similarly we can show that $T_{22} = 0$. Thus $T = 0$, that is,

$$\varphi(S_{11} + S_{12} + S_{21} + S_{22}) = \varphi(S_{11}) + \varphi(S_{12}) + \varphi(S_{21}) + \varphi(S_{22}).$$

□

Lemma 2.5. *For any $S_{12}, S'_{12} \in \mathcal{A}_{12}$ and $S_{21}, S'_{21} \in \mathcal{A}_{21}$ we have*

$$\varphi(S_{12} + S'_{12}) = \varphi(S_{12}) + \varphi(S'_{12}) \ \text{ and } \ \varphi(S_{21} + S'_{21}) = \varphi(S_{21}) + \varphi(S'_{21}).$$

Proof. Using the fact $P_n(\frac{I}{2}, \frac{I}{2}, \ldots, \frac{I}{2}, (P_2 + S^*_{12}), (P_1 + S'_{12})) = S_{12} + S'_{12} + S^*_{12} + S^*_{12}S'_{12}$ and Lemma 2.4, we have

$$\begin{aligned}
&\varphi(S_{12} + S'_{12}) + \varphi(S^*_{12}) + \varphi(S^*_{12}S'_{12}) \\
&= \varphi\Big(S_{12} + S'_{12} + S^*_{12} + S^*_{12}S'_{12}) \\
&= \varphi(P_n(\frac{I}{2}, \frac{I}{2}, \ldots, \frac{I}{2}, (P_2 + S^*_{12}), (P_1 + S'_{12}))) \\
&= P_n(\varphi(\frac{I}{2}), \frac{I}{2}, \ldots, \frac{I}{2}, (P_2 + S^*_{12}), (P_1 + S'_{12})) + P_n(\frac{I}{2}, \varphi(\frac{I}{2}), \ldots, \frac{I}{2}, (P_2 + S^*_{12}), (P_1 + S'_{12})) \\
&\quad + \cdots + P_n(\frac{I}{2}, \frac{I}{2}, \ldots, \varphi(\frac{I}{2}), (P_2 + S^*_{12}), (P_1 + S'_{12})) + P_n(\frac{I}{2}, \frac{I}{2}, \ldots, \frac{I}{2}, \varphi(P_2 + S^*_{12}), (P_1 + S'_{12})) \\
&\quad + P_n(\frac{I}{2}, \frac{I}{2}, \ldots, \frac{I}{2}, (P_2 + S^*_{12}), \varphi(P_1 + S'_{12})) \\
&= \varphi(P_n(\frac{I}{2}, \frac{I}{2}, \ldots, \frac{I}{2}, P_2, P_1)) + \varphi(P_n(\frac{I}{2}, \frac{I}{2}, \ldots, \frac{I}{2}, P_2, S'_{12})) + \varphi(P_n(\frac{I}{2}, \frac{I}{2}, \ldots, \frac{I}{2}, S^*_{12}, P_1)) \\
&\quad + \varphi(P_n(\frac{I}{2}, \frac{I}{2}, \ldots, \frac{I}{2}, S^*_{12}, S'_{12})) \\
&= \varphi(S_{12}) + \varphi(S'_{12}) + \varphi(S^*_{12}) + \varphi(S^*_{12}S'_{12}).
\end{aligned}$$

Hence $\varphi(S_{12} + S'_{12}) = \varphi(S_{12}) + \varphi(S'_{12})$ for any $S_{12}, S'_{12} \in \mathcal{A}_{12}$. Similarly, we can prove other part. □

Lemma 2.6. *For any $S_{ii}, S'_{ii} \in \mathcal{A}_{ii}$ for $(i = 1, 2)$, we have*

$$\varphi(S_{11} + S'_{11}) = \varphi(S_{11}) + \varphi(S'_{11}) \text{ and } \varphi(S_{22} + S'_{22}) = \varphi(S_{22}) + \varphi(S'_{22}).$$

Proof. Let $T = \varphi(S_{11} + S'_{11}) - \varphi(S_{11}) - \varphi(S'_{11})$, we show that $T = 0$. Using the fact that $P_n(I, I, \dots, I, P_2, S_{11}) = P_n(I, I, \dots, I, P_2, S'_{11}) = 0$ and Lemma 2.2, we obtain

$$\begin{aligned}
&\varphi(P_n(I, I, \dots, I, P_2, (S_{11} + S'_{11}))) \\
&= \varphi(P_n(I, I, \dots, I, P_2, S_{11})) + \varphi(P_n(I, I, \dots, I, P_2, S'_{11})) \\
&= P_n(\varphi(I), I, \dots, I, P_2, S_{11}) + P_n(I, \varphi(I), \dots, I, P_2, S_{11}) + \cdots + \\
&\quad P_n(I, I, \dots, \varphi(I), P_2, S_{11}) + P_n(I, I, \dots, I, \varphi(P_2), S_{11}) + P_n(I, I, \dots, I, P_2, \varphi(S_{11})) \\
&\quad + P_n(\varphi(I), I, \dots, I, P_2, S'_{11}) + P_n(I, \varphi(I), \dots, I, P_2, S'_{11}) + \cdots + \\
&\quad P_n(I, I, \dots, \varphi(I), P_2, S'_{11}) + P_n(I, I, \dots, I, \varphi(P_2), S'_{11}) + P_n(I, I, \dots, I, P_2, \varphi(S'_{11})) \\
&= P_n(\varphi(I), I, \dots, I, P_2, (S_{11} + S'_{11})) + P_n(I, \varphi(I), \dots, I, P_2, (S_{11} + S'_{11})) + \cdots + \\
&\quad P_n(I, I, \dots, \varphi(I), P_2, (S_{11} + S'_{11})) + P_n(I, I, \dots, I, \varphi(P_2), (S_{11} + S'_{11})) \\
&\quad + P_n(I, I, \dots, I, P_2, (\varphi(S_{11}) + \varphi(S'_{11}))).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\varphi(P_n(I, I, \dots, I, P_2, (S_{11} + S'_{11}))) \\
&= P_n(\varphi(I), I, \dots, I, P_2, (S_{11} + S'_{11})) + P_n(I, \varphi(I), \dots, I, P_2, (S_{11} + S'_{11})) + \cdots + \\
&\quad P_n(I, I, \dots, \varphi(I), P_2, (S_{11} + S'_{11})) + P_n(I, I, \dots, I, \varphi(P_2), (S_{11} + S'_{11})) \\
&\quad + P_n(I, I, \dots, I, P_2, (\varphi(S_{11} + S'_{11}))).
\end{aligned}$$

Comparing the above two expressions for $\varphi(P_n(I, I, \dots, I, P_2, (S_{11} + S'_{11})))$, we find that $P_n(I, I, \dots, I, P_2, T) = 0$, which in turn gives $T_{12} = T_{21} = T_{22} = 0$. Next, we show that $T_{11} = 0$. Let $X_{12} \in \mathcal{A}_{12}$ and it is easy to observe that $P_n(P_1, P_1, \dots, P_1, S_{11}, X_{12}), P_n(P_1, P_1, \dots, P_1, S'_{11}, X_{12}) \in \mathcal{A}_{12}$. Thus, using Lemma 2.5, we find that

$$\begin{aligned}
&\varphi(P_n(P_1, P_1, \dots, P_1, (S_{11} + S'_{11}), X_{12})) \\
&= \varphi(P_n(P_1, P_1, \dots, P_1, S_{11}, X_{12})) + \varphi(P_n(P_1, P_1, \dots, P_1, S'_{11}, X_{12})) \\
&= P_n(\varphi(P_1), P_1, \dots, P_1, S_{11}, X_{12}) + P_n(P_1, \varphi(P_1), \dots, P_1, S_{11}, X_{12}) \\
&\quad + \cdots + P_n(P_1, P_1, \dots, \varphi(P_1), S_{11}, X_{12}) + P_n(P_1, P_1, \dots, P_1, \varphi(S_{11}), X_{12}) \\
&\quad + P_n(P_1, P_1, \dots, P_1, S_{11}, \varphi(X_{12})) + P_n(\varphi(P_1), P_1, \dots, P_1, S'_{11}, X_{12}) \\
&\quad + P_n(P_1, \varphi(P_1), \dots, P_1, S'_{11}, X_{12}) + \cdots + P_n(P_1, P_1, \dots, \varphi(P_1), S'_{11}, X_{12}) \\
&\quad + P_n(P_1, P_1, \dots, P_1, \varphi(S'_{11}), X_{12}) + P_n(P_1, P_1, \dots, P_1, S'_{11}, \varphi(X_{12})) \\
&= P_n(\varphi(P_1), P_1, \dots, P_1, (S_{11} + S'_{11}), X_{12}) + P_n(P_1, \varphi(P_1), \dots, P_1, (S_{11} + S'_{11}), X_{12}) \\
&\quad + \cdots + P_n(P_1, P_1, \dots, \varphi(P_1), (S_{11} + S'_{11}), X_{12}) \\
&\quad + P_n(P_1, P_1, \dots, P_1, (\varphi(S_{11}) + \varphi(S'_{11})), X_{12}) + P_n(P_1, P_1, \dots, P_1, (S_{11} + S'_{11}), \varphi(X_{12})).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\varphi(P_n(P_1, P_1, \dots, P_1, (S_{11} + S'_{11}), X_{12})) \\
&= P_n(\varphi(P_1), P_1, \dots, P_1, (S_{11} + S'_{11}), X_{12}) + P_n(P_1, \varphi(P_1), \dots, P_1, (S_{11} + S'_{11}), X_{12}) \\
&\quad + \cdots + P_n(P_1, P_1, \dots, \varphi(P_1), (S_{11} + S'_{11}), X_{12}) \\
&\quad + P_n(P_1, P_1, \dots, P_1, (\varphi(S_{11} + S'_{11})), X_{12}) + P_n(P_1, P_1, \dots, P_1, (S_{11} + S'_{11}), \varphi(X_{12})).
\end{aligned}$$

From the last two expressions for $\varphi(P_n(P_1, P_1, \dots, P_1, (S_{11} + S'_{11}), X_{12}))$ we get $P_n(P_1, P_1, \dots, P_1, T, X_{12}) = 0$, on solving further, we obtain $P_1 T X_{12} + T P_1 X_{12} + X_{12} T^* P_1 = 0$. Multiplying it both sides by P_1 and P_2 from left and right, we obtain $P_1TP_1XP_2 = 0$ for all $H \in \mathcal{A}$. Application of the condition (♠) yields $T_{11} = 0$. Hence $T = 0$, that is, $\varphi(S_{11} + S'_{11}) = \varphi(S_{11}) + \varphi(S'_{11})$. Symmetrically, one can prove that $\varphi(S_{22} + S'_{22}) = \varphi(S_{22}) + \varphi(S'_{22})$. □

Lemma 2.7. *φ is additive on $\mathcal{A}$.*

Proof. For any $L, R \in \mathcal{A}$, we have $L = L_{11} + L_{12} + L_{21} + L_{22}$ and $R = R_{11} + R_{12} + R_{21} + R_{22}$. With the help of Lemmas 2.4, 2.5 and 2.6, we obtain

$$\begin{aligned}\varphi(L+R) &= \varphi(L_{11} + L_{12} + L_{21} + L_{22} + R_{11} + R_{12} + R_{21} + R_{22})\\ &= \varphi(L_{11} + R_{11}) + \varphi(L_{12} + R_{12}) + \varphi(L_{21} + R_{21}) + \varphi(L_{22} + R_{22})\\ &= \varphi(L_{11}) + \varphi(R_{11}) + \varphi(L_{12}) + \varphi(R_{12}) + \varphi(L_{21}) + \varphi(R_{21}) + \varphi(L_{22}) + \varphi(R_{22})\\ &= \varphi(L_{11} + L_{12} + L_{21} + L_{22}) + \varphi(R_{11} + R_{12} + R_{21} + R_{22})\\ &= \varphi(L) + \varphi(R).\end{aligned}$$

□

Lemma 2.8. *$\varphi(I) = 0$ and $\varphi(P_i)^* = \varphi(P_i)$ $(i = 1, 2)$.*

Proof. It follows from $2^{n-1}I = P_n(I, I, I, \ldots, I)$ and Lemma 2.7, we have

$$\begin{aligned}2^{n-1}\varphi(I) &= \varphi(P_n(I, I, I, \ldots, I)).\\ &= P_n(\varphi(I), I, I, \ldots, I) + P_n(I, \varphi(I), I, \ldots, I) + P_n(I, I, \varphi(I), \ldots, I)\\ &\quad + \cdots + P_n(I, I, I, \ldots, \varphi(I)).\end{aligned}$$

Simplifying and using the fact that $\varphi(I)$ is symmetric, we obtain $\varphi(I) = 0$. Using the fact that $P_n(I, I, \ldots, I, P_1, I) = P_n(I, I, \ldots, I, I, P_1)$ and $\varphi(I) = 0$

$$P_n(I, I, \ldots, I, \varphi(P_1), I) = P_n(I, I, \ldots, I, I, \varphi(P_1))$$

On solving, we obtain $\varphi(P_1)^* = \varphi(P_1)$, other part can also be proved in similar way. □

Lemma 2.9. (1) $P_1\varphi(P_1)P_2 = -P_1\varphi(P_2)P_2$.
(2) $P_2\varphi(P_1)P_1 = -P_2\varphi(P_2)P_1$.
(3) $P_1\varphi(P_2)P_1 = P_2\varphi(P_1)P_2 = 0$.

Proof. Using the fact that $P_n(I, I, \ldots, P_1, P_1, P_2) = 0$ and Lemmas 2.2 , we obtain

$$\begin{aligned}0 &= \varphi(P_n(I, I, \ldots, P_1, P_1, P_2))\\ &= P_n(I, I, \ldots, \varphi(P_1), P_1, P_2) + P_n(I, I, \ldots, P_1, \varphi(P_1), P_2) + P_n(I, I, \ldots, P_1, P_1, \varphi(P_2))\\ &= P_1\varphi(P_1)^*P_2 + P_2\varphi(P_1)P_1 + P_1\varphi(P_1)P_2 + P_2\varphi(P_1)^*P_1 + 2P_1\varphi(P_2) + 2\varphi(P_2)P_1.\end{aligned}$$

Multiplying the left and right sides by P_1 and P_2, respectively and using Lemma 2.8, we obtain

$$P_1\varphi(P_1)P_2 = -P_1\varphi(P_2)P_2.$$

(2) Since $P_n(I, I, \ldots, P_2, P_2, P_1) = 0$ and Lemma 2.2 , we obtain

$$\begin{aligned}0 &= \varphi(P_n(I, I, \ldots, P_2, P_2, P_1))\\ &= P_n(I, I, \ldots, \varphi(P_2), P_2, P_1) + P_n(I, I, \ldots, P_2, \varphi(P_2), P_1) + P_n(I, I, \ldots, P_2, P_2, \varphi(P_1))\\ &= P_2\varphi(P_2)^*P_1 + P_1\varphi(P_2)P_2 + P_2\varphi(P_2)P_1 + P_1\varphi(P_2)^*P_2 + 2P_2\varphi(P_1) + 2\varphi(P_1)P_2.\end{aligned}$$

Multiplying the last relation by P_2 from left and by P_1 from right and using Lemma 2.8, we obtain

$$P_2\varphi(P_1)P_1 = -P_2\varphi(P_2)P_1.$$

(3) In (1), we have

$$P_1\varphi(P_1)^*P_2 + P_2\varphi(P_1)P_1 + P_1\varphi(P_1)P_2 + P_2\varphi(P_1)^*P_1 + 2P_1\varphi(P_2) + 2\varphi(P_2)P_1 = 0.$$

Multiplying both sides by P_1 from left and right respectively, we obtain

$$P_1\varphi(P_2)P_1 = 0.$$

Similarly, in (2), we have

$$P_2\varphi(P_2)^*P_1 + P_1\varphi(P_2)P_2 + P_2\varphi(P_2)P_1 + P_1\varphi(P_2)^*P_2 + 2P_2\varphi(P_1) + 2\varphi(P_1)P_2 = 0.$$

Multiplying both sides by P_2 from left and right respectively, we obtain

$$P_2\varphi(P_1)P_2 = 0.$$

□

Lemma 2.10. *$P_1\varphi(P_1)P_1 = P_2\varphi(P_2)P_2 = 0$.*

Proof. For $S_{12} \in \mathcal{A}_{12}$, we have $2^{n-2}S_{12} = P_n(I, I, \ldots, P_1, P_1, S_{12})$ and applying Lemmas 2.7 and 2.8, we obtain

$$\begin{aligned} 2^{n-2}\varphi(S_{12}) &= \varphi(P_n(I, I, \ldots, P_1, P_1, S_{12})) \\ &= (P_n(I, I, \ldots, \varphi(P_1), P_1, S_{12})) + (P_n(I, I, \ldots, P_1, \varphi(P_1), S_{12})) \\ &\quad +(P_n(I, I, \ldots, P_1, P_1, \varphi(S_{12}))) \\ &= 2^{n-2}\{\varphi(P_1)P_1S_{12} + P_1\varphi(P_1)S_{12} + S_{12}\varphi(P_1)P_1 + P_1\varphi(S_{12}) + \varphi(S_{12})P_1\}. \end{aligned}$$

Multiplying both sides by P_1 and P_2 from left and right respectively, we obtain $P_1\varphi(P_1)P_1S_{12} = 0$, implies $P_1\varphi(P_1)P_1SP_2 = 0$ for all $S \in \mathcal{A}$. It follows from (♠) that $P_1\varphi(P_1)P_1 = 0$. Similarly, we can prove that $P_2\varphi(P_2)P_2 = 0$. □

Lemma 2.11. *For any $S \in \mathcal{A}$, we have $\varphi(S^*) = \varphi(S)^*$.*

Proof. Observe that $P_n(I, I, \ldots, S, I, I) = 2^{n-2}(S + S^*)$, for any $S \in \mathcal{A}$. Using Lemmas 2.7 and 2.8, we find that

$$\begin{aligned} 2^{n-2}(\varphi(S) + \varphi(S^*)) &= \varphi(P_n(I, I, \ldots, S, I, I)) \\ &= P_n(I, I, \ldots, \varphi(S), I, I) \\ &= 2^{n-2}(\varphi(S) + \varphi(S)^*) \end{aligned}$$

which implies

$$\varphi(S^*) = \varphi(S)^*.$$

□

Now, let $M = P_1\varphi(P_1)P_2 - P_2\varphi(P_1)P_1$, then $M^* = -M$. Defining a map $\delta : \mathcal{A} \to \mathcal{A}$ by $\delta(L) = \varphi(L) - (LM - ML)$ holds for all $L \in \mathcal{A}$. It is easy to verify that $\delta(P_n(L_1, L_2, \ldots, L_n)) = \sum_{i=1}^{n} P_n(L_1, \ldots, L_{i-1}, \delta(L_i), L_{i+1}, \ldots, L_n)$ for all $L_1, L_2, \ldots, L_n \in \mathcal{A}$,

Remark 2.12. *δ has the following properties*
(1) $\delta(L^*) = \delta(L)^*$.
(2) *δ is additive.*
(3) $\delta(P_1) = \delta(P_2) = 0$.
(4) $\delta(I) = 0$.
(5) *δ is a $*$-derivation if and only if φ is a $*$-derivation.*

Lemma 2.13. *$\delta(A_{ij}) \subseteq A_{ij}$ i,j=1,2.*

Proof. First we have for $i = 1, j = 2$, in view of the fact $P_n(I, I, \dots, I, P_1, A_{12}) = 2^{n-2}A_{12}$ and Remark 2.12, we have

$$\begin{aligned} 2^{n-2}\delta(A_{12}) &= \delta(P_n(I, I, \dots, I, P_1, A_{12})) \\ &= P_n(I, I, \dots, I, P_1, \delta(A_{12}) \\ &= 2^{n-2}\{P_1\delta(A_{12}) + \delta(A_{12})P_1\}. \end{aligned}$$

This implies that $P_1\delta(A_{12})P_1 = 0$ and $P_2\delta(A_{12})P_2 = 0$, using the fact $P_n(I, I, \dots, I, A_{12}, P_1) = 0$ and Remark 2.12, we have

$$\begin{aligned} 0 &= \delta(P_n(I, I, \dots, I, A_{12}, P_1)) \\ &= P_n(I, I, \dots, I, \delta(A_{12}), P_1) \\ &= 2^{n-2}\{\delta(A_{12})P_1 + P_1\delta(A_{12})^*\}. \end{aligned}$$

This implies that $P_2\delta(A_{12})P_1 = 0$, thus $\delta(A_{12}) \subseteq A_{12}$. Similarly, we can show that $\delta(A_{21}) \subseteq A_{21}$. Now we have for $i = 1, j = 1$, in view of the fact $P_n(I, I, \dots, I, P_2, A_{11}) = 0$ and Remark 2.12, we have

$$\begin{aligned} 0 &= \delta(P_n(I, I, \dots, I, P_2, A_{11})) \\ &= P_n(I, I, \dots, I, P_2, \delta(A_{11})) \\ &= 2^{n-2}\{P_2\delta(A_{12}) + \delta(A_{12})^*P_2\}. \end{aligned}$$

This implies that $P_2\delta(A_{11})P_2 = P_1\delta(A_{11})P_2 = P_2\delta(A_{11})P_1 = 0$, thus $\delta(A_{11}) \subseteq A_{11}$. Similarly, we can show that $\delta(A_{22}) \subseteq A_{22}$. □

Lemma 2.14. *For any $A_{ij}, B_{ij} \in \mathcal{A}_{ij}, 1 \leq i, j \leq 2$, we have*
(1) $\delta(A_{11}B_{12}) = \delta(A_{11})B_{12} + A_{11}\delta(B_{12})$ *and* $\delta(A_{22}B_{21}) = \delta(A_{22})B_{21} + A_{22}\delta(B_{21})$.
(2) $\delta(A_{12}B_{21}) = \delta(A_{12})B_{21} + A_{12}\delta(B_{21})$ *and* $\delta(A_{21}B_{12}) = \delta(A_{21})B_{12} + A_{21}\delta(B_{12})$.
(3) $\delta(A_{11}B_{11}) = \delta(A_{11})B_{11} + A_{11}\delta(B_{11})$ *and* $\delta(A_{22}B_{22}) = \delta(A_{22})B_{22} + A_{22}\delta(B_{22})$.
(4) $\delta(A_{12}B_{22}) = \delta(A_{12})B_{22} + A_{12}\delta(B_{22})$ *and* $\delta(A_{21}B_{11}) = \delta(A_{21})B_{11} + A_{21}\delta(B_{11})$.

Proof. (1) In view of the fact $P_n(I, I, \dots, A_{11}, B_{12}) = 2^{n-2}(A_{11}B_{12})$ and using Remark 2.12, we have

$$\begin{aligned} 2^{n-2}\delta(A_{11}B_{12}) &= \delta(P_n(I, I, \dots, A_{11}, B_{12})) \\ &= P_n(I, I, \dots, \delta(A_{11}), B_{12}) + P_n(I, I, \dots, A_{11}, \delta(B_{12})). \end{aligned}$$

Invoking Lemma 2.13, we obtain

$$\delta(A_{11}B_{12}) = \delta(A_{11})B_{12} + A_{11}\delta(B_{12}).$$

Similarly, we can prove that $\delta(A_{22}B_{21}) = \delta(A_{22})B_{21} + A_{22}\delta(B_{21})$.
(2) In view of the fact $P_n(I, I, \dots, A_{12}, B_{21}) = 2^{n-2}A_{12}B_{21}$ and using Remark 2.12, we have

$$\begin{aligned} 2^{n-2}\delta(A_{12}B_{21}) &= \delta(P_n(I, I, \dots, A_{12}, B_{21})) \\ &= P_n(I, I, \dots, \delta(A_{12}), B_{21}) + P_n(I, I, \dots, A_{12}, \delta(B_{21})) \end{aligned}$$

Invoking Lemma 2.13, we obtain

$$\delta(A_{12}B_{21}) = \delta(A_{12})B_{21} + A_{12}\delta(B_{21}).$$

Similarly, we can prove that $\delta(A_{21}B_{12}) = \delta(A_{21})B_{12} + A_{21}\delta(B_{12})$.
(3) For any $X_{12} \in \mathcal{A}_{12}$ and using the fact $P_n(I, I, \dots, A_{11}B_{11}, X_{12}) = 2^{n-2}(A_{11}B_{11}X_{12})$ and Remark 2.12, we have

$$\begin{aligned} 2^{n-2}\delta(A_{11}B_{11}X_{12}) &= \delta(P_n(I, I, \dots, A_{11}B_{11}, X_{12}) \\ &= P_n(I, I, \dots, \delta(A_{11}B_{11}), X_{12}) + P_n(I, I, \dots, A_{11}B_{11}, \delta(X_{12})) \\ &= 2^{n-2}\{\delta(A_{11}B_{11})X_{12} + X_{12}\delta(A_{11}B_{11}) + A_{11}B_{11}\delta(X_{12}) + \delta(X_{12})A_{11}B_{11}\}. \end{aligned}$$

Invoking Lemma 2.13, we obtain

$$\delta(A_{11}B_{11}X_{12}) = \delta(A_{11}B_{11})X_{12} + A_{11}B_{11}\delta(X_{12}).$$

In view of the fact $P_n(I, I, \ldots, A_{11}, B_{11}X_{12}) = 2^{n-2}(A_{11}B_{11}X_{12})$ and Remark 2.12, we have

$$\begin{aligned} 2^{n-2}\delta(A_{11}B_{11}X_{12}) &= \delta(P_n(I, I, \ldots, A_{11}, B_{11}X_{12}) \\ &= P_n(I, I, \ldots, \delta(A_{11}), B_{11}X_{12}) + P_n(I, I, \ldots, A_{11}, \delta(B_{11}X_{12})) \\ &= 2^{n-2}\{\delta(A_{11})B_{11}X_{12} + A_{11}\delta(B_{11}X_{12})\}. \end{aligned}$$

Using Lemma 2.14(1), we have

$$\delta(A_{11}B_{11}X_{12}) = \delta(A_{11})B_{11}X_{12} + A_{11}\delta(B_{11})X_{12} + A_{11}B_{11}\delta(X_{12}).$$

Comparing the above two expressions for $\delta(A_{11}B_{11}X_{12})$, we get $(\delta(A_{11}B_{11}) - \delta(A_{11})B_{11} - A_{11}\delta(B_{11}))X_{12} = 0$, implies $(\delta(A_{11}B_{11}) - \delta(A_{11})B_{11} - A_{11}\delta(B_{11}))XP_2 = 0$ for all $X \in \mathcal{A}$. It follows from (♠) that $\delta(A_{11}B_{11}) = \delta(A_{11})B_{11} + A_{11}\delta(B_{11})$. Similarly, we can prove that

$$\delta(A_{22}B_{22}) = \delta(A_{22})B_{22} + A_{22}\delta(B_{22}).$$

(4) In view of fact $P_n(I, I, \ldots, I, P_1, A_{12}, B_{22}) = 2^{n-3}(A_{12}B_{22} + B_{22}A_{12}^*)$ and using Remark 2.12, we have

$$\begin{aligned} 2^{n-3}\{\delta(A_{12}B_{22}) + \delta(B_{22}A_{12}^*)\} &= \delta(P_n(I, I, \ldots, I, P_1, A_{12}, B_{22})) \\ &= P_n(I, I, \ldots, I, P_1, \delta(A_{12}), B_{22}) + P_n(I, I, \ldots, I, P_1, A_{12}, \delta(B_{22})). \end{aligned}$$

Invoking Lemma 2.13, we obtain

$$\delta(A_{12}B_{22}) + \delta(B_{22}A_{12}^*) = \delta(A_{12})B_{22} + B_{22}\delta(A_{12})^* + A_{12}\delta(B_{22}) + \delta(B_{22})A_{12}^*.$$

Applying Lemmas 2.11 and 2.14(1), we obtain

$$\delta(A_{12}B_{22}) + \delta(B_{22})A_{12}^* + B_{22}\delta(A_{12}^*) = \delta(A_{12})B_{22} + B_{22}\delta(A_{12})^* + A_{12}\delta(B_{22}) + \delta(B_{22})A_{12}^*.$$

Hence

$$\delta(A_{12}B_{22}) = \delta(A_{12})B_{22} + A_{12}\delta(B_{22}).$$

Similarly, we can prove that $\delta(A_{21}B_{11}) = \delta(A_{21})B_{11} + A_{21}\delta(B_{11})$. □

Lemma 2.15. *$\delta(LR) = \delta(L)R + L\delta(R)$, for all $L, R \in \mathcal{A}$.*

Proof. For any $L, R \in \mathcal{A}$, write $L = L_{11} + L_{12} + L_{21} + L_{22}$ and $R = R_{11} + R_{12} + R_{21} + R_{22}$. Using the fact that δ is additive and using Lemma 2.14, we obtain

$$\begin{aligned} \delta(LR) &= \delta(L_{11}R_{11} + L_{11}R_{12} + L_{12}R_{21} + L_{12}R_{22} \\ &\quad + L_{21}R_{11} + L_{21}R_{12} + L_{22}R_{21} + L_{22}R_{22}) \\ &= \delta(L_{11}R_{11}) + \delta(L_{11}R_{12}) + \delta(L_{12}R_{21}) + \delta(L_{12}R_{22}) \\ &\quad + \delta(L_{21}R_{11}) + \delta(L_{21}R_{12}) + \delta(L_{22}R_{21}) + \delta(L_{22}R_{22}) \\ &= \delta(L_{11} + L_{12} + L_{21} + L_{22})(R_{11} + R_{12} + R_{21} + R_{22}) \\ &\quad + (L_{11} + L_{12} + L_{21} + L_{22})\delta(R_{11} + R_{12} + R_{21} + R_{22}). \\ &= \delta(L)R + L\delta(R). \end{aligned}$$

By Remark 2.12 and Lemma 2.15, we can conclude that δ qualifies as an additive $*$-derivation. Consequently, we can infer that φ is also an additive $*$-derivation. This, in turn, allows us to conclude the proof of Theorem 2.1. □

3. Applications

Remember that a ring $\mathcal{A}$ is considered prime if, $\mu_1, \mu_2 \in \mathcal{A}$, $\mu_1 \mathcal{A} \mu_2 = \{0\}$ implies that either $\mu_1 = 0$ or $\mu_2 = 0$. It's easy to confirm that every prime rings satisfies property (♠). Consequently, as a straightforward impact of Theorem 2.1, we can deduce the following result:

Corollary 3.1. *Consider $\mathcal{A}$ as a unital prime $*$-ring that is 2-torsion-free and contains a non-trivial symmetric idempotent. If a map $\varphi : \mathcal{A} \to \mathcal{A}$ satisfies*

$$\varphi(P_n(S_1, S_2, \ldots, S_n)) = \sum_{i=1}^{n} P_n(S_1, \ldots, S_{i-1}, \varphi(S_i), S_{i+1}, \ldots, S_n) \quad (2)$$

for all $S_1, S_2, \ldots, S_n \in \mathcal{A}$, then φ is additive. Furthermore, if $\varphi(I)$ is symmetric, then φ is a $*$-derivation.

Remember that an algebra $\mathcal{A}$ is considered prime if, $\mu_1, \mu_2 \in \mathcal{A}$, $\mu_1 \mathcal{A} \mu_2 = \{0\}$ implies that either $\mu_1 = 0$ or $\mu_2 = 0$. It's easy to confirm that every prime $*$-algebra satisfies property (♠). Consequently, as a straightforward impact of Theorem 2.1, we can deduce the following result:

Corollary 3.2. *Consider $\mathcal{A}$ as a unital prime $*$-algebras that contains a non-trivial projection. If a map $\varphi : \mathcal{A} \to \mathcal{A}$ satisfies*

$$\varphi(P_n(S_1, S_2, \ldots, S_n)) = \sum_{i=1}^{n} P_n(S_1, \ldots, S_{i-1}, \varphi(S_i), S_{i+1}, \ldots, S_n) \quad (3)$$

for all $S_1, S_2, \ldots, S_n \in \mathcal{A}$, then φ is additive. Furthermore, if $\varphi(I)$ is self-adjoint, then φ is a $*$-derivation.

Further, It is widely recognised that factor von Neumann algebra is also prime thus, it always satisfies (♠). Then, as a straightforward impact of Corollary 3.2, we get the following result:

Corollary 3.3. *Let $\mathcal{A}$ be a factor von Neumann algebra with dimension greater than or equal to 2. If a map $\varphi : \mathcal{A} \to \mathcal{A}$ satisfies*

$$\varphi(P_n(S_1, S_2, \ldots, S_n)) = \sum_{i=1}^{n} P_n(S_1, \ldots, S_{i-1}, \varphi(S_i), S_{i+1}, \ldots, S_n) \quad (4)$$

for all $S_1, S_2, \ldots, S_n \in \mathcal{A}$, then φ is additive. Furthermore, if $\varphi(I)$ is self-adjoint, then φ is a $*$-derivation.

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References

[1] M. Ashraf, M. S. Akhter and M. A. Ansari, *Nonlinear bi-skew Lie-type derivations on factor von Neumann algebras,* Comm. Algebra **50(11)** (2022), 4766–4780.
[2] M. Ashraf, M. S. Akhter, and M. A. Ansari, *Nonlinear bi-skew Jordan-type derivations on factor von Neumann algebras,* Filomat **37(17)** (2023), 5591–-5599.
[3] L. Dai and F. Lu, *Nonlinear maps preserving Jordan $*$-products,* J. Math. Anal. Appl. **409(1)** (2014), 180–188,
[4] A. N. Khan, *Multiplicative bi-skew Lie triple derivations on factor von Neumann algebras,* Rocky Mountain J. Math. **51(6)** (2021), 2103–2114.
[5] A. N. Khan and H. Alhazmi, *Multiplicative bi-skew jordan triple derivation on prime $*$-algebra,* Georgian Math. J. **30(3)** (2023), 389–396.
[6] L. Kong and J. Zhang, *Nonlinear skew-Lie derivations on prime $*$-rings,* Indian J. Pure and Applied Math. **54(2)** (2023), 475–484
[7] C. Li, F. Lu and X. Fang, *Nonlinear ξ-Jordan $*$-derivations on von Neumann algebras.* Linear Multilinear Algebra **62(4)** (2014), 466–473.
[8] C.J. Li, F. F. Zhao and Q.Y. Chen, *Nonlinear skew Lie triple derivations between factors,* Acta Math. Sin. (Engl. Ser.) **32(7)** (2016), 821–830.
[9] C. Li, Y. Zhao and F. Zhao, *Nonlinear $*$-Jordan-type derivations on $*$-algebras,* Rocky Mountain J. Math **51(2)** (2021), 601–612.
[10] C. Li and D. Zhang, *Nonlinear mixed jordan triple $*$-derivation on $*$-algebra,* Sib J. Math **63(4)** (2022), 735–742.
[11] W. Lin, *Nonlinear $*$-Lie type derivations on standard operator algebra,* Acta Mathematica Hungarica **156(2)** (2018), 480–500.
[12] W. Lin, *Nonlinear $*$-Lie type derivations on von neumann algebra,* Acta Mathematica Hungarica **156(1)** (2018), 112–131.

[13] Y. Pang, D. Zhang and D. Ma, *The second nonlinear mixed Jordan triple derivable mapping on factor von Neumann algebras*, Bull. Iran. Math. Soc. **48(3)** (2022), 951–962.

[14] N. Rehman, J. Nisar and M. Nazim, *A note on nonlinear mixed Jordan triple derivation on ∗-algebras,* Comm. Algebra **51(4)** (2023), 1334—1343.

[15] P. Šemrl, *On Jordan ∗-derivations and an application,* Colloq. Math. **59(2)** (1990), 241–251.

[16] P. Šemrl, *Additive derivations of some operator algebras,* Illinois J. Math. **35(2)** (1991), 234–240.

[17] P. Šemrl, *Jordan ∗-derivations of standard operator algebras,* Proc. Amer. Math. Soc. **120(2)** (1994), 515–519.

[18] A. Taghavi, H. Rohi and V. Darvish, *Nonlinear ∗-Jordan derivations on von neumann algebras,* Linear Multilinear Algebra **64(3)** (2016), 426–439.

[19] W. Yu and J. Zhang, *Nonlinear ∗-Lie derivations on factor von Neumann algebras,* Linear Algebra Appl. **437(8)** (2012), 1979–1991.

[20] F. Zhang, *Nonlinear skew Jordan derivable maps on factor von neumann algebras,* Linear Multilinear Algebra **64(10)** (2016), 2090–2103.

[21] F. F. Zhao and C. J. Li, *Nonlinear ∗-Jordan triple derivations on von Neumann algebras,* Math. Slovaca **68(1)** (2018), 163–170.

[22] X. Zhao and X. Fang, *The second nonlinear mixed Lie triple derivations on finite von Neumann algebras,* Bull. Iran. Math. Soc. **47(1)** (2021), 237–254.

[23] Y. Zhou, Z. Yang and J. Zhang, *Nonlinear mixed Lie triple derivations on prime ∗-algebra,* Comm. Algebra **27(11)** (2019), 4791–4796.