Filomat 38:19 (2024), 6897–6910
https://doi.org/10.2298/FIL2419897S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A bivariate geometric minification integer-valued autoregressive model

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Abstract. This manuscript introduces a new bivariate minification integer-valued autoregressive model of order one. The model is based on a modification of the negative binomial thinning operator. The basic features of the model are given and some of them are used for estimating the parameters. Two methods are used to estimate the unknown parameters: the conditional maximum likelihood method and the conditional least squares method. The characteristics of the estimates obtained using these two methods are checked through some Monte-Carlo simulations.

1. Introduction

Special attention has been paid to minification models in the last forty years. The first minification processes were introduced in 1980 by Tavares (see [8] and [9]). He observed the exponential minification model. The minification process with Weibull marginal distribution is introduced in [7]. The discrete minification model first appeared in [3]. Moreover, Littlejohn introduced the discrete minification model in [4] a year later, and considered cases with different discrete marginal distributions. Later, Aleksić and Ristić introduced a new minification integer-valued autoregressive model in [1] , using a modified negative binomial tinning operator constructed in [13]. They constructed this model to solve the problem arising from the use of the binomial tinning operator and the negative binomial tinning operator. Using these operators to construct the minification models can cause the model to become constant zero over time. Thus, the problem is solved by using the modified negative binomial tinning operator.

At the end of the 20th century, a more active study of bivariate and multivariate minification models began. The bivariate minification process with bivariate semi-Pareto distribution was introduced in [2]. Then, Thomas and Jose introduced multivariate minification processes in [11]. Also, Thomas and Jose constructed the Marshall-Olkin bivariate semi-Pareto AR (1) model with Marshall-Olkin bivariate semi-Pareto marginal distribution in [10]. Ristić considered a class of stationary bivariate minification processes in [5], while Ristić et al. presented a stationary bivariate minification process with Marshall and Olkin exponential distribution in [6].

This manuscript aims to contribute to the development of bivariate minification processes. Motivated by the model introduced in [1], we construct a new bivariate minification model with geometric marginal

2020 *Mathematics Subject Classification*. Primary 62M10.

Keywords. Bivariate integer-valued autoregressive models, Minification models, Modified negative binomial operator, Geometric marginal

Received: 11 October 2023; Revised: 16 December 2023; Accepted: 10 January 2024

Communicated by Aleksandar Nastić

Research supported by the grant of the Ministry of Education, Science and Technological Development of Serbia 451-03-65/2024-03/200124.

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distribution. We need this kind of model when we observe mutually correlated data that represent the realizations of two correlated random variables. The model is based on the modified negative binomial operator, which solves the problem of the model becoming consistently zero over time.

The paper is organized as follows. In Section 2, a new bivariate minification model is constructed and its features are listed. The model is defined and some of its properties, such as transition probabilities, conditional expectation, and conditional variances are presented. The estimates of the unknown parameters are given in Section 3. Two methods are considered, the conditional maximum likelihood method and the conditional least squares method. The effects of the estimates obtained by the mentioned methods are checked by simulations. The manuscript ends with concluding remarks stating what could be done in the future based on the results obtained.

2. Construction of the model and properties

In this section, we construct a new bivariate minification model and list its features. Aleksić and Ristić introduced the geometric minification INAR model with non-negative integer values of the following form $X_t = \min(\alpha \diamond X_{t-1}, \varepsilon_t),\ t \in \mathbb{Z},\ \alpha > 0,$ in [1]. This model is based on the modified negative binomial tinning operator defined as $\alpha \diamond X = \sum_{i=1}^{X+1} G_i$, where all the random variables of counting series $\{G_i\}$ are i.i.d. random variables with geometric marginal distribution $Geom\left(\frac{\alpha}{1+\alpha}\right)$ with pmf of the form $P(G_i = x) = \frac{\alpha^x}{(1+\alpha)^{x+1}},\ x \in \mathbb{N}_0$ and the random variables $\{G_i\}$ and X are mutually independent for all $i \geq 1$. Similarly, we can also consider the operator $\beta\diamond$, which is defined as $\beta \diamond Y = \sum_{i=1}^{Y+1} W_i$, where $\{W_i\}$ are i.i.d. random variables with geometric marginal distributions $Geom\left(\frac{\beta}{1+\beta}\right)$. So, using these two operators, we introduce a new bivariate minification model defined as

$$X_t = \begin{cases} \min(\alpha \diamond X_{t-1}, \varepsilon_t) & w.p. \quad p \\ \min(\alpha \diamond Y_{t-1}, \varepsilon_t), & w.p. \quad 1-p \end{cases} \tag{1}$$

$$Y_t = \begin{cases} \min(\beta \diamond X_{t-1}, \eta_t) & w.p. \quad q \\ \min(\beta \diamond Y_{t-1}, \eta_t), & w.p. \quad 1-q \end{cases}, \tag{2}$$

where:

i) $\alpha, \beta > 0, p, q \in [0, 1]$,

ii) $\{\varepsilon_t, t \in \mathbb{Z}\}$ and $\{\eta_t, t \in \mathbb{Z}\}$ are mutually independent sequences of i.i.d. random variables and ε_t and η_t are independent of X_s and Y_s for all $s < t$,

iii) the counting series incorporated in $\alpha \diamond X_{t-1}$, $\alpha \diamond Y_{t-1}$, $\beta \diamond X_{t-1}$, $\beta \diamond Y_{t-1}$ are mutually independent for all $t \in \mathbb{Z}$,

iv) the counting series contained in $\alpha \diamond X_{t-1}$, $\alpha \diamond Y_{t-1}$, $\beta \diamond X_{t-1}$, $\beta \diamond Y_{t-1}$ are independent of the random variables X_{t-1} and ε_t, Y_{t-1} and ε_t, X_{t-1} and η_t, Y_{t-1} and η_t, respectively, for all $t \in \mathbb{Z}$,

v) for all $t \neq k$, the counting series contained in $\alpha \diamond X_{t-1}$ and $\alpha \diamond X_{k-1}$ are mutually independent, as well as the counting series involved in $\alpha \diamond Y_{t-1}$ and $\alpha \diamond Y_{k-1}$, $\beta \diamond X_{t-1}$ and $\beta \diamond X_{s-1}$, $\beta \diamond Y_{t-1}$ and $\beta \diamond Y_{s-1}$,

vi) the random variables X_{t-l} and ε_t, as well as Y_{t-l} and η_t are independent, for all $l \in \mathbb{N}$ and for all $t \in \mathbb{Z}$.

Same as in [1], we consider a model based on the assumption that the random variables X_t and Y_t have geometric distributions $Geom\left(\frac{\mu}{1+\mu}\right)$, $\mu > 0$.

As we know, the model is completely determined if the distributions of the innovation sequences $\{\varepsilon_t, t \in \mathbb{Z}\}$ and $\{\eta_t, t \in \mathbb{Z}\}$ are given. Therefore, in the first proposition we determine the form of the distributions of the innovation sequences.

Proposition 2.1. *Let $\mu > 0$ and $\{(X_t, Y_t) \in \mathbb{Z}^2\}$ be the marginally stationary bivariate minification model defined in (1) and (2) with geometric marginal distribution $Geom\left(\frac{\mu}{1+\mu}\right)$, $\mu > 0$. Then, the random variables ε_t and η_t have the geometric distributions $Geom\left(\frac{\mu[1+\alpha(1+\mu)]}{\alpha(1+\mu)^2}\right)$ and $Geom\left(\frac{\mu[1+\beta(1+\mu)]}{\beta(1+\mu)^2}\right)$, respectively, if and only if $\alpha > \frac{\mu}{1+\mu}$ and $\beta > \frac{\mu}{1+\mu}$.*

Proof. Let $\{(X_t, Y_t) \in \mathbb{Z}^2\}$ be marginally stationary with geometric marginal distribution $Geom\left(\frac{\mu}{1+\mu}\right)$, $\mu > 0$. Let x be an arbitrary nonnegative integer. Then, from the definition of the model given in (1) and (2), we have that

$$P(X_t \geq x) = p \cdot P(\min(\alpha \diamond X_{t-1}, \varepsilon_t) \geq x) + (1-p) \cdot P(\min(\alpha \diamond Y_{t-1}, \varepsilon_t) \geq x) \tag{3}$$

and

$$P(Y_t \geq x) = q \cdot P(\min(\beta \diamond X_{t-1}, \eta_t) \geq x) + (1-q) \cdot P(\min(\beta \diamond Y_{t-1}, \eta_t) \geq x). \tag{4}$$

As can be seen from the definition of the model, $\alpha \diamond X_{t-1}$ and $\alpha \diamond Y_{t-1}$ are independent of ε_t, just as $\beta \diamond X_{t-1}$ and $\beta \diamond Y_{t-1}$ are independent of η_t. This implies the following

$$P(X_t \geq x) = p \cdot P(\alpha \diamond X_{t-1} \geq x) \cdot P(\varepsilon_t \geq x) + (1-p) \cdot P(\alpha \diamond Y_{t-1}) \geq x) \cdot P(\varepsilon_t \geq x) \tag{5}$$

and

$$P(Y_t \geq x) = q \cdot P(\beta \diamond X_{t-1} \geq x) \cdot P(\eta_t \geq x) + (1-q) \cdot P(\beta \diamond Y_{t-1}) \geq x) \cdot P(\eta_t \geq x). \tag{6}$$

It was shown in [1] that the random variable $\alpha \diamond X_{t-1}$ has the geometric distribution $Geom\left(\frac{\alpha(1+\mu)}{1+\alpha(1+\mu)}\right)$. Similarly, $\alpha \diamond Y_{t-1}$ has the geometric distribution $Geom\left(\frac{\alpha(1+\mu)}{1+\alpha(1+\mu)}\right)$, while $\beta \diamond X_{t-1}$ and $\beta \diamond Y_{t-1}$ have the geometric distribution $Geom\left(\frac{\beta(1+\mu)}{1+\beta(1+\mu)}\right)$. Based on the previous facts, we have that

$$P(X_t \geq x) = \left(\frac{\alpha(1+\mu)}{1+\alpha(1+\mu)}\right)^x \cdot P(\varepsilon_t \geq x) \tag{7}$$

and

$$P(Y_t \geq x) = \left(\frac{\beta(1+\mu)}{1+\beta(1+\mu)}\right)^x \cdot P(\eta_t \geq x). \tag{8}$$

Finally, we obtain

$$P(\varepsilon_t \geq x) = \frac{P(X_t \geq x)}{\left(\frac{\alpha(1+\mu)}{1+\alpha(1+\mu)}\right)^x} = \left(\frac{\mu[1+\alpha(1+\mu)]}{\alpha(1+\mu)^2}\right)^x \tag{9}$$

and

$$P(\eta_t \geq x) = \frac{P(Y_t \geq x)}{\left(\frac{\beta(1+\mu)}{1+\beta(1+\mu)}\right)^x} = \left(\frac{\mu[1+\beta(1+\mu)]}{\beta(1+\mu)^2}\right)^x. \tag{10}$$

In order to define distributions $Geom\left(\frac{\mu[1+\alpha(1+\mu)]}{\alpha(1+\mu)^2}\right)$ and $Geom\left(\frac{\mu[1+\beta(1+\mu)]}{\beta(1+\mu)^2}\right)$ in proper way, the conditions $0 < \frac{\mu[1+\alpha(1+\mu)]}{\alpha(1+\mu)^2} < 1$ and $0 < \frac{\mu[1+\beta(1+\mu)]}{\beta(1+\mu)^2} < 1$ have to be satisfied.

Therefore, we can conclude that ε_t and η_t have the geometric marginal distribution $Geom\left(\frac{\mu[1+\alpha(1+\mu)]}{\alpha(1+\mu)^2}\right)$ and $Geom\left(\frac{\mu[1+\beta(1+\mu)]}{\beta(1+\mu)^2}\right)$, respectively, if and only if $\alpha > \frac{\mu}{1+\mu}$ and $\beta > \frac{\mu}{1+\mu}$.
□

Now, we will show some features of the model that will be used later in unknown parameters estimation. First, let us consider the transition probabilities $P(X_t = x, Y_t = y \mid X_{t-1} = u, Y_{t-1} = v)$, $x, y, u, v \in \mathbb{N}$, that we will use later in the conditional maximum likelihood method in Section 3.

Proposition 2.2. *Let* $\{(X_t, Y_t) \in \mathbb{Z}^2\}$ *be the bivariate minification process defined in (1) and (2) with geometric marginal distribution* $Geom\left(\frac{\mu}{1+\mu}\right)$, $\mu > 0$. *Then,* $\{(X_t, Y_t) \in \mathbb{Z}^2\}$ *has the transition probabilities given by*

$$\begin{aligned} &P(X_t = x, Y_t = y \mid X_{t-1} = u, Y_{t-1} = v) \\ &\qquad = [p \cdot (f(\theta_1, x, u, \alpha) + g(\theta_1, x, u, \alpha)) + (1 - p) \cdot (f(\theta_1, x, v, \alpha) + g(\theta_1, x, v, \alpha))] \\ &\qquad \times [q \cdot (f(\theta_2, y, u, \beta) + g(\theta_2, y, u, \beta)) + (1 - q) \cdot (f(\theta_2, y, v, \beta) + g(\theta_2, y, v, \beta)], \end{aligned} \tag{11}$$

$x, y, u, v \in \mathbb{N}$, *where* $\theta_1 = \frac{\mu[1+\alpha(1+\mu)]}{\alpha(1+\mu)^2}$, $\theta_2 = \frac{\mu[1+\beta(1+\mu)]}{\beta(1+\mu)^2}$, $f(\theta, z, t, a) = \theta^z\binom{z+t}{z}\left(\frac{a}{1+a}\right)^z\left(\frac{1}{1+a}\right)^{t+1}$, $g(\theta, z, t, a) = (1-\theta)\theta^z\left[1 - \sum_{i=0}^{z}\binom{i+t}{i}\left(\frac{a}{1+a}\right)^i\left(\frac{1}{1+a}\right)^{t+1}\right]$.

Proof. Let x, y, u and v be arbitrary non-negative integers. Since X_t and Y_t are conditionally independent for known values of X_{t-1} and Y_{t-1}, the transition probabilities of the model defined by (1) and (2) have the following form:

$$\begin{aligned} &P(X_t = x, Y_t = y \mid X_{t-1} = u, Y_{t-1} = v) \\ &= P(X_t = x, \mid X_{t-1} = u, Y_{t-1} = v) \cdot P(Y_t = y \mid X_{t-1} = u, Y_{t-1} = v) \\ &= [p \cdot P(\min(\alpha \diamond X_{t-1}, \varepsilon_t) = x \mid X_{t-1} = u, Y_{t-1} = v) + (1 - p) \cdot P(\min(\alpha \diamond Y_{t-1}, \varepsilon_t) = x \mid X_{t-1} = u, Y_{t-1} = v)] \\ &\times [q \cdot P(\min(\beta \diamond X_{t-1}, \eta_t) = y \mid X_{t-1} = u, Y_{t-1} = v) + (1 - q) \cdot P(\min(\beta \diamond Y_{t-1}, \eta_t) = y \mid X_{t-1} = u, Y_{t-1} = v)]. \end{aligned} \tag{12}$$

It is proven in [1] that for given $X_{t-1} = u$ and $Y_{t-1} = v$, $\alpha \diamond X_{t-1}$, $\alpha \diamond Y_{t-1}$, $\beta \diamond X_{t-1}$ and $\beta \diamond Y_{t-1}$ have the negative binomial distribution $NB(u+1, \frac{\alpha}{1+\alpha})$, $NB(v+1, \frac{\alpha}{1+\alpha})$, $NB(u+1, \frac{\beta}{1+\beta})$ and $NB(v+1, \frac{\beta}{1+\beta})$, respectively, which, for $\theta_1 = \frac{\mu[1+\alpha(1+\mu)]}{\alpha(1+\mu)^2}$, $\theta_2 = \frac{\mu[1+\beta(1+\mu)]}{\beta(1+\mu)^2}$, implies that

$$\begin{aligned} &P(X_t = x, Y_t = y \mid X_{t-1} = u, Y_{t-1} = v) \\ &= \left\{p\left[\theta_1^x\binom{x+u}{x}\left(\frac{\alpha}{1+\alpha}\right)^x\left(\frac{1}{1+\alpha}\right)^{u+1} + (1-\theta_1)\theta_1^x\left(1 - \sum_{i=0}^{x}\binom{i+u}{i}\left(\frac{\alpha}{1+\alpha}\right)^i\left(\frac{1}{1+\alpha}\right)^{u+1}\right)\right]\right. \\ &\left. + (1-p)\left[\theta_1^x\binom{x+v}{x}\left(\frac{\alpha}{1+\alpha}\right)^x\left(\frac{1}{1+\alpha}\right)^{v+1} + (1-\theta_1)\theta_1^x\left(1 - \sum_{i=0}^{x}\binom{i+v}{i}\left(\frac{\alpha}{1+\alpha}\right)^i\left(\frac{1}{1+\alpha}\right)^{v+1}\right)\right]\right\} \\ &\times \left\{q\left[\theta_2^y\binom{y+u}{y}\left(\frac{\beta}{1+\beta}\right)^y\left(\frac{1}{1+\beta}\right)^{u+1} + (1-\theta_2)\theta_2^y\left(1 - \sum_{i=0}^{y}\binom{i+u}{i}\left(\frac{\beta}{1+\beta}\right)^i\left(\frac{1}{1+\beta}\right)^{u+1}\right)\right]\right. \\ &\left. + (1-q)\left[\theta_2^y\binom{y+v}{y}\left(\frac{\beta}{1+\beta}\right)^y\left(\frac{1}{1+\beta}\right)^{v+1} + (1-\theta_2)\theta_2^y\left(1 - \sum_{i=0}^{y}\binom{i+v}{i}\left(\frac{\beta}{1+\beta}\right)^i\left(\frac{1}{1+\beta}\right)^{v+1}\right)\right]\right\}, \end{aligned} \tag{13}$$

which gives (11). □

For random variables X_t and Y_t defined in (1) and (2) with geometric marginal distribution $Geom\left(\frac{\mu}{1+\mu}\right)$, we have that the expectations and the variances are $E(X_t) = E(Y_t) = \mu$ and $Var(X_t) = Var(Y_t) = \mu(1+\mu)$. In addition, random variables ε_t and η_t with geometric distributions $Geom\left(\frac{\mu[1+\alpha(1+\mu)]}{\alpha(1+\mu)^2}\right)$ and $Geom\left(\frac{\mu[1+\beta(1+\mu)]}{\beta(1+\mu)^2}\right)$ respectively, have the expectations and the variances of the form $E(\varepsilon_t) = \frac{\mu[1+\alpha(1+\mu)]}{\alpha+\alpha\mu-\mu}$, $E(\eta_t) = \frac{\mu[1+\beta(1+\mu)]}{\beta+\beta\mu-\mu}$, $Var(\varepsilon_t) = \frac{\alpha\mu(1+\mu)^2[1+\alpha(1+\mu)]}{(\alpha+\alpha\mu-\mu)^2}$ and $Var(\eta_t) = \frac{\beta\mu(1+\mu)^2[1+\beta(1+\mu)]}{(\beta+\beta\mu-\mu)^2}$.

In the following propositions, we will derive conditional expectation and conditional variance of the random variables X_t and Y_t. First, we will determine the conditional expectation, which we will use later to determine the conditional variance. Also, we will use the conditional expectation to estimate the unknown parameters by the conditional least squares method.

Proposition 2.3. *Let $\{(X_t, Y_t) \in \mathbb{Z}^2\}$ be the bivariate minification model defined in (1) and (2) with the geometric marginal distribution Geom $\left(\frac{\mu}{1+\mu}\right)$, $\mu > 0$. Then, the conditional expectation of the random variables X_t and Y_t , for given X_{t-1} and Y_{t-1}, are of the form*

$$E(X_t \mid X_{t-1}, Y_{t-1}) = \frac{\theta_1}{1-\theta_1}[1 - pA^{1+X_{t-1}} - (1-p)A^{1+Y_{t-1}}] \tag{14}$$

and

$$E(Y_t \mid X_{t-1}, Y_{t-1}) = \frac{\theta_2}{1-\theta_2}[1 - qB^{1+X_{t-1}} - (1-q)B^{1+Y_{t-1}}], \tag{15}$$

for $\theta_1 = \frac{\mu[1+\alpha(1+\mu)]}{\alpha(1+\mu)^2}$, $\theta_2 = \frac{\mu[1+\beta(1+\mu)]}{\beta(1+\mu)^2}$, $A = \frac{1}{1+\alpha-\alpha\theta_1}$ and $B = \frac{1}{1+\beta-\beta\theta_2}$.

Proof. Let X_t be a random variable defined in (1). The conditional expectation of the random variable X_t for given X_{t-1} and Y_{t-1} can be observed as

$$E(X_t \mid X_{t-1}, Y_{t-1}) = E(\min(\alpha \diamond X_{t-1}, \varepsilon_t) \mid X_{t-1}, Y_{t-1}) \cdot p + E(\min(\alpha \diamond Y_{t-1}, \varepsilon_t) \mid X_{t-1}, Y_{t-1}) \cdot (1-p). \tag{16}$$

Using the results presented in [1], we have

$$E(\min(\alpha \diamond X_{t-1}, \varepsilon_t) \mid X_{t-1}, Y_{t-1}) = \frac{\theta_1}{1-\theta_1}\Big[1 - \Big(\frac{1}{1+\alpha-\alpha\theta_1}\Big)^{1+X_{t-1}}\Big]$$

and

$$E(\min(\alpha \diamond Y_{t-1}, \varepsilon_t) \mid X_{t-1}, Y_{t-1}) = \frac{\theta_1}{1-\theta_1}\Big[1 - \Big(\frac{1}{1+\alpha-\alpha\theta_1}\Big)^{1+Y_{t-1}}\Big].$$

From the above, we conclude that

$$E(X_t \mid X_{t-1}, Y_{t-1}) = \frac{\theta_1}{1-\theta_1}\left[1 - p\Big(\frac{1}{1+\alpha-\alpha\theta_1}\Big)^{1+X_{t-1}} - (1-p)\Big(\frac{1}{1+\alpha-\alpha\theta_1}\Big)^{1+Y_{t-1}}\right]. \tag{17}$$

Using the fact that $A = \frac{1}{1+\alpha-\alpha\theta_1}$ we obtain equality (14). Analogously, we can conclude that equality (15) is also valid. □

Based on the previous theorem, we conclude that conditional expectations for X_t and Y_t are not linear functions of X_{t-1} and Y_{t-1}.

The following lemma gives the result needed to derive the conditional variance.

Lemma 2.4. *Let $\{(X_t, Y_t) \in \mathbb{Z}^2\}$ be the bivariate minification model defined in (1) and (2) with geometric marginal distribution Geom $\left(\frac{\mu}{1+\mu}\right)$, $\mu > 0$. Then ,*

$$\begin{aligned} E({X_t}^2 \mid X_{t-1}, Y_{t-1}) = &\frac{\theta_1(1+\theta_1)}{(1-\theta_1)^2}\Big(1 - pA^{1+X_{t-1}} - (1-p)A^{1+Y_{t-1}}\Big) \\ &- \frac{2\alpha{\theta_1}^2}{1-\theta_1}\Big[p(1+X_{t-1})A^{2+X_{t-1}} + (1-p)(1+Y_{t-1})A^{2+Y_{t-1}}\Big] \end{aligned} \tag{18}$$

and

$$\begin{aligned} E({Y_t}^2 \mid X_{t-1}, Y_{t-1}) = &\frac{\theta_2(1+\theta_2)}{(1-\theta_2)^2}\Big(1 - qB^{1+X_{t-1}} - (1-q)B^{1+Y_{t-1}}\Big) \\ &- \frac{2\beta{\theta_2}^2}{1-\theta_2}\Big[q(1+X_{t-1})B^{2+X_{t-1}} + (1-q)(1+Y_{t-1})B^{2+Y_{t-1}}\Big], \end{aligned} \tag{19}$$

for $\theta_1 = \frac{\mu[1+\alpha(1+\mu)]}{\alpha(1+\mu)^2}$, $\theta_2 = \frac{\mu[1+\beta(1+\mu)]}{\beta(1+\mu)^2}$, $A = \frac{1}{1+\alpha-\alpha\theta_1}$ and $B = \frac{1}{1+\beta-\beta\theta_2}$.

Proof. First, we have that

$$E({X_t}^2 \mid X_{t-1}, Y_{t-1}) = pE((\min(\alpha \diamond X_{t-1}, \varepsilon_t))^2) \mid X_{t-1}, Y_{t-1}) + (1-p)E((\min(\alpha \diamond Y_{t-1}, \varepsilon_t))^2) \mid X_{t-1}, Y_{t-1}). \tag{20}$$

From [1] we can conclude that

$$E((\min(\alpha \diamond X_{t-1}, \varepsilon_t))^2) \mid X_{t-1}) = \frac{\theta_1(1+\theta_1)}{(1-\theta_1)^2}\Big(1 - \Big(\frac{1}{1+\alpha-\alpha\theta_1}\Big)^{1+X_{t-1}}\Big) - \frac{2\alpha{\theta_1}^2}{1-\theta_1}(1+X_{t-1})\Big(\frac{1}{1+\alpha-\alpha\theta_1}\Big)^{2+X_{t-1}} \tag{21}$$

and

$$E((\min(\alpha \diamond Y_{t-1}, \varepsilon_t))^2) \mid Y_{t-1}) = \frac{\theta_1(1+\theta_1)}{(1-\theta_1)^2}\Big(1 - \Big(\frac{1}{1+\alpha-\alpha\theta_1}\Big)^{1+Y_{t-1}}\Big) - \frac{2\alpha{\theta_1}^2}{1-\theta_1}(1+Y_{t-1})\Big(\frac{1}{1+\alpha-\alpha\theta_1}\Big)^{2+Y_{t-1}}, \tag{22}$$

for $\theta_1 = \frac{\mu[1+\alpha(1+\mu)]}{\alpha(1+\mu)^2}$, $\theta_2 = \frac{\mu[1+\beta(1+\mu)]}{\beta(1+\mu)^2}$. By replacing (21) and (22) in (20) we have that

$$E({X_t}^2 \mid X_{t-1}, Y_{t-1}) = \frac{\theta_1(1+\theta_1)}{(1-\theta_1)^2}\Big(1 - pA^{1+X_{t-1}} - (1-p)A^{1+Y_{t-1}}\Big) - \frac{2\alpha{\theta_1}^2}{1-\theta_1}\Big[p(1+X_{t-1})A^{2+X_{t-1}} + (1-p)(1+Y_{t-1})A^{2+Y_{t-1}}\Big], \tag{23}$$

for $A = \frac{1}{1+\alpha-\alpha\theta_1}$, by which equality (18) is shown. Similarly, we demonstrate the validity of equality (19). □

The following proposition gives conditional variances of the random variables X_t and Y_t for given X_{t-1} and Y_{t-1}.

Proposition 2.5. *Let $\{(X_t, Y_t) \in \mathbb{Z}^2\}$ be the bivariate minification model defined in (1) and (2) with geometric marginal distribution $Geom\Big(\frac{\mu}{1+\mu}\Big)$, $\mu > 0$. Then, the conditional variances of the random variables X_t and Y_t, for given X_{t-1} and Y_{t-1}, are of the form*

$$Var(X_t \mid X_{t-1}, Y_{t-1}) = \frac{\theta_1}{(1-\theta_1)^2} - \frac{\theta_1}{1-\theta_1}\Big(pA^{1+X_{t-1}} + (1-p)A^{1+Y_{t-1}}\Big) - \frac{2\alpha{\theta_1}^2}{1-\theta_1}\Big(pA^{2+X_{t-1}}(1+X_{t-1}) + (1-p)A^{2+Y_{t-1}}(1+Y_{t-1})\Big) - \frac{{\theta_1}^2}{(1-\theta_1)^2}\Big(p^2A^{2+2X_{t-1}} + 2p(1-p)A^{2+X_{t-1}+Y_{t-1}} + (1-p)^2A^{2+2Y_{t-1}}\Big) \tag{24}$$

and

$$Var(Y_t \mid X_{t-1}, Y_{t-1}) = \frac{\theta_2}{(1-\theta_2)^2} - \frac{\theta_2}{1-\theta_2}\Big(qB^{1+X_{t-1}} + (1-q)B^{1+Y_{t-1}}\Big) - \frac{2\beta{\theta_2}^2}{1-\theta_2}\Big(qB^{2+X_{t-1}}(1+X_{t-1}) + (1-q)B^{2+Y_{t-1}}(1+Y_{t-1})\Big) - \frac{{\theta_2}^2}{(1-\theta_2)^2}\Big(q^2B^{2+2X_{t-1}} + 2q(1-q)B^{2+X_{t-1}+Y_{t-1}} + (1-q)^2B^{2+2Y_{t-1}}\Big), \tag{25}$$

for $\theta_1 = \frac{\mu[1+\alpha(1+\mu)]}{\alpha(1+\mu)^2}$, $\theta_2 = \frac{\mu[1+\beta(1+\mu)]}{\beta(1+\mu)^2}$, $A = \frac{1}{1+\alpha-\alpha\theta_1}$ and $B = \frac{1}{1+\beta-\beta\theta_2}$.

Proof. Let X_t be a random variable defined in (1). The conditional variance of the random variable X_t , for given X_{t-1} and Y_{t-1}, is of the form

$$Var(X_t \mid X_{t-1}, Y_{t-1}) = E({X_t}^2 \mid X_{t-1}, Y_{t-1}) - (E(X_t \mid X_{t-1}, Y_{t-1}))^2. \tag{26}$$

Replacing the (18) and (14) in (26) results in the following equality

$$\begin{aligned} Var(X_t \mid X_{t-1}, Y_{t-1}) = &\frac{\theta_1(1+\theta_1)}{(1-\theta_1)^2}\Big(1 - pA^{1+X_{t-1}} - (1-p)A^{1+Y_{t-1}}\Big) \\ &- \frac{2\alpha{\theta_1}^2}{1-\theta_1}\Big[p(1+X_{t-1})\frac{2\alpha{\theta_1}^2}{1-\theta_1} + (1-p)(1+Y_{t-1})A^{2+Y_{t-1}}\Big] \\ &- \frac{{\theta_1}^2}{1-\theta_1}\Big[1 - 2pA^{1+X_{t-1}} - 2(1-p)A^{1+Y_{t-1}} \\ &+ 2p(1-p)A^{2+X_{t-1}Y_{t-1}} + p^2A^{2+2X_{t-1}} + (1-p)^2A^{2+2Y_{t-1}}\Big]. \end{aligned} \tag{27}$$

By arranging the previous equality, we get the expression (24). The proof of equality (25) is similar. □

From the previous theorem, we obtain the same feature for the conditional variance as for the conditional expectation. Namely, the conditional variances of the random variables X_t and Y_t , for given X_{t-1} and Y_{t-1} , are also non-linear functions of X_{t-1} and Y_{t-1}.

3. Estimation of the parameters

In this section, we will derive the estimators of the unknown parameters μ, α, β, p and q of the bivariate minification model with geometric marginal distribution defined in (1) and (2). We consider two methods for estimating unknown parameters: the conditional maximum likelihood method and the conditional least squares method. Let $(X_1, Y_1), (X_2, Y_2), ..., (X_n, Y_n)$ be a bivariate sample of the bivariate minification model.

Let us first consider the conditional maximum likelihood method. Since the introduced model is a first-order Markov model, we can use the conditional probability given in (11). Thus, for observed values $(x_1, y_1), ..., (x_n, y_n)$, the conditional log-likelihood function $logL(\mu, \alpha, \beta, p, q)$ is given as

$$\begin{aligned} logL(\mu,\alpha,\beta,p,q) &= \sum_{i=2}^{n} logP(X_i = x_i, Y_i = y_i \mid X_{i-1} = x_{i-1}, Y_{i-1} = y_{i-1}) \\ &= \sum_{i=2}^{n} log\Bigg\{\Bigg\{p\Bigg[\theta_1^{x_i}\binom{x_i+x_{i-1}}{x_i}\Big(\frac{\alpha}{1+\alpha}\Big)^{x_i}\Big(\frac{1}{1+\alpha}\Big)^{x_{i-1}+1} + (1-\theta_1)\theta_1^{x_i}\Bigg(1 - \sum_{i=0}^{x_i}\binom{i+x_{i-1}}{i}\Big(\frac{\alpha}{1+\alpha}\Big)^{i}\Big(\frac{1}{1+\alpha}\Big)^{x_{i-1}+1}\Bigg)\Bigg] \\ &+ (1-p)\Bigg[\theta_1^{x_i}\binom{x_i+y_{i-1}}{x_i}\Big(\frac{\alpha}{1+\alpha}\Big)^{x_i}\Big(\frac{1}{1+\alpha}\Big)^{y_{i-1}+1} + (1-\theta_1)\theta_1^{x_i}\Bigg(1 - \sum_{i=0}^{x_i}\binom{i+y_{i-1}}{i}\Big(\frac{\alpha}{1+\alpha}\Big)^{i}\Big(\frac{1}{1+\alpha}\Big)^{y_{i-1}+1}\Bigg)\Bigg]\Bigg\} \\ &\times \Bigg\{q\Bigg[\theta_2^{y_i}\binom{y_i+x_{i-1}}{y_i}\Big(\frac{\beta}{1+\beta}\Big)^{y_i}\Big(\frac{1}{1+\beta}\Big)^{x_{i-1}+1} + (1-\theta_2)\theta_2^{y_i}\Bigg(1 - \sum_{i=0}^{y_i}\binom{i+x_{i-1}}{i}\Big(\frac{\beta}{1+\beta}\Big)^{i}\Big(\frac{1}{1+\beta}\Big)^{x_{i-1}+1}\Bigg)\Bigg] \\ &+ (1-q)\Bigg[\theta_2^{y_i}\binom{y_i+y_{i-1}}{y_i}\Big(\frac{\beta}{1+\beta}\Big)^{y_i}\Big(\frac{1}{1+\beta}\Big)^{y_{i-1}+1} + (1-\theta_2)\theta_2^{y_i}\Bigg(1 - \sum_{i=0}^{y_i}\binom{i+y_{i-1}}{i}\Big(\frac{\beta}{1+\beta}\Big)^{i}\Big(\frac{1}{1+\beta}\Big)^{y_{i-1}+1}\Bigg)\Bigg]\Bigg\}\Bigg\}. \end{aligned} \tag{28}$$

The estimators of unknown parameters cannot be derived analytically. We use the statistical software R to calculate the estimates numerically.

Next, we consider the estimators of the parameters μ, α, β, p and q obtained by the conditional least squares method. Let $Z_t = (X_t, Y_t)$. The estimator of the unknown parameters are determined by minimizing the following sum of squares

$$S(\mu,\alpha,\beta,p,q) = \sum_{t=2}^{n}(Z_t - E(Z_t \mid Z_{t-1}))'(Z_t - E(Z_t \mid Z_{t-1})), \tag{29}$$

which is further equal to

$$\begin{aligned} S(\mu,\alpha,\beta,p,q) &= \sum_{t=2}^{n}\left(\begin{bmatrix} X_t \\ Y_t \end{bmatrix} - \begin{bmatrix} E(X_t \mid X_{t-1}, Y_{t-1}) \\ E(Y_t \mid X_{t-1}, Y_{t-1}) \end{bmatrix}\right)'\left(\begin{bmatrix} X_t \\ Y_t \end{bmatrix} - \begin{bmatrix} E(X_t \mid X_{t-1}, Y_{t-1}) \\ E(Y_t \mid X_{t-1}, Y_{t-1}) \end{bmatrix}\right) \\ &= \sum_{t=2}^{n}\begin{bmatrix} X_t - E(X_t \mid X_{t-1}, Y_{t-1}) & Y_t - E(Y_t \mid X_{t-1}, Y_{t-1}) \end{bmatrix}\begin{bmatrix} X_t - E(X_t \mid X_{t-1}, Y_{t-1}) \\ Y_t - E(Y_t \mid X_{t-1}, Y_{t-1}) \end{bmatrix} \\ &= \sum_{t=2}^{n}(X_t - E(X_t \mid X_{t-1}, Y_{t-1}))^2 + \sum_{t=2}^{n}(Y_t - E(Y_t \mid X_{t-1}, Y_{t-1}))^2. \end{aligned} \tag{30}$$

Replacing (14) and (15) in 30, the function S takes the following form :

$$\begin{aligned} S(\mu,\alpha,\beta,p,q) &= \sum_{t=2}^{n}\left(X_t - \frac{\theta_1}{1-\theta_1}[1 - pA^{1+X_{t-1}} - (1-p)A^{1+Y_{t-1}}]\right)^2 \\ &+ \sum_{t=2}^{n}\left(Y_t - \frac{\theta_2}{1-\theta_2}[1 - qB^{1+X_{t-1}} - (1-q)B^{1+Y_{t-1}}]\right)^2, \end{aligned} \tag{31}$$

for $\theta_1 = \frac{\mu[1+\alpha(1+\mu)]}{\alpha(1+\mu)^2}$, $\theta_2 = \frac{\mu[1+\beta(1+\mu)]}{\beta(1+\mu)^2}$, $A = \frac{1}{1+\alpha-\alpha\theta_1}$ and $B = \frac{1}{1+\beta-\beta\theta_2}$.

The estimates obtained by the conditional least squares method, like those obtained by the previous method, cannot be calculated using analytical solutions; they can only be calculated numerically. We also use statistical software R to find these estimates.

4. Simulations

In this section, we demonstrate the effectiveness of the observed methods for estimating the unknown parameters of the new bivariate minification model defined in (1) and (2). We simulate 100 samples of length 1000 based on equations (1) and (2) and observe subsamples of length 100, 200, 500 and 1000. We generate samples based on the following parameters:
Table 1: a) $\mu = 4.5$, $\alpha = 0.88$, $\beta = 0.9$, $p = 0.5$, $q = 0.45$; b) $\mu = 4.5$, $\alpha = 0.88$, $\beta = 0.9$, $p = 0.8$, $q = 0.8$; c) $\mu = 4.5$, $\alpha = 0.88$, $\beta = 0.9$, $p = 0.2$, $q = 0.2$; d) $\mu = 4.5$, $\alpha = 0.88$, $\beta = 0.9$, $p = 0.2$, $q = 0.8$;
Table 2: a) $\mu = 4.5$, $\alpha = 2.1$, $\beta = 1.8$, $p = 0.5$, $q = 0.45$; b) $\mu = 4.5$, $\alpha = 2.1$, $\beta = 1.8$, $p = 0.8$, $q = 0.8$; c) $\mu = 4.5$, $\alpha = 2.1$, $\beta = 1.8$, $p = 0.2$, $q = 0.2$; d) $\mu = 4.5$, $\alpha = 2.1$, $\beta = 1.8$, $p = 0.2$, $q = 0.8$;
Table 3: a) $\mu = 2$, $\alpha = 0.7$, $\beta = 0.68$, $p = 0.5$, $q = 0.45$; b) $\mu = 2$, $\alpha = 0.7$, $\beta = 0.68$, $p = 0.8$, $q = 0.8$; c) $\mu = 2$, $\alpha = 0.7$, $\beta = 0.68$, $p = 0.2$, $q = 0.2$; d) $\mu = 2$, $\alpha = 0.7$, $\beta = 0.68$, $p = 0.2$, $q = 0.8$;
Table 4: c) $\mu = 2$, $\alpha = 1.55$, $\beta = 1.45$, $p = 0.5$, $q = 0.45$; d) $\mu = 2$, $\alpha = 1.55$, $\beta = 1.45$, $p = 0.8$, $q = 0.8$; c) $\mu = 2$, $\alpha = 1.55$, $\beta = 1.45$, $p = 0.2$, $q = 0.2$; d) $\mu = 2$, $\alpha = 1.55$, $\beta = 1.45$, $p = 0.2$, $q = 0.8$.

We do not choose parameters randomly, because the goal is to display as many different cases as possible with the appropriate selection of parameters. We choose parameters in order to check the behavior of the methods when series are with relatively large and relatively small values. Based on the previous discussion, for the value of the parameter μ we take $\mu = 4.5$ and $\mu = 2$. Table 1 and Table 2 show the results when $\mu = 4.5$, while Table 3 and Table 4 show the results for $\mu = 2$.

a) $\mu = 4.5$, $\alpha = 0.88$, $\beta = 0.9$, $p = 0.5$ and $q = 0.45$										
n	$\widehat{\mu}_{CML}$	$\widehat{\alpha}_{CML}$	$\widehat{\beta}_{CML}$	$\widehat{p}_{CML}$	$\widehat{q}_{CML}$	$\widehat{\mu}_{CLS}$	$\widehat{\alpha}_{CLS}$	$\widehat{\beta}_{CLS}$	$\widehat{p}_{CLS}$	$\widehat{q}_{CLS}$
100	3.7809	0.8814	0.9039	0.4821	0.4430	4.1550	1.0333	1.0938	0.4710	0.4349
	(0.9975)	(0.1321)	(0.1549)	(0.1087)	(0.1314)	(1.0247)	(0.3466)	(0.4325)	(0.1589)	(0.1590)
200	4.0412	0.8666	0.8919	0.4885	0.4471	4.2186	0.9463	0.9583	0.4896	0.4434
	(0.9441)	(0.0791)	(0.0956)	(0.0743)	(0.0838)	(0.8467)	(0.1870)	(0.2097)	(0.1080)	(0.1128)
500	4.3115	0.8761	0.8993	0.4952	0.4445	4.2480	0.9169	0.9272	0.4856	0.4436
	(0.5929)	(0.0482)	(0.0514)	(0.0390)	(0.0480)	(0.6444)	(0.1397)	(0.1318)	(0.0638)	(0.0726)
1000	4.3135	0.8733	0.8951	0.4981	0.4472	4.2072	0.9003	0.9015	0.4875	0.4492
	(0.4647)	(0.0401)	(0.0423)	(0.0335)	(0.0387)	(0.6171)	(0.1160)	(0.1050)	(0.0531)	(0.0569)
b) $\mu = 4.5$, $\alpha = 0.88$, $\beta = 0.9$, $p = 0.8$ and $q = 0.8$										
n	$\widehat{\mu}_{CML}$	$\widehat{\alpha}_{CML}$	$\widehat{\beta}_{CML}$	$\widehat{p}_{CML}$	$\widehat{q}_{CML}$	$\widehat{\mu}_{CLS}$	$\widehat{\alpha}_{CLS}$	$\widehat{\beta}_{CLS}$	$\widehat{p}_{CLS}$	$\widehat{q}_{CLS}$
100	3.9356	0.8760	0.8938	0.8184	0.8065	4.2052	1.0452	0.9236	0.8237	0.8426
	(1.0537)	(0.1143)	(0.1161)	(0.1457)	(0.1221)	(1.1576)	(0.3042)	(0.2728)	(0.1555)	(0.1412)
200	4.2686	0.8827	0.8944	0.8090	0.8007	4.2413	0.9877	0.8952	0.8221	0.8234
	(0.8247)	(0.0759)	(0.0838)	(0.0741)	(0.0812)	(0.7977)	(0.1917)	(0.1902)	(0.1086)	(0.1053)
500	4.4634	0.8863	0.8981	0.8071	0.8069	4.2671	0.9245	0.8770	0.8140	0.8209
	(0.4787)	(0.0421)	(0.0442)	(0.0522)	(0.0530)	(0.5601)	(0.1207)	(0.1305)	(0.0706)	(0.0709)
1000	4.4548	0.8811	0.8992	0.8025	0.8053	4.3143	0.8979	0.8967	0.8082	0.8180
	(0.3404)	(0.0296)	(0.0357)	(0.0324)	(0.0362)	(0.5046)	(0.0830)	(0.0966)	(0.0507)	(0.0546)
c) $\mu = 4.5$, $\alpha = 0.88$, $\beta = 0.9$, $p = 0.2$ and $q = 0.2$										
n	$\widehat{\mu}_{CML}$	$\widehat{\alpha}_{CML}$	$\widehat{\beta}_{CML}$	$\widehat{p}_{CML}$	$\widehat{q}_{CML}$	$\widehat{\mu}_{CLS}$	$\widehat{\alpha}_{CLS}$	$\widehat{\beta}_{CLS}$	$\widehat{p}_{CLS}$	$\widehat{q}_{CLS}$
100	4.0634	0.8862	0.9255	0.1962	0.1575	4.2082	0.9243	1.0512	0.1729	0.1548
	(1.0517)	(0.1045)	(0.2140)	(0.1091)	(0.1488)	(1.0502)	(0.2716)	(0.3129)	(0.1316)	(0.1511)
200	4.2460	0.8832	0.9041	0.1963	0.1844	4.1779	0.9110	0.9962	0.1757	0.1660
	(0.7874)	(0.0807)	(0.0794)	(0.0811)	(0.0819)	(0.7628)	(0.1929)	(0.2063)	(0.1047)	(0.1032)
500	4.4344	0.8823	0.9020	0.2024	0.1913	4.3325	0.8951	0.9362	0.1843	0.1765
	(0.5642)	(0.0434)	(0.0458)	(0.0440)	(0.0447)	(0.6352)	(0.1249)	(0.1254)	(0.0668)	(0.0599)
1000	4.4994	0.8817	0.9069	0.2019	0.1979	4.3798	0.8848	0.9277	0.1908	0.1917
	(0.3489)	(0.0291)	(0.0311)	(0.0321)	(0.0283)	(0.5585)	(0.0927)	(0.1006)	(0.0502)	(0.0475)
d) $\mu = 4.5$, $\alpha = 0.88$, $\beta = 0.9$, $p = 0.2$ and $q = 0.8$										
n	$\widehat{\mu}_{CML}$	$\widehat{\alpha}_{CML}$	$\widehat{\beta}_{CML}$	$\widehat{p}_{CML}$	$\widehat{q}_{CML}$	$\widehat{\mu}_{CLS}$	$\widehat{\alpha}_{CLS}$	$\widehat{\beta}_{CLS}$	$\widehat{p}_{CLS}$	$\widehat{q}_{CLS}$
100	3.9249	0.8815	0.9352	0.1851	0.8086	4.2388	0.9869	1.0349	0.1753	0.8372
	(0.9969)	(0.1083)	(0.1596)	(0.1141)	(0.1211)	(0.9653)	(0.2738)	(0.3151)	(0.1240)	(0.1228)
200	4.0320	0.8638	0.8998	0.2033	0.7934	4.4661	0.9641	0.9831	0.1813	0.8209
	(0.9223)	(0.0899)	(0.1193)	(0.0710)	(0.0842	(0.8358)	(0.1913)	(0.2439)	(0.1029)	(0.0952)
500	4.2366	0.8715	0.8936	0.2001	0.7978	4.3338	0.9219	0.9401	0.1871	0.8168
	(0.6533)	(0.0565)	(0.0607)	(0.0443)	(0.0436	(0.5744)	(0.1259)	(0.1286)	(0.0673)	(0.0527)
1000	4.3357	0.8738	0.9035	0.2014	0.7987	4.2598	0.8871	0.9042	0.1912	0.8091
	(0.5063)	(0.0430)	(0.0461)	(0.0353)	(0.0363	(0.4964)	(0.0988)	(0.0998)	(0.0542)	(0.0453)

Table 1: Estimates for different cases of true values of the parameters obtained using the conditional maximum likelihood method and the conditional least squares method

a) $\mu = 4.5, \alpha = 2.1, \beta = 1.8, p = 0.5$ and $q = 0.45$										
n	$\widehat{\mu}_{CML}$	$\widehat{\alpha}_{CML}$	$\widehat{\beta}_{CML}$	$\widehat{p}_{CML}$	$\widehat{q}_{CML}$	$\widehat{\mu}_{CLS}$	$\widehat{\alpha}_{CLS}$	$\widehat{\beta}_{CLS}$	$\widehat{p}_{CLS}$	$\widehat{q}_{CLS}$
100	4.4996	2.4761	1.7742	0.4667	0.4402	4.4442	2.4941	1.9936	0.4791	0.4310
	(0.5684)	(2.4616)	(0.4203)	(0.2960)	(0.1658)	(0.5647)	(1.5727)	(0.8908)	(0.2267)	(0.1863)
200	4.4571	2.1553	1.8040	0.4815	0.4539	4.4169	2.2747	1.8755	0.4829	0.4580
	(0.3941)	(0.4239)	(0.2784)	(0.1116)	(0.1113)	(0.4201)	(0.8261)	(0.5075)	(0.1400)	(0.1405)
500	4.4652	2.1688	1.7931	0.4932	0.4447	4.4591	2.2495	1.8693	0.4890	0.4413
	(0.2628)	(0.2531)	(0.1870)	(0.0770)	(0.0606)	(0.2718)	(0.4731)	(0.4049)	(0.1007)	(0.0789)
1000	4.4897	2.1181	1.7821	0.4985	0.4447	4.4812	2.1665	1.8481	0.5009	0.4468
	(0.1896)	(0.1716)	(0.1214)	(0.0518)	(0.0427)	(0.1933)	(0.4572)	(0.2955)	(0.0637)	(0.0544)
b) $\mu = 4.5, \alpha = 2.1, \beta = 1.8, p = 0.8$ and $q = 0.8$										
n	$\widehat{\mu}_{CML}$	$\widehat{\alpha}_{CML}$	$\widehat{\beta}_{CML}$	$\widehat{p}_{CML}$	$\widehat{q}_{CML}$	$\widehat{\mu}_{CLS}$	$\widehat{\alpha}_{CLS}$	$\widehat{\beta}_{CLS}$	$\widehat{p}_{CLS}$	$\widehat{q}_{CLS}$
100	4.5029	2.4827	1.9255	0.8315	0.8263	4.4384	2.5995	1.9798	0.7932	0.8302
	(0.5730)	(1.5224)	(0.5522)	(0.3626)	(0.1724)	(0.6241)	(1.3790)	(0.8661)	(0.2682)	(0.1716)
200	4.5446	2.1574	1.8366	0.8000	0.8092	4.5187	2.2823	1.8568	0.7951	0.8054
	(0.3942)	(0.4297)	(0.3281)	(0.1210)	(0.1198)	(0.4174)	(0.9266)	(0.5257)	(0.1528)	(0.1309)
500	4.5190	2.1191	1.8161	0.8001	0.7940	4.5052	2.1812	1.7950	0.8042	0.7944
	(0.2191)	(0.2317)	(0.1847)	(0.0718)	(0.0731)	(0.2316)	(0.4667)	(0.3361)	(0.0961)	(0.0849)
1000	4.4831	2.1257	1.7998	0.8017	0.7964	4.4845	2.1687	1.7742	0.7981	0.7977
	(0.1868)	(0.2001)	(0.1101)	(0.0575)	(0.0472)	(0.1910)	(0.2785)	(0.2298)	(0.0683)	(0.0584)
c) $\mu = 4.5, \alpha = 2.1, \beta = 1.8, p = 0.2$ and $q = 0.2$										
n	$\widehat{\mu}_{CML}$	$\widehat{\alpha}_{CML}$	$\widehat{\beta}_{CML}$	$\widehat{p}_{CML}$	$\widehat{q}_{CML}$	$\widehat{\mu}_{CLS}$	$\widehat{\alpha}_{CLS}$	$\widehat{\beta}_{CLS}$	$\widehat{p}_{CLS}$	$\widehat{q}_{CLS}$
100	4.4162	2.3442	1.9733	0.1165	0.1348	4.3782	4.0544	2.2307	0.0887	0.1541
	(0.5817)	(0.7397)	(0.7115)	(0.3145)	(0.2240)	(0.6160)	(13.0328)	(1.1650)	(0.4058)	(0.2280)
200	4.4541	2.1571	1.8844	0.1839	0.1704	4.4241	2.2944	1.9559	0.1830	0.1826
	(0.3818)	(0.3699)	(0.359)	(0.1369)	(0.1192)	(0.3992)	(0.8274)	(0.5578)	(0.1443)	(0.1327)
500	4.4650	2.1251	1.8528	0.1926	0.1832	4.4511	2.1930	1.8898	0.1925	0.1786
	(0.2509)	(0.2433)	(0.1996)	(0.0744)	(0.0683)	(0.2512)	(0.6626)	(0.3519)	(0.1028)	(0.0892)
1000	4.4661	2.1104	1.8073	0.1985	0.1919	4.4520	2.1099	1.8114	0.2014	0.1909
	(0.1734)	(0.1675)	(0.1364)	(0.0510)	(0.0446)	(0.1815)	(0.3015)	(0.2443)	(0.0642)	(0.0601)
d) $\mu = 4.5, \alpha = 2.1, \beta = 1.8, p = 0.2$ and $q = 0.8$										
n	$\widehat{\mu}_{CML}$	$\widehat{\alpha}_{CML}$	$\widehat{\beta}_{CML}$	$\widehat{p}_{CML}$	$\widehat{q}_{CML}$	$\widehat{\mu}_{CLS}$	$\widehat{\alpha}_{CLS}$	$\widehat{\beta}_{CLS}$	$\widehat{p}_{CLS}$	$\widehat{q}_{CLS}$
100	4.5110	2.3653	2.1221	0.1300	0.8724	4.4855	2.6287	2.1809	0.1526	0.8535
	(0.5351)	(1.2689)	(2.3449)	(0.3245)	(0.3543)	(0.5443)	(2.2505)	(1.3486)	(0.3311)	(0.1757)
200	4.4930	2.1528	1.8693	0.1771	0.8235	4.4814	2.3608	1.9918	0.1624	0.8249
	(0.3597)	(0.4400)	(0.3150)	(0.1179)	(0.0929)	(0.3653)	(0.8732)	(0.6896)	(0.1517)	(0.1220)
500	4.5338	2.1111	1.8377	0.2065	0.8051	4.5189	2.1825	1.8799	0.1974	0.8151
	(0.2429)	(0.2868)	(0.1798)	(0.0697)	(0.0509)	(0.2498)	(0.6240)	(0.3055)	(0.1056)	(0.0651)
1000	4.5038	2.0928	1.8090	0.2030	0.7995	4.4915	2.1375	1.7957	0.2030	0.8010
	(0.1714)	(0.1456)	(0.1221)	(0.0424)	(0.0395)	(0.1758)	(0.3465)	(0.2418)	(0.0595)	(0.0484)

Table 2: Estimates for different cases of true values of the parameters obtained using the conditional maximum likelihood method and the conditional least squares method

a) $\mu = 2$, $\alpha = 0.7$, $\beta = 0.68$, $p = 0.5$ and $q = 0.45$

n	$\widehat{\mu}_{CML}$	$\widehat{\alpha}_{CML}$	$\widehat{\beta}_{CML}$	$\widehat{p}_{CML}$	$\widehat{q}_{CML}$	$\widehat{\mu}_{CLS}$	$\widehat{\alpha}_{CLS}$	$\widehat{\beta}_{CLS}$	$\widehat{p}_{CLS}$	$\widehat{q}_{CLS}$
100	2.0003	0.7934	0.7884	0.4964	0.4663	2.0198	0.8539	0.8451	0.4652	0.4576
	(0.3877)	(0.2322)	(0.2165)	(0.1455)	(0.1401)	(0.6067)	(0.4222)	(0.4065)	(0.1692)	(0.1862)
200	2.0485	0.7342	0.7318	0.4937	0.4538	1.9755	0.7897	0.7677	0.4616	0.4450
	(0.3016)	(0.0888)	(0.0973)	(0.0860)	(0.1047)	(0.2942)	(0.2471)	(0.2155)	(0.1110)	(0.1330)
500	2.0262	0.7138	0.7092	0.4945	0.4463	1.9735	0.7521	0.7295	0.4792	0.4407
	(0.2005)	(0.0502)	(0.0523)	(0.0661)	(0.0612)	(0.1770)	(0.1390)	(0.1363)	(0.0770)	(0.0809)
1000	2.0472	0.7141	0.6918	0.4921	0.4475	2.002	0.7381	0.7077	0.4834	0.4428
	(0.1826)	(0.0421)	(0.0286)	(0.0425)	(0.0450)	(0.1241)	(0.1028)	(0.0958)	(0.0572)	(0.0580)

b) $\mu = 2$, $\alpha = 0.7$, $\beta = 0.68$, $p = 0.8$ and $q = 0.8$

n	$\widehat{\mu}_{CML}$	$\widehat{\alpha}_{CML}$	$\widehat{\beta}_{CML}$	$\widehat{p}_{CML}$	$\widehat{q}_{CML}$	$\widehat{\mu}_{CLS}$	$\widehat{\alpha}_{CLS}$	$\widehat{\beta}_{CLS}$	$\widehat{p}_{CLS}$	$\widehat{q}_{CLS}$
100	1.9332	0.8231	0.7881	0.8666	0.8620	1.9916	0.9406	0.8139	0.8473	0.8630
	(0.4046)	(0.4771)	(0.2216)	(0.3173)	(0.1756)	(0.6364)	(0.6750)	(0.4729)	(0.1980)	(0.1843)
200	1.9369	0.7190	0.7185	0.8131	0.8260	1.8888	0.8152	0.6758	0.8201	0.8270
	(0.3182)	(0.1039)	(0.0879)	(0.0983)	(0.1028)	(0.3395)	(0.2687)	(0.2154)	(0.1264)	(0.1081)
500	1.9766	0.7112	0.6939	0.8051	0.8195	1.9691	0.7611	0.6740	0.8063	0.8171
	(0.1682)	(0.0514)	(0.0475)	(0.0650)	(0.0530)	(0.1928)	(0.1317)	(0.1237)	(0.0791)	(0.0685)
1000	2.0000	0.7077	0.6913	0.8006	0.8133	1.9861	0.7277	0.6897	0.8055	0.8113
	(0.1457)	(0.0407)	(0.0310)	(0.0448)	(0.0435)	(0.1310)	(0.0858)	(0.0808)	(0.0541)	(0.0518)

c) $\mu = 2$, $\alpha = 0.7$, $\beta = 0.68$, $p = 0.2$ and $q = 0.2$

n	$\widehat{\mu}_{CML}$	$\widehat{\alpha}_{CML}$	$\widehat{\beta}_{CML}$	$\widehat{p}_{CML}$	$\widehat{q}_{CML}$	$\widehat{\mu}_{CLS}$	$\widehat{\alpha}_{CLS}$	$\widehat{\beta}_{CLS}$	$\widehat{p}_{CLS}$	$\widehat{q}_{CLS}$
00	2.1466	0.8217	0.7746	0.1609	0.1480	2.0915	0.8062	0.8448	0.1509	0.1410
	(0.4583)	(0.2288)	(0.1953)	(0.2245)	(0.1497)	(1.1920)	(0.3865)	(0.4333)	(0.1818)	(0.1754)
200	2.1729	0.7687	0.7307	0.2036	0.1863	1.9843	0.7778	0.7651	0.1788	0.1823
	(0.3858)	(0.1316)	(0.0678)	(0.1176)	(0.0899)	(0.3156)	(0.2454)	(0.1964)	(0.1308)	(0.1232)
500	2.1510	0.7323	0.7151	0.2060	0.1925	2.0001	0.7318	0.7249	0.1904	0.1929
	(0.2747)	(0.0630)	(0.0484)	(0.0617)	(0.0550)	(0.1798)	(0.1334)	(0.1223)	(0.0661)	(0.0746)
1000	2.1130	0.7206	0.7017	0.2012	0.1946	2.0057	0.7296	0.7056	0.1908	0.1962
	(0.2218)	(0.0374)	(0.0360)	(0.0435)	(0.0406)	(0.1364)	(0.1021)	(0.0930)	(0.0474)	(0.0550)

d) $\mu = 2$, $\alpha = 0.7$, $\beta = 0.68$, $p = 0.2$ and $q = 0.8$

n	$\widehat{\mu}_{CML}$	$\widehat{\alpha}_{CML}$	$\widehat{\beta}_{CML}$	$\widehat{p}_{CML}$	$\widehat{q}_{CML}$	$\widehat{\mu}_{CLS}$	$\widehat{\alpha}_{CLS}$	$\widehat{\beta}_{CLS}$	$\widehat{p}_{CLS}$	$\widehat{q}_{CLS}$
100	1.9683	0.8303	0.7661	0.1515	0.8177	1.9604	0.9286	0.8789	0.1361	0.8069
	(0.3906)	(0.2367)	(0.1806)	(0.1456)	(0.1395)	(0.5372)	(0.4762)	(0.4113)	(0.1295)	(0.1613)
200	1.9990	0.7731	0.7326	0.1770	0.8091	1.9322	0.8108	0.7813	0.1685	0.8067
	(0.3115)	(0.1256)	(0.1011)	(0.0792)	(0.0820)	(0.3424)	(0.2530)	(0.2477)	(0.0969)	(0.1006)
500	2.0299	0.7262	0.7040	0.1941	0.8036	1.9691	0.7389	0.7170	0.1885	0.8062
	(0.2123)	(0.0726)	(0.0479)	(0.0490)	(0.0468)	(0.1887)	(0.1400)	(0.1198)	(0.0679)	(0.0684)
1000	2.0742	0.7150	0.7003	0.2005	0.8035	1.9966	0.7293	0.7110	0.1963	0.8069
	(0.1886)	(0.0423)	(0.0298)	(0.0383)	(0.0323)	(0.1437)	(0.0948)	(0.0902)	(0.0487)	(0.0441)

Table 3: Estimates for different cases of true values of the parameters obtained using the conditional maximum likelihood method and the conditional least squares method

a) $\mu = 2$, $\alpha = 1.55$, $\beta = 1.45$, $p = 0.5$ and $q = 0.45$										
n	$\widehat{\mu}_{CML}$	$\widehat{\alpha}_{CML}$	$\widehat{\beta}_{CML}$	$\widehat{p}_{CML}$	$\widehat{q}_{CML}$	$\widehat{\mu}_{CLS}$	$\widehat{\alpha}_{CLS}$	$\widehat{\beta}_{CLS}$	$\widehat{p}_{CLS}$	$\widehat{q}_{CLS}$
100	2.0036	1.8048	2.1343	0.4593	0.4925	1.9798	2.1395	2.4074	0.4349	0.4901
	(0.2329)	(0.9395)	(3.9253)	(0.2604)	(0.2665)	(0.2295)	(1.7311)	(3.6484)	(0.2695)	(0.3036)
200	1.9794	1.6669	1.7035	0.5046	0.4516	1.9624	1.6913	1.9148	0.4819	0.4545
	(0.1669)	(0.6425)	(0.8250)	(0.1662)	(0.1325)	(0.1679)	(0.8421)	(1.1688)	(0.1799)	(0.1705)
500	1.9965	1.5562	1.5432	0.5106	0.4543	1.9894	1.5701	1.5562	0.5008	0.4508
	(0.0947)	(0.2843)	(0.2633)	(0.0847)	(0.0782)	(0.0980)	(0.4263)	(0.3763)	(0.0933)	(0.0891)
1000	2.0008	1.5512	1.5086	0.5094	0.4496	1.9968	1.5778	1.5216	0.5008	0.4487
	(0.0707)	(0.2074)	(0.1738)	(0.0569)	(0.0566)	(0.0735)	(0.3220)	(0.2701)	(0.0678)	(0.0695)
b) $\mu = 2$, $\alpha = 1.55$, $\beta = 1.45$, $p = 0.8$ and $q = 0.8$										
n	$\widehat{\mu}_{CML}$	$\widehat{\alpha}_{CML}$	$\widehat{\beta}_{CML}$	$\widehat{p}_{CML}$	$\widehat{q}_{CML}$	$\widehat{\mu}_{CLS}$	$\widehat{\alpha}_{CLS}$	$\widehat{\beta}_{CLS}$	$\widehat{p}_{CLS}$	$\widehat{q}_{CLS}$
100	2.0137	1.8566	2.0294	0.8529	0.8343	1.9769	2.2310	1.8214	0.8688	0.8242
	(0.2803)	(0.8454)	(2.8367)	(0.2927)	(0.3542)	(0.2943)	(1.5556)	(1.4830)	(0.3441)	(0.2721)
200	2.0080	1.6468	1.5407	0.8263	0.7963	1.9897	1.8065	1.5569	0.8370	0.7971
	(0.1837)	(0.5127)	(0.4542)	(0.1689)	(0.1457)	(0.1823)	(0.7773)	(0.5996)	(0.1900)	(0.1402)
500	2.0208	1.5577	1.5183	0.7988	0.8000	2.0081	1.6342	1.5256	0.8047	0.8055
	(0.1101)	(0.2547)	(0.2695)	(0.0754)	(0.0745)	(0.1137)	(0.4160)	(0.4202)	(0.0946)	(0.0914)
1000	2.0137	1.5402	1.4729	0.7921	0.7990	2.0071	1.5826	1.4885	0.7925	0.8026
	(0.0729)	(0.1729)	(0.1732)	(0.0585)	(0.0591)	(0.0731)	(0.3057)	(0.2912)	(0.0789)	(0.0726)
c) $\mu = 2$, $\alpha = 1.55$, $\beta = 1.45$, $p = 0.2$ and $q = 0.2$										
n	$\widehat{\mu}_{CML}$	$\widehat{\alpha}_{CML}$	$\widehat{\beta}_{CML}$	$\widehat{p}_{CML}$	$\widehat{q}_{CML}$	$\widehat{\mu}_{CLS}$	$\widehat{\alpha}_{CLS}$	$\widehat{\beta}_{CLS}$	$\widehat{p}_{CLS}$	$\widehat{q}_{CLS}$
100	1.9537	1.8928	1.8643	0.1663	0.1556	1.9436	2.6808	2.105	0.0971	0.1601
	(0.2893)	(1.1360)	(1.0949)	(0.2924)	(0.3006)	(0.3099)	(6.4100)	(1.4172)	(0.8390)	(0.2459)
200	1.9699	1.7271	1.5233	0.1635	0.2049	1.9577	1.7824	1.7669	0.1634	0.1922
	(0.2084)	(0.7133)	(0.4296)	(0.2354)	(0.1379)	(0.2136)	(0.9255)	(0.8487)	(0.1882)	(0.1511)
500	1.9811	1.5331	1.4719	0.2036	0.1955	1.9780	1.5514	1.5566	0.2023	0.1892
	(0.1210)	(0.2627)	(0.2542)	(0.0836)	(0.0779)	(0.1259)	(0.3705)	(0.4238)	(0.0891)	(0.0912)
1000	1.9903	1.5402	1.4679	0.1994	0.1918	1.9886	1.5582	1.5060	0.2007	0.1856
	(0.0776)	(0.1808)	(0.1896)	(0.0653)	(0.0645)	(0.0790)	(0.2708)	(0.2491)	(0.0694)	(0.0755)
d) $\mu = 2$, $\alpha = 1.55$, $\beta = 1.45$, $p = 0.2$ and $q = 0.8$										
n	$\widehat{\mu}_{CML}$	$\widehat{\alpha}_{CML}$	$\widehat{\beta}_{CML}$	$\widehat{p}_{CML}$	$\widehat{q}_{CML}$	$\widehat{\mu}_{CLS}$	$\widehat{\alpha}_{CLS}$	$\widehat{\beta}_{CLS}$	$\widehat{p}_{CLS}$	$\widehat{q}_{CLS}$
100	1.9981	9.9492	2.0570	0.6249	0.9482	1.9803	5.9522	1.9492	0.2329	0.8626
	(0.2321)	(70.9957)	(3.1647)	(8.6664)	(0.7300)	(0.2415)	(35.1240)	(1.9149)	(1.6462)	(0.3405)
200	1.9929	1.7254	1.5742	0.1635	0.8183	1.9875	1.7345	1.6515	0.1696	0.8233
	(0.1749)	(0.4795)	(0.4969)	(0.1589)	(0.1439)	(0.1861)	(0.6525)	(0.6447)	(0.1626)	(0.1557)
500	2.0003	1.6013	1.4826	0.1919	0.8048	1.9983	1.6562	1.4878	0.1858	0.7991
	(0.1123)	(0.2485)	(0.2557)	(0.0734)	(0.0885)	(0.1163)	(0.3998)	(0.3898)	(0.1093)	(0.1038)
1000	2.0057	1.5788	1.465	0.1951	0.8006	2.0045	1.6070	1.4653	0.1929	0.7985
	(0.0754)	(0.1975)	(0.1686)	(0.0557)	(0.0592)	(0.0797)	(0.2868)	(0.2652)	(0.0709)	(0.0676)

Table 4: Estimates for different cases of true values of the parameters obtained using the conditional maximum likelihood method and the conditional least squares method

The parameters α and β belong to the interval $(\frac{\mu}{1+\mu}, \infty)$, and therefore, we choose values that are close to the lower limit and values that are distant from the boundary. For values close to the limit, we use $\alpha = 0.88; \beta = 0.9$ (Table 1) and $\alpha = 0.7; \beta = 0.68$ (Table 3), while for values that are not close to the limit, we use $\alpha = 2.1; \beta = 1.8$ (Table 2) and $\alpha = 1.55; \beta = 1.45$ (Table 4). We choose values for the parameters p and q to cover four different cases. First, we consider the case when p=0.5 and q=0.45. Then both, X_{t-1} and Y_{t-1} equally often participate in the construction of sequences X_t and Y_t. The second scenario is where X_{t-1} is the more dominant branch in constructing X_t and Y_t. This happens when the values for p and q are large and we will observe here the case p=q=0.8. Contrary to the second case, we have the third case where, e.g., p=q=0.2 and then Y_{t-1} participates much more often in the formation of the X_t and Y_t series. Finally, we consider the case when p is small and q is large. Then, more often X_t is formed on the basis of Y_{t-1} and Y_t on the basis of X_{t-1}. For the values of the parameters p and q we will here take p=0.2 and q=0.8. In this way, we have covered all the interesting cases of construction of the process in the sense of cross-correlation inference.

Based on the data from Table 1, Table 2, Table 3 and Table 4 we can conclude that both methods give solid estimates with small standard deviations, except for some cases where the sample size is small, which can happen for such a sample size. For sample lengths of 100 and 200, we cannot decide which of the methods is more suitable. When the sample lengths are 500 and 1000, we observe a difference in the standard deviation of the estimations of the parameters α, β, p and q, in favor of the CML methods. Finally, we can conclude that as the sample size increases, the estimated values converge towards the real values in both methods.

5. Concluding remarks

In this manuscript, a new bivariate minification model based on a modified negative binomial tinning operator was presented. Transition probabilities, conditional expectations and conditional variances of the model were observed. The estimates were obtained using two parameter estimation methods: the conditional maximum likelihood method and the conditional least squares method. The effectiveness of both methods was proven on simulated data.

For future research, observing four different tinning operators instead of two would be interesting, as was done in this manuscript.

Acknowledgements

The author acknowledge the grant of the Ministry of Education, Science and Technological Development of Serbia 451-03-65/2024-03/200124 for carrying out this research.

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