Filomat 38:19 (2024), 6663–6680
https://doi.org/10.2298/FIL2419663I

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On a new approach to Fredholm theory in unital C^* -algebras

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Abstract. Motivated by Fredholm theory on the standard Hilbert module over a unital C^* -algebra introduced by Mishchenko and Fomenko, we provide a new approach to axiomatic Fredholm theory in unital C^* -algebras established by Kečkić and Lazović in [16]. Our approach is equivalent to the approach introduced by Kečkić and Lazović, however, we provide new proofs which are motivated by the proofs given by Mishchenko and Fomenko in [18]. Next,we extend Fredholm theory in von Neumann algebras established by Breuer in [4] and [5] to spectral Fredholm theory. We consider 2 by 2 upper triangular operator matrices with coefficients in a von Neumann algebra and give the relationship between the generalized essential spectra in the sense of Breuer of such matrices and of their diagonal entries, thus generalizing in this setting the result by Đorđević in [6]. Finally, we prove that if a generalized Fredholm operator in the sense of Breuer has 0 as an isolated point of its spectrum, then the corresponding spectral projection is finite.

1. Introduction

The Fredholm and semi-Fredholm theory on Hilbert and Banach spaces started by studying the integral equations introduced in the pioneering work by Fredholm in 1903 in [7]. After that, the abstract theory of Fredholm and semi-Fredholm operators on Hilbert and Banach spaces was further developed in numerous papers and books such as [2], [3]. In addition to classical semi-Fredholm theory on Hilbert and Banach spaces, several generalizations of this theory have been considered. Breuer for example started the development of Fredholm theory in von-Neumann algebras as a generalization of the classical Fredholm theory for operators on Hilbert spaces. In [4] and [5] he introduced the notion of a Fredholm operator in a von Neumann algebra and established its main properties. On the other hand, Fredholm theory on Hilbert C^*-modules as another generalization of the classical Fredholm theory on Hilbert spaces was started by Mishchenko and Fomenko. In [18] they introduced the notion of a Fredholm operator on the standard Hilbert C^*-module and proved a generalization in this setting of some of the main results from the classical Fredholm theory.

The interest for considering these generalizations comes from the theory of pseudo differential operators acting on manifolds. The classical theory can be applied in the case of compact manifolds, but not in the case of non-compact ones. Even operators on Euclidian spaces are hard to study, for example Laplacian is not Fredholm. Kernels and cokernels of many operators are infinite dimensional Banach spaces, however,

2020 *Mathematics Subject Classification*. Primary MSC 47A53 mandatory; Secondary MSC 46L08.
Keywords. Fredholm theory, Hilbert module,von Neumann algebra, finite projections, K-group, index
Received: 22 October 2023; Accepted: 05 April 2024
Communicated by Dragan S. Djordjević
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they may also at the same time be finitely generated Hilbert modules over some appropriate C^*-algebra. Similarly, orthogonal projections onto kernels and cokernels of many bounded linear operators on Hilbert spaces are not finite rank projections in the classical sense, but they are still finite projections in an appropriate von Neumann algebra. Therefore, many operators that are not semi-Fredholm in the classical sense may become semi-Fredholm in a more general sense if we consider them as operators on Hilbert C^*-modules or as elements of an appropriate von Neumann algebra. Hence, by studying these generalized semi-Fredholm operators, we get a proper extension of the classical semi-Fredholm theory to new classes of operators.

Now, Kečkić and Lazović in [16] established an axiomatic approach to Fredholm theory. They introduced the notion of a finite type element in a unital C^*-algebra which generalizes the notion of the compact operator on the standard Hilbert C^*-module and the notion of a finite operator in a properly infinite von Neumann algebra. They also introduced the notion of a Fredholm type element with respect to the ideal of these finite type elements. This notion is at a same time a generalization of the classical Fredholm operator on a Hilbert space, Fredholm C^*-operator on the standard Hilbert C^*-module defined by Mishchenko and Fomenko and the Fredholm operator on a properly infinite von Neumann algebra defined by Breuer. The index of this Fredholm type element takes values in the K-group. They showed that the set of Fredholm type elements in a unital C^*-algebra is open in the norm topology and they proved a generalization of the Atkinson theorem. Moreover, they proved the multiplicativity of the index in the K-group and that the index is invariant under perturbations of Fredholm type elements by finite type elements.
In [15] we went further in this direction and defined semi-Fredholm and semi-Weyl type elements in a unital C^*-algebra. We investigated and proved several properties of these elements as a generalization of the results from the classical semi-Fredholm and semi-Weyl theory on Hilbert and Banach spaces.

In Section 3 of this paper we introduce a new approach to axiomatic Fredholm theory in unital C^*-algebras and we prove that this approach is in fact equivalent to the above mentioned approach developed by Kečkić and Lazović. In this new approach we use the fact that a unital C^*-algebra $\mathcal{A}$ is isometrically isomorphic to the algebra of all $\mathcal{A}-$ linear bounded operators on $\mathcal{A}$ when $\mathcal{A}$ is considered as a Hilbert module over itself. This enables us to apply some known results from operator theory on Hilbert C^*-modules, such as the result concerning the complementability of the kernel and the image of a closed range C^*-operator (for more details, see [17, Theorem 2.3.3]) and in that way we bypass several technical lemmas from the paper by Kečkić and Lazović [16] which require long proofs.

Next, in Section 4 we extend Fredholm theory in von Neumann algebras established in [4] and [5] to spectral Fredholm theory in von Neumann algebras generalizing in this setting the results from the classical spectral semi-Fredholm theory for operators on Hilbert and Banach spaces. The concept of invertibility up to a pair of orthogonal projections given in [16] plays the key role in this section. In Proposition 4.9 in Section 4 we consider 2 by 2 upper triangular operator matrices with coefficients in a von Neumann algebra and describe the relationship between the essential spectra of such matrices and of their diagonal entries,generalizing in this setting the result by Đorđević in [6]. These essential spectra which we consider are induced by the class of generalized Fredholm operators in the sense of Breuer. Next, in Section 4 we consider isolated points of the spectrum of an operator F in a von Neumann algebra $\mathcal{A}$. We prove that if F is generalized Fredholm operator in the sense of Breuer and has 0 as an isolated point of its spectrum, then the spectral projection corresponding to 0 is a finite operator in $\mathcal{A}$. Then we introduce a concept of generalized Browder operators in $\mathcal{A}$ as a proper generalization of the classical Browder operators on Hilbert spaces (Fredholm operators with finite ascent and descent), and we show that the class of these generalized Browder operators is a subclass of generalized Fredholm operators in the sense of Breuer. As a consequence of our result regarding finiteness of spectral projections corresponding to isolated points of the spectrum, we prove that if a generalized Fredholm operator in the sense of Breuer has 0 as an isolated point of its spectrum, then it is generalized Browder in the sense of our definition. This is a generalization of the well known result from the classical Fredholm theory on Hilbert spaces given in [19, Theorem 3.1].

2. Preliminaries

Throughout this paper $\mathcal{A}$ always stands for a unital C^* -algebra and $B(\mathcal{A})$ denotes the set of all $\mathcal{A}$ - linear bounded operators on $\mathcal{A}$ when $\mathcal{A}$ is considered as a right Hilbert module over itself. Since $\mathcal{A}$ is self-dual

Hilbert module over itself, by [17, Proposition 2.5.2] all operators that belong to $B(\mathcal{A})$ are adjointable. Moreover, by [17, Corollary 2.5.3] the set $B(\mathcal{A})$ is a unital C^* -algebra.
Let V be the map from $\mathcal{A}$ into $B(\mathcal{A})$ given by $V(a) = L_a$ for all $a \in \mathcal{A}$ where L_a is the corresponding left multiplier by a. Then V is an isometric *-homomorphism, and, since $\mathcal{A}$ is unital, it follows that V is in fact an isomorphism. Thus, $B(\mathcal{A})$ can be identified with $\mathcal{A}$ by considering the left multipliers.
We recall now the following definition.

Definition 2.1. *[16, Definition 1.1] Let $\mathcal{A}$ be an unital C^*-algebra, and $\mathcal{F} \subseteq \mathcal{A}$ be a subalgebra which satisfies the following conditions:*
(i) $\mathcal{F}$ is a selfadjoint ideal in $\mathcal{A}$, i.e. for all $a \in \mathcal{A}, b \in \mathcal{F}$ there holds $ab, ba \in \mathcal{F}$, and $a \in \mathcal{F}$ implies $a^* \in \mathcal{F}$;
(ii) There is an approximate unit $p_\alpha \in \mathcal{F}$ consisting of projections;
(iii) If $p, q \in \mathcal{F}$ are projections, then there exists $v \in \mathcal{A}$, such that $vv^* = q$ and $v^*v \perp p$, i.e. $v^*v + p$ is a projection as well.
We shall call the elements of such an ideal finite type elements. Henceforward we shall denote this ideal by $\mathcal{F}$.

Let V be the isometric *-isomorphism given above. If $\mathcal{F}$ is an ideal of finite type elements in $\mathcal{A}$, then it is not hard to see that $V(\mathcal{F})$ is an ideal of finite type elements in $B(\mathcal{A})$, so we may identify $\mathcal{F}$ with $V(\mathcal{F})$.

Definition 2.2. *[16, Definition 1.2] Let $\mathcal{A}$ be a unital C^*-ideal, and let $\mathcal{F} \subseteq \mathcal{A}$ be an algebra of finite type elements. In the set $Proj(\mathcal{F})$ we define the equivalence relation:*

$$p \sim q \Leftrightarrow \exists v \in \mathcal{A}\; vv^* = p,\; v^*v = p,$$

i.e. Murray - von Neumann equivalence. The set $S(\mathcal{F}) = Proj(\mathcal{F}) \;/\; \sim$ is a commutative semigroup with respect to addition, and the set,$K(\mathcal{F}) = G(S(\mathcal{F}))$, where G denotes the Grothendic functor, is a commutative group.

The following concept will be of crucial importance in the rest of the paper.

Definition 2.3. *[16, Definition 2.1] Let $a \in \mathcal{A}$ and p, q be projections in $\mathcal{A}$. We say that a is invertible up to pair (p, q) if there exists some $b \in \mathcal{A}$ such that*

$$(1 - q)a(1 - p)b = 1 - q,\; b(1 - q)a(1 - p) = 1 - p.$$

We refer to such b as almost inverse of a, or (p, q)-inverse of a.

Finally, we recall the definitions of Fredholm and semi-Fredholm elements in a unital C^*- algebra.

Definition 2.4. *[16, Definition 2.2] [15, Definition 5] We say that $a \in \mathcal{A}$ is of Fredholm type (or abstract Fredholm element) with respect to the ideal $\mathcal{F}$ if there are projections $p, q \in \mathcal{F}$ such that a is invertible up to (p, q). The index of the element a (or abstract index) is the element of the group $K(\mathcal{F})$ defined by*

$$ind(a) = ([p], [q]) \in K(\mathcal{F}),$$

or less formally

$$ind(a) = [p] - [q].$$

Definition 2.5. *[15, Definition 5] Let $a \in \mathcal{A}$. We say that a is an upper semi-Fredholm type element if a is invertible up to pair (p, q) where $p \in \mathcal{F}$. Similarly, we say that a is a lower semi-Fredholm type element, however in this case we assume that $q \in \mathcal{F}$ (and not p).*

Next, we recall the following two lemmas.

Lemma 2.6. *[15, Lemma 1] Let $a \in \mathcal{A}$ and $p, q \in \mathcal{F}$. Then a is invertible up to pair (p, q) if and only if a^* is invertible up to pair (q, p).*

Lemma 2.7. *[15, Lemma 2] Let $a \in \mathcal{A}$ and p, q, p', q' be projections in $\mathcal{A}$. Suppose that $p, q, p' \in \mathcal{F}$. If a is invertible up to pair (p, q) and also invertible up to pair (p', q'), then $q' \in \mathcal{F}$. Similarly, if instead of p, q, p' we have that $p, q, q' \in \mathcal{F}$, then we must have that $p' \in \mathcal{F}$ as well.*

From the proof of Lemma 2.7 we can also deduce the following corollaries.

Corollary 2.8. *[15, Corollary 4] Let $a \in \mathcal{A}$. If a is invertible both up to pair (p, q) and up to pair (p, q'), then $q \sim q'$.*

Corollary 2.9. *Let $a \in \mathcal{A}$. If a is invertible up to pairs (p, q) and (p', q') where $p, p' \in \mathcal{F}$, then there exist projections $\tilde{q}, \tilde{q}', \tilde{\tilde{q}}$ and $\tilde{\tilde{q}}'$ in $\mathcal{A}$ such that $\tilde{\tilde{q}}, \tilde{\tilde{q}}' \in \mathcal{F}$, $\tilde{q}\tilde{\tilde{q}} = \tilde{q}'\tilde{\tilde{q}} = 0$, $q \sim \tilde{q}, q' \sim \tilde{q}'$, $\tilde{q} + \tilde{\tilde{q}} \sim \tilde{q}' + \tilde{\tilde{q}}'$ and a is invertible up to pairs $(p, \tilde{q})$ and $(p'', \tilde{q}')$ for some projection $p'' \sim p'$. A similar statement holds if we instead of p and p' have that $q, q' \in \mathcal{F}$, however, in this case there exist projections $\tilde{p}, \tilde{p}', \tilde{\tilde{p}}, \tilde{\tilde{p}}'$ in $\mathcal{A}$ such that $\tilde{\tilde{p}}, \tilde{\tilde{p}}' \in \mathcal{F}$, $\tilde{p}\tilde{\tilde{p}} = \tilde{p}'\tilde{\tilde{p}}' = 0$, $p \sim \tilde{p}, p' \sim \tilde{p}', \tilde{p} + \tilde{\tilde{p}} \sim \tilde{p}' + \tilde{\tilde{p}}'$ and a is invertible up to pairs $(\tilde{p}, q)$ and $(\tilde{p}', q'')$ for some projection $q'' \sim q'$.*

Proof. By [16, Proposition 2.8] there exists projection $\tilde{q}$ in $\mathcal{A}$ such that $\tilde{q} \sim q$, $\tilde{q}a(1 - p) = 0$ and a is invertible up to pair $(p, \tilde{q})$. Then, by the proof of Lemma 2.7, there is an approximate unit $\{p_\alpha\}$ for $\mathcal{F}$, projections p'', q'' in $\mathcal{A}$ and nets of projections $\{q_\alpha\}$ and $\{q''_\alpha\}$ in $\mathcal{A}$ such that $p' \sim p'' \leq p_\alpha$ for all α, $q'' \sim q'$, a is invertible up to pair (p'', q'') and $q_\alpha - \tilde{q} \sim p_\alpha - p, q''_\alpha - q'' \sim p_\alpha - p''$ and $q''_\alpha \sim q_\alpha$. For any fixed α, set $\tilde{\tilde{q}} = q_\alpha - \tilde{q}$, $\tilde{q}' = q''$, $\tilde{\tilde{q}}' = q''_\alpha - q''$. This proves the first statement.
The second statement can be proved by passing to the adjoints and applying Lemma 2.6. □

Furthermore, we recall also the following definition regarding Hilbert modules.

Definition 2.10. *[17, Definition 2.3.1] A closed submodule $\mathcal{N}$ in a Hilbert C^*-module $\mathcal{M}$ is called (topologically) complementable if there exists a closed submodule $\mathcal{L}$ in $\mathcal{M}$ such that $\mathcal{N} + \mathcal{L} = \mathcal{M}, \mathcal{N} \cap \mathcal{L} = 0$.*

By the symbol $\tilde{\oplus}$ we denote the direct sum of modules as given in [17].

Thus, if M is a Hilbert C^*-module and M_1, M_2 are two closed submodules of M, we write $M = M_1 \tilde{\oplus} M_2$ if $M_1 \cap M_2 = \{0\}$ and $M_1 + M_2 = M$. If, in addition M_1 and M_2 are mutually orthogonal, then we write $M = M_1 \oplus M_2$.

Remark 2.11. *If $\sqcap \in B(\mathcal{A})$ is a (skew) projection, then, since $Im\sqcap$ is closed, by [17, Theorem 2.3.3] we get that $Im\sqcap$ is orthogonally complementable. Hence, every closed and complementable submodule M of $\mathcal{A}$ is orthogonally complementable. The corresponding orthogonal projection onto M will be denoted by P_M throughout the paper.*
However, the assumption that $\mathcal{A}$ is unital is indeed necessary. As the reviewer pointed out, for the counter example, let $M_2(\mathcal{A})$ denote the C^*-algebra consisting of 2 by 2 matrices with coefficients in $\mathcal{A}$ and consider the C^*-subalgebra B of $M_2(\mathcal{A})$ consisting of the diagonal matrices. Suppose that $\mathcal{A}$ has a proper two-sided ideal I. Take the Hilbert B-submodule M consisting of all diagonal matrices with the upper entry from $\mathcal{A}$ and the lower entry from I. Then it can be decomposed into the direct topological sum of two Hilbert B-submodules, one consisting of all diagonal matrices with equal entries from I, and one consisting of all diagonal matrices having arbitrary entries from $\mathcal{A}$ in the upper corner and zero in the lower corner. The image of the skew projection from M to the first summand has closed range, but the image of it is not an orthogonal direct summand.

Remark 2.12. *If $\sqcap \in B(\mathcal{A})$ has closed range and $P_{Im\sqcap} \in \mathcal{F}$, then, since $P_{Im\sqcap}\sqcap = \sqcap$ and $\mathcal{F}$ is an ideal, we get that $\sqcap \in \mathcal{F}$.*

At the end of this section we give also the following technical results.

Lemma 2.13. *Let $T \in B(\mathcal{A})$ and suppose that ImT is closed. Then $Im(T^*T)^{1/2}$ is closed.*

Proof. By the proof of [17, Theorem 2.3.3] we have that ImT^* is closed when ImT is closed. In addition, ImT is orthogonally complementable in $\mathcal{A}$ by [17, Theorem 2.3.3] . Let P denote the orthogonal projection onto ImT. Then $T = PT$, hence $T^* = T^*P$. It follows that $ImT^* = ImT^*P = ImT^*T$, so ImT^*T is closed. Hence,

$$\mathcal{A} = ImT^*T \oplus \ker T^*T$$

by [17, Theorem 2.3.3], and T^*T maps ImT^*T isomorphically onto itself, which gives that $((T^*T)^{1/2})_{|ImT^*T}$ is bounded below.
Next, it is obvious that $\ker T^*T = \ker(T^*T)^{1/2}$. Indeed, if $x \in \ker T^*T$, then

$$\langle (T^*T)^{1/2}x, (T^*T)^{1/2}x \rangle = \langle (T^*T)x, x \rangle = 0,$$

so $\ker T^*T \subseteq \ker(T^*T)^{1/2}$, whereas the opposite inclusion is obvious. Thus we obtain that

$$\mathcal{A} = ImT^*T \oplus \ker(T^*T)^{1/2},$$

so $(T^*T)^{1/2}(ImT^*T) = Im(T^*T)^{1/2}$. However, since $((T^*T)^{1/2})_{|ImT^*T}$ is bounded below, we must have that $Im(T^*T)^{1/2}$ is closed. $\square$

Corollary 2.14. *Let $T \in B(\mathcal{A})$ and suppose that ImT is closed. Then T admits polar decomposition.*

Proof. By Lemma 2.13 we have that $Im(T^*T)^{1/2}$ is closed. Hence, by [17, Theorem 2.3.3] we get that ImT and $Im(T^*T)^{1/2}$ are orthogonally complementable, so T admits polar decomposition. $\square$

3. Axiomatic Fredholm theory in unital C^* -algebras

We start with the following lemma.

Lemma 3.1. *Let $\tilde{P}, \tilde{Q} \in Proj(\mathcal{A})$. Then $\tilde{P} \sim \tilde{Q}$ if and only if $Im\tilde{P} \cong Im\tilde{Q}$.*

Proof. Suppose that $Im\tilde{P} \cong Im\tilde{Q}$ and let $\mathcal{U}$ be an isomorphism from $Im\tilde{P}$ onto $Im\tilde{Q}$. Set $T := J\mathcal{U}\tilde{P}$ where $J : Im\tilde{Q} \to \mathcal{A}$ is inclusion. Then $T \in B(\mathcal{A})$ and $ImT = Im\tilde{Q}$ is closed. Hence, by Corollary 2.14 we deduce that T admits polar decomposition. The partial isometry V from this decomposition satisfies that $V^*V = P_{\ker T^{\perp}} = \tilde{P}$, and $VV^* = P_{ImT} = \tilde{Q}$.
Conversely, if $\tilde{P} \sim \tilde{Q}$, then there exists some $V \in \mathcal{A}$ such that $VV^* = \tilde{Q}$ and $V^*V = \tilde{P}$. Then $\tilde{Q}V\tilde{P}$ is the desired isomorphism. $\square$

The next proposition will be the key tool for proving the results in the rest of this section.

Proposition 3.2. *Let $\{P_\alpha\}_\alpha$ be an approximate unit for $\mathcal{F}$ consisting of orthogonal projections and N be a closed, complementable submodule of $\mathcal{A}$ such that $P_N \in \mathcal{F}$. Then there exists some element α_0 of the index net of the approximate unit for $\mathcal{F}$ and a closed submodule M of $\mathcal{A}$ such that $Im(I - P_{\alpha_0}) \subseteq M$ and $\mathcal{A} = M\tilde{\oplus}N$.*

Proof. Choose α_0 sufficiently large such that $\| P_N - P_{\alpha_0}P_N \|< 1$. Then we get that $\| P_N - P_N P_{\alpha_0} P_N \|< 1$ which gives that $P_N P_{\alpha_0} P_N$ is invertible in the corner C^* - algebra $P_N\mathcal{A}P_N$. It is not hard to deduce then that $P_{\alpha_0|_N}$ must be bounded below. So $ImP_{\alpha_0}P_N$ is closed, thus orthogonally complementable by [17, Theorem 2.3.3]. Let $M = (ImP_{\alpha_0}P_N)^{\perp}$. Then $Im(I - P_{\alpha_0}) \subseteq M$.
Set $\tilde{P}$ to be the orthogonal projection onto $ImP_{\alpha_0}P_N$. Then $\tilde{P} \leq P_{\alpha_0}$ and therefore $\tilde{P}_{|_N} = \tilde{P}P_{\alpha_0|_N} = P_{\alpha_0|_N}$. Since $P_{\alpha_0|_N}$ is an isomorphism onto $ImP_{\alpha_0}P_N$, it follows that $\tilde{P}_{|_N}$ an isomorphism onto $ImP_{\alpha_0}P_N = Im\tilde{P}$. It is then not hard to deduce that $\mathcal{A} = M\tilde{\oplus}N$. $\square$

The following three technical lemmas are also needed for the proofs of the main results later in this section.

Lemma 3.3. *Let N be a closed complementable submodule of $\mathcal{A}$ such that $P_N \in \mathcal{F}$. Suppose that $F \in B(\mathcal{A})$ is such that $F_{|_N}$ is an isomorphism. Then $F(N)$ is complementable and $P_{F(N)} \in \mathcal{F}$.*

Proof. Since N is complementable, it is orthogonally complementable by Remark 2.11. Now, $F(N) = ImFP_N$, so by [17, Theorem 2.3.3], $F(N)$ is complementable in $\mathcal{A}$. Since $F_{|_N}$ is an isomorphism onto $F(N)$, we have that $P_N \sim P_{F(N)}$ by Lemma 3.1. $\square$

Lemma 3.4. *Let M, N be two closed, complementable submodules of $\mathcal{A}$ such that $P_N, P_M \in \mathcal{F}$. Suppose that $M \cap N = \{0\}$ and that $M + N$ is closed and complementable in $\mathcal{A}$. Then $P_{M\tilde{\oplus}N} \in \mathcal{F}$ and $[P_{M\tilde{\oplus}N}] = [P_M] + [P_N]$.*

Proof. Note first that by Remark 2.11 the submodules M, N and $M\tilde{\oplus}N$ are orthogonally complementable in $\mathcal{A}$. Since M is orthogonally complementable and $M \subseteq M\tilde{\oplus}N$, by [9, Lemma 2.6] we have that M is orthogonally complementable in $M\tilde{\oplus}N$. Let R be the orthogonal complement of M in $M\tilde{\oplus}N$. Then, since $M\tilde{\oplus}N = M \oplus R$, it is clear that $R \cong N$. By Lemma 3.1 $P_R \sim P_N \in \mathcal{F}$. Indeed, since $M\tilde{\oplus}N$ is orthogonally complementable in $\mathcal{A}$, then R is orthogonally complementable in $\mathcal{A}$. Now, we have $P_{M\tilde{\oplus}N} = P_M + P_R \in \mathcal{F}$. Moreover, $[P_{M\tilde{\oplus}N}] = [P_M] + [P_R] = [P_M] + [P_N]$, as $P_R \sim P_N$. $\square$

Lemma 3.5. *Let $F \in B(\mathcal{A})$ and suppose that*

$$\mathcal{A} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = \mathcal{A}$$

is a decomposition with respect to which F has the matrix $\begin{pmatrix} F_1 & 0 \\ 0 & F_4 \end{pmatrix}$ where F_1 is an isomorphism. Then, with respect to the decomposition

$$\mathcal{A} = N_1^{\perp} \oplus N_1 \xrightarrow{F} F(N_1^{\perp}) \tilde{\oplus} N_2 = \mathcal{A},$$

F has the matrix $\begin{pmatrix} \tilde{F}_1 & 0 \\ 0 & F_4 \end{pmatrix}$ where $\tilde{F}_1$ is an isomorphism.

Proof. Let $\mathcal{A} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = \mathcal{A}$ be a decomposition with respect to which F has the matrix $\begin{pmatrix} F_1 & 0 \\ 0 & F_4 \end{pmatrix}$ where F_1 is an isomorphism. Observe first that, since N_1 is orthogonally complementable by Remark 2.11, then

$$\mathcal{A} = M_1 \tilde{\oplus} N_1 = N_1 \oplus N_1^{\perp},$$

so $\sqcap_{M_1|_{N_1^{\perp}}}$ is an isomorphism from $N_1^{\perp}$ onto M_1, where $\sqcap_{M_1|_{N_1^{\perp}}}$ stands for the projection onto M_1 along N_1 restricted to $N_1^{\perp}$. Observe next that, since $F(M_1) = M_2$ and $F(N_1) \subseteq N_2$, we have $\sqcap_{M_2} F_{|_{N_1^{\perp}}} = F \sqcap_{M_1|_{N_1^{\perp}}}$, where $\sqcap_{M_2}$ stands for the projection onto M_2 along N_2. Since $F_{|_{M_1}}$ is an isomorphism, it follows that $\sqcap_{M_2} F_{|_{N_1^{\perp}}} = F \sqcap_{M_1|_{N_1^{\perp}}}$ is an isomorphism as a composition of isomorphisms. Hence, with respect to the decomposition

$$\mathcal{A} = N_1^{\perp} \oplus N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = \mathcal{A},$$

F has the matrix $\begin{pmatrix} \tilde{F}_1 & 0 \\ \tilde{F}_3 & F_4 \end{pmatrix}$ where $\tilde{F}_1 = \sqcap_{M_2} F_{|_{N_1^{\perp}}}$ is an isomorphism. Using the technique of diagonalization as in the proof of [17, Lemma 2.7.10], we deduce that there exists an isomorphism V such that

$$\mathcal{A} = N_1^{\perp} \oplus N_1 \xrightarrow{F} V(M_2) \tilde{\oplus} V(N_2) = \mathcal{A}$$

is a decomposition with respect to which F has the matrix $\begin{pmatrix} \tilde{\tilde{F}}_1 & 0 \\ 0 & F_4 \end{pmatrix}$ where $\tilde{\tilde{F}}_1$ is an isomorphism. Moreover, by the construction of V we have $V(N_2) = N_2$. Hence

$$\mathcal{A} = F(N_1^{\perp}) \tilde{\oplus} N_2.$$

Thus, with respect to the decomposition

$$\mathcal{A} = N_1^{\perp} \oplus N_1 \xrightarrow{F} F(N_1^{\perp}) \tilde{\oplus} N_2 = \mathcal{A},$$

F has the desired matrix. □

In exactly the same way we can prove the following corollary.

Corollary 3.6. *Let $F \in B(\mathcal{A})$ and suppose that*

$$\mathcal{A} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = \mathcal{A}$$

is a decomposition with respect to which F has the matrix $\begin{pmatrix} F_1 & 0 \\ 0 & F_4 \end{pmatrix}$ where F_1 is an isomorphism. If there exists a closed submodule $\tilde{M}$ of $\mathcal{A}$ such that $\mathcal{A} = \tilde{M} \tilde{\oplus} N_1$, then F has the matrix $\begin{pmatrix} \tilde{F}_1 & 0 \\ 0 & F_4 \end{pmatrix}$ with respect to the decomposition

$$\mathcal{A} = \tilde{M} \tilde{\oplus} N_1 \xrightarrow{F} F(\tilde{M}) \tilde{\oplus} N_2 = \mathcal{A}$$

where $\tilde{F}_1$ is an isomorphism.

Thanks to Lemma 3.5 we obtain the following useful characterization of invertibility up to a pair of orthogonal projections.

Lemma 3.7. *Let $F \in \mathcal{A}$. Then F has the matrix $\begin{pmatrix} F_1 & 0 \\ 0 & F_4 \end{pmatrix}$ with respect to the decomposition*

$$\mathcal{A} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = \mathcal{A}$$

where F_1 is an isomorphism if and only if F is invertible up to (P, Q) where $P \sim P_{N_1}$ and $Q \sim P_{N_2}$.

Proof. By the proof of Lemma 3.5, if F has a decomposition

$$\mathcal{A} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = \mathcal{A},$$

with respect to which F has the matrix then $\begin{pmatrix} F_1 & 0 \\ 0 & F_4 \end{pmatrix}$ where F_1 is an isomorphism, then F is invertible up to pair ($P_{N_1}, P_{F(N_1^{\perp})^{\perp}}$). However, we have

$$\mathcal{A} = F(N_1^{\perp}) \oplus F(N_1^{\perp})^{\perp} = F(N_1^{\perp}) \tilde{\oplus} N_2,$$

hence $N_2 \cong F(N_1^{\perp})^{\perp}$. By Lemma 3.1 , $P_{N_2} \sim P_{F(N_1^{\perp})^{\perp}}$.
Conversely, if F is invertible up to pair of orthogonal projections (P, Q), then by the proof of [17, Lemma 2.7.10], F has decomposition

$$\mathcal{A} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = \mathcal{A}$$

where $N_1 \cong ImP$ and $N_2 \cong ImQ$. By Lemma 3.1 we have that $P_{N_1} \sim P$ and $P_{N_2} \sim Q$. □

We introduce now the following definition.

Definition 3.8. *Let $F \in B(\mathcal{A})$. We say that $F \in \mathcal{MK}\Phi(\mathcal{A})$ if there exists a decomposition*

$$\mathcal{A} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = \mathcal{A}$$

with respect to which F has the matrix $\begin{pmatrix} F_1 & 0 \\ 0 & F_4 \end{pmatrix}$ where F_1 is an isomorphism and $P_{N_1}, P_{N_2} \in \mathcal{F}$. We put then

$$indexF = [P_{N_1}] - [P_{N_2}]$$

in $K(\mathcal{F})$.

Notice that since N_1 and N_2 are closed and complementable, by Remark 2.11 they are orthogonally complementable, hence P_{N_1} and P_{N_2} are well defined. It remains to prove that the index is well defined.

Theorem 3.9. *The index is well defined.*

Proof. Let

$$\mathcal{A} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = \mathcal{A}$$
$$\mathcal{A} = M_1' \tilde{\oplus} N_1' \xrightarrow{F} M_2' \tilde{\oplus} N_2' = \mathcal{A}$$

be two $\mathcal{MK}\Phi$ -decompositions for F. By Proposition 3.2 there exist closed submodules $\tilde{M}_1, \tilde{M}_1{}'$ of $\mathcal{A}$ such that $\mathcal{A} = \tilde{M}_1 \tilde{\oplus} N_1 = \tilde{M}_1{}' \tilde{\oplus} N_1'$ and $Im(I - P_\alpha) \subseteq \tilde{M}_1 \cap \tilde{M}_1{}'$ for sufficiently large α. By Corollary 3.6 the operator F has the matrices $\begin{pmatrix} F_1 & 0 \\ 0 & F_4 \end{pmatrix}, \begin{pmatrix} F_1' & 0 \\ 0 & F_4' \end{pmatrix}$, with respect to the decompositions

$$\tilde{M}_1 \tilde{\oplus} N_1 \xrightarrow{F} F(\tilde{M}_1) \tilde{\oplus} N_2,$$

$$\tilde{M}_1{}'\tilde{\oplus}N_1' \xrightarrow{F} F(\tilde{M}_1{}')\tilde{\oplus}N_2',$$

respectively, where F_1 and F_1' are isomorphisms.
Now, since $Im(I-P_\alpha) \subseteq \tilde{M}_1 \cap \tilde{M}_1{}'$, by [9, Lemma 2.6] there exist closed submodules $\mathcal{R}$ and $\mathcal{R}'$ of $\mathcal{A}$ such that $\tilde{M}_1 = Im(I-P_\alpha) \oplus \mathcal{R}$ and $\tilde{M}_1{}' = Im(I-P_\alpha) \oplus \mathcal{R}'$. As in the proof of [17, Lemma 2.7.13] we obtain new $\mathcal{MK}\Phi$ - decompositions

$$\mathcal{A} = Im(I-P_\alpha)\tilde{\oplus}\mathcal{R}\tilde{\oplus}N_1 \xrightarrow{F} F(Im(I-P_\alpha))\tilde{\oplus}F(\mathcal{R})\tilde{\oplus}N_2 = \mathcal{A},$$

$$\mathcal{A} = Im(I-P_\alpha)\tilde{\oplus}\mathcal{R}'\tilde{\oplus}N_1' \xrightarrow{F} F(Im(I-P_\alpha))\tilde{\oplus}F(\mathcal{R}')\tilde{\oplus}N_2' = \mathcal{A}$$

for the operator F. Indeed, since

$$\mathcal{A} = \tilde{M}_1\tilde{\oplus}N_1 = Im(I-P_\alpha)\tilde{\oplus}\mathcal{R}\tilde{\oplus}N_1 = Im(I-P_\alpha) \oplus Im\ P_\alpha,$$

we have that $\mathcal{R}\tilde{\oplus}N_1 \cong Im\ P_\alpha$. Hence $P_{\mathcal{R}\tilde{\oplus}N_1} \sim P_\alpha$ by Lemma 3.1, so $P_{\mathcal{R}\tilde{\oplus}N_1} \in \mathcal{F}$. Since $P_{\mathcal{R}} \leq P_{\mathcal{R}\tilde{\oplus}N_1}$, it follows that $P_{\mathcal{R}} \in \mathcal{F}$ (as $P_{\mathcal{R}} = P_{\mathcal{R}}P_{\mathcal{R}\tilde{\oplus}N_1}$ and $\mathcal{F}$ is an ideal). Similarly, $P_{\mathcal{R}'\tilde{\oplus}N_1'} \sim P_\alpha$ and thus $P_{\mathcal{R}'}, P_{\mathcal{R}'\tilde{\oplus}N_1'} \in \mathcal{F}$. Then, by Lemma 3.3, as $F_{|\mathcal{R}}$ and $F_{|\mathcal{R}'}$ are isomorphisms, we get that $P_{F(\mathcal{R})}, P_{F(\mathcal{R}')} \in \mathcal{F}$. Hence, by Lemma 3.4 we obtain that $P_{F(\mathcal{R})\tilde{\oplus}N_2}, P_{F(\mathcal{R})\tilde{\oplus}N_2'} \in \mathcal{F}$ and $[P_{F(\mathcal{R})\tilde{\oplus}N_2}] = [P_{F(\mathcal{R})}] + [P_{N_2}]$, $[P_{F(\mathcal{R}')\tilde{\oplus}N_2'}] = [P_{F(\mathcal{R}')}] + [P_{N_2'}]$.
Next, since $P_{\mathcal{R}} \sim P_{F(\mathcal{R})}$ and $P_{\mathcal{R}'} \sim P_{F(\mathcal{R}')}$ we have that $[P_{F(\mathcal{R})}] = [P_{\mathcal{R}}]$ and $[P_{F(\mathcal{R}')}] = [P_{\mathcal{R}'}]$. By Lemma 3.4 we also have $[P_\alpha] = [P_{\mathcal{R}\tilde{\oplus}N_1}] = [P_{\mathcal{R}}] + [P_{N_1}]$ and $[P_\alpha] = [P_{\mathcal{R}'\tilde{\oplus}N_1'}] = [P_{\mathcal{R}'}] + [P_{N_1'}]$.

On the other hand, since

$$\mathcal{A} = F(Im\ P_\alpha)\tilde{\oplus}F(\mathcal{R})\tilde{\oplus}N_2 = F(Im\ P_\alpha)\tilde{\oplus}F(\mathcal{R}')\tilde{\oplus}N_2',$$

we have that $F(\mathcal{R})\tilde{\oplus}N_2 \cong F(\mathcal{R}')\tilde{\oplus}N_2'$, hence, by Lemma 3.1 and Lemma 3.4 we get that $[P_{F(R)}] + [P_{N_2}] = [P_{F(R')}] + [P_{N_2'}]$.

Putting all this together, we obtain that

$$[P_{\mathcal{R}\tilde{\oplus}N_1}] - [P_{F(\mathcal{R})\tilde{\oplus}N_2}] = [P_{\mathcal{R}'\tilde{\oplus}N_1'}] - [P_{F(\mathcal{R}')\tilde{\oplus}N_2'}],$$

however,

$$[P_{\mathcal{R}\tilde{\oplus}N_1}] - [P_{F(\mathcal{R})\tilde{\oplus}N_2}] = [P_{\mathcal{R}}] + [P_{N_1}] - [P_{F(\mathcal{R})}] - [P_{N_2}] = [P_{N_1}] - [P_{N_2}]$$

and similarly

$$[P_{\mathcal{R}'\tilde{\oplus}N_1'}] - [P_{F(\mathcal{R}')\tilde{\oplus}N_2'}] = [P_{N_1'}] - [P_{N_2'}].$$

□

Thanks to the technical results which we derived so far, we can prove the next result in a similar way as [17, Lemma 2.7.10] . For the convenience of readers, we give the full proof here.

Proposition 3.10. *Let $F, D \in \mathcal{MK}\Phi(\mathcal{A})$. Then $DF \in \mathcal{MK}\Phi(\mathcal{A})$ and*

$$\text{index } DF = \text{index } D + \text{index } F.$$

Proof. Let

$$\mathcal{A} = M_1\tilde{\oplus}N_1 \xrightarrow{F} M_2\tilde{\oplus}N_2 = \mathcal{A}$$

be an $\mathcal{MK}\Phi$-decomposition for F. By Proposition 3.2 there exists some α_0 and a closed submodule $\tilde{M}$ such that $Im(I-P_{\alpha_0}) \subseteq \tilde{M}$ and $\mathcal{A} = \tilde{M}\tilde{\oplus}N_2$. If $\sqcap$ denotes the projection onto $\tilde{M}$ along N_2, then $\sqcap_{|M_2}$ is an isomorphism onto $\tilde{M}$. Let V be the operator with the matrix $\begin{pmatrix} \sqcap & 0 \\ 0 & 1 \end{pmatrix}$ with respect to the decomposition

$$\mathcal{A} = M_2\tilde{\oplus}N_2 \xrightarrow{V} \tilde{M}\tilde{\oplus}N_2 = \mathcal{A}.$$

Then V is an isomorphism on $\mathcal{A}$, and with respect to the decomposition

$$\mathcal{A} = M_1\tilde{\oplus}N_1 \longrightarrow \tilde{M}\tilde{\oplus}N_2 = \mathcal{A},$$

the operator VF has the matrix $\begin{pmatrix} (VF)_1 & 0 \\ 0 & (VF)_4 \end{pmatrix}$ where $(VF)_1$ is an isomorphism. Hence, index VF = index F. Note that if

$$\mathcal{A} = M_1'\tilde{\oplus}N_1' \xrightarrow{D} M_2'\tilde{\oplus}N_2' = \mathcal{A}$$

is an $\mathcal{MK}\Phi$-decomposition for D, then

$$\mathcal{A} = V(M_1')\tilde{\oplus}V(N_1') \longrightarrow M_2'\tilde{\oplus}N_2' = \mathcal{A}$$

an $\mathcal{MK}\Phi$-decomposition for DV^{-1} and $indexDV^{-1} = indexD$. This follows from Lemma 3.3 since $V(N_1') \cong N_1'$, hence $P_{V(N_1')} \sim P_{N_1'} \in \mathcal{F}$. Now, since $P_{V(N_1')} \in \mathcal{F}$, by Proposition 3.2 we can find some $\alpha_1 \geq \alpha_0$ and a closed submodule $\tilde{M}'$ such that $\mathcal{A} = \tilde{M}'\tilde{\oplus}V(N_1')$ and $Im(I - P_{\alpha_1}) \subseteq \tilde{M}'$. Then, by Corollary 3.6, the decomposition

$$\mathcal{A} = \tilde{M}'\tilde{\oplus}V(N_1') \longrightarrow DV^{-1}(\tilde{M}')\tilde{\oplus}N_2' = \mathcal{A}$$

is also an $\mathcal{MK}\Phi$-decomposition for DV^{-1}. Moreover, $Im(I - P_{\alpha_1}) \subseteq \tilde{M} \cap \tilde{M}'$. By [9, Lemma 2.6] there exist closed submodules $R, R' \subseteq \mathcal{A}$ such that

$$\tilde{M} = Im(I - P_{\alpha_1}) \oplus R, \tilde{M}' = Im(I - P_{\alpha_1}) \oplus R'.$$

As in the first part of the proof of Theorem 3.9, we obtain $\mathcal{MK}\Phi$-decompositions

$$\mathcal{A} = (VF)_1^{-1}(Im(I - P_{\alpha_1}))\tilde{\oplus}((VF)_1^{-1}(R)\tilde{\oplus}N_1) \xrightarrow{VF} Im(I - P_{\alpha_1})\tilde{\oplus}(R\tilde{\oplus}N_2) = \mathcal{A},$$

$$\mathcal{A} = Im(I - P_{\alpha_1})\tilde{\oplus}(R'\tilde{\oplus}V(N_1')) \xrightarrow{DV^{-1}} DV^{-1}(Im(I - P_{\alpha_1}))\tilde{\oplus}(DV^{-1}(R')\tilde{\oplus}N_2') = \mathcal{A}$$

for the operators VF and DV^{-1}, respectively, where $(VF)_1^{-1}(R) \cong R,\ R' \cong DV^{-1}(R')$. Finally, since

$$\mathcal{A} = Im(I - P_{\alpha_1})\tilde{\oplus}R\tilde{\oplus}N_2 = Im(I - P_{\alpha_1})\tilde{\oplus}R'\tilde{\oplus}V(N_1'),$$

we get that $R\tilde{\oplus}N_2 \cong R'\tilde{\oplus}V(N_1')$. By Lemma 3.1 and Lemma 3.4 we deduce that index D+ index F=index DV^{-1}+ index $VF = [P_{R'\tilde{\oplus}V(N_1')}] - [P_{DV^{-1}(R')\tilde{\oplus}N_2'}] + [P_{VF_1^{-1}(R)\tilde{\oplus}N_1}] - [P_{R\tilde{\oplus}N_2}] = [P_{VF_1^{-1}(R)\tilde{\oplus}N_1}] - [P_{DV^{-1}(R')\tilde{\oplus}N_2'}]$.

On the other hand, it is clear that the operator DF has the matrix $\begin{pmatrix} (DF)_1 & (DF)_2 \\ 0 & (DF)_4 \end{pmatrix}$ with respect to the decomposition

$$\mathcal{A} = (VF)_1^{-1}(Im(I - P_{\alpha_1}))\tilde{\oplus}((VF)_1^{-1}(R)\tilde{\oplus}N_1) \xrightarrow{DF} DV^{-1}(Im(I - P_{\alpha_1}))\tilde{\oplus}(DV^{-1}(R')\tilde{\oplus}N_2') = \mathcal{A},$$

where $(DF)_1$ is an isomorphism (because $DF = DV^{-1}VF$). Hence, as in the proof of [17, Lemma 2.7.10] we can find an isomorphism U such that DF has the matrix $\begin{pmatrix} (DF)_1 & 0 \\ 0 & (\tilde{DF})_4 \end{pmatrix}$ with respect to the decomposition

$$\mathcal{A} = (VF)_1^{-1}(Im(I - P_{\alpha_1}))\tilde{\oplus}U((VF)_1^{-1}(R)\tilde{\oplus}N_1) \xrightarrow{DF} DV^{-1}(Im(I - P_{\alpha_1}))\tilde{\oplus}(DV^{-1}(R')\tilde{\oplus}N_2') = \mathcal{A}.$$

Since $U((VF)_1^{-1}(R)\tilde{\oplus}N_1) \cong (VF)_1^{-1}(R)\tilde{\oplus}N_1$, by Lemma 3.1 we conclude that
index DF = index D+ index F. □

Next, in a similar way as in the proof of [17, Lemma 2.7.13], we can prove the following lemma. For the convenience of readers, we give the full proof here.

Lemma 3.11. *Let $F \in \mathcal{MK}\Phi(\mathcal{A})$ and $K \in \mathcal{F}$. Then $F + K \in \mathcal{MK}\Phi(\mathcal{A})$ and* index $(F + K)$ = index F.

Proof. Let $F \in \mathcal{MK}\Phi(\mathcal{A})$ and

$$\mathcal{A} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = \mathcal{A}$$

be an $\mathcal{MK}\Phi$-decomposition for F. By Proposition 3.2, there exists some α_0 and a closed submodule $\tilde{M}$ such that $Im(I - P_{\alpha_0}) \subseteq \tilde{M}$ and $\mathcal{A} = \tilde{M} \tilde{\oplus} N_1$. Then, by Corollary 3.6, F has the matrix $\begin{pmatrix} F_1 & 0 \\ 0 & F_4 \end{pmatrix}$ with respect to the decomposition

$$\mathcal{A} = \tilde{M} \tilde{\oplus} N_1 \xrightarrow{F} F(\tilde{M}) \tilde{\oplus} N_2 = \mathcal{A},$$

where F_1 is an isomorphism. Let $K \in \mathcal{F}$. Since $\{P_\alpha\}$ is an approximate unit for $\mathcal{F}$, we can find some $\alpha_1 \geq \alpha_0$ such that $\| KP_{\alpha_1} \| \leq \| F_1^{-1} \|^{-1}$.
We have that $Im(I - P_{\alpha_1}) \subseteq Im(I - P_{\alpha_0}) \subseteq \tilde{M}$, hence, by [9, Lemma 2.6] we obtain that $\tilde{M} = Im(I - P_{\alpha_1}) \oplus \mathcal{R}$ where $\mathcal{R} = ImP_{\alpha_1} \cap \tilde{M}$. Since $P_{\mathcal{R}} \leq P_{\alpha_1}$, we have $P_{\mathcal{R}} \in \mathcal{F}$. We get a decomposition $\mathcal{A} = F_1(Im(I - P_{\alpha_1})) \tilde{\oplus} F_1(\mathcal{R}) \tilde{\oplus} N_2$. Since F_1 is an isomorphism, by Lemma 3.1 we get that $P_{F_1(\mathcal{R})} \sim P_{\mathcal{R}}$, so $P_{F_1(\mathcal{R})} \in \mathcal{F}$ as $P_{\mathcal{R}} \in \mathcal{F}$. Moreover, by Lemma 3.4 we deduce that $P_{F_1(\mathcal{R}) \tilde{\oplus} N_2} \in \mathcal{F}$ and

$$[P_{F_1(\mathcal{R}) \tilde{\oplus} N_2}] = [P_{F_1(\mathcal{R})}] + [P_{N_2}] = [P_{\mathcal{R}}] + [P_{N_2}].$$

With respect to the decomposition

$$\mathcal{A} = Im(I - P_{\alpha_1}) \tilde{\oplus} \mathcal{R} \tilde{\oplus} N_1 \xrightarrow{F} F_1(Im(I - P_{\alpha_1})) \oplus F_1(Im(I - P_{\alpha_1}))^{\perp} = \mathcal{A},$$

F has the matrix $\begin{pmatrix} F_1 & \tilde{F_2} \\ 0 & \tilde{F_4} \end{pmatrix}$. Let $\begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \end{pmatrix}$ be the matrix of K with respect to the same decomposition. Then

$$\| K_1 \| \leq \| K_{|_{Im\, P_{\alpha_1}}} \| = \| KP_{\alpha_1} \| \leq \| F_1^{-1} \|^{-1} .$$

As in the proof of [17, Lemma 2.7.13] we can find isomorphisms $\mathcal{U}$ and V such that

$$\mathcal{A} = Im(I - P_{\alpha_1}) \tilde{\oplus} (\mathcal{U}(\mathcal{R}) \tilde{\oplus} \mathcal{U}(N_1)) \xrightarrow{F+K} V(Im(I - P_{\alpha_1})) \tilde{\oplus} (F_1(\mathcal{R}) \tilde{\oplus} N_2) = \mathcal{A}$$

is an $\mathcal{MK}\Phi$-decomposition for the operator $F + K$. Indeed, by Lemma 3.4 we have that $P_{\mathcal{R} \tilde{\oplus} N_1} \in \mathcal{F}$ and $[P_{\mathcal{R} \tilde{\oplus} N_1}] = [P_{\mathcal{R}}] + [P_{N_1}]$, whereas by Lemma 3.1 we get that $P_{\mathcal{U}(\mathcal{R} \tilde{\oplus} N_1)} \sim P_{\mathcal{R} \tilde{\oplus} N_1}$. Hence, we deduce that

$$index(F + K) = [P_{\mathcal{U}(\mathcal{R} \tilde{\oplus} N_1)}] - [P_{F_1(\mathcal{R}) \tilde{\oplus} N_2}] = [P_{\mathcal{R}}] - [P_{N_1}] - [P_{\mathcal{R}}] - [P_{N_2}] = indexF.$$

□

Finally, in a similar way as in the proof of [17, Theorem 2.7.14], we can prove the next theorem. For the convenience of readers, we give the full proof here.

Theorem 3.12. *Let $F, D, D' \in B(\mathcal{A})$. If there exist some $K_1, K_2 \in \mathcal{F}$ such that*

$$FD = I + K_1, \; D'F = I + K_2,$$

then $F \in \mathcal{MK}\Phi(\mathcal{A})$.

Proof. As in the proof of [17, Theorem 2.7.14] we obtain from Lemma 3.11 an $\mathcal{MK}\Phi$-decomposition for $I + K_1$

$$\mathcal{A} = M_1 \tilde{\oplus} N_1 \xrightarrow{I+K_1} M_2 \tilde{\oplus} N_2 = \mathcal{A}$$

and we let $\sqcap$ be the projection onto N_2 along M_2. Then $(I - \sqcap)F$ is an epimorphism onto M_2 and $D'(I - \sqcap)F = I + \tilde{K}_2$ for some $\tilde{K}_2 \in \mathcal{F}$. This follows since $\sqcap \in \mathcal{F}$ by Remark 2.12, so $D' \sqcap F \in \mathcal{F}$ because $\mathcal{F}$ is an ideal. Hence $D'(I - \sqcap)F \in \mathcal{MK}\Phi(\mathcal{A})$ by Lemma 3.11, so there exists an $\mathcal{MK}\Phi$-decomposition

$$\mathcal{A} = \overline{M}_1 \tilde{\oplus} \overline{N}_1 \xrightarrow{D'(I-\sqcap)F} \overline{M}_2 \tilde{\oplus} \overline{N}_2 = \mathcal{A}$$

for $D'(I-\sqcap)F$. By the same arguments as in the proof of [17, Theorem 2.7.14] we get that $(I-\sqcap)F_{|_{\overline{M}_1}}$ is an isomorphism onto $(I-\sqcap)F(\overline{M}_1)$ and $\ker(I-\sqcap)F \subseteq \overline{N}_1$. Since $Im(I-\sqcap)F = M_2$ and $(I-\sqcap)F$ is adjointable by [17, Corollary 2.5.3], by [17, Theorem 2.3.3] we have that $\ker(I-\sqcap)F$ is orthogonally complementable in $\mathcal{A}$. Hence, by [9, Lemma 2.6] we have that $\ker(I-\sqcap)F$ is orthogonally complementable in $\overline{N}_1$. Thus,

$$\mathcal{A} = \overline{M}_1 \oplus \mathcal{R} \oplus \ker(I-\sqcap)F,$$

where $\mathcal{R}$ is the orthogonal complement of $\ker(I-\sqcap)F$ in $\overline{N}_1$. However, since $\ker(I-\sqcap)F$ closed and complementable in $\mathcal{A}$, by Remark 2.11 $\ker(I-\sqcap)F$ is orthogonally complementable in $\mathcal{A}$. Hence, since $\ker(I-\sqcap)F \subseteq \overline{N}_1$ and $P_{\overline{N}_1} \in \mathcal{F}$, we get that $P_{\ker(I-\sqcap)F} \in \mathcal{F}$. Since $Im(I-\sqcap)F$, (which is equal to M_2), is closed, it follows that $(I-\sqcap)F_{|_{(\overline{M}_1 \oplus \mathcal{R})}}$ is an isomorphism onto $M_2 = Im(I-\sqcap)$. This gives that F has the matrix $\begin{pmatrix} F_1 & F_2 \\ F_3 & F_4 \end{pmatrix}$ with respect to the decomposition

$$\mathcal{A} = (\overline{M}_1 \oplus \mathcal{R}) \tilde{\oplus} \ker((I-\sqcap)F) \xrightarrow{F} M_2 \tilde{\oplus} N_2 = \mathcal{A},$$

where F_1 is an isomorphism. By the same arguments as in the proof of [17, Lemma 2.7.10] we can find isomorphisms $\mathcal{U}$ and V of $\mathcal{A}$ such that

$$\mathcal{A} = (\overline{M}_1 \tilde{\oplus} \mathcal{R}) \tilde{\oplus} \mathcal{U}(\ker((I-\sqcap)F)) \xrightarrow{F} V(M_2) \tilde{\oplus} N_2 = \mathcal{A}$$

is a decomposition with respect to which F has the matrix $\begin{pmatrix} \tilde{F}_1 & 0 \\ 0 & \tilde{F}_4 \end{pmatrix}$ where $\tilde{F_1}$ is an isomorphism. Since $P_{\ker((I-\sqcap)F)} \in \mathcal{F}$, by Lemma 3.3 we have that $\mathcal{U}(\ker((I-\sqcap)F))$ is orthogonally complementable in $\mathcal{A}$ and $P_{\mathcal{U}(\ker((I-\sqcap)F))} \in \mathcal{F}$. Thus we have obtained an $\mathcal{MK}\Phi$-decomposition for the operator F. □

Remark 3.13. *By Lemma 3.7 it follows that our approach to Fredholm theory in unital C^* -algebras is equivalent to the approach established in [16].*

4. Spectral Fredholm theory in von Neumann algebras

From now on and in the rest of this paper, $\mathcal{A}$ denotes a properly infinite von Neumann algebra acting on a Hilbert space H. We let $Proj_0(\mathcal{A})$ denote the set of all finite projections in $\mathcal{A}$, (i.e. those projections that are not Murray von Neumann equivalent to any of its subprojections).
We recall the notion of $\mathcal{A}$-Fredholm operator, originally introduced by Breuer in [4], [5].

Definition 4.1. *[16, Definition 3.1] A linear operator $T \in \mathcal{A}$ is said to be $\mathcal{A}$-Fredholm if the following holds.*
(i) $P_{\ker T} \in Proj_0(\mathcal{A})$, where $P_{\ker T}$ is the projection onto the subspace $\ker T$.
(ii) There is a projection $E \in Proj_0(\mathcal{A})$ such that $Im(I-E) \subseteq ImT$.
The second condition ensures that $P_{\ker T^*}$ also belongs to $Proj_0(\mathcal{A})$.

The index of an $\mathcal{A}$-Fredholm operator T is defined as

$$indexT = dim(\ker T) - dim(\ker T^*) \in I(\mathcal{A}).$$

Here, $I(\mathcal{A})$ is the so called index group of a von Neumann algebra $\mathcal{A}$ defined as the Grothendieck group of the commutative monoid of all representations of the commutant $\mathcal{A}'$ generated by representations of the form $\mathcal{A}' \ni S \mapsto ES = \pi_E(S)$ for some $E \in Proj_0(\mathcal{A})$. For a subspace L, its dimension $dimL$ is defined as the class$[\pi_{P_L}] \in I(\mathcal{A})$ of the representation π_{P_L}, where P_L is the projection onto L.

Next we recall the following characterization of $\mathcal{A}$-Fredholm operators.

Lemma 4.2. *[15, Lemma 22] Let $\mathcal{A}$ be a properly infinite von Neumann algebra. Then an operator $T \in \mathcal{A}$ is $\mathcal{A}-$Fredholm in the sense of Breuer if and only if there exist projections $P, Q \in Proj_0(\mathcal{A})$ such that T is invertible up to (P, Q).*

Let $\mathcal{F} = \mathfrak{m}$ where $\mathfrak{m}$ is the norm closure of the set of all $S \in \mathcal{A}$ for which $P_{\overline{ImS}} \in Proj_0(\mathcal{A})$. Then

$$\{P \in \mathfrak{m} \mid P \text{ is projection } \} = Proj_0(\mathcal{A}). \tag{1}$$

This relation has been used in the proof of Lemma 4.2.

In the rest of this section, we will denote by $\mathcal{K}\Phi(\mathcal{A})$ the set of all $\mathcal{A}-$Fredholm operators. We recall also the following definition.

Definition 4.3. *[15, Definition 9] Let $\mathcal{A}$ be a properly infinite von Neumann algebra and $T \in \mathcal{A}$. We say that T is upper semi$-\mathcal{A}-$Fredholm if there exist projections P, Q in $\mathcal{A}$ such that T is invertible up to (P, Q) where $P \in Proj_0(\mathcal{A})$. Similarly we say that T is lower semi$-\mathcal{A}-$Fredholm, however, in this case we assume that $Q \in Proj_0(\mathcal{A})$.*

Thanks to the relation (1) we obtain the following useful characterization of semi$-\mathcal{A}-$Fredholm operators.

Corollary 4.4. *[15, Corollary 23] Let $T \in \mathcal{A}$. Then T is upper (respectively lower) semi-Fredholm type element in $\mathcal{A}$ with respect to $\mathfrak{m}$ if and only if T is upper (respectively lower) semi$-\mathcal{A}-$Fredholm.*

In the sequel, we let $M_2(\mathcal{A})$ denote the von Neumann algebra consisting of 2 by 2 matrices with coefficients in $\mathcal{A}$. If $\mathcal{A}$ is a properly infinite von Neumannn algebra, then $M_2(\mathcal{A})$ is also a properly infinite von Neumannn algebra. For an operator $T' \in M_2(\mathcal{A})$ we shall simply say that T' is $\mathcal{A}-$Fredholm if T' is $M_2(\mathcal{A})-$Fredholm.
Now we will derive some technical properties of 2 by 2 matrices in a von Neumann algebra that are needed for proving Proposition 4.9, which is the main result in this section. We start with the following auxiliary technical lemma.

Lemma 4.5. *Let $\mathcal{A}$ be a properly infinite von Neumann algebra and $P, Q \in Proj(\mathcal{A})$. Then $\left(\begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & Q \end{pmatrix}\right) \in Proj_0(M_2(\mathcal{A}))$ if and only if $P, Q \in Proj_0(\mathcal{A})$.*

Proof. Let $P \in Proj_0(\mathcal{A})$ and assume that there exists a subprojection $\tilde{P}'$ of $\begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix}$ such that $\tilde{P}' \sim \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix}$. Since $\tilde{P}$ is a subprojection of $\begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix}$, then $\tilde{P}' = \begin{pmatrix} P' & 0 \\ 0 & 0 \end{pmatrix}$ for some $P' \leq P$. Let $V \in M_2(\mathcal{A})$ such that $VV^* = \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix}$ and $V^*V = \begin{pmatrix} P' & 0 \\ 0 & 0 \end{pmatrix}$. Then, if we put $\tilde{P} = \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix}$, we get that $\tilde{P}V\tilde{P}'V^*\tilde{P} = \tilde{P}VV^*VV^*\tilde{P} = \tilde{P}$ and $\tilde{P}'V^*\tilde{P}V\tilde{P}' = \tilde{P}'V^*VV^*V\tilde{P}' = \tilde{P}'$. Moreover, if we write V as $\begin{pmatrix} V_1 & V_2 \\ V_3 & V_4 \end{pmatrix}$ for some $V_1, V_2, V_3, V_4 \in \mathcal{A}$, we get that $\tilde{P}V\tilde{P}' = \begin{pmatrix} PV_1P' & 0 \\ 0 & 0 \end{pmatrix}$. Hence $P = (PV_1P')(PV_1P')^*$ and $P' = (PV_1P')^*(PV_1P')$, so $P \sim P'$ which is a contradiction. Thus, we must have that $\begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix} \in Proj_0(M_2(\mathcal{A}))$. Conversely, it is obvious that if $P \in Proj(\mathcal{A})$ and $P \sim P'$ for some $P' \leq P$, then $\begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix} \sim \begin{pmatrix} P' & 0 \\ 0 & 0 \end{pmatrix}$. Hence, if $\begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix} \in Proj_0(M_2(\mathcal{A}))$, then $P \in Proj_0(\mathcal{A})$. Similarly we treat the case with $\begin{pmatrix} 0 & 0 \\ 0 & Q \end{pmatrix}$. $\square$

Next we recall also the following properties of finite operators in von Neumann algebras.

Lemma 4.6. *[4],[5] Let $\mathcal{A}$ be a properly infinite von Neumannn algebra and $T \in \mathcal{A}$. Then $P_{\overline{ImT}} \in Proj_0(\mathcal{A})$ if and only if $P_{\overline{ImT^*}} \in Proj_0(\mathcal{A})$ and in this case $P_{\overline{ImS_1TS_2}} \in Proj_0(\mathcal{A})$ for all $S_1, S_2 \in \mathcal{A}$.*

Proof. If $T \in \mathcal{A}$, then $P_{\overline{ImT}} \sim P_{\ker T^\perp}$. Since $P_{\ker T^\perp} = P_{\overline{ImT^*}}$, the first statement follows. Now, if $S_2 \in \mathcal{A}$, then $P_{\overline{ImTS_2}} \leq P_{\overline{ImT}}$, hence we must have $P_{\overline{ImTS_2}} \in Proj_0(\mathcal{A})$ if $P_{\overline{ImT}} \in Proj_0(\mathcal{A})$. By the first statement we also get that $P_{\overline{ImS_2^*T^*}} \in Proj_0(\mathcal{A})$. Hence, if in addition $S_1 \in \mathcal{A}$, by repeating the same argument we get that $P_{\overline{ImS_2^*T^*S_1^*}} \in Proj_0(\mathcal{A})$, so $P_{\overline{ImS_1TS_2}} \in Proj_0(\mathcal{A})$. □

From Lemma 4.6 we deduce the following useful corollary.

Corollary 4.7. *Let $T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$ be an element of $M_2(\mathcal{A})$ where $\mathcal{A}$ is a properly infinite von Neumann algebra. If $P_{\overline{ImT}} \in Proj_0(M_2(\mathcal{A}))$, then $P_{\overline{ImT_1}}, P_{\overline{ImT_4}} \in Proj_0(\mathcal{A})$.*

Proof. By Lemma 4.6, if $\tilde{T}_1 = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix}$, then $P_{\overline{Im\tilde{T}_1}} \in Proj_0(M_2(\mathcal{A}))$ if $P_{\overline{ImT}} \in Proj_0(M_2(\mathcal{A}))$. This is because $\tilde{T}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. However, $P_{\overline{Im\tilde{T}_1}} = \begin{pmatrix} P_{\overline{ImT_1}} & 0 \\ 0 & 0 \end{pmatrix}$, hence by Lemma 4.5, $P_{\overline{ImT_1}} \in Proj_0(\mathcal{A})$. Similarly we can prove that $P_{\overline{ImT_4}} \in Proj_0(\mathcal{A})$. □

We have also the following corollary.

Corollary 4.8. *Let $\mathcal{A}$ be a properly infinite von Neumann algebra and $T, S \in \mathcal{A}$. If T is $\mathcal{A}$−Fredholm, then $\begin{pmatrix} T & 0 \\ 0 & 1 \end{pmatrix}$ is $\mathcal{A}$−Fredholm. Similarly, if S is $\mathcal{A}$−Fredholm, then $\begin{pmatrix} 1 & 0 \\ 0 & S \end{pmatrix}$ is $\mathcal{A}$−Fredholm.*

Proof. If T is invertible up to (P, Q) for some projections P and Q, then $\begin{pmatrix} T & 0 \\ 0 & 1 \end{pmatrix}$ is invertible up to $\left(\begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix} \right)$. Similarly, if S is invertible up to (P', Q'), for some projections P' and Q', then $\begin{pmatrix} 1 & 0 \\ 0 & S \end{pmatrix}$ is invertible up to $\left(\begin{pmatrix} 0 & 0 \\ 0 & P' \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & Q' \end{pmatrix} \right)$. Hence, by applying Lemma 4.5, we deduce the statements in the corollary. □

Now we are ready to give the main result in this section, which is a generalization of the result by Đorđević in [6] in the setting of Fredholm operators in von Neumann algebras.

Proposition 4.9. *Let $\mathcal{A}$ be a properly infinite von Neumann algebra and $T, S \in \mathcal{A}$. For fixed $S, T \in \mathcal{A}$, let $M_C \in M_2(\mathcal{A})$ be given by*

$M_C = \begin{pmatrix} T & C \\ 0 & S \end{pmatrix}$ where C varies over $\mathcal{A}$. Set

$$\sigma_{ef}(M_C) = \{\lambda \in \mathbb{C} \mid M_C - \lambda I \text{ is not } \mathcal{A} - \text{Fredholm}\},$$

$$\sigma_{ef}(T) = \{\lambda \in \mathbb{C} \mid T - \lambda I \text{ is not } \mathcal{A} - \text{Fredholm}\},$$

$$\sigma_{ef}(S) = \{\lambda \in \mathbb{C} \mid S - \lambda I \text{ is not } \mathcal{A} - \text{Fredholm}\}.$$

Then $\sigma_{ef}(T) \cup \sigma_{ef}(S) = \sigma_{ef}(M_C) \cup (\sigma_{ef}(T) \cap \sigma_{ef}(S))$ for all $C \in \mathcal{A}$.

Proof. By applying Corollary 4.7, we may show that $\sigma_{ef}(M_C) \subseteq \sigma_{ef}(T) \cup \sigma_{ef}(S)$ for all $C \in \mathcal{A}$ in a similar way as in the proof of [12, Proposition 3.1].
If $\sigma_{ef}(M_C) = \sigma_{ef}(T) \cup \sigma_{ef}(S)$, then there is nothing to prove. Suppose now that there exists some $C \in \mathcal{A}$ such that the inclusion $\sigma_{ef}(M_C) \subset \sigma_{ef}(T) \cup \sigma_{ef}(S)$ is proper, As in [6], we write M_C as $M_C = S'C'T'$ where

$S' = \begin{pmatrix} 1 & 0 \\ 0 & S \end{pmatrix}$, $C' = \begin{pmatrix} 1 & C \\ 0 & 1 \end{pmatrix}$ and $T' = \begin{pmatrix} T & 0 \\ 0 & 1 \end{pmatrix}$. Assume that M_C is $\mathcal{A}$−Fredholm and let $P, Q \in Proj_0(M_2(\mathcal{A}))$ such that M_C is invertible up to (P, Q). By [15, Corollary 6], there is some $R \in Proj(M_2(\mathcal{A}))$ such that T' is invertible up to (P, R), $S'C'$ is invertible up to (R, Q) and $(I - R)T'(I - P) = T'(I - P)$. Since C' is invertible, we have that $\tilde{R} \sim R$, where $\tilde{R}$ is the orthogonal projection onto $C'R(H^2)$. Moreover, $H^2 = C'(I - R)(H^2) \tilde{\oplus} C'R(H^2)$. If $\tilde{R}'$ denotes the orthogonal projection onto $C'(I - R)(H^2)^{\perp}$, then obviously $\tilde{R}'$ maps $C'R(H^2)$ isomorphically onto $\tilde{R}'(H^2)$. Thus $\tilde{R}' \sim \tilde{R} \sim R$. Now, C' is invertible up to $(R, \tilde{R}')$ and $(I - \tilde{R}')C'(I - R) = C'(I - R)$. Since $S'C'$ is invertible up to (R, Q), we can deduce that S' is invertible up to $(\tilde{R}', Q)$. Indeed, let B be $(R, \tilde{R}')$-inverse of C' and $\tilde{B}$ be (R, Q)-inverse of $S'C'$. Then we get

$$C'(I - R)\tilde{B}(I - Q)S'(I - \tilde{R}') = C'(I - R)\tilde{B}(I - Q)S'(I - \tilde{R}')C'(I - R)B =$$

$$C'(I - R)\tilde{B}(I - Q)S'C'(I - R)B = C'(I - R)B = (I - \tilde{R}')C'(I - R)B = I - \tilde{R}',$$

and

$$(I - Q)S'(I - \tilde{R}')C'(I - R)\tilde{B} = (I - Q)S'C'(I - R)\tilde{B} = (I - Q),$$

so $C'(I - R)\tilde{B}$ is an $(\tilde{R}', Q)$−inverse of S'. Hence, in particular we have that T' is left invertible up to P and S' is right invertible up to Q. If we write P and Q as $P = \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix}$, $Q = \begin{pmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{pmatrix}$, where $P_j, Q_j \in \mathcal{A}$ for $j \in \{1, \ldots 4\}$, then it follows that $FT = 1 - P_1$ and $SD = 1 - Q_4$ for some operators $F, D \in \mathcal{A}$. By Corollary 4.7 we have that $P_1, Q_4 \in \mathfrak{m}$, hence, by [15, Lemma 10] and Corollary 4.4 we deduce that T and S are upper semi−$\mathcal{A}$−Fredholm and lower semi−$\mathcal{A}$−Fredholm, respectively.

If $p, q \in Proj_0(\mathcal{A})$ and $r, r' \in Proj(\mathcal{A})$ such that T invertible up to (p, r) and S is invertible up to (r', q), then, obviously, T' is invertible up to $\left(\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix}\right)$ and S' is invertible up to $\left(\begin{pmatrix} 0 & 0 \\ 0 & r' \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix}\right)$. By Lemma 4.5 it follows that

$$\left(\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix}\right) \in Proj_0(M_2(\mathcal{A})).$$

Hence, by Corollary 2.9, we deduce that there exist projections

$$E, \tilde{E}, E', \tilde{E}', L, \tilde{L}, L', \tilde{L}' \in \mathcal{A}$$

such that $\tilde{E}, \tilde{E}', \tilde{L}, \tilde{L}'$ are finite, $E\tilde{E} = E'\tilde{E}' = L\tilde{L} = L'\tilde{L}' = 0$, $R \sim E$, $\begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix} \sim E'$, $\tilde{R}' \sim L$, $\begin{pmatrix} 0 & 0 \\ 0 & r' \end{pmatrix} \sim L'$, $E + \tilde{E} \sim E' + \tilde{E}'$ and $L + \tilde{L} \sim L' + \tilde{L}'$.

Suppose now that T is $\mathcal{A}$−Fredholm. Then by Lemma 2.7 and by (1), we must have that $r \in Proj_0(\mathcal{A})$. Hence, by Lemma 4.5 we get that $\begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix}$ is finite, so $E' \in Proj_0(M_2(\mathcal{A}))$, as $E' \sim \begin{pmatrix} 0 & 0 \\ 0 & r \end{pmatrix}$, which gives that $E' + \tilde{E}'$ is finite. Therefore, $E + \tilde{E} \in Proj_0(M_2(\mathcal{A}))$, hence $E \in Proj_0(M_2(\mathcal{A}))$ since $\tilde{E}$ is finite. However, $L \sim \tilde{R}' \sim R \sim E$, so we get that $L + \tilde{L}$ is finite since $L, \tilde{L} \in Proj_0(M_2(\mathcal{A}))$. Thus, $L' + \tilde{L}' \in Proj_0(M_2(\mathcal{A}))$ since $L + \tilde{L} \sim L' + \tilde{L}'$, so we must have that $L' \in Proj_0(M_2(\mathcal{A}))$ because $\tilde{L}' \in Proj_0(M_2(\mathcal{A}))$. It follows that $\begin{pmatrix} 0 & 0 \\ 0 & r' \end{pmatrix} \in Proj_0(M_2(\mathcal{A}))$ as $\begin{pmatrix} 0 & 0 \\ 0 & r' \end{pmatrix} \sim L'$, hence, by Lemma 4.5 we obtain that $r' \in Proj_0(\mathcal{A})$. Since S is invertible up to (r', q), we get that S is $\mathcal{A}$−Fredholm. Similarly we can show that T is $\mathcal{A}$−Fredholm if S is $\mathcal{A}$−Fredholm. If now $\lambda \in \mathbb{C}$, we can apply previous arguments to deduce that if $M_C - \lambda I$ and $T - \lambda 1$ are $\mathcal{A}$−Fredholm, then $S - \lambda 1$ is $\mathcal{A}$−Fredholm, and, similarly, if $M_C - \lambda I$ and $S - \lambda 1$, are $\mathcal{A}$−Fredholm, then $T - \lambda 1$ is $\mathcal{A}$−Fredholm. This is because $M_C - \lambda I = \begin{pmatrix} T - \lambda 1 & 0 \\ 0 & S - \lambda 1 \end{pmatrix}$, so we can apply the previous arguments for arbitrary $\lambda \in \mathbb{C}$. Hence we can deduce that

$$(\sigma_{ef}(T) \setminus \sigma_{ef}(S)) \setminus \sigma_{ef}(M_C) = \emptyset, (\sigma_{ef}(S) \setminus \sigma_{ef}(T)) \setminus \sigma_{ef}(M_C) = \emptyset,$$

which gives that $\sigma_{ef}(T) \cup \sigma_{ef}(S) = (\sigma_{ef}(T) \cap \sigma_{ef}(S)) \cup \sigma_{ef}(M_C)$. $\square$

Next, we will consider isolated points of the spectra of operators in $\mathcal{A}$. We wish to show that if 0 is an isolated point of the spectrum of an $\mathcal{A}-$ Fredholm operator, then the corresponding spectral projection is finite. To this end, we give first the following auxiliary technical lemma.

Lemma 4.10. *Let N, M be a closed subspaces of H such that $P_N, P_M \in \mathcal{A}$ and $D \in \mathcal{A}$ such that D has the matrix $\begin{pmatrix} D_1 & D_2 \\ D_1 & D_4 \end{pmatrix}$ with respect to the decomposition $N \oplus N^\perp \to M \oplus M^\perp$, where D_1 is an isomorphism. If S is the operator with matrix $\begin{pmatrix} D_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}$ with respect to the decomposition $M \oplus M^\perp \to N \oplus N^\perp$, then $S \in \mathcal{A}$.*

Proof. We have that $\begin{pmatrix} D_1 & 0 \\ 0 & 0 \end{pmatrix} = P_M D P_N \in \mathcal{A}$. Let U be the partial isometry from the polar decomposition of D_1. The operator $\tilde{U}$ given by the operator matrix $\begin{pmatrix} U & 0 \\ 0 & 0 \end{pmatrix}$ with respect to the decomposition $N \oplus N^\perp \to M \oplus M^\perp$ is obviously the partial isometry from the polar decomposition of the operator $\begin{pmatrix} D_1 & 0 \\ 0 & 0 \end{pmatrix}$, hence $\tilde{U} \in \mathcal{A}$. Since D_1 is an isomorphism, then $|D_1|$ is invertible in $B(M)$. Now, $\left|\begin{pmatrix} D_1 & 0 \\ 0 & 0 \end{pmatrix}\right| = \begin{pmatrix} |D_1| & 0 \\ 0 & 0 \end{pmatrix}$, so $\begin{pmatrix} |D_1| & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{A}$. Hence $\begin{pmatrix} |D_1| & 0 \\ 0 & 0 \end{pmatrix} + P_{M^\perp} = \begin{pmatrix} |D_1| & 0 \\ 0 & 1 \end{pmatrix} \in \mathcal{A}$. The operator $\begin{pmatrix} |D_1| & 0 \\ 0 & 1 \end{pmatrix}$ is positive, invertible operator in $\mathcal{A}$ with its inverse $\begin{pmatrix} |D_1|^{-1} & 0 \\ 0 & 1 \end{pmatrix}$. This follows from the functional calculus. Hence $\begin{pmatrix} |D_1|^{-1} & 0 \\ 0 & 1 \end{pmatrix} \in \mathcal{A}$ since $\begin{pmatrix} |D_1|^{-1} & 0 \\ 0 & 1 \end{pmatrix} = P_M \begin{pmatrix} |D_1| & 0 \\ 0 & 1 \end{pmatrix}^{-1} P_M$. Next, notice that $D_1^{-1} = |D_1|^{-1} U^*$. Hence $\begin{pmatrix} D_1^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} |D_1|^{-1} & 0 \\ 0 & 1 \end{pmatrix} \tilde{U}^* \in \mathcal{A}$.
$\square$

Corollary 4.11. *Let N, M be a closed subspaces of H such that $P_N, P_M \in \mathcal{A}$ and $D \in \mathcal{A}$ such that D has the matrix $\begin{pmatrix} D_1 & D_2 \\ D_1 & D_4 \end{pmatrix}$ with respect to the decomposition $N \oplus N^\perp \to M \oplus M^\perp$. Then D is invertible up to $(I - P_N, I - P_M)$ in $\mathcal{A}$ if and only if D_1 is an isomorphism.*

We present also the following proposition.

Proposition 4.12. *Let F be $\mathcal{A}-$ Fredholm and P_0 be some skew or orthogonal projection in $\mathcal{A}$. Suppose that F has the matrix $\begin{pmatrix} F_1 & 0 \\ 0 & F_4 \end{pmatrix}$ with respect to the decomposition $H = \ker P_0 \tilde{\oplus} Im P_0$, where F_1 is an isomorphism. If P_0 is not a finite operator, then $P_{ImP_0} F P_{ImP_0} \in \mathcal{K}\Phi(P_{ImP_0} \mathcal{A} P_{ImP_0})$.*

Proof. Since F is $\mathcal{A}-$ Fredholm, there exist some orthogonal projections $\tilde{P}, \tilde{Q}$ in $\mathcal{A}$ such that $I - \tilde{P}$ and $I - \tilde{Q}$ are finite and such that F is invertible up to $(I - \tilde{P}, I - \tilde{Q})$. By [16, Proposition 2.8] we may without loss of generality assume that $(I - \tilde{Q}) F \tilde{P} = 0$. We have $\tilde{P}(H)^\perp \oplus (ImP_0 \cap \tilde{P}(H)) = \ker((I - P_0)\tilde{P})$. Hence $P_{\tilde{P}(H)^\perp \oplus (ImP_0 \cap \tilde{P}(H))} \in \mathcal{A}$. Since $ImP_0 \cap \tilde{P}(H) = Im((I - \tilde{P}) P_{\tilde{P}(H)^\perp \oplus (ImP_0 \cap \tilde{P}(H))})$, we get that $P_{ImP_0 \cap \tilde{P}(H)} \in \mathcal{A}$.

Let N be the orthogonal complement of $ImP_0 \cap \tilde{P}(H)$ in ImP_0. Then $P_N = P_{ImP_0} - P_{ImP_0 \cap \tilde{P}(H)} \in \mathcal{A}$. Now, $I - \tilde{P}$ is injective on N, hence $\ker((I - \tilde{P})P_N)^\perp = N$. Therefore, $P_N \sim P_{\overline{Im(I-\tilde{P})P_N}}$. Since $(I - \tilde{P})P_N$ is a finite operator because $(I - \tilde{P})$ is a finite operator, we get that $P_{\overline{Im(I-\tilde{P})P_N}} \in Proj_0(\mathcal{A})$. Hence $P_N \in Proj_0(\mathcal{A})$.

Notice that, since $(I - \tilde{Q}) F \tilde{P} = 0$ and F is invertible up to $(I - \tilde{P}, I - \tilde{Q})$, we have that F maps $Im\tilde{P}$ isomorphically onto $F(\tilde{P}(H)) = \tilde{Q}(H)$. It follows that F maps $ImP_0 \cap \tilde{P}(H)$ isomorphically onto $F(ImP_0 \cap \tilde{P}(H))$, so $F(ImP_0 \cap \tilde{P}(H))$ is closed. Since $F(ImP_0 \cap \tilde{P}(H)) = ImFP_{ImP_0 \cap \tilde{P}(H)}$, we have that $P_{F(ImP_0 \cap \tilde{P}(H))} \in \mathcal{A}$. If M denotes

the orthogonal complement of $F(ImP_0 \cap \tilde{P}(H))$ in ImP_0, then, since $P_M = P_{ImP_0} - P_{F(ImP_0 \cap \tilde{P}(H))}$, we have that $P_M \in \mathcal{A}$.
Observe now that F has the matrix $\begin{pmatrix} F_1 & F_2 \\ 0 & F_4 \end{pmatrix}$, with respect to the decomposition $(\ker P_0 \tilde{\oplus} (ImP_0 \cap \tilde{P}(H))) \tilde{\oplus} N \to (\ker P_0 \tilde{\oplus} F(ImP_0 \cap \tilde{P}(H))) \tilde{\oplus} M$, where F_1 is an isomorphism. Set

$$\tilde{N} = \ker P_0 \tilde{\oplus} (ImP_0 \cap \tilde{P}(H)),$$

$$\tilde{M} = \ker P_0 \tilde{\oplus} F(ImP_0 \cap \tilde{P}(H)).$$

Then, since $\tilde{M} = Im(I - P_0 + P_{F(ImP_0 \cap \tilde{P}(H))} P_0)$ and $\tilde{N} = Im(I - P_0 + P_{ImP_0 \cap \tilde{P}(H)} P_0)$, we have that $P_{\tilde{M}}, P_{\tilde{N}} \in \mathcal{A}$. Hence $P_{\tilde{M}^\perp}, P_{\tilde{N}^\perp} \in \mathcal{A}$. Since $H = \tilde{N} \tilde{\oplus} N$, we have that $P_{\tilde{N}^\perp}$ is injective on N and $\tilde{N}^\perp = P_{\tilde{N}^\perp}(N)$. Hence, we get that $\ker P_{\tilde{N}^\perp} P_N = N^\perp$ and $ImP_{\tilde{N}^\perp} P_N = \tilde{N}^\perp$. Therefore, $P_N \sim P_{\tilde{N}^\perp}$, so $P_{\tilde{N}^\perp} \in Proj_0(\mathcal{A})$. Likewise, $P_M \sim P_{\tilde{M}^\perp}$. Now, since F maps $\tilde{N}$ isomorphically onto $\tilde{M}$, then by Corollary 4.11 we have that F is invertible up to $(P_{\tilde{N}^\perp}, P_{\tilde{M}^\perp})$ in $\mathcal{A}$. Since F is $\mathcal{A}-$ Fredholm and $P_{\tilde{N}^\perp} \in Proj_0(\mathcal{A})$, by Lemma 2.7 and (1) we must have that $P_{\tilde{M}^\perp} \in Proj_0(\mathcal{A})$. Thus, $P_M \in Proj_0(\mathcal{A})$.
Consider next the von Neumann algebra $P_{ImP_0} \mathcal{A} P_{ImP_0}$. If P_{ImP_0} is not finite, then $P_{ImP_0} \mathcal{A} P_{ImP_0}$ is also a properly infinite von Neumann algebra. Now,

$$P_M, P_N \in Proj_0(P_{ImP_0} \mathcal{A} P_{ImP_0}).$$

Since $P_{ImP_0 \cap \tilde{P}(H)} = P_{ImP_0} - P_N, P_{F(ImP_0 \cap \tilde{P}(H))} = P_{ImP_0} - P_M$ and F maps $ImP_0 \cap \tilde{P}(H)$ isomorphically onto $F(ImP_0 \cap \tilde{P}(H))$, it follows by Corollary 4.11 that $P_{ImP_0} F P_{ImP_0}$, which is equal to $F P_{ImP_0}$, is invertible up to (P_N, P_M) in $P_{ImP_0} \mathcal{A} P_{ImP_0}$. Hence

$$P_{ImP_0} F P_{ImP_0} \in \mathcal{K}\Phi(P_{ImP_0} \mathcal{A} P_{ImP_0}).$$

□

We can now deduce the desired result concerning spectral projections as a corollary of Proposition 4.12.

Corollary 4.13. *Let $F \in \mathcal{A}$ and α be an isolated point of $\sigma(F)$. If $F - \alpha I$ is $\mathcal{A}-$ Fredholm and P_0 is the spectral projection corresponding to α, then P_0 is finite operator.*

Proof. Note that $\sigma(F)$ in $\mathcal{A}$ is the same as the spectrum of F in $B(H)$ since $\mathcal{A}$ is a von Neumann algebra. By [19, Section 3] it follows that $F - \alpha I$ satisfies the conditions of Proposition 4.12 with respect to the decomposition $\ker P_0 \tilde{\oplus} ImP_0 = H$. Moreover, $F - \lambda I$ maps ImP_0 isomorphically onto ImP_0 for all $\lambda \neq \alpha$. If P_{ImP_0} is not a finite projection, by Proposition 4.12 we have that $P_{ImP_0}(F - \lambda I) P_{ImP_0} \in \mathcal{K}\Phi(P_{ImP_0} \mathcal{A} P_{ImP_0})$ for all $\lambda \in \mathbb{C}$. By the similar arguments as in the proof of [19, Corollary 2.8] we can deduce that $P_{ImP_0} \mathcal{A} P_{ImP_0}$ consists only of finite operators, so P_{ImP_0} is a finite operator, which contradicts the assumption in the beginning of this proof that P_{ImP_0} is not finite. □

The theory regarding isolated points of the spectrum of Fredholm operators on Hilbert and Banach spaces is closely connected to the concept of Browder operators, as illustrated in [19, Theorem 3.1]. Motivated by [14, Definition 5.7] we give now the following definition of generalized $\mathcal{A}$-Browder operators.

Definition 4.14. *Let $F \in \mathcal{A}$. We say that F is generalized $\mathcal{A}$-Browder if there exists a decomposition*

$$H = M \tilde{\oplus} N \xrightarrow{F} M \tilde{\oplus} N = H$$

with respect to which F has the matrix $\begin{pmatrix} F_1 & 0 \\ 0 & F_4 \end{pmatrix}$, where F_1 is an isomorphism and such that $P_N \in Proj_0(\mathcal{A})$.

We have the following lemma.

Lemma 4.15. *Let $F \in \mathcal{A}$. If F is generalized $\mathcal{A}$-Browder, then F is $\mathcal{A}$-Fredholm.*

Proof. Let

$$H = M\tilde{\oplus}N \xrightarrow{F} M\tilde{\oplus}N = H$$

be an $\mathcal{A}$-Browder decomposition for F. By the proof of Lemma 3.5, F has the matrix $\begin{pmatrix} F_1 & 0 \\ 0 & F_4 \end{pmatrix}$ with respect to the decomposition

$$H = N^{\perp} \oplus N \xrightarrow{F} F(N^{\perp})\tilde{\oplus}N = H,$$

where F_1 is an isomorphism. Hence, $F(N^{\perp})$ is closed and F has he matrix $\begin{pmatrix} F_1 & \tilde{F_2} \\ 0 & \tilde{F_4} \end{pmatrix}$, with respect to the decomposition

$$H = N^{\perp} \oplus N \xrightarrow{F} F(N^{\perp})\tilde{\oplus}F(N^{\perp})^{\perp} = H.$$

Now, $F(N^{\perp}) = ImF(I - P_N)$, so $P_{F(N^{\perp})} \in \mathcal{A}$ since $P_N \in \mathcal{A}$. By Corollary 4.11 we deduce that F is invertible up to $(P_N, I - P_{F(N^{\perp})})$ in $\mathcal{A}$. Since $H = F(N^{\perp})\tilde{\oplus}N$, it follows that $I - P_{F(N^{\perp})}$ maps N isomorphically onto $F(N^{\perp})^{\perp}$. Thus we get that $F(N^{\perp})^{\perp} = Im(I - P_{F(N^{\perp})})P_N$ and $N^{\perp} = \ker((I - P_{F(N^{\perp})})P_N)$, which gives that $P_N \sim P_{F(N^{\perp})^{\perp}}$. Hence $I - P_{F(N^{\perp})} \in Proj_0(\mathcal{A})$ because $P_N \in Proj_0(\mathcal{A})$, so F is $\mathcal{A}$-Fredholm by Lemma 4.2. □

The next corollary is motivated by [19, Theorem 3.1].

Corollary 4.16. *Let $F \in \mathcal{A}$ and suppose that 0 is an isolated point of $\sigma(F)$. Then F is $\mathcal{A}$-Fredholm if and only if F is generalized $\mathcal{A}$-Browder.*

Proof. The implication in one direction follows from Corollary 4.13, whereas the implication in the other direction follows from Lemma 4.15. □

Acknowledgement

I am grateful to Professor Dragan S. Đorđević for suggesting Fredholm theory on Hilbert C^*-modules as the topic for my research and for introducing to me the relevant literature. Also, I am very grateful to the Reviewer for careful reading of the paper and for inspiring suggestions and useful comments that led to the improved version of the paper.

References

[1] T. Aghasizadeh, S. Hejazian, *Maps preserving semi-Fredholm operators on Hilbert C*-modules*, J. Math. Anal. Appl. **354**(2009), 625-629.
[2] P. Aiena, *Fredholm and Local Spectral Theory, with Applications to Multipliers*, Kluwer 2004, ISBN 978-1-4020-2525-9
[3] P. Aiena, *Fredholm and Local spectral Theory II, Lecture Notes in Mathematics*, 2235, 2018, ISBN 978-3-030-02266-2
[4] M. Breuer, *Fredholm theories in von Neumann algebras. I*, Math. Ann. **178**, 243â€"254 (1968). https://doi.org/10.1007/BF01350663
[5] M. Breuer, *Fredholm theories in von Neumann algebras. II*, Math. Ann. **180**, 313â€"325 (1969). https://doi.org/10.1007/BF01351884
[6] Dragan S. Đorđević, *Perturbations of spectra of operator matrices*, J. Operator Theory 48(2002), 467-486.
[7] E. I. Fredholm, *Sur une classe d'equations fontionnelles*, Acta Math. **27** (1903), 365–390.
[8] S. Ivković, *Semi-Fredholm theory on Hilbert C*-modules,*Banach J. Math. Anal, Vol. **13** (2019) no. 4 2019, 989-1016 doi:10.1215/17358787-2019-0022. https://projecteuclid.org/euclid.bjma/1570608171
[9] S. Ivković, *On operators with closed range and semi-Fredholm operators over W*-algebras*, Russ. J. Math. Phys. 27, 4860 (2020) http://link.springer.com/article/10.1134/S1061920820010057
[10] S. Ivković, *On various generalizations of semi-A-Fredholm operators*, Complex Anal. Oper. Theory **14**, 41 (2020). https://doi.org/10.1007/s11785-020-00995-3
[11] S. Ivković, *Semi-Fredholm operators on Hilbert C*-modules*, Doctoral dissertation, Faculty of Mathematics, University of Belgrade (2022),
$www.matf.bg.ac.rs/files/Stefan_Ivkovic_Doktorska_disertacija.pdf$
[12] S. Ivković, *On upper triangular operator* 2×2 *matrices over C*-algebras*, Filomat, (2020), vol. 34 no. 3, 691-706 https://doi.org/10.2298/FIL2003691I
[13] S. Ivković, *On non-adjointable semi-C*-Fredholm and semi-C*-Weyl operators*, Filomat (2023) vol. 37 no. 17, 5523â€"5539 https://doi.org/10.2298/FIL2317523I
[14] S. Ivković, *On Drazin invertible C*-operators and generalized C*-Weyl operators*, Ann. Funct. Anal. 14, 36 (2023). https://doi.org/10.1007/s43034-023-00258-0
[15] S. Ivković, *Semi-Fredholm theory in C* -algebras*, Banach J. Math. Anal. **17**, 51 (2023). https://doi.org/10.1007/s43037-023-00277-y

[16] D. J. Kečkić, Z. Lazović, Fredholm operators on C*-algebras. Acta Sci. Math. 83, 629â€“655 (2017). https://doi.org/10.14232/actasm-015-526-5

[17] V. M. Manuilov, E. V. Troitsky, *Hilbert C*-modules*, Translations of Mathematical Monographs. 226, American Mathematical Society, Providence, RI, 2005.

[18] A. S. Mishchenko, A.T. Fomenko, *The index of eliptic operators over C*-algebras*, Izv. Akad. Nauk SSSR Ser. Mat. **43** (1979), 831–859; English transl., Math. USSR-Izv.**15** (1980) 87–112.

[19] S. Živković Zlatanović, *An Introduction into Fredholm theory and Generalized Drazin-Riesz Invertible Operators*, Zbornik Radova 20(28) (2022), 113 - 198 Issue 20(28) - Zbornik Radova - eLibrary of Mathematical Institute SASA (sanu.ac.rs)