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A strong limit theorem of the largest entries of sample correlation matrices under a φ-mixing assumption

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Abstract. Let $\{X_{k,i}; k \geq 1, i \geq 1\}$ be an array of random variables, $\{\mathbf{X}_k; k \geq 1\}$ be a strictly stationary φ-mixing sequence, where $\mathbf{X}_k = (X_{k,1}, X_{k,2}, \cdots, X_{k,p})$. Let $\{p_n; n \geq 1\}$ be a sequence of positive integers such that $0 < c_1 \leq p_n/n^\tau \leq c_2 < \infty$, where $\tau > 0$, $c_2 \geq c_1 > 0$. In this paper, we obtain a strong limit theorem of $L_n = \max_{1\leq i<j\leq p_n}|\rho_{ij}|$, where ρ_{ij} denotes the Pearson correlation coefficient between $\mathbf{X}^{(i)}$ and $\mathbf{X}^{(j)}$, $\mathbf{X}^{(i)} = (X_{1,i}, X_{2,i}, \cdots, X_{n,i})'$. The strong limit theorem is derived by using Chen-Stein Poisson approximation method.

1. Introduction

Random matrix theory has demonstrated its efficacy across diverse domains such as statistics, high-energy physics, electrical engineering, and number theory. The correlation coefficient matrix holds significance as a crucial statistic in multivariate analysis, playing pivotal roles in the statistical test of multivariate data. The maximum likelihood estimator is the sample correlation matrix.

Consider a p-dimensional population represented by a random vector $\mathbf{X} = (X^{(1)}, \cdots, X^{(p)})$ with unknown mean $\boldsymbol{\mu} = (\mu_1, \cdots, \mu_p)$, unknown covariance matrix $\boldsymbol{\Sigma}$ and unknown correlation coefficient matrix $\mathbf{R}$. Let $\boldsymbol{X}_n = (X_{k,i})$ be an $n \times p$ matrix whose rows are an observed random sample of size n from the $\mathbf{X}$ population; that is, the rows of $\boldsymbol{X}_n$ are independent copies of $\mathbf{X}$. Set $\bar{X}^{(i)} = \sum_{k=1}^n X_{k,i}/n$, $1 \leq i \leq p$. $\mathbf{X}^{(i)}$ denotes the ith column of $\boldsymbol{X}_n$. Let

$$L_n = \max_{1\leq i<j\leq p} |\rho_{ij}|,$$

where

$$\rho_{ij} = \frac{\sum_{k=1}^n (X_{k,i}-\bar{X}^{(i)})(X_{k,j}-\bar{X}^{(j)})}{\sqrt{\sum_{k=1}^n (X_{k,i}-\bar{X}^{(i)})^2}\sqrt{\sum_{k=1}^n (X_{k,j}-\bar{X}^{(j)})^2}}$$

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is the Pearson correlation coefficient between the ith and jth columns of $\mathbf{X}_n$. Then $\mathbf{\Gamma}_n := (\rho_{ij})$ is a p by p symmetric matrix. It is called the sample correlation matrix generated by $\mathbf{X}_n$. This paper investigates a logarithmic law of the largest entries of a sample correlation matrices under a φ-mixing assumption.

This investigation is the promotion of the statistical hypothesis testing problem studied by [11]. When both n and p are large, [11] considered the statistical test with null hypothesis $H_0 : \mathbf{R} = \mathbf{I}$, where $\mathbf{I}$ is the $p \times p$ identity matrix. In general, this null hypothesis asserts that the components of $\mathbf{X} = (\mathbf{X}^{(1)}, \cdots, \mathbf{X}^{(p)})$ are uncorrelated whereas when $\mathbf{X}$ has a p-variate normal distribution, this null hypothesis asserts that these components are independent.

[11]'s test statistic is L_n. Let $\mathbf{e} = (1, \cdots, 1)' \in \mathbb{R}^n$, $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^n$. We can rewrite ρ_{ij} as

$$\rho_{ij} = \frac{(\mathbf{X}^{(i)} - \bar{X}^{(i)}\mathbf{e})'(\mathbf{X}^{(j)} - \bar{X}^{(j)}\mathbf{e})}{\|\mathbf{X}^{(i)} - \bar{X}^{(i)}\mathbf{e}\| \cdot \|\mathbf{X}^{(j)} - \bar{X}^{(j)}\mathbf{e}\|}, \tag{1}$$

[11] proved the following strong limit theorem concerning the test statistic L_n when $p = p_n$ and $\{X_{k,i}; k \geq 1, i \geq 1\}$ is an array of independent and identically distributed (i.i.d) random variables.

Theorem 1.1. *Suppose $\{\xi, X_{k,i}; k \geq 1, i \geq 1\}$ are i.i.d random variables. Let $X_n = (X_{k,i})$ be an $n \times p$ matrix. $E|\xi|^{30-\varepsilon} < \infty$ for any $\varepsilon > 0$. If $n/p \to \gamma \in (0, \infty)$, then*

$$\lim_{n\to\infty} \sqrt{\frac{n}{\log n}} L_n = 2 \qquad a.s.$$

Under the i.i.d. assumption, [11] found an asymptotic distribution of L_n.

Theorem 1.2. *Suppose $\{\xi, X_{k,i}; k \geq 1, i \geq 1\}$ are i.i.d random variables. Let $X_n = (X_{k,i})$ be an $n \times p$ matrix. $E|\xi|^{30+\varepsilon} < \infty$ for some $\varepsilon > 0$. If $n/p \to \gamma$, then*

$$P(nL_n^2 - 4\log n + \log(\log n) \leq y) \to e^{-Ke^{-y/2}}$$

as $n \to \infty$ for any $y \in \mathbb{R}$, where $K = (\gamma^2 \sqrt{8\pi})^{-1}$.

Subsequently, [24] showed the asymptotic distributions of L_n that the moment condition $E|X_{1,1}|^r < \infty$ for some $r > 30$ can be weakened to $x^6 P(|X_{1,1}X_{1,2}| \geq x) \to 0$ as $x \to \infty$ under $\limsup_{n\to\infty} p/n < \infty$. Another moment condition for the asymptotic distributions of L_n to hold has been obtained by [18] who showed that the asymptotic distributions of L_n holds under the condition $(x^6/\log^3 x)P(|X_{1,1}X_{1,2}| \geq x) \to 0$ as $x \to \infty$ and $p = O(n^\alpha)$ as $p/n \to \infty$. As for the strong limit, [13] established the strong limit theorems of L_n under some more relaxed assumption, [14], [15] had further improved the assumption of the result, under the assumption the p/n bounded away from zero to infinity. They actually obtained some necessary and sufficient conditions under which the limit theorem holds. As $p/n \to \infty$, [7] considered the ultra-high dimensional case where p can be as large as e^{n^α} for some $0 < \alpha \leq 1$ and they extended the result to dependent case. Afterwards, [8] derived the limiting theorem of L_n under the assumption that the population has a spherical distribution. In fact, a phase transition phenomenon occurs at three different regimes: $(\log p)/n \to 0$, $(\log p)/n \to \alpha \in (0, \infty)$ and $(\log p)/n \to \infty$. Without the Gaussian assumption, [21] obtained the limit theorem as $\log p = o(n^\alpha)$ for some $0 < \alpha \leq 1$. As for the dependent case, [10] investigated the limiting distribution of the largest off-diagonal entry of the sample correlation matrix in the high-dimensional setting when the correlation matrix admits a compound symmetry structure, namely, is of equi-correlation. [17] showed the asymptotic distribution of L_n for φ-mixing assumptions as follow.

Assumption 1.1. *Let $\mathbf{X}_k = (X_{k,1}, X_{k,2}, \cdots)$ be an infinite dimensional random vector, suppose that $\{\mathbf{X}_k; k \geq 1\}$ is a sequence of strictly stationary φ−mixing random vector, satisfied with the $Var(X_{1,1}) = 1$ and for some $T > 3$, $\varphi(n) = O(1/n^T)$.*

Assumption 1.2. *Let $\mathbf{X}^{(i)} = (X_{1,i}, X_{2,i}, \cdots)'$ be an infinite dimensional random vector, suppose that $\{\mathbf{X}^{(i)}; i \geq 1\}$ is a sequence of independent and identically distributed random vector.*

Theorem 1.3. *Under the Assumption 1.1 and 1.2, let* $EX_{1,1}^6 I\{|X_{1,1}| \geq n\} = o((\log n)^{-3})$*, and suppose* $c_1 \leq n/p_n \leq c_2$*, where* $c_1, c_2 > 0$*. Then*

$$P\left(nL_n^2 - 4ES_n^2 \frac{\log p_n}{n} + ES_n^2 \frac{\log(\log p_n)}{n} \leq y\right) \to e^{-Ke^{-y/(2\sigma^2)}} \tag{2}$$

for any $y \in \mathbb{R}$*, where* $K = (\sqrt{8\pi})^{-1}$*,* $S_n = \Sigma_{k=1}^n X_{k,1}X_{k,2}$*,* $\sigma^2 := \lim_{n\to\infty} ES_n^2/n$*.*

Afterwards, [22] showed the asymptotic distribution of L_n for α-mixing assumption, also under the α-mixing assumption [23] get the logarithmic law of L_n . [12] showed the limiting behavior of largest entry of random tensor constructed by high-dimensional data. Most of the aforementioned work mainly focus on the improvement of the moment assumption on $X_{1,1}$ from the data matrix $(X_{i,j})_{n\times p}$ as well as relaxing the range of p relative to n to obtain the asymptotic distributions of L_n. The strong law of large numbers for L_n remains largely unknown. [7], [8] show that under regularity conditions

$$\sqrt{\frac{n}{\log p}} L_n \xrightarrow{P} 2 \qquad as \quad n \to \infty$$

where $\xrightarrow{P}$ denotes convergence in probability.

In this paper, we will give a strong limit theorem for L_n almost sure convergence under φ-mixing assumptions. Let $\{X_n; n \geq 1\}$ be a sequence of random variables on some probability space (Ω, F, P). Let $\mathcal{F}_a^b$ denote the σ-field generated by the random variables $X_a, X_{a+1}, \cdots, X_b$. For any two σ-fields $\mathcal{A}, \mathcal{B} \subset F$, put

$$\varphi(\mathcal{A}, \mathcal{B}) := \sup\{|P(B|A) - P(B)|; A \in \mathcal{A}, B \in \mathcal{B}\},$$
$$\rho(\mathcal{A}, \mathcal{B}) := \sup\left\{\frac{Cov(X, Y)}{||X||_2||Y||_2}; X \in L^2(\mathcal{A}), Y \in L^2(\mathcal{B})\right\},$$

where, and in the sequel $||X||_p = (E|X|^p)^{1/p}$ for $1 \leq p < \infty$. The mixing coefficients of the sequence $\{X_n; n \geq 1\}$ are defined as usual:

$$\varphi(n) := \sup_{k\geq 1} \varphi(\mathcal{F}_1^k, \mathcal{F}_{k+n}^\infty), \quad \rho(n) := \sup_{k\geq 1} \rho(\mathcal{F}_1^k, \mathcal{F}_{k+n}^\infty),$$

the sequence $\{X_n; n \geq 1\}$ is called φ-mixing if $\varphi(n) \to 0$, is called ρ-mixing if $\rho(n) \to 0$. It is easy to know that $\rho(n) \leq 2\varphi^{1/2}(n)$, thus φ-mixing sequence is ρ-mixing sequence (See [20]).

The rest sections of this paper are organized as follows. Our main result is present in Section 2. Section 3 gives detailed proof of our main results. In section 4, we give the significance of the main result and its applications.

2. Main result

Assumption 2.1. *Let* $\mathbf{X}_k = (X_{k,1}, X_{k,2}, \cdots, X_{k,p})$ *be an random vector, suppose that* $\{\mathbf{X}_k; k \geq 1\}$ *is a sequence of strictly stationary* $\varphi-$*mixing random vector, satisfied with the* $Var(X_{1,1}) = 1$ *and* $\varphi(n) = O\left(1/n^T\right)$*, for some* $T > 6 + 8\tau + \varepsilon$*,* $\varepsilon > 0$*,* $\tau > 0$*, the definition of* τ *is in the Theorem 2.1*

Assumption 2.2. *Let* $\mathbf{X}^{(i)} = (X_{1,i}, X_{2,i}, \cdots, X_{n,i})'$ *be an random vector, suppose that* $\{\mathbf{X}^{(i)}; i \geq 1\}$ *is a sequence of independent and identically distributed random vector.*

Remark 2.1. *Let* $\mathscr{B}_i := \{X_{k,i}; 1 \leq k \leq n\}$ *be a random sampling of* $\mathbf{X}^{(i)}$*, under the condition of* $H_0 : \mathbf{R} = \mathbf{I}$*, it is reasonable to suppose* $\{\mathscr{B}_i; 1 \leq i \leq p\}$ *is independent. Therefore in order to obtain the strong limit theorem of* L_n*, Assumption 2.2 is reasonable.*

The following theorem is our main result.

Theorem 2.1. *Under the Assumption 2.1 and 2.2, let $S_n = \Sigma_{k=1}^n X_{k,1}X_{k,2}$, define $\sigma^2 := \lim_{n\to\infty} ES_n^2/n$. Suppose that $EX_{1,1} = 0$, and $0 < c_1 \le p_n/n^\tau \le c_2 < \infty$, where $\tau > 0$, $c_2 \ge c_1 > 0$. $E|X_{1,1}|^{4+4\tau+\varepsilon} < \infty$, for some $\varepsilon > 0$. Then*

$$\lim_{n\to\infty} \sqrt{\frac{n}{\log p_n}} L_n = 2\sigma \qquad a.s. \tag{3}$$

Remark 2.2. *Compare with [17], who showed the asymptotic distribution of L_n under $EX_{1,1}^6 I\{|X_{1,1}| \ge n\} = o((\log n)^{-3})$. We proved the logarithm law of L_n under $E|X_{1,1}|^{4+4\tau+\varepsilon} < \infty$, for some $\varepsilon > 0$.*

3. Proofs

The proof of Theorem 2.1 is intricate and complex. In this section, we will gather and establish several technical tools that contribute to the proof of Theorems 2.1. The following lemmas are useful for the proof of our result. First, we present Lemma 3.1, from [6].

Lemma 3.1. *Let $\{(S_k, \sigma_k); k \ge 1\}$ be a sequence of complete separable metric spaces. Let $\{X_k; k \ge 1\}$ be a sequence of random variables with values in S_k and let $\{L_k; k \ge 1\}$ be a sequence of σ-fields such that X_k is L_k-measurable. Suppose that for some $\phi_k \ge 0$*

$$|P(AB) - P(A)P(B)| \le \phi_k P(A)$$

for all $A \in \vee_{j<k} L_j$ and $B \in L_k$. Then without changing its distribution we can redefine the sequence $\{X_k; k \ge 1\}$ on a richer probability space together with a sequence $\{Y_k; k \ge 1\}$ of independent random variables such that Y_k has the same distribution as X_k and

$$P\{\sigma_k(X_k, Y_k) \ge 6\phi_k\} \le 6\phi_k \qquad k = 1, 2, \cdots.$$

The following result provides some inequalities. It will be applied to the proofs later.

Lemma 3.2. *Let $\{\xi_n; n \ge 1\}$ be a φ-mixing sequence. Put $T_k(n) = \sum_{i=k+1}^{k+n} \xi_i$. Suppose that there exists an array $\{C_{k,n}\}$ of positive numbers such that*

$$\max_{1\le k\le n} ET_k^2(n) \le C_{k,n} \quad for\ every\ \ k \ge 0,\ n \ge 1.$$

Then for every $q \ge 2$, there exists a constant K depending only on q and $\varphi(\cdot)$ such that

$$E \max_{i\le n} |T_k(i)|^q \le K \left(C_{k,n}^{q/2} + E \max_{k<i\le k+n} |\xi_i|^q \right)$$

for every $k \ge 0$, $n \ge 1$.

Proof. See [20]. □

The following Lemma is the Rosenthal type maximal inequality.

Lemma 3.3. *Suppose that $\{X_n; n \ge 1\}$ is a sequence of independent random variables and $E|X_n|^q < \infty$ for any $q \ge 2$, $n \ge 1$ then there exists a positive constant $C(q)$ depending only on q such that*

$$\begin{aligned} & E\left(\max_{1\le i\le n}\left|\sum_{k=1}^{i}(X_k - EX_k)\right|\right)^q \\ \le\ & C(q)\left(\sum_{k=1}^{n} E|X_k - EX_k|^q + \left(\sum_{k=1}^{n} E|X_k - EX_k|^2\right)^{q/2}\right). \end{aligned}$$

Proof. See [1]. □

This is an inequality with an atmosphere of the Marcinkieiwicz-Zygmund inequalities. The following is a corollary of the Marcinkiewicz-Zygmund inequality.

Lemma 3.4. *If $\{\eta_n; n \geq 1\}$ are i.i.d. random variables with $E\eta_1 = 0$, $E|\eta_1|^p < \infty$, $p \geq 1$, and $S_n = \sum_{i=1}^n \eta_i$. Then*

$$E|S_n|^p = \begin{cases} O(n^{p/2}), & \text{if } p \geq 2, \\ O(n), & \text{if } 1 \leq p < 2. \end{cases}$$

Proof. See [9]. □

Lemma 3.5. *For any sequence of independent random variables $\{\xi_n; n \geq 1\}$ with mean zero and finite variance, there exists a sequence of independent normal variables $\{\eta_n; n \geq 1\}$ with $E\eta_n = 0$, $E\eta_n^2 = E\xi_n^2$ such that for all $Q > 2$ and $y > 0$,*

$$P\left(\max_{k\leq n}\left|\sum_{i=1}^k \xi_i - \sum_{i=1}^k \eta_i\right| \geq y\right) \leq (AQ)^Q y^{-Q} \sum_{i=1}^n E|\xi_i|^Q$$

whenever $E|\xi_i|^Q < \infty$, $i = 1, 2, \cdots, n$. Here A is a universal constant.

Proof. See [19]. □

Lemma 3.6. *Let $\{\eta_k; 1 \leq k \leq n\}$ be independent symmetric random variables and $S_n = \sum_{k=1}^n \eta_k$. Then, for each integer $j \geq 1$, there exist positive numbers C_j and D_j depending only on j such that for all $t > 0$,*

$$P\Big(|S_n| \geq 2jt\Big) \leq C_j P\left(\max_{1\leq k\leq n} |\eta_j| \geq t\right) + D_j \left(P(|S_n| \geq t)\right)^j.$$

Proof. See [16]. □

The next one is the Chen-Stein Poisson approximation method, which is a special case of Theorem 1 from [3].

Lemma 3.7. *Let $\{\eta_\alpha; \alpha \in \mathbb{I}\}$ be random variables on an index set $\mathbb{I}$ and $\{B_\alpha; \alpha \in \mathbb{I}\}$ be a set of subsets of $\mathbb{I}$, that is, for each $\alpha \in \mathbb{I}, B_\alpha \subset \mathbb{I}$. For any $t \in \mathbb{R}$, set $\lambda = \sum_{\alpha\in\mathbb{I}} P(\eta_\alpha > t)$, Then we have*

$$\left|P\left(\max_{\alpha\in\mathbb{I}} \eta_\alpha \leq t\right) - e^{-\lambda}\right| \leq (1 \wedge \lambda^{-1})(b_1 + b_2 + b_3),$$

where

$$\begin{aligned} b_1 &= \sum_{\alpha\in\mathbb{I}} \sum_{\beta\in B_\alpha} P(\eta_\alpha > t) P\left(\eta_\beta > t\right), \\ b_2 &= \sum_{\alpha\in\mathbb{I}} \sum_{\alpha\neq\beta\in B_\alpha} P\left(\eta_\alpha > t, \eta_\beta > t\right), \\ b_3 &= \sum_{\alpha\in\mathbb{I}} E\left|P\left(\eta_\alpha > t|\sigma(\eta_\beta, \beta \notin B_\alpha)\right) - P(\eta_\alpha > t)\right|, \end{aligned}$$

and $\sigma(\eta_\beta; \beta \notin B_\alpha)$ is the $\sigma-$algebra generated by $\{\eta_\beta; \beta \notin B_\alpha\}$. In particular, if η_α is independent of $\{\eta_\beta; \beta \notin B_\alpha\}$, for each α, then b_3 vanishes.

The next one is Ottaviani's inequality.

Lemma 3.8. *Let $\{X_n; n \geq 1\}$ is a sequence of independent random variables. $S_k = \sum_{i=1}^k X_i$. For $\forall\ x > 0$,*

$$P\left(\max_{1\leq k\leq n} |S_k| > 2x\right) \leq \frac{P(|S_n| > x)}{\min_{1\leq k\leq n} P(|S_n - S_k| \leq x)}.$$

Proof. See [9]. □

The following lemma refers to Lemma 2 in [4], we generalize his result to the φ-mixing condition.

Lemma 3.9. *Under the Assumption 2.1 and 2.2, let $a > 1/2$, $b \geq 0$ and $M > 0$ be constants.*

(1) $E|X_{1,1}|^{(1+b)/a} < \infty$;

(2) $c = \begin{cases} EX_{1,1}, & \text{if } a \leq 1, \\ \text{any number}, & \text{if } a > 1 \end{cases}$

is the sufficient condition for

$$\max_{j \leq Mn^b} \left| n^{-a} \sum_{i=1}^{n} (X_{i,j} - c) \right| \to 0 \quad a.s. \quad as \ \ n \to \infty.$$

Proof. Without loss of generality, assume that $c = 0$. Since, for $\varepsilon > 0$ and $N \geq 1$,

$$\begin{aligned} & P\left\{ \max_{j \leq Mn^b} \left| \frac{1}{n^a} \sum_{i=1}^{n} X_{i,j} \right| \geq \varepsilon, i.o. \right\} \\ \leq & \sum_{k \geq N} P\left\{ \max_{2^{k-1} < n \leq 2^k} \max_{j \leq M2^{kb}} \left| \sum_{i=1}^{n} X_{i,j} \right| \geq \varepsilon' 2^{ka} \right\} \\ \leq & \sum_{k \geq N} M2^{kb} P\left\{ \max_{2^{k-1} < n \leq 2^k} \left| \sum_{i=1}^{n} X_{i,1} \right| \geq 2^{ka} \varepsilon' \right\}, \end{aligned}$$

where $\varepsilon' = 2^{-a}\varepsilon$, to conclude that the probability on the left-hand side of this inequality is equal to zero, it is sufficient to show that

$$\sum_{k=1}^{\infty} 2^{kb} P\left\{ \max_{n \leq 2^k} \left| \sum_{i=1}^{n} X_{i,1} \right| \geq 2^{ka} \varepsilon \right\} < \infty. \tag{4}$$

Let $Y_{i,k} = X_{i,1} I\{|X_{i,1}| < 2^{ka}\}$ and $Z_{i,k} = Y_{i,k} - EY_{i,k}$. Then $|Z_{i,k}| \leq 2^{ka+1}$ and $EZ_{i,k} = 0$. Let g be an even integer such that $g(a - 1/2) > b + 2a$. It is easy to see

$$\begin{aligned} X_{i,1} &= X_{i,1} I\{|X_{i,1}| < 2^{ka}\} + X_{i,1} I\{|X_{i,1}| \geq 2^{ka}\} \\ &= Z_{i,k} + EY_{i,k} + X_{i,1} I\{|X_{i,1}| \geq 2^{ka}\}. \end{aligned}$$

We have that

$$\begin{aligned} & \sum_{k=1}^{\infty} 2^{kb} P\left\{ \max_{n \leq 2^k} \left| \sum_{i=1}^{n} X_{i,1} \right| \geq 2^{ka} \varepsilon \right\} \\ \leq & \sum_{k=1}^{\infty} 2^{kb} P\left\{ \max_{n \leq 2^k} \left| \sum_{i=1}^{n} Z_{i,k} \right| \geq \frac{2^{ka}\varepsilon}{4} \right\} + \sum_{k=1}^{\infty} 2^{kb} P\left\{ \max_{n \leq 2^k} \left| \sum_{i=1}^{n} EY_{i,k} \right| \geq \frac{2^{ka}\varepsilon}{2} \right\} \\ + & \sum_{k=1}^{\infty} 2^{kb} P\left\{ \max_{n \leq 2^k} \left| \sum_{i=1}^{n} X_{i,1} I\{|X_{i,1}| \geq 2^{ka}\} \right| \geq \frac{2^{ka}\varepsilon}{4} \right\}. \end{aligned}$$

Then, note that $\{Z_{i,k}\}$ is a sequence of φ-mixing random variables by Assumption 2.1. By Lemma 3.2, we have

$$P\left\{ \max_{n \leq 2^k} \left| \sum_{i=1}^{n} Z_{i,k} \right| \geq \frac{\varepsilon 2^{ka}}{4} \right\} \leq C \frac{E\left| \sum_{i=1}^{2^k} Z_{i,k} \right|^g}{2^{kga}} \leq C \frac{2^k E|Z_{1,k}^g|}{2^{kga}} + C \frac{2^{kg/2} (EZ_{1,k}^2)^{g/2}}{2^{kga}}, \tag{5}$$

by the following bounds:

$$\begin{aligned}
&\sum_{k=1}^{\infty} 2^{kb-kga+k} E\left|Z_{1,k}^{g}\right| \leq C \sum_{k=1}^{\infty} 2^{k(b-ga+1)} E\left|X_{1,1}^{g}\right| I\left\{\left|X_{1,1}\right| < 2^{ka}\right\} \\
\leq\ & C \sum_{k=1}^{\infty} 2^{k(b-ga+1)} \left(\sum_{l=1}^{k} E|X_{1,1}|^{g} I\left\{2^{a(l-1)} \leq |X_{1,1}| < 2^{al}\right\} + 1 \right) \\
\leq\ & C \sum_{l=1}^{\infty} \left(E|X_{1,1}|^{g} I\left\{2^{a(l-1)} \leq |X_{1,1}| < 2^{al}\right\} + 1 \right) \sum_{k=l}^{\infty} 2^{k(b-ga+1)} \\
\leq\ & C \sum_{l=1}^{\infty} 2^{l(b-ga+1)} \left(E|X_{1,1}|^{g} I\{2^{a(l-1)} \leq |X_{1,1}| < 2^{al}\} + 1 \right) \\
\leq\ & C \sum_{l=1}^{\infty} E|X_{1,1}|^{(b+1)/a} I\left\{2^{a(l-1)} \leq |X_{1,1}| < 2^{al}\right\} + C_1 < \infty.
\end{aligned}$$

Note that $ga - b - 1 > g(a - 1/2) - (b + 2a) > 0$ and when $(1 + b)/a \geq 2$, $EZ_{1,k}^2 \leq EX_{1,1}^2 < \infty$,

$$\sum_{k=1}^{\infty} 2^{kb-kga+kg/2} (EZ_{1,k}^2)^{g/2} \leq C \sum_{k=1}^{\infty} 2^{k(b+2a-g(a-1/2))} < \infty.$$

If $(1 + b)/a < 2$, we have

$$\begin{aligned}
&\sum_{k=1}^{\infty} 2^{kb-kga+gk/2} \left(EZ_{1,k}^2\right)^{g/2} \\
\leq\ & \sum_{k=1}^{\infty} 2^{kb-kga+kg/2} \left(EZ_{1,k}^{(1+b)/a+2-(1+b)/a}\right)^{g/2-1} \left(EZ_{1,k}^2\right) \\
\leq\ & C \sum_{k=1}^{\infty} 2^{kb-kga+kg/2} 2^{ka(2-(1+b)/a)(g/2-1)} EX_{1,1}^2 I\left\{\left|X_{1,1}\right| < 2^{ka}\right\} \\
\leq\ & C \sum_{k=1}^{\infty} 2^{k(b+\frac{g}{2}-2a-\frac{(1+b)g}{2}+1+b)} \left(\sum_{l=1}^{k} EX_{1,1}^2 I\left\{2^{(l-1)a} \leq |X_{1,1}| < 2^{la}\right\} + 1 \right) \\
\leq\ & C \sum_{k=1}^{\infty} 2^{k(b-2a-bg/2+1+b)} \sum_{l=1}^{k} E|X_{1,1}|^2 I\left\{2^{(l-1)a} \leq |X_{1,1}| < 2^{la}\right\} + C_1 \\
=\ & C \sum_{l=1}^{\infty} 2^{l(b-2a-bg/2+1+b)} E|X_{1,1}|^2 I\left\{2^{(l-1)a} \leq |X_{1,1}| < 2^{la}\right\} + C_1 \\
\leq\ & C \sum_{l=1}^{\infty} 2^{l(b-2a-\frac{bg}{2}+1+b)} E|X_{1,1}|^{\frac{(1+b)}{a}+2-\frac{(1+b)}{a}} I\{2^{(l-1)a} \leq |X_{1,1}| < 2^{la}\} + C_1 \\
\leq\ & C \sum_{l=1}^{\infty} 2^{l(b-2a-\frac{bg}{2}+1+b)} 2^{la(2-\frac{(1+b)}{a})} E|X_{1,1}|^{\frac{(1+b)}{a}} I\{2^{(l-1)a} \leq |X_{1,1}| < 2^{la}\} + C_1 \\
\leq\ & CE|X_{1,1}|^{\frac{1+b}{a}} + C_1 < \infty.
\end{aligned}$$

We obtain that

$$\sum_{k=N}^{\infty} 2^{kb} P\left\{ \max_{n \leq 2^k} \left| \sum_{i=1}^{n} Z_{i,k} \right| \geq \varepsilon 2^{ka} \right\} < \infty. \tag{6}$$

Now we estimate $EY_{i,k}$ for large k. We have

$$\begin{aligned}
&\max_{n\le 2^k}\left|\sum_{i=1}^{n} EY_{ik}\right| \le 2^k\,|EY_{1k}| \\
\le &\begin{cases} 2^k E|X_{11}|I\{|X_{11}|\ge 2^{ka}\} \le 2^{k(a-b)}E|X_{11}|^{\frac{(1+b)}{a}}I\{|X_{11}|\ge 2^{ka}\}, & \text{if } a\le 1+b \\ 2^k\log k + 2^{k(a-b)}E|X_{11}|^{\frac{(1+b)}{a}}I\{|X_{11}|>\log k\}, & \text{if } a>1+b \end{cases} \\
\le &2^{-1}\varepsilon 2^{ka}
\end{aligned} \tag{7}$$

for all $\varepsilon > 0$, Hence,

$$\sum_{k=N}^{\infty} 2^{kb} P\left\{\max_{n\le 2^k}\left|\sum_{i=1}^{n} EY_{ik}\right| \ge \frac{\varepsilon 2^{ka}}{2}\right\} < \infty. \tag{8}$$

Finally, since $E|X_{11}|^{(b+1)/a} < \infty$, we have

$$\begin{aligned}
\sum_{k=1}^{\infty} 2^{kb} P\left(\bigcup_{i=1}^{2^k}\left\{|X_{i1}|\ge 2^{ka}\right\}\right) &\le \sum_{k=1}^{\infty} 2^{k(b+1)} P\left\{|X_{11}|\ge 2^{ka}\right\} \\
&< CE|X_{1,1}|^{\frac{b+1}{a}} < \infty.
\end{aligned} \tag{9}$$

Hence,

$$\sum_{k=N}^{\infty} 2^{kb} P\left\{\max_{n\le 2^k}\left|\sum_{i=1}^{n} X_{i1} I\{|X_{i1}|\ge 2_{ka}\}\right| \ge \frac{\varepsilon 2^{ka}}{4}\right\} < \infty. \tag{10}$$

Then, (4) follows from (6), (8) and (10). □

Now we define

$$W_n = \max_{1\le i<j\le p_n}\left|\sum_{k=1}^{n} X_{k,i}X_{k,j}\right|, \quad n\ge 1. \tag{11}$$

For any square matrix $A = (a_{i,j})$, define $|||A||| = \max_{1\le i\ne j\le n}|a_{i,j}|$; that is, the maximum of the absolute values of the off-diagonal entries of A.

Lemma 3.10. *Recall $\mathbf{X}^{(i)}$ in (1). Let $h_i = \|\mathbf{X}^{(i)} - \bar{X}^{(i)}\mathbf{e}\|/\sqrt{n}$ for each i. Then*

$$|||n\boldsymbol{\Gamma}_n - \mathbf{X}_n'\mathbf{X}_n||| \le (b_{n,1}^2 + 2b_{n,1})W_n b_{n,3}^{-2} + nb_{n,3}^{-2}b_{n,4}^2,$$

where

$$b_{n,1} = \max_{1\le i\le p_n}|h_i - 1|, \quad W_n = \max_{1\le i<j\le p_n}|(\mathbf{X}^{(i)})'\mathbf{X}^{(j)}|$$
$$b_{n,3} = \min_{1\le i\le p_n} h_i, \qquad b_{n,4} = \max_{1\le i\le p_n}|\bar{X}^{(i)}|.$$

Proof. See [7]. □

Lemma 3.11. *Under the Assumption 2.1 and 2.2, $EX_{1,1} = 0$, $Var(X_{1,1}) = 1$. Suppose that $0 < c_1 \le p_n/n^\tau \le c_2 < \infty$, where $\tau > 0$, $c_2 \ge c_1 > 0$. If $E|X_{1,1}|^{(2+2\tau)/(1-a)} < \infty$ for some $a \in (0, 1/2)$, then*

$$n^a b_{n,1} \to 0 \quad a.s., \quad b_{n,3} \to 1 \quad a.s. \quad and \quad n^a b_{n,4} \to 0 \quad a.s.$$

as $n \to \infty$.

Proof. The second limit follows from the first one. Easily, $\|\mathbf{X}^{(i)} - \bar{X}^{(i)}\|^2 = (\mathbf{X}^{(i)})'\mathbf{X}^{(i)} - n|\bar{X}^{(i)}|^2$. Using the fact that $|x-1| \le |x^2-1|$ for any $x > 0$, we have that

$$\begin{aligned} n^a b_{n,1} &= n^a \max_{1\le i\le p_n} \left| \sqrt{\frac{(\mathbf{X}^{(i)})'\mathbf{X}^{(i)}}{n} - |\bar{X}^{(i)}|^2} - 1 \right| \\ &\le n^a \max_{1\le i\le p_n} \left| \frac{(\mathbf{X}^{(i)})'\mathbf{X}^{(i)} - n}{n} - |\bar{X}^{(i)}|^2 \right| \\ &\le \max_{1\le i\le p_n} \left| \frac{(\mathbf{X}^{(i)})'\mathbf{X}^{(i)} - n}{n^{1-a}} \right| + \left(n^{\frac{a}{2}} \max_{1\le i\le p_n} |\bar{X}^{(i)}| \right)^2 . \end{aligned} \tag{12}$$

Note as $(\mathbf{X}^{(i)})'\mathbf{X}^{(i)} = \sum_{k=1}^n X_{k,i}^2$. By Lemma 3.9 the first and second maximum above go to zero when $E|X_{1,1}|^{(2+2\tau)/(1-a)} < \infty$. It is true under the assumptions of Theorem 2.1. So the first limit is proved. Under the condition that $E|X_{1,1}|^{(1+\tau)/(1-a)} < \infty$, we have

$$n^a b_{n,4} = n^a \max_{1\le i\le p_n} |\bar{X}^{(i)}| = \max_{1\le i\le p_n} \left| \frac{\sum_{k=1}^n X_{k,i}}{n^{1-a}} \right|$$

the limit that $n^a b_{n,4} \to 0$ a.s. is proved by noting the relationship between $n^a b_{n,4}$ and the right most term in (12). □

We introduce some notations now. Let $1/2 - \delta < \mu < 1/2$, where $\delta > 0$ sufficiently small, and $\delta < \frac{1}{2} - \frac{4+8\tau}{(4+4\tau+\varepsilon)(2+4\tau+\varepsilon)}$,

$$\begin{aligned} S_{n,i,j} &= \sum_{k=1}^n X_{k,i}X_{k,j}, \quad Y_{k,i,j} = X_{k,i}X_{k,j}I\{|X_{k,i}X_{k,j}| \le n^\mu\}, \\ S'_{n,i,j} &= \sum_{k=1}^n (Y_{k,i,j} - EY_{k,i,j}). \end{aligned}$$

Let $\rho = 1/2$, $\alpha = 1/2$, for some $\varepsilon' > 0$, $T\alpha\rho + \rho - 1 - 2\tau > 1 + \varepsilon'$. Set $z = z_n = [n^\rho]$, $q = q_n = [n^{\alpha\rho}]$, $m_n = [n/(z_n + q_n)] \sim n^{1-\rho}$, $N_n = m_n(z_n + q_n)$. Define

$$\begin{aligned} H_{i,n} &= \{j : i(z+q) + 1 \le j \le (i+1)z + iq\}, \\ I_{i,n} &= \{j : (i+1)z + iq + 1 \le j \le (i+1)(z+q)\}, \end{aligned}$$

and $u_{m,i,j} = \sum_{k\in H_{m,n}} (Y_{k,i,j} - EY_{k,i,j})$, $v_{m,i,j} = \sum_{k\in I_{m,n}} (Y_{k,i,j} - EY_{k,i,j})$, $1 \le m \le m_n$.

Lemma 3.12. *Under the condition of Theorem 2.1. Let $\{Y^*_{m,i,j}; m = 1, 2, \cdots, m_n\}$ be i.i.d. normal random variables with mean 0 and variance $Eu^2_{m,i,j}$, $\sigma^2 = \lim_{n\to\infty} ES_n^2/n$. Then*

$$\frac{ES_n^2}{\sum_{m=1}^{m_n} EY^{*2}_{m,i,j}} \to 1, \quad as \quad n \to \infty.$$

Proof. We have $ES_n^2 = E\left(S'_{n,1,2} + \sum_{k=1}^n X_{k1}X_{k2}I\{|X_{k1}X_{k2}| \ge n^\mu\}\right)^2$, where we have $S'_{n,1,2} = \sum_{m=1}^{m_n} u_{m,1,2} + \sum_{m=1}^{m_n} v_{m,1,2} +$

$\sum_{k=N_{m_n}+1}^{n}(Y_{k,1,2} - EY_{k,1,2})$. Since $E|X_{1,1}|^{4+4\tau+\varepsilon} < \infty$, for some $\varepsilon > 0$, we have that

$$\begin{aligned}
&\sum_{k=1}^{n} E|X_{k,1}X_{k,2}|I\{|X_{k,1}X_{k,2}| \geq n^{\mu}\} \\
\leq\ & \sum_{k=1}^{n} E|X_{k,1}X_{k,2}|I\{|X_{k,1}| \geq n^{\frac{\mu}{2}}\} + \sum_{k=1}^{n} E|X_{k,1}X_{k,2}|I\{|X_{k,2}| \geq n^{\frac{\mu}{2}}\} \\
\leq\ & \sum_{k=1}^{n} E|X_{k,1}|I\{|X_{k,1}| \geq n^{\frac{\mu}{2}}\}E|X_{k,2}| \\
+\ & \sum_{k=1}^{n} E|X_{k,1}|E|X_{k,2}|I\{|X_{k,2}| \geq n^{\frac{\mu}{2}}\} \\
\leq\ & \sum_{k=1}^{n} E|X_{k,1}|^{4+4\tau+\varepsilon}|X_{k,1}|^{1-(4+4\tau+\varepsilon)}I\{|X_{k,1}| \geq n^{\frac{\mu}{2}}\}E|X_{k,2}| \\
+\ & \sum_{k=1}^{n} E|X_{k,1}|E|X_{k,2}|^{4+4\tau+\varepsilon}|X_{k,2}|^{1-(4+4\tau+\varepsilon)}I\{|X_{k,2}| \geq n^{\frac{\mu}{2}}\} \\
\leq\ & Cn^{1-\frac{\mu(3+4\tau+\varepsilon)}{2}} = o\left(\sqrt{\frac{n}{\log n}}\right),
\end{aligned}$$

as $n \to \infty$. Therefore,

$$\sum_{k=1}^{n} E|X_{k,1}X_{k,2}|I\{|X_{k,1}X_{k,2}| \geq n^{\mu}\} = o\left(\sqrt{\frac{n}{\log n}}\right),$$

as $n \to \infty$. By Lemma 3.2, we have that

$$\begin{aligned}
E\left(\sum_{m=1}^{m_n} v_{m,1,2}\right)^2 &\leq Cm_nq_nE|X_{k1}X_{k2}|^2I\{|X_{k1}X_{k2}| \leq n^{\mu}\} \\
&\leq Cn^{1-\rho+\alpha\rho} = o\left(\frac{n}{\log n}\right),
\end{aligned}$$

as $n \to \infty$. And we obtain

$$\begin{aligned}
E\left(\sum_{k=N_{m_n}+1}^{n}(Y_{k,1,2} - EY_{k,1,2})\right)^2 &\leq C(z_n + q_n)E|X_{k1}X_{k2}|^2I\{|X_{k1}X_{k2}| \leq n^{\mu}\} \\
&\leq C(n^{\rho} + n^{\alpha\rho}) = o\left(\frac{n}{\log n}\right)
\end{aligned}$$

as $n \to \infty$. $\sum_{m=1}^{m_n} Y^*_{m,i,j}$, $i, j \geq 1$ is a sum of m_n i.i.d normal random variables with mean zero and variance $\sum_{m=1}^{m_n} Eu^2_{m,i,j}$. We have that $ES_n^2 = E\left(\sum_{m=1}^{m_n} u_{m,1,2}\right)^2 + o\left(\frac{n}{\log n}\right)$, using the fact that $|EXY - EXEY| \leq 2(\varphi(n))^{\frac{1}{2}}\|X\|_2\|Y\|_2$, we obtain

$$\begin{aligned}
\left|E\left(\sum_{m=1}^{m_n} u_{m,1,2}\right)^2 - \sum_{m=1}^{m_n} EY^{*2}_{m,1,2}\right| &\leq C\sum_{j=1}^{m_n} |H_{j,n}|^{\frac{1}{2}} \sum_{i=1}^{j-1} \varphi^{\frac{1}{2}}(|I_{i,n}|)|H_{i,n}|^{\frac{1}{2}} \\
&\leq Cn^{-T\alpha\rho/2+2-\rho} = o\left(\frac{n}{\log n}\right),
\end{aligned}$$

as $n \to \infty$. Hence, $\left|ES_n^2 - \sum_{m=1}^{m_n} EY_{m,i,j}^{*2}\right| = o\left(\frac{n}{\log n}\right)$, $n \to \infty$. Therefore,

$$\frac{ES_n^2}{\sum_{m=1}^{m_n} EY_{m,i,j}^{*2}} \to 1,$$

as $n \to \infty$, where $EY_{m,i,j}^{*2} = Eu_{m,i,j}^2$.
□

Lemma 3.13. *Let $\{Y_{m,i,j}^*; m = 1, 2, \cdots, m_n\}$ be i.i.d. normal random variables with mean 0 and variance $Eu_{m,i,j}^2$. Then*

$$\limsup_{n\to\infty} \frac{\max_{1\le i<j\le p_n} \left|\sum_{m=1}^{m_n} Y_{m,i,j}^*\right|}{\sqrt{n\log p_n}} \le 2\sigma \qquad a.s. \tag{13}$$

Proof. Given $t \in (0,1)$, let $\omega_n = (2+t)\sigma\sqrt{n\log p_n}$, we can suppose $\sigma = 1$. By Lemma 3.12, we have $ES_n^2/\sum_{m=1}^{m_n} EY_{m,i,j}^{*2} \to 1$, as $n \to \infty$, where $EY_{m,i,j}^{*2} = Eu_{m,i,j}^2$. Then we can obtain,

$$\begin{aligned}
&\max_{1\le i\ne j<\infty} P\left(\left|\sum_{m=1}^{m_n} Y_{m,i,j}^*\right| > \omega_n\right) = \max_{1\le i\ne j<\infty} P\left(\frac{\left|\sum_{m=1}^{m_n} Y_{m,i,j}^*\right|}{\sqrt{\sum_{m=1}^{m_n} Eu_{m,i,j}^{*2}}} > \frac{w_n}{\sqrt{\sum_{m=1}^{m_n} Eu_{m,i,j}^{*2}}}\right)\\
=&2\left[1-\Phi\left(\frac{(2+t)\sqrt{n\log p_n}}{\sqrt{\sum_{m=1}^{m_n} Eu_{m,i,j}^{*2}}}\right)\right]\\
\le& C\frac{\sqrt{\sum_{m=1}^{m_n} Eu_{m,i,j}^{*2}}}{\sqrt{2\pi}(2+t)\sqrt{n\log p_n}}\exp\left(-\frac{(2+t)^2 n\log p_n}{2\sum_{m=1}^{m_n} Eu_{m,i,j}^{*2}}\right)\\
\le& C\frac{1}{(2+t)\sqrt{2\pi\log p_n}\,p_n^{(2+t)^2/2}} = O\left(\frac{1}{n^{\tau(2+t)^2/2}}\right),
\end{aligned} \tag{14}$$

as n is large, where we use the fact that

$$1-\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-t^2/2}dt \sim \frac{1}{\sqrt{2\pi}x}e^{-x^2/2} \tag{15}$$

as $x \to +\infty$ (see e.g., page 49 from [9]). And we define $W_n' = \max_{1\le i<j\le p_n}\left|\sum_{m=1}^{m_n} Y_{m,i,j}^*\right|$, $n_k = k^g$, for any integer $g > \left(2+\tau(2+t)^2\right)/(t^2+4t)\tau$,

$$\begin{aligned}
\max_{n_k\le n\le n_{k+1}} W_n' \le& \max_{1\le i\ne j\le p_{n_{k+1}}}\left(\max_{n_k\le n\le n_{k+1}}\left|\sum_{m=1}^{m_n} Y_{m,i,j}^*\right|\right)\\
\le& \max_{1\le i\ne j\le p_{n_{k+1}}}\left|\sum_{m=1}^{m_{n_k}} Y_{m,i,j}^*\right| + r_n,
\end{aligned} \tag{16}$$

where

$$r_n = \max_{1\le i\ne j\le p_{n_{k+1}}}\max_{n_k\le n\le n_{k+1}}\left|\sum_{m=1}^{m_n} Y_{m,i,j}^* - \sum_{m=1}^{m_{n_k}} Y_{m,i,j}^*\right|. \tag{17}$$

By (14),

$$
\begin{aligned}
& P\left(\max_{1\le i\neq j\le p_{n_{k+1}}}\left|\sum_{m=1}^{m_{n_k}} Y^*_{m,i,j}\right| > \omega_{n_k}\right) \le p^2_{n_{k+1}} P\left(\left|\sum_{m=1}^{m_{n_k}} Y^*_{m,1,2}\right| > \omega_{n_k}\right) \\
\le \;& p^2_{n_{k+1}} P\left(\frac{\left|\sum_{m=1}^{m_{n_k}} Y^*_{m,1,2}\right|}{\sqrt{\sum_{m=1}^{m_{n_k}} Eu^{*2}_{m,i,j}}} > \frac{\omega_{n_k}}{\sqrt{\sum_{m=1}^{m_{n_k}} Eu^{*2}_{m,i,j}}}\right) \\
\le \;& 2p^2_{n_{k+1}}\left(1-\Phi\left(\frac{(2+t)\sqrt{n_k\log p_{n_k}}}{\sqrt{\sum_{m=1}^{m_{n_k}} Eu^{*2}_{m,i,j}}}\right)\right) \\
\le \;& C(k+1)^{2g\tau}\frac{\sqrt{\sum_{m=1}^{m_{n_k}} Eu^{*2}_{m,i,j}}}{\sqrt{2\pi}(2+t)\sqrt{n_k\log p_{n_k}}}\exp\left(-\frac{(2+t)^2 n_k\log p_{n_k}}{2\sum_{m=1}^{m_{n_k}} Eu^{*2}_{m,i,j}}\right) \\
= \;& O\left(k^{-\frac{(t^2+4t)g\tau}{2}}\right).
\end{aligned}
$$

Since $\sum_k k^{-(t^2+4t)g\tau/2} < \infty$, by the Borel-Cantelli lemma,

$$
\limsup_{n\to\infty}\frac{\max_{1\le i\neq j\le p_{n_k}}\left|\sum_{m=1}^{m_{n_k}} Y^*_{m,i,j}\right|}{\sqrt{n_k\log p_{n_k}}} \le 2+t \quad a.s. \tag{18}
$$

Now let us estimate r_n as in (17).

Let partial sums $S_0 = 0$ and $S_l = \sum_{m=1}^{l} Y^*_{m,i,j}$. Observe that the distribution of $\sum_{m=1}^{m_n} Y^*_{m,i,j} - \sum_{m=1}^{m_{n_k}} Y^*_{m,i,j}$ is equal to that of $S_{m_n - m_{n_k}}$ for all $m_n \ge m_{n_k}$. Thus, by Lemma 3.8, we have

$$
\begin{aligned}
& P\left(r_n \ge t\sqrt{n_k\log p_{n_k}}\right) \\
\le\;& p^2_{n_{k+1}} P\left(\max_{1\le l\le n_{k+1}-n_k}|S_l| \ge t\sqrt{n_k\log p_{n_k}}\right) \\
\le\;& 2p^2_{n_{k+1}} P\left(\left|S_{n_{k+1}-n_k}\right| \ge (t/2)\sqrt{n_k\log p_{n_k}}\right)
\end{aligned} \tag{19}
$$

as n is sufficiently large, since $\min_{1\le l\le n_{k+1}-n_k} P\left(\left|S_{n_{k+1}-n_k} - S_l\right| \le (t/2)\sqrt{n_k\log p_{n_k}}\right) \ge 1/2$, where Ottaviani's inequality in Lemma 3.8 is used in the last inequality. Note that, for fixed g and t, $(t/2)\sqrt{n_k\log p_{n_k}} \ge$

$(2+t)\sqrt{(n_{k+1}-n_k)\log(p_{n_{k+1}}-p_{n_k})}$ as n is sufficiently large. By (15), we have

$$\begin{aligned}
& 2p_{n_{k+1}}^2 P\left(\left|S_{n_{k+1}-n_k}\right| \geq (t/2)\sqrt{n_k \log p_{n_k}}\right) \\
\leq\ & 2p_{n_{k+1}}^2 P\left(\left|S_{n_{k+1}-n_k}\right| \geq (2+t)\sqrt{(n_{k+1}-n_k)\log(p_{n_{k+1}}-p_{n_k})}\right) \\
\leq\ & 2p_{n_{k+1}}^2 P\left(\frac{|S_{n_{k+1}-n_k}|}{\sqrt{\sum_{m=1}^{n_{k+1}-n_k} Eu_{m,i,j}^{*2}}} \geq \frac{(2+t)\sqrt{(n_{k+1}-n_k)\log(p_{n_{k+1}}-p_{n_k})}}{\sqrt{\sum_{m=1}^{n_{k+1}-n_k} Eu_{m,i,j}^{*2}}}\right) \\
\leq\ & 4p_{n_{k+1}}^2 \left(1-\Phi\left(\frac{(2+t)\sqrt{(n_{k+1}-n_k)\log(p_{n_{k+1}}-p_{n_k})}}{\sqrt{\sum_{m=1}^{n_{k+1}-n_k} Eu_{m,i,j}^{*2}}}\right)\right) \\
\leq\ & C(k+1)^{2g\tau}\exp\left(-\frac{(2+t)^2(n_{k+1}-n_k)\log(p_{n_{k+1}}-p_{n_k})}{2\sum_{m=1}^{n_{k+1}-n_k} Eu_{m,i,j}^{*2}}\right) \\
=\ & O\left(k^{-(\tau g-1)(2+t)^2/2+2g\tau}\right).
\end{aligned}$$

Therefore,

$$P\left(r_n \geq t\sqrt{n_k \log p_{n_k}}\right) = O\left(k^{-\mu}\right),$$

where $u = (\tau g-1)(2+t)^2/2 - 2g\tau$, since g is chosen such that $u > 1$. By the Borel-Cantelli lemma again, we have

$$\limsup_{n\to\infty} \frac{r_n}{\sqrt{n_k \log p_{n_k}}} \leq t \quad a.s. \tag{20}$$

By (16), (18) and (20), we obtain that

$$\limsup_{n\to\infty} \frac{\max_{n_k\leq n\leq n_{k+1}} W_n'}{\sqrt{n_k \log p_{n_k}}} \leq 2+2t \quad a.s.$$

for any sufficiently small $t > 0$. This implies inequality (13) in Lemma 3.13.
□

Lemma 3.14. *Let $\{Y_{m,i,j}^*; m = 1, 2, \cdots, m_n\}$ be i.i.d. normal random variables with mean 0 and variance $Eu_{m,i,j}^2$. Then*

$$\liminf_{n\to\infty} \frac{\max_{1\leq i<j\leq p_n}\left|\sum_{m=1}^{m_n} Y_{m,i,j}^*\right|}{\sqrt{n\log p_n}} \geq 2\sigma \qquad a.s. \tag{21}$$

Proof. We continue to use the notations in the proof of (13) of Lemma 3.13. For any $t \in (0, 1)$, define $v_n = (2-t)\sigma\sqrt{n\log p_n}$, $\sigma^2 = \lim_{n\to\infty} ES_n^2/n$, we can suppose $\sigma = 1$. We first claim that

$$P\left(W_n' \leq v_n\right) = O\left(\frac{1}{n^{t'}}\right) \tag{22}$$

as $n \to \infty$, for some positive constant t' depending on t and the distribution of $X_{1,1}X_{1,2}$ only. If this is true, take an integer g such that $g > 1/t'$. Then $P(W_{n_k}' \leq v_{n_k}) = O\left(1/k^{t'g}\right)$. Since $\sum_k k^{-t'g} < \infty$, by the Borel-Cantelli lemma, we have that

$$\liminf_{n\to\infty} \frac{W_{n_k}'}{\sqrt{n_k \log p_{n_k}}} \geq 2-t \quad a.s. \tag{23}$$

for any $t \in (0,1)$. Recalling the definition of r_n in (17), we have that

$$\inf_{n_k \le n \le n_{k+1}} W'_n \ge W'_{n_k} - r_n.$$

By (20) and (23), we have that

$$\liminf_{n\to\infty} \frac{\inf_{n_k \le n \le n_{k+1}} W'_n}{\sqrt{n_k \log p_{n_k}}} \ge 2 - 2t \quad a.s.$$

for any $t > 0$ small enough. This implies (21) of Lemma 3.14.

Now we turn to prove claim (22) by Lemma 3.7.

Take $I = \{(i,j);\ 1 \le i < j \le p\}$. For $\alpha = (i,j) \in I$, set $B_\alpha = \{(k,l) \in I;\ one\ of\ k\ and\ l = i\ or\ j\ but\ (k,l) \ne \alpha\}$, $\eta_\alpha = |\sum_{m=1}^{m_n} Y^*_{m,i,j}|$, $t = v_n$ and $A_\alpha = A_{ij} = \{|\sum_{m=1}^{m_n} Y^*_{m,i,j}| > v_n\}$. By Lemma 3.7,

$$P(W'_n \le v_n) \le e^{-\lambda_n} + b_{1,n} + b_{2,n}. \tag{24}$$

Evidently

$$\begin{aligned} &\lambda_n = \frac{p(p-1)}{2} P(A_{12}), \\ &b_{1,n} \le 2p^3 P(A_{12})^2 \quad and \quad b_{2,n} \le 2p^3 P(A_{12}A_{13}). \end{aligned} \tag{25}$$

Remember that $\sum_{m=1}^{m_n} Y^*_{m,i,j}$ is a sum of i.i.d. normal random variables with mean zero and variance $\sum_{m=1}^{m_n} Eu^2_{m,i,j}$. Recall (15). We have

$$\begin{aligned} P(A_{12}) &= P\left(\left|\sum_{m=1}^{m_n} Y^*_{m,1,2}\right| > v_n\right) = P\left(\left|\sum_{m=1}^{m_n} Y^*_{m,1,2}\right| > (2-t)\sqrt{n \log p_n}\right) \\ &= P\left(\frac{\left|\sum_{m=1}^{m_n} Y^*_{m,1,2}\right|}{\sqrt{\sum_{m=1}^{m_n} Eu^{*2}_{m,1,2}}} > \frac{(2-t)\sqrt{n\log p_n}}{\sqrt{\sum_{m=1}^{m_n} Eu^{*2}_{m,1,2}}}\right) \\ &= 2\left[1 - \Phi\left(\frac{(2-t)\sqrt{n\log p_n}}{\sqrt{\sum_{m=1}^{m_n} Eu^{*2}_{m,1,2}}}\right)\right] \\ &\le C \frac{\sqrt{\sum_{m=1}^{m_n} Eu^{*2}_{m,1,2}}}{(2-t)\sqrt{2\pi n \log p_n}} \exp\left(-\frac{(2-t)^2 n \log p_n}{2\sum_{m=1}^{m_n} Eu^{*2}_{m,1,2}}\right) = O\left(\frac{1}{n^{\tau(2-t)^2/2}}\right) \end{aligned} \tag{26}$$

as $n \to \infty$. Provided $E|X_{1,1}|^{4+4\tau+\varepsilon} < \infty$ and $v_n/\sqrt{n \log p_n} \to 2 - t$,

$$P(A_{12}A_{13}) = P\left(\left|\sum_{m=1}^{m_n} Y^*_{m,1,2}\right| \ge v_n, \left|\sum_{m=1}^{m_n} Y^*_{m,1,3}\right| \ge v_n\right). \tag{27}$$

The two events in (27) are conditionally independent given $Y^*_{m,i,j}$'s. P^1 and E^1 represent the conditional probability and expectation of $\{Y^*_{m,i,j}; 1 \le m \le m_n, 1 \le i,j \le p_n\}$, respectively. Then the probability in (27) is

$$E\left[P^1\left(\left|\sum_{m=1}^{m_n} Y^*_{m,1,2}\right| \ge (2-t)\sqrt{n \log p_n}\right)^2\right]. \tag{28}$$

Set

$$A_n(s) := \left\{ \frac{1}{n} \left| \sum_{m=1}^{m_n} \left(\left| Y^*_{m,1,2} \right|^s - E \left| Y^*_{m,1,2} \right|^s \right) \right| \le \tilde{\delta} \right\}$$

for $s \ge 2$ and $\tilde{\delta} \in (1, \frac{1}{2})$. Choose $\beta \in (a^2+2, q/(a^2+1))$ and $r = a^2+1$. Let $\zeta_m = \left| Y^*_{m,1,2} \right|^\beta - E \left| Y^*_{m,1,2} \right|^\beta$ for $m = 1, 2, \cdots, m_n$. Then $E\left|\zeta_1\right|^r < \infty$. By the Chebyshev inequality and Lemma 3.4,

$$P(A_n(\beta)^c) = P\left(\left| \sum_{m=1}^{m_n} \zeta_m \right| > n\tilde{\delta} \right) \le \frac{E\left|\sum_{m=1}^{m_n} \zeta_m\right|^r}{(n\tilde{\delta})^r} = O(n^{-f(r)}) \tag{29}$$

as $n \to \infty$, where $f(r) = r/2$ if $r \ge 2$, and $f(r) = r-1$ if $1 < r \le 2$. Let $\{\zeta'_m; 1 \le m \le m_n\}$ be an independent copy of $\{\zeta_m; 1 \le m \le m_n\}$. Then since (29), $P(|\sum_{m=1}^{m_n} \zeta_m| \le n\tilde{\delta}/2) \ge 1/2$ for sufficiently large n, it follows that

$$P\left(\left| \sum_{m=1}^{m_n} \zeta_m \right| > n\tilde{\delta} \right) \le 2P\left(\left| \sum_{m=1}^{m_n} (\zeta_m - \zeta'_m) \right| > n\tilde{\delta}/2 \right) = O\left(n^{-f(r)}\right) \tag{30}$$

by repeating (29). Given an integer $j \ge 1$, let $\nu = n\tilde{\delta}/4j$. Then by Lemma 3.6, there are positive constants C_j and D_j such that

$$\begin{aligned} P\left(\left| \sum_{m=1}^{m_n} (\zeta_m - \zeta'_m) \right| > n\tilde{\delta}/2 \right) &= P\left(\left| \sum_{m=1}^{m_n} (\zeta_m - \zeta'_m) \right| > 2j\nu \right) \\ &\le C_j P\left(\max_{1 \le m \le m_n} \left| \zeta_m - \zeta'_m \right| > \nu \right) + D_j P\left(\left| \sum_{m=1}^{m_n} (\zeta_m - \zeta'_m) \right| > \nu \right)^j . \end{aligned}$$

Since $E|\zeta_1|^r < \infty$, $P(\max_{1 \le m \le m_n} |\zeta_m - \zeta'_m| > \nu) \le m_n P(|\zeta_1 - \zeta'_1| > \nu) = O(n^{1-r})$. By the same argument as the equality in (30), we obtain

$$\left(P\left(\left| \sum_{m=1}^{m_n} (\zeta_m - \zeta'_m) \right| > \nu \right) \right)^j = O(n^{-jf(r)}).$$

Take $j = [(r-1)/f(r)] + 1$. It follows that

$$P\left(\left| \sum_{m=1}^{m_n} (\zeta_m - \zeta'_m) \right| > n\tilde{\delta}/2 \right) = O(n^{1-r}) \tag{31}$$

as $n \to \infty$. Combining (29), (30) and (31), we obtain that

$$P(A_n(\beta)^c) = O(n^{1-r})$$

as $n \to \infty$. By the same arguments the above still holds if β is replaced by 2. Consequently,

$$\begin{aligned} P\left(A_{12}A_{13}\right) &= E\left[P^1\left(\left| \sum_{m=1}^{m_n} Y^*_{m,1,2} \right| \ge (2-t)\sqrt{n \log p_n} \right)^2 \right] \\ &\le E\left[P^1\left(\left| \sum_{m=1}^{m_n} Y^*_{m,1,2} \right| \ge (2-t)\sqrt{n \log p_n} \right)^2 I_{A_n(s) \cap A_n(2)} \right] + P(A_n(s)^c). \end{aligned} \tag{32}$$

Since

$$\begin{aligned}
&P^1\left(\left|\sum_{m=1}^{m_n} Y^*_{m,1,2}\right| \geq (2-t)\sqrt{n\log p_n}\right)\\
=&P^1\left(\frac{\left|\sum_{m=1}^{m_n} Y^*_{m,1,2}\right|}{\sqrt{\sum_{m=1}^{m_n} Eu^{*2}_{m,1,2}}} \geq \frac{(2-t)\sqrt{n\log p_n}}{\sqrt{\sum_{m=1}^{m_n} Eu^{*2}_{m,1,2}}}\right) \qquad (33)\\
\leq&C\exp\left(-\frac{(2-t)^2 n\log p_n}{2\sum_{m=1}^{m_n} Eu^{*2}_{m,1,2}}\right),
\end{aligned}$$

we can obtain

$$\begin{aligned}
&P^1\left(\left|\sum_{m=1}^{m_n} Y^*_{m,1,2}\right| \geq (2-t)\sqrt{n\log p_n}\right)^2 I_{A_n(s)\cap A_n(2)}\\
\leq&C\exp\left(-\frac{(2-t)^2 n\log p_n}{\sum_{m=1}^{m_n} Eu^{*2}_{m,1,2}}\right) = O\left(n^{b-(2-t)^2}\right)
\end{aligned} \qquad (34)$$

for any $b > 0$. Choosing both b and t small enough, we obtain

$$e^{-\lambda_n} \leq e^{-n^t}, \qquad b_{1,n} \leq \frac{1}{\sqrt{n}} \quad \textit{and} \quad b_{2,n} \leq \frac{1}{\sqrt{n}} \qquad (35)$$

for sufficiently large n. Then (22) follows from (24) and (35). □

Lemma 3.15. *Under the condition of Theorem 2.1, take* $T_n = \max_{1\leq i<j\leq p_n}\left|S'_{n,i,j}\right|$, *then*

$$\limsup_{n\to\infty} \frac{T_n}{\sqrt{n\log p_n}} \leq 2\sigma \qquad a.s. \qquad (36)$$

$$\liminf_{n\to\infty} \frac{T_n}{\sqrt{n\log p_n}} \geq 2\sigma \qquad a.s. \qquad (37)$$

Proof. $S'_{n,i,j} = \sum_{k=1}^{n}(Y_{k,i,j} - EY_{k,i,j}) = \sum_{m=1}^{m_n} u_{m,i,j} + \sum_{m=1}^{m_n} v_{m,i,j} + \sum_{k=N_{m_n}+1}^{n}(Y_{k,i,j} - EY_{k,i,j})$. By Markov inequality and Lemma 3.2, for $\forall\ \delta' > 0$, we obtain

$$\begin{aligned}
&P\left(\max_{1\leq i<j\leq p_n}\left|\sum_{m=1}^{m_n} v_{m,i,j}\right| \geq \delta'\sqrt{n\log p_n}\right) \leq C\frac{p_n^2 E\left|\sum_{i=1}^{m_n} v_{i,1,2}\right|^q}{(n\log p_n)^{q/2}}\\
\leq\ & C\frac{p_n^2 m_n^{\frac{q}{2}} q_n^{\frac{q}{2}}\left(E\left|X_{1,1}X_{1,2}\right|^2 I\{|X_{1,1}X_{1,2}| \leq n^{\mu}\}\right)^{\frac{q}{2}}}{(n\log p_n)^{q/2}}\\
&+C\frac{p_n^2 m_n q_n E|X_{1,1}X_{1,2}|^q I\{|X_{1,1}X_{1,2}| \leq n^{\mu}\}}{(n\log p_n)^{q/2}}\\
\leq\ & C\frac{n^{2\tau+\frac{q}{2}(1-\rho+\alpha\rho)}}{(n\log p_n)^{q/2}} + C\frac{n^{2\tau} n^{1-\rho+\alpha\rho} n^{\mu q}}{(n\log p_n)^{q/2}}\\
\leq\ & C\frac{1}{(\log p_n)^{\frac{q}{2}}\, n^{(\rho-\alpha\rho)\frac{q}{2}-2\tau}} + C\frac{1}{(\log p_n)^{\frac{q}{2}}\, n^{(1-2\mu)\frac{q}{2}+\rho-\alpha\rho-1-2\tau}} = O\left(\frac{1}{n^{1+\varepsilon'}}\right),
\end{aligned}$$

for $\varepsilon' > 0$ and sufficiently large q, and for $\forall\ \delta' > 0$,

$$\begin{aligned}
& P\left(\max_{1\le i<j\le p_n}\left|\sum_{k=N_{m_n}+1}^{n}(Y_{k,i,j}-EY_{k,i,j})\right|\ge \delta'\sqrt{n\log p_n}\right)\\
\le\ & C\frac{p_n^2 E|\sum_{k=N_{m_n}+1}^{n}(Y_{k,1,2}-EY_{k,1,2})|^q}{(n\log p_n)^{q/2}}\\
\le\ & C\frac{p_n^2(z_n+q_n)^{q/2}\left(E\left|X_{1,1}X_{1,2}\right|^2 I\{|X_{1,1}X_{1,2}|\le n^{\mu}\}\right)^{\frac{q}{2}}}{(n\log p_n)^{q/2}}\\
& +C\frac{p_n^2(z_n+q_n)E\left|X_{1,1}X_{1,2}\right|^q I\{|X_{1,1}X_{1,2}|\le n^{\mu}\}}{(n\log p_n)^{q/2}}\\
\le\ & C\frac{n^{2\tau}\,(n^{\rho}+n^{\alpha\rho})^{q/2}}{n^{q/2}\,(\log p_n)^{q/2}}+C\frac{n^{2\tau}(n^{\rho}+n^{\alpha\rho})n^{\mu q}}{n^{q/2}\,(\log p_n)^{q/2}}\\
\le\ & C\frac{(n^{\rho}+n^{\alpha\rho})^{q/2}}{n^{\frac{q}{2}-2\tau}\,(\log p_n)^{q/2}}+C\frac{n^{\rho}+n^{\alpha\rho}}{n^{(1-2\mu)\frac{q}{2}-2\tau}(\log p_n)^{q/2}}=O\left(\frac{1}{n^{1+\varepsilon'}}\right),
\end{aligned}$$

for sufficiently large q.

By Lemma 3.1, we can construct the independent random variables $\{u^*_{m,i,j}; 1\le m\le m_n\}$, $\{u^*_{m,i,j}; 1\le m\le m_n\}$ has the same distribution as $\{u_{m,i,j}; 1\le m\le m_n\}$, and$P(|u_{m,i,j}-u^*_{m,i,j}|\ge 6\varphi(|I_{m,n}|))\le 6\varphi(|I_{m,n}|)$. We have that

$$\begin{aligned}
& P\left(\max_{1\le i<j\le p_n}\left|\sum_{m=1}^{m_n}\left(u_{m,i,j}-u^*_{m,i,j}\right)\right|\ge \delta'\sqrt{n\log p_n}\right)\\
\le\ & p_n^2 P\left(\left|\sum_{m=1}^{m_n}\left(u_{m,i,j}-u^*_{m,i,j}\right)\right|\ge \delta'\sqrt{n\log p_n}\right)\\
\le\ & p_n^2 m_n P(|u_{i,1,2}-u^*_{i,1,2}|>6\varphi(|I_{1,n}|))\\
\le\ & Cn^{2\tau+1-\rho}n^{-T\alpha\rho}\le\frac{1}{n^{T\alpha\rho+\rho-1-2\tau}}=o\left(\frac{1}{n^{1+\varepsilon'}}\right),
\end{aligned}$$

for $T > 6+8\tau+\varepsilon$. Let $u^*_{m,i,j}=\sum_{k\in H_{m,n}}(Y^i_{k,i,j}-EY^i_{k,i,j})$, $1\le m\le m_n$, where $Y^i_{k,i,j}=X^i_{k,i}X^i_{k,j}I\{|X^i_{k,i}X^i_{k,j}|\le n^{\mu}\}$, $\{X^i_{k,j}; k\in H_{i,n}\}$ is an independent replication of $\{X_{k,j}; k\in H_{i,n}\}$. Thus, we only need to prove

$$(1)\quad \limsup_{n\to\infty}\frac{\max_{1\le i<j\le p_n}\left|\sum_{m=1}^{m_n}u^*_{m,i,j}\right|}{\sqrt{n\log p_n}}\le 2\sigma \qquad a.s.$$

$$(2)\quad \liminf_{n\to\infty}\frac{\max_{1\le i<j\le p_n}\left|\sum_{m=1}^{m_n}u^*_{m,i,j}\right|}{\sqrt{n\log p_n}}\ge 2\sigma \qquad a.s.$$

$\sum_{m=1}^{m_n}u^*_{m,i,j}=\sum_{m=1}^{m_n}\left(u^*_{m,i,j}-Y^*_{m,i,j}+Y^*_{m,i,j}\right)$, where $Y^*_{m,i,j}, 1\le m\le m_n$ is a sequence of independent normal random variables with variance $Var\left(u^*_{m,i,j}\right), 1 \le m \le m_n$. Since Lemma 3.5, we have that

$$\begin{aligned}
& P\left(\max_{1\le i<j\le p_n}\left|\sum_{m=1}^{m_n}\left(u^*_{m,i,j}-Y^*_{m,i,j}\right)\right| \ge \delta' \sqrt{n\log p_n}\right) \\
\le\ & p_n^2 P\left(\left|\sum_{i=1}^{m_n}\left(u^*_{i,1,2}-Y^*_{i,1,2}\right)\right| \ge \delta' \sqrt{n\log p_n}\right) \le C\frac{p_n^2\sum_{i=1}^{m_n} E\left|u^*_{i,1,2}\right|^q}{(n\log n)^{\frac{q}{2}}} \\
\le\ & C\frac{p_n^2 m_n z_n^{\frac{q}{2}}(E|X^i_{k,1}X^i_{k,2}|^2 I\{|X^i_{k,1}X^i_{k,2}| \le n^{\mu}\})^{\frac{q}{2}}}{(n\log p_n)^{\frac{q}{2}}} \\
& +C\frac{p_n^2 m_n z_n E|X^i_{k,1}X^i_{k,2}|^q I\{|X^i_{k,1}X^i_{k,2}| \le n^{\mu}\}}{(n\log p_n)^{\frac{q}{2}}} \\
\le\ & C\frac{p_n^2 m_n n^{\rho\frac{q}{2}}}{(n\log p_n)^{\frac{q}{2}}} + C\frac{p_n^2 m_n n^{\rho} n^{\mu q}}{(n\log p_n)^{\frac{q}{2}}} \\
\le\ & C\frac{1}{(\log p_n)^{\frac{q}{2}} n^{(1-\rho)\frac{q}{2}+\rho-1-2\tau}} + C\frac{1}{(\log p_n)^{\frac{q}{2}} n^{(1-2\mu)\frac{q}{2}-1-2\tau}} = O\left(\frac{1}{n^{1+\varepsilon'}}\right),
\end{aligned}$$

for sufficiently large q. Thus, since Lemma 3.13, Lemma 3.14 and Borel-Cantelli lemma, we can obtain the result. □

Now we are ready to give the proof of Theorem 2.1.

Proof of Theorem 2.1. Recall W_n in (11). Choose $a = 1/3$, Under the condition that $E|X_{1,1}|^{4+4\tau+\varepsilon} < \infty$, We have from the triangle inequality, Lemma 3.10 and Lemma 3.11 that

$$|nL_n - W_n| \le |||n\mathbf{\Gamma}_n - \mathbf{X}_n'\mathbf{X}_n||| \le 4n^{-1/3}W_n + 2n^{1/3} \qquad a.s. \tag{38}$$

as n is sufficiently large. To prove Theorem 2.1, we need to show that

$$\lim_{n\to\infty}\frac{W_n}{\sqrt{n\log p_n}} = 2\sigma \qquad a.s. \tag{39}$$

Take $T_n = \max_{1\le i<j\le p_n}\left|S'_{n,i,j}\right|$, we can observe that

$$|W_n - T_n| \le \max_{1\le i<j\le p_n}\left|\sum_{k=1}^{n} X_{k,i}X_{k,j} I\left\{|X_{k,i}X_{k,j}| \ge n^{\mu}\right\}\right| =: U_n.$$

Recall $1/2-\delta < \mu < 1/2$, where $\delta > 0$ sufficiently small, and $0 < \delta < \frac{1}{2} - \frac{4+8\tau}{(4+4\tau+\varepsilon)(2+4\tau+\varepsilon)}$. By Lemma 3.2, let

$q = 4 + 4\tau + \varepsilon$, for any $\delta' > 0$,

$$\begin{aligned}
&P\left(U_n \geq \delta' \sqrt{n \log p_n}\right) \\
\leq &P\left(\max_{1\leq i<j\leq p_n} \left|\sum_{k=1}^{n} X_{k,i}X_{k,j} I\left\{|X_{k,i}X_{k,j}| \geq n^{\mu}\right\}\right| \geq \delta' \sqrt{n \log p_n}\right) \\
\leq &p_n^2 \frac{E\left|\sum_{k=1}^{n} X_{k,1}X_{k,2} I\{|X_{k,1}X_{k,2}\right| \geq n^{\mu}\}|^q}{\left(\delta' \sqrt{n \log p_n}\right)^q} \\
\leq &\frac{Cn^{2\tau} n^{\frac{q}{2}} \left(E\left|X_{1,1}X_{1,2}\right|^2 I\{|X_{1,1}X_{1,2}| \geq n^{\mu}\}\right)^{\frac{q}{2}}}{n^{q/2}(\log p_n)^{q/2}} \\
&+ \frac{Cn^{2\tau+1} E\left|X_{1,1}X_{1,2}\right|^q I\{|X_{1,1}X_{1,2}| \geq n^{\mu}\}}{n^{q/2}(\log p_n)^{q/2}} \\
\leq &\frac{Cn^{2\tau} \left(E\left|X_{1,1}\right|^2 I\left\{|X_{1,1}| \geq n^{\frac{\mu}{2}}\right\} E|X_{1,2}|^2 + E\left|X_{1,2}\right|^2 I\left\{|X_{1,2}| \geq n^{\frac{\mu}{2}}\right\} E|X_{1,1}|^2\right)^{\frac{q}{2}}}{(\log p_n)^{q/2}} \\
&+ \frac{Cn^{2\tau+1} E\left|X_{1,1}\right|^q I\left\{|X_{1,1}| \geq n^{\frac{\mu}{2}}\right\} E|X_{1,2}|^q + Cn^{2\tau+1} E\left|X_{1,2}\right|^q I\left\{|X_{1,2}| \geq n^{\frac{\mu}{2}}\right\} E|X_{1,1}|^q}{n^{q/2}(\log p_n)^{q/2}} \\
\leq &\frac{C\left(E|X_{1,2}|^2 + E|X_{1,1}|^2\right)^{\frac{q}{2}}}{(\log p_n)^{\frac{q}{2}} n^{\frac{\mu(2+4\tau+\varepsilon)(4+4\tau+\varepsilon)}{4} - 2\tau}} + \frac{C}{n^{\frac{4+4\tau+\varepsilon}{2} - 2\tau - 1}(\log p_n)^{q/2}} = o\left(\frac{1}{n^{1+\varepsilon'}}\right).
\end{aligned} \tag{40}$$

By the Borel-Cantelli lemma,

$$\frac{U_n}{\sqrt{n \log p_n}} \to 0 \qquad a.s. \quad as \quad n \to \infty.$$

To prove (39), we need to show that

$$\lim_{n\to\infty} \frac{T_n}{\sqrt{n \log p_n}} = 2\sigma \qquad a.s.$$

Lemma 3.15 actually says that $\lim_{n\to\infty} T_n/\sqrt{n \log p_n} = 2\sigma$. The reason we did not combine Lemma 3.15 (36) and (37) as a single limit is that the proof of the combined one is relatively long. (39) then follows immediately from Lemma 3.15, Applying (39), it follows that $4n^{-1/3}W_n = O(n^{1/6} \log p_n)$ almost surely. Hence $nLn - W_n = O(n^{1/3})$ a.s. Theorem 2.1 then follows immediately from (39), then we complete our proof of Theorem 2.1.

4. Examples

In certain applications such as the construction of compressed sensing matrices, the means $\mu_i = E\mathrm{X}^{(i)}$ and $\mu_j = E\mathrm{X}^{(j)}$ are given and one is interested in

$$\tilde{\rho}_{ij} = \frac{(\mathrm{X}^{(i)} - \mu_i)^T(\mathrm{X}^{(j)} - \mu_j)}{||\mathrm{X}^{(i)} - \mu_i|| \cdot ||\mathrm{X}^{(j)} - \mu_j||}, \ 1 \leq i, j \leq p$$

and the corresponding coherence is defined by

$$\tilde{L}_n = \max_{1\leq i<j\leq p} |\tilde{\rho}_{ij}|.$$

Compressed sensing is a fast developing field which provides a novel and efficient data acquisition technique that enables accurate reconstruction of highly undersampled sparse signals. It has a wide range of applications including signal processing, medical imaging, and seismology. In addition, the development of the compressed sensing theory also provides crucial insights into high-dimensional regression in statistics. One of the main goals of compressed sensing is to construct measurement matrices $X_{n\times p}$, such that for any k-sparse signal $\beta \in \mathbb{R}^p$, one can recover β exactly from linear measurements $y = X\beta$ using a computationally efficient recovery algorithm.

Two commonly used conditions are called restricted isometry property (RIP) and mutual incoherence property (MIP). Roughly speaking, the RIP requires subsets of certain cardinality of the columns of X to be close to an orthonormal system and the MIP requires the pairwise correlations among the column vectors of X to be small.

Example 4.1. *Given a matrix Φ and any set T of column indices, we denote by Φ_T the $n \times \#(T)$ matrix composed of these columns. Similarly, for a vector $x \in \mathbb{R}^N$, we denote by x_T the vector obtained by retaining only the entries in x corresponding to the column indices T. We say that a matrix Φ satisfies the Restricted Isometry Property (RIP) of order k if there exists a $\delta_k \in (0,1)$ such that*

$$(1-\delta_k)\,\|x_T\|^2_{\ell_2^N} \le \|\Phi_T x_T\|^2_{\ell_2^n} \le (1+\delta_k)\,\|x_T\|^2_{\ell_2^N}$$

holds for all sets T with $\#T \le k$. This condition is equivalent to requiring that the Grammian matrix $\Phi_T^t\Phi_T$ has all of its eigenvalues in $[1-\delta_k, 1+\delta_k]$ (here Φ_T^t denotes the transpose of Φ_T). This was shown by [5].

Example 4.2. *A commonly used condition is the mutual incoherence property (MIP) which requires the pairwise correlations among the column vectors of X_n to be small. $X_n = (X^{(1)}, X^{(2)}, \cdots, X^{(p)}) = (X_{k,i})_{n\times p}$. It has been shown that the condition*

$$(2k-1)\tilde{L}_n < 1$$

ensures the exact recovery of k-sparse signal β in the noiseless case where $y = X\beta$, and stable recovery of sparse signal in the noisy case where $y = X\beta + z$. Here z is an error vector, not necessarily random. The limiting laws derived in this paper can be used to show how likely a random matrix satisfies the MIP condition. This was shown by [8].

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Data Availability

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations

All authors declare that they have no conflicts of interest.

References

[1] G. Allan, Probability: A graduate course, *Springer*, New York, 2005.
[2] N. Amosova, On limit theorems for the probabilities of moderate deviations, *Vestnik Leningrad. Univ.* 13 (1972), 5-14.
[3] R. Arratia, L. Goldstein, L. Gordon, Two moments suffice for Poisson approximations: the Chen-Stein method, *Ann. Probab.* 17 (1989), 9-25.
[4] Z. Bai, Y. Yin, Limit of the smallest eigenvalue of a large-dimensional sample covariance matrix, *Ann. Probab.* 21 (1993), 1275-1294.
[5] R. Baraniuk, M. Davenport, R. DeVore, M. Wakin, A simple proof of the restricted isometry property for random matrices, *Constr. Approx.* 28 (2008), 253–263.
[6] I. Berkes, W. Philipp, Approximation theorems for independent and weakly dependent random vectors, *Ann. Probab.* 7 (1979), 29-54.
[7] T. Cai, T. Jiang, Limiting laws of coherence of random matrices with applications to testing covariance structure and construction of compressed sensing matrices, *Ann. Statist.* 39 (2011), 1496-1525.
[8] T. Cai, T. Jiang, Phase transition in limiting distributions of coherence of high-dimensional random matrices, *J. Multivariate Anal.* 107 (2012), 24-39.
[9] Y. Chow, H. Teicher, Probability Theory, Independence, Interchangeability, Martingales, (2nd edition), *Springer*, New York, 1988.
[10] J. Fan, T. Jiang, Largest entries of sample correlation matrices from equi-correlated normal populations, *Ann. Probab.* 47 (2019), 3321-3374.
[11] T. Jiang, The asymptotic distributions of the largest entries of sample correlation matrices, *Ann. Appl. Probab.* 14 (2004), 865-880.
[12] T. Jiang, J. Xie, Limiting behavior of largest entry of random tensor constructed by high-dimensional data, *Journal of Theoretical Probability.* 33 (2020), 2380–2400.
[13] D. Li, A. Rosalsky, Some strong limit theorems for the largest entries of sample correlation matrices, *Ann. Appl. Probab.* 16 (2006), 423-447.
[14] D. Li, W. Liu, A. Rosalsky, Necessary and sufficient conditions for the asymptotic distribution of the largest entry of a sample correlation matrix, *Probab. Theory Related Fields* 148 (2010), 5-35.
[15] D. Li, Y. Qi, A. Rosalsky, On Jiang's asymptotic distribution of the largest entry of a sample correlation matrix, *J. Multivariate Anal.* 111 (2012), 256-270.
[16] D. Li, M. Rao, T. Jiang, X. Wang, Complete convergence and almost sure convergence of weighted sums of random variables, *J. Theoret. Probab.* 8 (1995), 49-76.
[17] W. Liu, Z. Lin, Asymptotic distributions of the largest entries of sample correlation matrices under dependence assumptions, *Chinese Ann. Math. Ser.* 29A (2008), 543-556.
[18] W. Liu, Z. Lin, Q. Shao, The asymptotic distribution and Berry-Esseen bound of a new test for independence in high dimension with an application to stochastic optimization, *Ann. Appl. Probab.* 18 (2008), 2337-2366.
[19] A. Sakhanenko, On the accuracy of normal approximation in the invariance principle, *Siberian Adv. Math.* 1 (1991), 58-91.
[20] Q. Shao, Almost sure invariance principles for mixing sequences of random variables, *Stochastic Process. Appl.* 48 (1993), 319-334.
[21] Q. Shao, W. Zhou, Necessary and sufficient conditions for the asymptotic distributions of coherence of ultra-high dimensional random matrices, *Ann. Probab.* 42 (2014), 623-648.
[22] H. Zhao, and Y. Zhang. The asymptotic distributions of the largest entries of sample correlation matrices under an α-mixing assumption, *Acta. Math. Sin.-English Ser.* 38 (2022), 2039-2056.
[23] H. Zhao, and Y. Zhang. A strong limit theorem of the largest entries of a sample correlation matrices under a strong mixing assumption. *Axioms* 12 (2023), 657.
[24] W. Zhou, Asymptotic distribution of the largest off-diagonal entry of correlation matrices, *Trans. Amer. Math. Soc.* 359 (2007), 5345-5363.