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Necessary and sufficient conditions of Schur m-power convexity of a new mixed mean

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Abstract. In the paper, the authors construct a new class of mixed multi-variable means in terms of the arithmetic, geometric, and harmonic means, and determine necessary and sufficient conditions for the mixed two-variable mean to be Schur m-power convex.

1. Preliminaries

We first recall definitions of the majorization, the Schur convexity, and the Schur m-power convexity.

Definition 1 ([9, 23]). For $\ell \ge 2$, let

$$\boldsymbol{s} = (s_1, s_2, \dots, s_\ell) \in \mathbb{R}^\ell \quad \text{and} \quad \boldsymbol{t} = (t_1, t_2, \dots, t_\ell) \in \mathbb{R}^\ell$$

be two ℓ-tuples.

1. The ℓ-tuple $\boldsymbol{s}$ is said to be majorized by $\boldsymbol{t}$, denoted by $\boldsymbol{s} \prec \boldsymbol{t}$, if

$$\sum_{i=1}^{k} s_{[i]} \le \sum_{i=1}^{k} t_{[i]} \quad \text{and} \quad \sum_{i=1}^{\ell} s_i = \sum_{i=1}^{\ell} t_i$$

for $1 \le k \le \ell - 1$, where

$$s_{[1]} \ge s_{[2]} \ge \dots \ge s_{[\ell]} \quad \text{and} \quad t_{[1]} \ge t_{[2]} \ge \dots \ge t_{[\ell]}$$

are rearrangements of $\boldsymbol{s}$ and $\boldsymbol{t}$ in descending order.

2. A set $\Omega \subseteq \mathbb{R}^\ell$ is called to be convex if

$$(\lambda s_1 + \mu t_1, \lambda s_2 + \mu t_2, \dots, \lambda s_\ell + \mu t_\ell) \in \Omega$$

for all $\boldsymbol{s}$ and $\boldsymbol{t} \in \Omega$, where $\lambda, \mu \in [0, 1]$ with $\lambda + \mu = 1$.

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3. A function $\varphi : \Omega \to \mathbb{R}$ is said to be Schur-convex if the majorizing relation $\boldsymbol{s} \prec \boldsymbol{t}$ on Ω implies the inequality $\varphi(\boldsymbol{s}) \le \varphi(\boldsymbol{t})$. If the majorizing relation $\boldsymbol{s} \prec \boldsymbol{t}$ on Ω implies the inequality $\varphi(\boldsymbol{s}) \ge \varphi(\boldsymbol{t})$, then we say that the function $\varphi : \Omega \to \mathbb{R}$ is Schur-concave.

Definition 2 ([31–33])**.** Let $f : \mathbb{R} \times (0, \infty) \to \mathbb{R}$ be defined by

$$f(m,s) = \begin{cases} \dfrac{s^m - 1}{m}, & m \neq 0; \\ \ln s, & m = 0. \end{cases} \tag{1.1}$$

A function $\varphi : \Omega \subseteq (0, \infty)^\ell \to \mathbb{R}$ is said to be Schur m-power convex on Ω if the majorizing relation

$$f(m, \boldsymbol{s}) = (f(m,s_1), f(m,s_2), \dots, f(m,s_\ell)) \prec f(m,\boldsymbol{t}) = (f(m,t_1), f(m,t_2), \dots, f(m,t_\ell)) \tag{1.2}$$

on Ω implies the inequality $\varphi(\boldsymbol{s}) \le \varphi(\boldsymbol{t})$. If the relation (1.2) on Ω implies the inequality $\varphi(\boldsymbol{s}) \ge \varphi(\boldsymbol{t})$, then we say that the function $\varphi : \Omega \subseteq (0, \infty)^\ell \to \mathbb{R}$ is Schur m-power concave on Ω.

Remark 1. The function $f(m, s)$ defined in (1.1) can be reformulated as

$$f(m,s) = \int_1^s u^{m-1} \,\mathrm{d}\, u. \tag{1.3}$$

This function and its reciprocal have been being systematically investigated and extensively applied from the late 1990s to current. The first two papers dedicating to initially studying of the function $f(m, s)$ are [18, 19] and the latest three papers relating to this function are [1, 13]. This function has been applied in the theory of mean values, analytic number theory, and differential geometry (see [6–8, 14, 16, 21] and a number of closely-related references therein).

Remark 2. The definitions of the Schur-convexity (see [9, 23]), Schur-geometric convexity (see [4, 37]), and Schur-harmonic convexity (see [2, 28, 29]) correspond to $f(1, s) = s - 1$, $f(0, s) = \ln s$, and $f(-1, s) = 1 - \frac{1}{s}$ in Definition 2 respectively.

For $\ell \ge 2$ and $q \in \mathbb{R}$, let

$$\boldsymbol{s} = (s_1, s_2, \dots, s_\ell) \quad \text{and} \quad \boldsymbol{s}^q = (s_1^q, s_2^q, \dots, s_\ell^q).$$

When $s_i > 0$ for $1 \le i \le \ell$, by virtue of the arithmetic, harmonic, and geometric means $A_\ell(\boldsymbol{s})$, $H_\ell(\boldsymbol{s})$, and $G_\ell(\boldsymbol{s})$, we define a new mixed mean

$$B_\ell(\boldsymbol{s}; \omega; q) = \begin{cases} \left[\dfrac{A_\ell(\boldsymbol{s}^q) + \omega H_\ell(\boldsymbol{s}^q)}{1 + \omega}\right]^{1/q}, & q \neq 0 \\ G_\ell(\boldsymbol{s}), & q = 0 \end{cases} \tag{1.4}$$

for $\omega \ge 0$. For $\ell = 2$, the mean $B_\ell(\boldsymbol{s}; \omega; q)$ in (1.4) can be formulated as

$$B_2(\alpha, \beta; \omega; q) = \begin{cases} \left[\dfrac{A(\alpha^q, \beta^q) + \omega H(\alpha^q, \beta^q)}{1 + \omega}\right]^{1/q}, & q \neq 0 \\ G(\alpha, \beta), & q = 0 \end{cases} \tag{1.5}$$

for $(\alpha, \beta) \in (0, \infty)^2$, $\omega \in [0, \infty)$, and $q \in \mathbb{R}$.

In this paper, we will determine necessary and sufficient conditions on (m, q, ω) for the mixed mean $B_2(\alpha, \beta; \omega; q)$ to be Schur m-power convex (or Schur m-power concave, respectively) with respect to $(\alpha, \beta) \in (0, \infty)^2$.

2. Lemmas

We need the following lemmas.

Lemma 1 ([31–33]). *Let $\Omega \subset (0,\infty)^\ell$ be a symmetric set with nonempty interior Ω° and let $\varphi : \Omega \to (0,\infty)$ be continuous and symmetric on Ω and differentiable on Ω°. Then φ is Schur m-power convex on Ω if and only if*

$$\frac{s_1^m - s_2^m}{m}\left[s_1^{1-m}\frac{\partial\varphi(\boldsymbol{s})}{\partial s_1} - s_2^{1-m}\frac{\partial\varphi(\boldsymbol{s})}{\partial s_2}\right] \ge 0, \quad m \ne 0 \tag{2.1}$$

and

$$(\ln s_1 - \ln s_2)\left[s_1\frac{\partial\varphi(\boldsymbol{s})}{\partial s_1} - s_2\frac{\partial\varphi(\boldsymbol{s})}{\partial s_2}\right] \ge 0, \quad m = 0 \tag{2.2}$$

for $\boldsymbol{s} \in \Omega^\circ$.

Remark 3. If letting $m = 1, 0, -1$ in Lemma 1 respectively, then we deduce criteria theorems for the Schur-convexity (see [9, 23]), the Schur-geometric convexity (see [4, 37]), and the Schur-harmonic convexity (see [2, 28, 29]) respectively.

Remark 4. Basing on the integral representation (1.3), we unify (2.1) and (2.2) as

$$\left[\frac{1}{s_1^{m-1}}\frac{\partial\varphi(\boldsymbol{s})}{\partial s_1} - \frac{1}{s_2^{m-1}}\frac{\partial\varphi(\boldsymbol{s})}{\partial s_2}\right]\int_{s_2}^{s_1} u^{m-1}\,\mathrm{d}\,u \ge 0, \quad m \in \mathbb{R}. \tag{2.3}$$

Remark 5. An anonymous referee pointed out that the inequalities (2.1) and (2.2) were equivalently written in [24, Remark 2.7] as

$$(s_1 - s_2)\left[s_1^{1-m}\frac{\partial\varphi(\boldsymbol{s})}{\partial s_1} - s_2^{1-m}\frac{\partial\varphi(\boldsymbol{s})}{\partial s_2}\right] \ge 0, \quad m \in \mathbb{R}. \tag{2.4}$$

The different expressions in Lemma 1, (2.3), and (2.4) are all useful and can not be replaced by each other.

Lemma 2. *Let $r \in (-1,1)$ and*

$$V_r(x) = \frac{1}{(x+1)^2}\frac{x^{r+1}-1}{x^{r-1}-1}, \quad x \in (0,1). \tag{2.5}$$

Then the function $V_r(x)$ is decreasing in $x \in (0,1)$ for $r \in (-1,1)$ and the double inequality

$$\frac{r+1}{4(r-1)} < V_r(x) < 0$$

is valid for $x \in (0,1)$ and $r \in (-1,1)$.

Proof. A direct differentiation gives

$$V_r'(x) = \frac{h_r(x)}{x^r(x+1)^3(1-x^{1-r})^2},$$

where

$$h_r(x) = 2x^{1+r} - 2x^{2-r} + (1-r)x^3 - (1+r)x^2 + (r+1)x + r - 1. \tag{2.6}$$

When $r \in (-1,0] \cup \left(\frac{1}{2},1\right)$, we have

$$h_r'(x) = 2(1+r)x^r - 2(2-r)x^{1-r} + 3(1-r)x^2 - 2(1+r)x + r + 1,$$

$$\begin{aligned} h_r''(x) &= 2r(1+r)x^{r-1} - 2(1-r)(2-r)x^{-r} + 6(1-r)x - 2(1+r), \\ h_r^{(3)}(x) &= 2r(1-r)\Big[(2-r)x^{-r-1} - (1+r)x^{r-2}\Big] + 6(1-r), \\ h_r^{(4)}(x) &= 2r(1-r)(1+r)(2-r)x^{-3}\Big(x^r - x^{1-r}\Big) \\ &\le 0. \end{aligned}$$

Hence, the third derivative $h_r^{(3)}(x)$ is decreasing in $x \in (0,1)$ and

$$h_r^{(3)}(x) \ge \lim_{x\to1^-} h_r^{(3)}(x) \ge 0, \quad x \in (0,1).$$

Thus, the second derivative $h_r''(x)$ is increasing in $x \in (0,1)$. From $\lim_{x\to1^-} h_r''(x) = 0$, we deduce $h_r''(x) < 0$ on $(0,1)$. Accordingly, the first derivative $h_r'(x)$ is decreasing on $(0,1)$. From $\lim_{x\to1^-} h_r'(x) = 0$, it follows that $h_r'(x) > 0$ on $(0,1)$. Therefore, the function $h_r(x)$ is increasing in $x \in (0,1)$. As a result, we acquire that $h_r(x) \le \lim_{x\to1^-} h_r(x) = 0$ for $x \in (0,1)$.

When $r \in \left(0, \frac{1}{2}\right]$, let $\phi_r(x) = (1-2r)(1-x) - x^r + x^{1-r}$ for $x \in (0,1]$. It is easy to see that

$$\phi_r'(x) = -(1-2r) - rx^{r-1} + (1-r)x^{-r} \quad \text{and} \quad \phi_r''(x) = r(1-r)x^{-r-1}\Big(x^{2r-1} - 1\Big) \ge 0.$$

Then $\phi_r(x)$ is decreasing in $x \in (0,1]$ and $\phi_r(x) \ge \phi_r(1) = 0$ for $x \in (0,1]$. Combining this with the function in (2.6) leads to

$$\begin{aligned} h_r(x) &= 2x^{1+r} - 2x^{2-r} + (1-r)x^3 - (1+r)x^2 + (r+1)x + r - 1 \\ &\le 2(1-2r)x(1-x) + (1-r)x^3 - (1+r)x^2 + (r+1)x + r - 1 \\ &= (1-r)(x-1)^3 \\ &\le 0 \end{aligned}$$

for $x \in (0,1)$.

In a word, the derivative $V_r'(x)$ is negative and the function $V_r(x)$ is decreasing in $x \in (0,1)$, with the limits

$$\lim_{x\to0^+} V_r(x) = 0 \quad \text{and} \quad \lim_{x\to1^-} V_r(x) = \frac{r+1}{4(r-1)}.$$

The proof of Lemma 2 is thus complete. □

Lemma 3. *For $\omega \ge 0$ and $r \in \mathbb{R}$, define*

$$F_{r,\omega}(x) = (x+1)^2\Big(1 - x^{1-r}\Big) + 4\omega x^{1-r}\Big(x^{r+1} - 1\Big), \quad x \in (0,1]. \tag{2.7}$$

Then $F_{r,\omega}(x) \ge 0$ in $x \in (0,1]$ if and only if

$$(r,\omega) \in \left\{(r,\omega) : r < 1, \frac{r+1}{r-1}\omega \ge -1, \omega \ge 0\right\},$$

while $F_{r,\omega}(x) \le 0$ in $x \in (0,1]$ if and only if $(r,\omega) \in \{(r,\omega) : r \ge 1, \omega \ge 0\}$.

Proof. It is easy to show that $F_{r,\omega}(1) = 0$ for $\omega \ge 0$ and $r \in \mathbb{R}$ and

$$F_{r,\omega}(x) \begin{cases} \ge 0, & r \le -1 \\ \le 0, & r \ge 1 \end{cases}$$

for $x \in (0,1]$ and $\omega \ge 0$.

When $r \in (-1,1)$, we can write the function $F_{r,\omega}(x)$ in (2.7) as

$$F_{r,\omega}(x) = (x+1)^2\Big(1 - x^{1-r}\Big)[1 + 4\omega V_r(x)], \quad x \in (0,1),$$

where $V_r(x)$ is defined as in (2.5). Applying Lemma 2 reveals that $F_{r,\omega}(x) \ge 0$ for $x \in (0,1)$ if and only if $1 + 4\omega V_r(x) \ge 1 + \omega\frac{r+1}{r-1} \ge 0$ for $x \in (0,1)$. If $\omega > 0$, by the limit $\lim_{x\to0^+} V_r(x) = 0$, the negativity $F_{r,\omega}(x) < 0$ is not necessarily true for $x \in (0,1)$. The proof of Lemma 3 is thus complete. □

3. Necessary and sufficient conditions

Our main results are the following necessary and sufficient conditions.

Theorem 1. *For fixed $q \in \mathbb{R}$ and $\omega \ge 0$, the mixed mean $B_2(\alpha,\beta;\omega;q)$ is Schur 0-power convex with respect to $(\alpha,\beta) \in (0,\infty)^2$ if and only if*

$$(q,\omega) \in E_1 = \{(q,\omega) : q = 0, \omega \ge 0\} \cup \{(q,\omega) : q > 0, 0 \le \omega \le 1\},$$

while it is Schur 0-power concave with respect to $(\alpha,\beta) \in (0,\infty)^2$ if and only if

$$(q,\omega) \in E_2 = \{(q,\omega) : q = 0, \omega \ge 0\} \cup \{(q,\omega) : q < 0, 0 \le \omega \le 1\}.$$

Proof. When $q = 0$, by the definition in (1.5), we have $B_2(\alpha,\beta;\omega;q) = G(\alpha,\beta)$ for $(\alpha,\beta) \in (0,\infty)^2$. Using Lemma 1 gives the trivial result

$$(\ln\alpha - \ln\beta)\left[\alpha\frac{\partial G(\alpha,\beta)}{\partial\alpha} - \beta\frac{\partial G(\alpha,\beta)}{\partial\beta}\right] = 0.$$

When $q \ne 0$, considering the expression (1.5) and differentiating yield

$$\frac{\partial B_2(\alpha,\beta;\omega;q)}{\partial\alpha} = \left[\alpha^{q-1}(\alpha^q+\beta^q)^2 + 4\omega\alpha^{q-1}\beta^{2q}\right]\mathfrak{B}(\alpha,\beta;\omega;q) \tag{3.1}$$

and

$$\frac{\partial B_2(\alpha,\beta;\omega;q)}{\partial\beta} = \left[\beta^{q-1}(\alpha^q+\beta^q)^2 + 4\omega\alpha^{2q}\beta^{q-1}\right]\mathfrak{B}(\alpha,\beta;\omega;q), \tag{3.2}$$

where

$$\mathfrak{B}(\alpha,\beta;\omega;q) = \frac{[B_2(\alpha,\beta;\omega;q)]^{1-q}}{2(1+\omega)(\alpha^q+\beta^q)^2}. \tag{3.3}$$

By the partial derivatives in (3.1) and (3.2) and in light of Lemma 1, we acquire

$$\begin{aligned}\Theta_1(\alpha,\beta;\omega;q) &= (\ln\alpha - \ln\beta)\left[\alpha\frac{\partial B_2(\alpha,\beta;\omega;q)}{\partial\alpha} - \beta\frac{\partial B_2(\alpha,\beta;\omega;q)}{\partial\beta}\right]\\ &= (\ln\alpha - \ln\beta)(\alpha^q-\beta^q)\left[(\alpha^q+\beta^q)^2 - 4\omega\alpha^q\beta^q\right]\mathfrak{B}(\alpha,\beta;\omega;q).\end{aligned}$$

When $q > 0$, we assume $\alpha \ge \beta > 0$ and put $x = \left(\frac{\beta}{\alpha}\right)^q$. Then $0 < x \le 1$ and

$$\Theta_1(\alpha,\beta;\omega;q) = (\ln\alpha - \ln\beta)(\alpha^q-\beta^q)\beta^{2q}\left[(x+1)^2 - 4\omega x\right]\mathfrak{B}(\alpha,\beta;\omega;q) \ge 0$$

for $\omega \le 1$. When $\omega > 1$, we define $Q_\omega(x) = (x+1)^2 - 4\omega x$ for $x \in (0,1]$. It is easy to show that

$$Q_\omega\left(\frac{1}{4\omega}\right) = \left(\frac{1}{4\omega}+1\right)^2 - 1 > 0 \quad\text{and}\quad Q_\omega\left(\frac{1}{\omega}\right) = \left(\frac{1}{\omega}+1\right)^2 - 4 < 0.$$

This means that, for $\omega > 1$, the sign of $\Theta_1(\alpha,\beta;\omega;q)$ does not keep the same.

When $q < 0$, by similar arguments to the above, we obtain

$$\Theta_1(\alpha,\beta;\omega;q) = (\ln\alpha - \ln\beta)(\alpha^q-\beta^q)\beta^{2q}Q_\omega(x)\mathfrak{B}(\alpha,\beta;\omega;q) \le 0, \quad \omega \le 1.$$

Meanwhile, for $\omega > 1$, the sign of $\Theta_1(\alpha,\beta;\omega;q)$ does not keep the same too.

In conclusion, the mixed mean $B_2(\alpha,\beta;\omega;q)$ is Schur 0-power convex (or Schur 0-power concave, respectively) with respect to $(\alpha,\beta) \in (0,\infty)^2$ if and only if $(q,\omega) \in E_1$ (or $(q,\omega) \in E_2$, respectively). The proof of Theorem 1 is finished. □

Theorem 2. *For $q \in \mathbb{R}$, $m \neq 0$, and $\omega \geq 0$, the mixed mean $B_2(\alpha, \beta; \omega; q)$ is Schur m-power convex (or Schur m-power concave, respectively) with respect to $(\alpha, \beta) \in (0, \infty)^2$ if and only if $(m, q, \omega) \in S_1$ (or $(m, q, \omega) \in S_2$, respectively), where*

$$S_1 = \left\{(m, q, \omega) : q > 0, m < q, \omega \geq 0, \frac{q+m}{q-m}\omega \geq -1\right\} \cup \{(m, q, \omega) : m \leq q \leq 0, \omega \geq 0\}$$

and

$$S_2 = \left\{(m, q, \omega) : q < 0, m > q, \omega \geq 0, \frac{q+m}{q-m}\omega \leq -1\right\} \\ \cup \{(m, q, \omega) : m \geq q \geq 0, \omega \geq 0\} \cup \{(m, q, \omega) : m = -q > 0, \omega \geq 0\}.$$

Proof. When $q = 0$, using the definition $B_2(\alpha, \beta; \omega; q) = G(\alpha, \beta)$ for $(\alpha, \beta) \in (0, \infty)^2$ and Lemma 1, we deduce

$$\frac{\alpha^m - \beta^m}{m}\left[\alpha\frac{\partial G(\alpha, \beta)}{\partial \alpha} - \beta\frac{\partial G(\alpha, \beta)}{\partial \beta}\right] = \frac{\alpha^m - \beta^m}{2m}(\alpha^{-m} - \beta^{-m})\begin{cases} \leq 0, & m > 0, \\ \geq 0, & m < 0. \end{cases}$$

When $q \neq 0$, by the partial derivatives in (3.1) and (3.2), we arrive at

$$\begin{aligned} \Theta_2(\alpha, \beta; \omega, q; m) &= \frac{\alpha^m - \beta^m}{m}\left[\alpha^{1-m}\frac{\partial B_2(\alpha, \beta; \omega; q)}{\partial \alpha} - \beta^{1-m}\frac{\partial B_2(\alpha, \beta; \omega; q)}{\partial \beta}\right] \\ &= \frac{\alpha^m - \beta^m}{m}\left[(\alpha^q + \beta^q)^2(\alpha^{q-m} - \beta^{q-m}) + 4\omega(\alpha\beta)^{q-m}(\beta^{q+m} - \alpha^{q+m})\right]\mathfrak{B}(\alpha, \beta; \omega; q), \end{aligned} \tag{3.4}$$

where $\mathfrak{B}(\alpha, \beta; \omega; q)$ is defined in (3.3).

In what follows, without loss of generality, we assume $\alpha > \beta > 0$ and $r = \frac{m}{q}$. Then $\frac{\alpha^m - \beta^m}{m} \geq 0$.

When $q > 0$, let $x = \left(\frac{\beta}{\alpha}\right)^q$. Then $0 < x \leq 1$ and, by virtue of the equation (3.4),

$$\Theta_2(\alpha, \beta; \omega; q; m) = \frac{(\alpha^m - \beta^m)\alpha^{2q-m}}{m}\mathfrak{B}(\alpha, \beta; \omega; q)\left[(x+1)^2\left(1 - x^{1-r}\right) + 4\omega x^{1-r}\left(x^{r+1} - 1\right)\right].$$

Making use of Lemma 3 reveals that,

1. the non-negativity $\Theta_2(\alpha, \beta; \omega; q; m) \geq 0$ for $(\alpha, \beta) \in (0, \infty)^2$ is true if and only if

$$\begin{aligned} (m, q, \omega) \in S_{11} &= \left\{(r, \omega) : r < 1, \omega \geq 0, \frac{r+1}{r-1}\omega \geq -1\right\} \\ &= \left\{(m, q, \omega) : q > 0, m < q, \omega \geq 0, \frac{q+m}{q-m}\omega \geq -1\right\}; \end{aligned}$$

2. the non-positivity $\Theta_2(\alpha, \beta; \omega; q; m) \leq 0$ for $(\alpha, \beta) \in (0, \infty)^2$ is valid if and only if

$$(m, q, \omega) \in S_{21} = \{(r, \omega) : r \geq 1, \omega \geq 0\} = \{(m, q, \omega) : m \geq q > 0, \omega \geq 0\}.$$

When $q < 0$, let $y = \left(\frac{\alpha}{\beta}\right)^q$. Then $0 < y \leq 1$ and, by virtue of the equality (3.4),

$$\Theta_2(\alpha, \beta; \omega, q; m) = -\frac{(\alpha^m - \beta^m)\beta^{2q-m}}{m}\mathfrak{B}(\alpha, \beta; \omega; q)\left[(y+1)^2\left(1 - y^{1-r}\right) + 4\omega y^{1-r}\left(y^{r+1} - 1\right)\right].$$

By Lemma 3, we reveal that,

1. the non-negativity $\Theta_2(\alpha, \beta; \omega; q; m) \geq 0$ for $(\alpha, \beta) \in (0, \infty)^2$ holds if and only if

$$(m, q, \omega) \in S_{12} = \{(r, \omega) : r \geq 1, \omega \geq 0\} = \{(m, q, \omega) : m \leq q < 0, \omega \geq 0\};$$

2. the non-positivity $\Theta_2(\alpha,\beta;\omega;q;m)\le 0$ for $(\alpha,\beta)\in(0,\infty)^2$ is valid if and only if

$$(m,q,\omega)\in S_{22}=\left\{(r,\omega): r<1,\omega\ge 0,\frac{r+1}{r-1}\omega\ge -1\right\}$$
$$=\left\{(m,q,\omega): q<0, m>q,\omega\ge 0,\frac{q+m}{q-m}\omega\le -1\right\}.$$

Consequently, we conclude that,

1. the non-negativity $\Theta_2(\alpha,\beta;\omega;q;m)\ge 0$ for $(\alpha,\beta)\in(0,\infty)^2$ validates if and only if

$$(m,q,\omega)\in S_1=S_{11}\cup S_{12}\cup\{(m,q,\omega): m\le q=0,\omega\ge 0\};$$

2. the non-positivity $\Theta_2(\alpha,\beta;\omega;q;m)\le 0$ for $(\alpha,\beta)\in(0,\infty)^2$ holds if and only if

$$(m,q,\omega)\in S_2=S_{21}\cup S_{22}\cup\{(m,q,\omega): m\ge q=0,\omega\ge 0\}.$$

The proof of Theorem 2 is complete. □

4. Conclusions

In this paper, our authors constructed the new mixed mean $B_\ell(\mathbf{s};\omega;q)$ in (1.4) and determined in Theorems 1 and 2 necessary and sufficient conditions for the mixed mean $B_2(\alpha,\beta;\omega;q)$ in (1.5) to be Schur m-convex or Schur m-convex for $m\in\mathbb{R}$.

The integral representations (1.3) and (2.3) in Remarks 1 and 4 are new observations which base on our experienced research since the late 1990s.

It seems that the investigations of various Schur convexities of mathematical means such as the extended mean values $E(r,s;\alpha,\beta)$ started off from the early 2000s in the preprints [11, 15], which were later formally published in [12, 16] respectively, while the second author of this paper was visiting the Research Group in Mathematical Inequalities and Applications (RGMIA) by invitation and partly financial support from Professor Dr. Sever Silvestru Dragomir at the Victoria University of Technology in Australia. This visit was the first time that the second author of this paper went abroad and this visit was also practically, physically, and financially supported by Professor Shi-Ying Yuan, the President of the Jiaozuo Institute of Technology in China.

The researches of various Schur convexities of mathematical means by the ideas, methods, tools, and techniques in the theory of majorization have been attracting more and more mathematicians and have been producing a number of literature such as [2–5, 10, 17, 20–22, 25–36], for example.

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References

[1] J. Cao, J. L. López-Bonilla, and F. Qi, *Three identities and a determinantal formula for differences between Bernoulli polynomials and numbers*, Electron. Res. Arch. **32** (2024), no. 1, 224–240; available online at `https://doi.org/10.3934/era.2024011`.
[2] Y.-M. Chu, G.-D. Wang, and X.-H. Zhang, *The Schur multiplicative and harmonic convexities of the complete symmetric function*, Math. Nachr. **284** (2011), no. 5-6, 653–663; available online at `https://doi.org/10.1002/mana.200810197`.
[3] Y. Chu and X. Zhang, *Necessary and sufficient conditions such that extended mean values are Schur-convex or Schur-concave*, J. Math. Kyoto Univ. **48** (2008), no. 1, 229–238; available online at `https://doi.org/10.1215/kjm/1250280982`.
[4] Y. Chu, X. Zhang, and G. Wang, *The Schur geometrical convexity of the extended mean values*, J. Convex Anal. **15** (2008), no. 4, 707–718.
[5] L.-L. Fu, B.-Y. Xi, and H. M. Srivastava, *Schur-convexity of the generalized Heronian means involving two positive numbers*, Taiwanese J. Math. **15** (2011), no. 6, 2721–2731; available online at `https://doi.org/10.11650/twjm/1500406493`.
[6] B.-N. Guo and F. Qi, *A simple proof of logarithmic convexity of extended mean values*, Numer. Algorithms **52** (2009), no. 1, 89–92; available online at `https://doi.org/10.1007/s11075-008-9259-7`.
[7] B.-N. Guo and F. Qi, *The function* $(b^x-a^x)/x$*: Logarithmic convexity and applications to extended mean values*, Filomat **25** (2011), no. 4, 63–73; available online at `https://doi.org/10.2298/FIL1104063G`.

[8] Y.-F. Li, D. Lim, and F. Qi, *Closed-form formulas, determinantal expressions, recursive relations, power series, and special values of several functions used in Clark–Ismail's two conjectures*, Appl. Comput. Math. **22** (2023), no. 4, 443–465; available online at https://doi.org/10.30546/1683-6154.22.4.2023.443.

[9] A. W. Marshall, I. Olkin, and B. C. Arnold, *Inequalities: Theory of Majorization and its Applications*, 2nd Ed., Springer Verlag, New York/Dordrecht/Heidelberg/London, 2011; available online at https://doi.org/10.1007/978-0-387-68276-1.

[10] K. Murali and K. M. Nagaraja, *Schur convexity of Stolarsky's extended mean values*, J. Math. Inequal. **10** (2016), no. 3, 725–735; available online at https://doi.org/10.7153/jmi-10-59.

[11] F. Qi, *A note on Schur-convexity of extended mean values*, RGMIA Res. Rep. Coll. **4** (2001), no. 4, Art. 4, 529–533; available online at http://rgmia.org/v4n4.php.

[12] F. Qi, *A note on Schur-convexity of extended mean values*, Rocky Mountain J. Math. **35** (2005), no. 5, 1787–1793; available online at https://doi.org/10.1216/rmjm/1181069663.

[13] F. Qi and R. P. Agarwal, *Several functions originating from Fisher–Rao geometry of Dirichlet distributions and involving polygamma functions*, Mathematics **12** (2024), no. 1, Art. 44, 21 pages; available online at https://doi.org/10.3390/math12010044.

[14] F. Qi and R. J. Chapman, *Two closed forms for the Bernoulli polynomials*, J. Number Theory **159** (2016), 89–100; available online at https://doi.org/10.1016/j.jnt.2015.07.021.

[15] F. Qi, J. Sándor, S. S. Dragomir, and A. Sofo, *Notes on the Schur-convexity of the extended mean values*, RGMIA Res. Rep. Coll. **5** (2002), no. 1, Art. 3, 19–27; available online at http://rgmia.vu.edu.au/v5n1.html.

[16] F. Qi, J. Sándor, S. S. Dragomir, and A. Sofo, *Notes on the Schur-convexity of the extended mean values*, Taiwanese J. Math. **9** (2005), no. 3, 411–420; available online at https://doi.org/10.11650/twjm/1500407849.

[17] F. Qi, X.-T. Shi, M. Mahmoud, and F.-F. Liu, *Schur-convexity of the Catalan–Qi function related to the Catalan numbers*, Tbilisi Math. J. **9** (2016), no. 2, 141–150; available online at https://doi.org/10.1515/tmj-2016-0026.

[18] F. Qi and S.-L. Xu, *Refinements and extensions of an inequality, II*, J. Math. Anal. Appl. **211** (1997), no. 2, 616–620; available online at https://doi.org/10.1006/jmaa.1997.5318.

[19] F. Qi and S.-L. Xu, *The function* $(b^x - a^x)/x$*: Inequalities and properties*, Proc. Amer. Math. Soc. **126** (1998), no. 11, 3355–3359; available online at https://doi.org/10.1090/S0002-9939-98-04442-6.

[20] J. Sándor, *The Schur-convexity of Stolarsky and Gini means*, Banach J. Math. Anal. **1** (2007), no. 2, 212–215; available online at https://doi.org/10.15352/bjma/1240336218.

[21] H.-N. Shi, S.-H. Wu, and F. Qi, *An alternative note on the Schur-convexity of the extended mean values*, Math. Inequal. Appl. **9** (2006), no. 2, 219–224; available online at https://doi.org/10.7153/mia-09-22.

[22] J. Sun, Z.-L. Sun, B.-Y. Xi, and F. Qi, *Schur-geometric and Schur-harmonic convexity of an integral mean for convex functions*, Turkish J. Anal. Number Theory **3** (2015), no. 3, 87–89; available online at https://doi.org/10.12691/tjant-3-3-4.

[23] B.-Y. Wang, *Foundations of Majorization Inequalities*, Beijing Normal Univ. Press, Beijing, China, 1990. (Chinese)

[24] W. Wang and S. Yang, *Schur m-power convexity of generalized Hamy symmetric function*, J. Math. Inequal. **8** (2014), no. 3. 661–667; available online at https://doi.org/10.7153/jmi-08-48.

[25] Y. Wu and F. Qi, *Schur-harmonic convexity for differences of some means*, Analysis (Munich) **32** (2012), no. 4, 263–270. https://doi.org/10.1524/anly.2012.1171.

[26] Y. Wu, F. Qi, and H.-N. Shi, *Schur-harmonic convexity for differences of some special means in two variables*, J. Math. Inequal. **8** (2014), no. 2, 321–330; available online at https://doi.org/10.7153/jmi-08-23.

[27] Y.-T. Wu and F. Qi, *Schur m-power convexity for general geometric Bonferroni mean of multiple parameters and comparison inequalities between means*, Math. Slovaca **73** (2023), no. 1, 3–14; available online at https://doi.org/10.1515/ms-2023-0002.

[28] W. Xia and Y. Chu, *The Schur convexity of Gini mean values in the sense of harmonic mean*, Acta Math. Sci. Ser. B Engl. Ed. **31** (2011), no. 3, 1103–1112; available online at https://doi.org/10.1016/S0252-9602(11)60301-9.

[29] W. F. Xia and Y. M. Chu, *Schur-convexity for a class of symmetric functions and its applications*, J. Inequal. Appl. **2009**, Art. ID 493759, 15 pages; available online at https://doi.org/10.1155/2009/493759.

[30] W.-F. Xia, Y.-M. Chu, and G.-D. Wang, *Necessary and sufficient conditions for the Schur harmonic convexity or concavity of the extended mean values*, Rev. Un. Mat. Argentina **52** (2011), no. 1, 121–132.

[31] Z.-H. Yang, *Schur power convexity of Gini means*, Bull. Korean Math. Soc. **50** (2013), no. 2, 485–498; available online at https://doi.org/10.4134/BKMS.2013.50.2.485.

[32] Z.-H. Yang, *Schur power convexity of Stolarsky means*, Publ. Math. Debrecen **80** (2012), no. 1-2, 43–66; available online at https://doi.org/10.5486/PMD.2012.4812.

[33] Z.-H. Yang, *Schur power convexity of the Daróczy means*, Math. Inequal. Appl. **16** (2013), no. 3, 751–762; available online at https://doi.org/10.7153/mia-16-57.

[34] H.-P. Yin, X.-M. Liu, H.-N. Shi, and F. Qi, *Necessary and sufficient conditions for a bivariate mean of three parameters to be the Schur m-power convex*, Contrib. Math. **6** (2022), 21–24; available online at https://doi.org/10.47443/cm.2022.023.

[35] H.-P. Yin, X.-M. Liu, J.-Y. Wang, and B.-N. Guo, *Necessary and sufficient conditions on the Schur convexity of a bivariate mean*, AIMS Math. **6** (2021), no. 1, 296–303; available online at https://doi.org/10.3934/math.2021018.

[36] H.-P. Yin, H.-N. Shi, and F. Qi, *On Schur m-power convexity for ratios of some means*, J. Math. Inequal. **9** (2015), no. 1, 145–153; available online at https://doi.org/10.7153/jmi-09-14.

[37] X.-M. Zhang, *Geometrically Convex Functions*, Anhui University Press, Anhui, China, 2004. (Chinese)