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On a trace inequaliy due to Ando-Hiai-Okubo trace inequalities

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Abstract.

In this article, we obtain another proof of the following classical trace inequality which says that if A is a positive semidefinite matrix and B is a Hermitian matrix, then

$$\operatorname{tr}\ A^{\alpha}BA^{\beta}B \leq \operatorname{tr}\ A^{\gamma}BA^{\delta}B$$

for all non-negative real numbers $\alpha, \beta, \gamma, \delta$ for which $\alpha+\beta=\gamma+\delta$ and

$$\max\{\alpha,\beta\} \leq \max\{\gamma,\delta\}.$$

This is a generalization of trace inequalities due to T. Ando, F. Hiai, and K. Okubo for the special cases when $\gamma=\alpha+\beta, \delta=0$ and when $\alpha=\beta=\frac{\gamma+\delta}{2}$, namely

$$\operatorname{tr}\left(A^{\frac{\alpha+\beta}{2}}B\right)^{2} \leq \operatorname{tr}\ A^{\alpha}BA^{\beta}B \leq \operatorname{tr}\ A^{\alpha+\beta}B^{2}.$$

1. Introduction

Let $M_n(\mathbb{C})$ be the algebra of all $n\times n$ complex matrices. In [1], T. Ando, F. Hiai, and K. Okubo proved that if A and B are positive semidefinite matrices in $M_n(\mathbb{C})$, then

$$\operatorname{tr}\left(A^{\frac{\alpha+\beta}{2}}B\right)^{2} \leq \operatorname{tr}\ A^{\alpha}BA^{\beta}B \leq \operatorname{tr}\ A^{\alpha+\beta}B^{2}. \tag{1}$$

The inequalities (1) can be generalized by proving that the inequality

$$\operatorname{tr}\ A^{\alpha}BA^{\beta}B \leq \operatorname{tr}\ A^{\gamma}BA^{\delta}B \tag{2}$$

holds for all non-negative real numbers $\alpha, \beta, \gamma, \delta$ for which $\alpha+\beta=\gamma+\delta$ and

$$\max\{\alpha,\beta\} \leq \max\{\gamma,\delta\}, \tag{3}$$

where A is a positive semidefinite matrix and B is a Hermitian matrix.

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The inequality (2), based on log convexity, has been proved in [3]. Another proof of the inequality (2) can be concluded from Lemma 2 in [2].

In Section 2, using the Pigeonhole Principle and some algorithmic calculations, we obtain another proof of the inequality (2).

It is important to see that the inequality (2) under the condition (3) is a generalization of the inequalities (1) for the special cases when $\gamma = \alpha + \beta, \delta = 0$ and when $\alpha = \beta = \frac{\gamma+\delta}{2}$.

One of the most important ingredients of our new alternative proof of the inequality (2) is the following interesting trace inequality

$$\text{tr } A^{\alpha}BA^{\beta}B \leq \frac{1}{2}\,\text{tr}\left(A^{\alpha+\eta}BA^{\beta-\eta}B + A^{\alpha-\eta}BA^{\beta+\eta}B\right)$$

for $\alpha, \beta \geq \eta \geq 0$, where A is a positive semidefinite matrix and B is a Hermitian matrix. This is the starting point of our proof.

2. The inequality $\text{tr } A^{\alpha}BA^{\beta}B \leq \frac{1}{2}\,\text{tr}\left(A^{\alpha+\eta}BA^{\beta-\eta}B + A^{\alpha-\eta}BA^{\beta+\eta}B\right)$

This section is devoted to proving the inequality

$$\text{tr } A^{\alpha}BA^{\beta}B \leq \frac{1}{2}\,\text{tr}\left(A^{\alpha+\eta}BA^{\beta-\eta}B + A^{\alpha-\eta}BA^{\beta+\eta}B\right),$$

where A is a positive semidefinite matrix, B is a Hermitian matrix and $\alpha, \beta \geq \eta \geq 0$.

The proof of the inequality (2) is based on the following trace inequality. Related trace inequalities can be found in [4]. In the sequel, we frequently use the cyclicity property of the trace, which says that if X and Y are any two matrices, then $\text{tr } XY = \text{tr } YX$.

Lemma 2.1. *Let A be a positive semidefinite matrix and B be a Hermitian matrix, and let $\alpha, \beta \geq \eta \geq 0$. Then*

$$\text{tr } A^{\alpha}BA^{\beta}B \leq \frac{1}{2}\,\text{tr}\left(A^{\alpha+\eta}BA^{\beta-\eta}B + A^{\alpha-\eta}BA^{\beta+\eta}B\right).$$

Proof. Let $C = BA^{\frac{\beta+\eta}{2}} - A^{\eta}BA^{\frac{\beta-\eta}{2}}$ and $R = A^{\alpha-\eta}$. Since CC^* is a positive semidefinite matrix, it follows that $\text{tr } RCC^* \geq 0$, and so we have

$$\text{tr } A^{\alpha-\eta}\left(BA^{\frac{\beta+\eta}{2}} - A^{\eta}BA^{\frac{\beta-\eta}{2}}\right)\left(A^{\frac{\beta+\eta}{2}}B - A^{\frac{\beta-\eta}{2}}BA^{\eta}\right) \geq 0,$$

which is equivalent to

$$\text{tr}\left(A^{\alpha+\eta}BA^{\beta-\eta}B + A^{\alpha-\eta}BA^{\beta+\eta}B\right) \geq 2\,\text{tr } A^{\alpha}BA^{\beta}B.$$

This completes the proof of the lemma. □

Lemma 2.2. *Let A be a positive semidefinite matrix and B be a Hermitian matrix. Then for all non-negative real numbers $\alpha, \beta, \gamma, \delta$ with $\gamma \geq \alpha \geq \beta \geq \delta$ and $\gamma + \delta = \alpha + \beta$, we have*

$$\text{tr } A^{\alpha}BA^{\beta}B \leq \frac{1}{2}\,\text{tr}\left(A^{\gamma}BA^{\delta}B + A^{2\alpha-\gamma}BA^{2\beta-\delta}B\right).$$

Proof. Let $\eta = \beta - \delta = \gamma - \alpha$. Since $\alpha, \beta, \gamma, \delta \geq 0$ with $\gamma \geq \alpha \geq \beta \geq \delta$, it follows that $\alpha, \beta \geq \eta \geq 0$. Note that $\alpha - \eta = 2\alpha - \gamma$, $\beta + \eta = 2\beta - \delta$, $\beta - \eta = \delta$ and $\alpha + \eta = \gamma$. Replacing η by $\beta - \delta = \gamma - \alpha$ in Lemma 2.1, completes the proof of the lemma. □

3. Two interesting sequences of nonnegative rational numbers

To prove the inequality (2), we need to define two sequences of nonnegative rational numbers and prove some properties related to them. These two sequences play an important role in proving the inequality (2).

We introduce some notations regarding our main result. For $\gamma \in \mathbb{Q}^+ \cup \{0\}$ and $k \in \mathbb{N}$, we define the following set:

$$\mathcal{A}_{\gamma,k} = \left\{ x \in \mathbb{Q}^+ \cup \{0\} : x \leq \gamma; x = \frac{m}{k} \text{ for some } m \in \mathbb{N} \cup \{0\} \right\}.$$

The following lemma computes the cardinality of $\mathcal{A}_{\gamma,k}$.

Lemma 3.1. *For $\gamma \in \mathbb{Q}^+ \cup \{0\}$ and $k \in \mathbb{N}$, we have $\left|\mathcal{A}_{\gamma,k}\right| = l_{\gamma,k}$, where $l_{\gamma,k} = [\gamma k] + 1$ and $[\gamma k]$ is the greatest integer less than or equal to γk.*

Proof. Note that

$$\begin{aligned} \left|\mathcal{A}_{\gamma,k}\right| &= \left|\{m \in \mathbb{N} \cup \{0\} : m \leq \gamma k\}\right| \\ &= \left|\{0, 1, 2, \ldots, [\gamma k]\}\right| \\ &= [\gamma k] + 1 \\ &= l_{\gamma,k}. \end{aligned}$$

This completes the proof of the lemma. □

We remark here that the family $\mathcal{A}_{\gamma,k}$ covers $\mathbb{Q}^+ \cup \{0\}$. In fact, if $x \in Q^+ \cup \{0\}$, then $x = \frac{m}{k}$ for some $m, k \in \mathbb{N} \cup \{0\}$ with $k \neq 0$. Thus, $x \in \mathcal{A}_{x,k}$.

Let $\gamma, \delta, x_0, y_0 \in \mathbb{Q}^+ \cup \{0\}$ with $\gamma \geq x_0 \geq y_0 \geq \delta$ and $\gamma + \delta = x_0 + y_0$. We define the two sequences x_j, y_j for $j \geq 0$ by

$$x_{j+1} = \max\left(f\left(x_j\right), g\left(y_j\right)\right) \tag{4}$$

$$y_{j+1} = \min\left(f\left(x_j\right), g\left(y_j\right)\right), \tag{5}$$

where $f(x) = 2x - \gamma$ and $g(x) = 2x - \delta$.

The following lemma is needed in the proof of Lemma 3.3.

Lemma 3.2. *Let $\gamma, \delta, x_0, y_0 \in \mathbb{Q}^+ \cup \{0\}$ with $\gamma \geq x_0 \geq y_0 \geq \delta$ and $x_0 + y_0 = \gamma + \delta$. Let x_j, y_j be the two sequences as in (4) and (5), respectively. Then for all $j \in \mathbb{N} \cup \{0\}$, we have*

1. $x_j + y_j = \gamma + \delta$
2. $\gamma \geq x_j \geq y_j \geq \delta$.

Proof. Part (1) can be seen as follows. First note that for $j = 0$, $x_0 + y_0 = \gamma + \delta$. Now suppose that it is true for some j, i.e., $x_j + y_j = \gamma + \delta$. To prove that it is true for $j + 1$, we have

$$\begin{aligned} x_{j+1} + y_{j+1} &= \max\left(f\left(x_j\right), g\left(y_j\right)\right) + \min\left(f\left(x_j\right), g\left(y_j\right)\right) \\ &= f\left(x_j\right) + g\left(y_j\right) \\ &= 2\left(x_j + y_j\right) - (\gamma + \delta) \\ &= \gamma + \delta \,(\text{by the induction assumption}). \end{aligned}$$

Part (2) can be seen as follows. First note that for $j = 0$, $\gamma \geq x_0 \geq y_0 \geq \delta$. Now suppose it is true for some j, i.e. $\gamma \geq x_j \geq y_j \geq \delta$. To prove that it is true for $j + 1$, we have $2y_j \geq 2\delta$ and $2x_j \geq x_j + y_j$ by the induction assumption. This implies that $2y_j - \delta \geq \delta$ and $2x_j \geq \gamma + \delta$ by Part (1). Thus, $f\left(x_j\right) \geq \delta$ and $g\left(y_j\right) \geq \delta$, and so $y_{j+1} \geq \delta$. Note also that

$$\begin{aligned} x_{j+1} &= \gamma + \delta - y_{j+1} \text{ (by part (1))} \\ &\leq \gamma + \delta - \delta \\ &= \gamma. \end{aligned}$$

Therefore, $\gamma \geq x_{j+1} \geq y_{j+1} \geq \delta$. □

Our goal in the following lemma is to prove that the ranges of the sequences x_j and y_j are finite sets.

Lemma 3.3. *Let $\gamma, \delta, x_0, y_0 \in \mathbb{Q}^+ \cup \{0\}$ with $\gamma \geq x_0 \geq y_0 \geq \delta$ and $x_0 + y_0 = \gamma + \delta$. Let x_j, y_j be the two sequences as in (4) and (5), respectively. Then there exists $k \in \mathbb{N}$ such that $x_j, y_j \in \mathcal{A}_{\gamma,k}$ for every $j \geq 0$ (and hence the sequences x_j, y_j have finite ranges).*

Proof. Let k be the smallest common denominator of the rational numbers x_0, y_0, γ, δ. Thus, we can write

$$x_0 = \frac{x_0'}{k}, y_0 = \frac{y_0'}{k}, \gamma = \frac{\gamma'}{k}, \delta = \frac{\delta'}{k},$$

for some $x_0', y_0', \gamma', \delta' \in \mathbb{N} \cup \{0\}$.

Our result follows by induction on the parameter j for both x_j and y_j. First note that for $j = 0$, we have $x_0, y_0 \in \mathbb{Q}^+ \cup \{0\}$, $x_0, y_0 \leq \gamma$, and $x_0 = \frac{x_0'}{k}, y_0 = \frac{y_0'}{k}$. Thus, $x_0, y_0 \in \mathcal{A}_{\gamma,k}$.

Now suppose that it is true for some j, i.e.,

$$x_j, y_j \in \mathcal{A}_{\gamma,k} \text{ with } x_j = \frac{m_j}{k}, y_j = \frac{n_j}{k} \text{ for some } m_j, n_j \in \mathbb{N} \cup \{0\}. \tag{6}$$

We are in a position to prove that it is true for $j+1$. Since $x_{j+1}, y_{j+1} \in \mathbb{Q}^+ \cup \{0\}$ and $\gamma \geq x_{j+1} \geq y_{j+1} \geq \delta$ by Part (2) of Lemma 3.2, it follows that $f\left(x_j\right), g\left(y_j\right) \in \mathbb{Q}^+ \cup \{0\}$ and $\gamma \geq f\left(x_j\right), g\left(y_j\right) \geq \delta$. Using (6), we have

$$f\left(x_j\right) = 2x_j - \gamma = 2\frac{m_j}{k} - \frac{\gamma'}{k} = \frac{2m_j - \gamma'}{k}$$

and

$$g\left(y_j\right) = 2y_j - \delta = 2\frac{n_j}{k} - \frac{\delta'}{k} = \frac{2n_j - \delta'}{k}$$

for some $m_j, n_j \in \mathbb{N} \cup \{0\}$. Since $x_{j+1} = \max\left(f\left(x_j\right), g\left(y_j\right)\right)$ and $y_{j+1} = \min\left(f\left(x_j\right), g\left(y_j\right)\right)$, it follows that $x_{j+1}, y_{j+1} \in \mathcal{A}_{\gamma,k}$. This completes the proof of the lemma. □

The following lemma plays an important role in the proof of Theorem 4.2.

Lemma 3.4. *Let x_j, y_j be the two sequences as in (4) and (5), respectively with $\gamma \geq x_0 \geq y_0 \geq \delta$ and $x_0 + y_0 = \gamma + \delta$. Then there exist $n, m \in \mathbb{N} \cup \{0\}$ such that $x_n = x_m$ and $n \neq m$.*

Proof. By Lemma 3.1, the cardinality of $\mathcal{A}_{\gamma,k}$ is finite and equals $l_{\gamma,k}$. By Lemma 3.3, there exists $k \in \mathbb{N}$ such that $x_j, y_j \in \mathcal{A}_{\gamma,k}$ for every $j \geq 0$. Thus, it is enough to show that there exist $n, m \in \left\{0, 1, \ldots l_{\gamma,k}\right\}$ such that $x_n = x_m$ and $n \neq m$. The argument is based on the Pigeonhole Principle. Suppose on the contrary that $x_0, x_1, \ldots, x_{l_{\gamma,k}}$ are distinct. Since $\left\{x_0, x_1, \ldots, x_{l_{\gamma,k}}\right\} \subseteq \mathcal{A}_{\gamma,k}$, it follows that $\left|\left\{x_0, x_1, \ldots, x_{l_{\gamma,k}}\right\}\right| \leq l_{\gamma,k}$. This contradicts the fact that $\left|\left\{x_0, x_1, \ldots, x_{l_{\gamma,k}}\right\}\right| = l_{\gamma,k} + 1$. □

4. The inequality tr $A^{\alpha}BA^{\beta}B \leq$ tr $A^{\gamma}BA^{\delta}B$

In this section, we present an alternative proof of the following inequality

$$\text{tr } A^{\alpha}BA^{\beta}B \leq \text{tr } A^{\gamma}BA^{\delta}B,$$

where A is a positive semidefinite matrix, B is a Hermitian matrix and $\alpha, \beta, \gamma, \delta$ are non-negative real numbers for which $\alpha + \beta = \gamma + \delta$ and $\max\{\alpha, \beta\} \leq \max\{\gamma, \delta\}$. We start with the following lemma.

Lemma 4.1. *Let A be a positive semidefinite matrix and B be a Hermitian matrix. Let $\gamma, \delta, x_0, y_0 \in \mathbb{Q}^{+} \cup \{0\}$ with $\gamma \geq x_0 \geq y_0 \geq \delta$ and $x_0 + y_0 = \gamma + \delta$. Let x_j, y_j be the two sequences as in (4) and (5), respectively.Then for $m, j \in \mathbb{N} \cup \{0\}$, we have*

$$\text{tr } A^{x_j}BA^{y_j}B \leq \left(1 - \frac{1}{2^m}\right)\text{tr } A^{\gamma}BA^{\delta}B + \frac{1}{2^m}\text{ tr } A^{x_{j+m}}BA^{y_{j+m}}B.$$

Proof. The result follows by induction on m. First note that for $m = 0$, we are done.

Now suppose that it is true for some m.

We are in a position to prove that it is true for $m+1$. Since $\gamma \geq x_{j+m} \geq y_{j+m} \geq \delta$, $x_{j+m} + y_{j+m} = \gamma + \delta$ by Lemma 3.2, it follows that

$$\begin{aligned}
\text{tr } A^{x_j}BA^{y_j}B &\leq \left(1 - \frac{1}{2^m}\right)\text{tr } A^{\gamma}BA^{\delta}B + \frac{1}{2^m}\text{ tr } A^{x_{j+m}}BA^{y_{j+m}}B \\
&\quad \text{(by the induction assumption)} \\
&\leq \left(1 - \frac{1}{2^m}\right)\text{tr } A^{\gamma}BA^{\delta}B + \frac{1}{2^m}\left(\frac{1}{2}\text{ tr}\left(A^{\gamma}BA^{\delta}B + A^{f(x_{j+m})}BA^{g(y_{j+m})}B\right)\right) \\
&\quad \left(\text{by Lemma 2.2 and taking } \alpha = x_{j+m}, \beta = y_{j+m}\right) \\
&= \left(1 - \frac{1}{2^{m+1}}\right)\text{tr } A^{\gamma}BA^{\delta}B + \frac{1}{2^{m+1}}\text{ tr } A^{f(x_{j+m})}BA^{g(y_{j+m})}B \\
&= \left(1 - \frac{1}{2^{m+1}}\right)\text{tr } A^{\gamma}BA^{\delta}B + \frac{1}{2^{m+1}}\text{ tr } A^{x_{j+m+1}}BA^{y_{j+m+1}}B \\
&\quad \left(\text{if } f\left(x_{j+m}\right) < g\left(y_{j+m}\right), \text{ then we use the cyclicity property of the trace.}\right)
\end{aligned}$$

This completes the proof of the lemma. □

Now, we are ready to state and prove the inequality (2) for the case of rational numbers.

Theorem 4.2. *Let A be a positive semidefinite matrix and B be a Hermitian matrix. Then for $\alpha, \beta, \gamma, \delta \in \mathbb{Q}^{+} \cup \{0\}$ with $\alpha + \beta = \gamma + \delta$ and $\max\{\alpha, \beta\} \leq \max\{\gamma, \delta\}$, we have*

$$\text{tr } A^{\alpha}BA^{\beta}B \leq \text{tr } A^{\gamma}BA^{\delta}B.$$

Proof. Note that $\alpha + \beta = \gamma + \delta$ and $\max\{\alpha, \beta\} \leq \max\{\gamma, \delta\}$ is equivalent to saying that $\gamma \geq \alpha \geq \beta \geq \delta$ if $\gamma \geq \delta$ and $\alpha \geq \beta$. Let $x_0 = \alpha$ and $y_0 = \beta$, where $\alpha, \beta \in \mathbb{Q}^{+} \cup \{0\}$ with $\gamma \geq \alpha \geq \beta \geq \delta$ and $\gamma + \delta = \alpha + \beta$, and let x_j, y_j be the two sequences as in (4) and (5), respectively. Note that using Lemma 3.4, there exist $s, d \in \mathbb{N} \cup \{0\}$ such that $x_s = x_d$ with $s \neq d$ (say $s < d$). Let $k = d - s$. Now applying Lemma 4.1 with $j = 0$ and $m = s$, we get

$$\text{tr } A^{\alpha}BA^{\beta}B \leq \left(1 - \frac{1}{2^s}\right)\text{tr } A^{\gamma}BA^{\delta}B + \frac{1}{2^s}\text{ tr } A^{x_s}BA^{y_s}B. \tag{7}$$

Now using Lemma 4.1 with $j = s$ and $m = k$, we get

$$\text{tr } A^{x_s}BA^{y_s}B \leq \left(1 - \frac{1}{2^k}\right)\text{tr } A^{\gamma}BA^{\delta}B + \frac{1}{2^k}\text{ tr } A^{x_{s+k}}BA^{y_{s+k}}B. \tag{8}$$

Since $x_s = x_{s+k}$, it follows that $y_s = \gamma + \delta - x_s = \gamma + \delta - x_{s+k} = y_{s+k}$. Therefore, using (8), we have

$$\text{tr } A^{x_s}BA^{y_s}B \leq \left(1 - \frac{1}{2^k}\right)\text{tr } A^{\gamma}BA^{\delta}B + \frac{1}{2^k}\text{ tr } A^{x_s}BA^{y_s}B,$$

which is equivalent to saying that

$$\text{tr } A^{x_s}BA^{y_s}B \leq \text{tr } A^{\gamma}BA^{\delta}B. \tag{9}$$

From (7) and (9), we have

$$\text{tr } A^{\alpha}BA^{\beta}B \leq \left(1 - \frac{1}{2^s}\right)\text{tr } A^{\gamma}BA^{\delta}B + \frac{1}{2^s}\text{ tr } A^{\gamma}BA^{\delta}B.$$

This implies that

$$\text{tr } A^{\alpha}BA^{\beta}B \leq \text{tr } A^{\gamma}BA^{\delta}B.$$

This completes the proof of the theorem. □

Now, the inequality (2) is an immediate consequence of Theorem 4.2 as we will see in the following corollary.

Corollary 4.3. *Let A be a positive semidefinite matrix and B be a Hermitian matrix. Then for all non-negative real numbers $\alpha, \beta, \gamma, \delta$ with $\alpha + \beta = \gamma + \delta$ and*

$$\max\{\alpha, \beta\} \leq \max\{\gamma, \delta\},$$

we have

$$\text{tr } A^{\alpha}BA^{\beta}B \leq \text{tr } A^{\gamma}BA^{\delta}B.$$

Proof. Since Theorem 4.2 is true for all non-negative rational numbers, it follows that the result is also true for all non-negative real numbers, and so we are done. Here we use the continuity of the mapping $x \longrightarrow A^x$ and the fact that the set of non-negative rational numbers is dense in the set of non-negative real numbers. □

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