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# Nonlinear mixed Jordan and bi-skew Jordan derivations on prime $*$-algebras

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**Abstract.** Let $\mathcal{A}$ be a prime $*$-algebra. In this paper, we suppose that $\Phi : \mathcal{A} \to \mathcal{A}$ satisfies

$\Phi[[A, B], C]_{\dagger} = [[\Phi(A), B], C]_{\dagger} + [[A, \Phi(B)], C]_{\dagger} + [[A, B], \Phi(C)]_{\dagger}$

where $[A, B] = AB + BA$ and $[A, B]_{\dagger} = A^*B + B^*A$ for all $A, B \in \mathcal{A}$. Then, $\Phi$ is additive $*$-derivation.

## 1. Introduction

Let $\mathcal{R}$ and $\mathcal{R}'$ be rings. We say the map $\Phi : \mathcal{R} \to \mathcal{R}'$ preserves product or is multiplicative if $\Phi(AB) = \Phi(A)\Phi(B)$ for all $A, B \in \mathcal{R}$. The question of when a product preserving or multiplicative map is additive was discussed by several authors, see [17] and references therein. Motivated by this, many authors pay more attention to the map on rings (and algebras) preserving Lie product $AB - BA$ or Jordan product $AB + BA$ (for example, see [1–3, 5–7, 9, 12, 15, 16, 20, 21]). These results show that, in some sense, Jordan product or Lie product structure is enough to determine the ring or algebraic structure. Historically, many mathematicians devoted themselves to the study of additive or linear Jordan or Lie product preservers between rings or operator algebras. Such maps are always called Jordan homomorphism or Lie homomorphism. Here we only list several results [10, 11, 13, 17–19].

Recall that a map $\Phi : \mathcal{R} \to \mathcal{R}$ is said to be an additive derivation if

$$\Phi(A + B) = \Phi(A) + \Phi(B)$$

and

$$\Phi(AB) = \Phi(A)B + A\Phi(B)$$

for all $A, B \in \mathcal{R}$. A map $\Phi$ is additive $*$-derivation if it is an additive derivation and $\Phi(A^*) = \Phi(A)^*$. Derivations are very important maps both in theory and applications, and have been studied intensively ([4, 22–24]).

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Recently the authors of [8] discussed some bijective maps preserving the bi-skew Jordan product $A^*B + B^*A$ between von Neumann algebras with no central abelian projections. In other words, $\Phi$ holds in the following condition

$$\Phi(A^*B + B^*A) = \Phi(A)^*\Phi(B) + \Phi(B)^*\Phi(A).$$

They showed that such a map is sum of a linear $*$-isomorphism and a conjugate linear $*$-isomorphism.

Moreover, The authors of [14] introduced the concept of Lie triple derivations. A map $\Phi : \mathcal{A} \to \mathcal{A}$ is a nonlinear skew Lie triple derivations if

$$\Phi([[A, B]_*, C]_*) = [[\Phi(A), B]_*, C]_* + [[A, \Phi(B)]_*, C]_* + [[A, B]_*, \Phi(C)]_*$$

for all $A, B, C \in \mathcal{A}$ where $[A, B]_* = AB - BA^*$. They showed that if $\Phi$ preserves the above characterizations on factor von Neumann algebras then $\Phi$ is additive $*$-derivation. In [25], we proved the above problem on prime $*$-algebras.

In [26], we considered a map $\Phi$ on prime $*$-algebra $\mathcal{A}$ which holds in the following conditions

$$\Phi(A \diamond_\lambda B \diamond_\lambda C) = \Phi(A) \diamond_\lambda B \diamond_\lambda C + A \diamond_\lambda \Phi(B) \diamond_\lambda C) + A \diamond_\lambda B \diamond_\lambda \Phi(C)$$

where $A \diamond_\lambda B = AB + \lambda BA^*$ such that a complex scalar $|\lambda| \neq 0, 1$, then $\Phi$ is additive. Also, if $\Phi(I)$ is self-adjoint then $\Phi$ is $*$-derivation.

On the other hand, in [27], the author considered the nonlinear mixed Lie triple derivations on prime $*$-algebra. They showed that the map $\Phi$ which satisfies

$$\Phi([[A, B]_*, C]) = [[\Phi(A), B]_*, C] + [[A, \Phi(B)]_*, C] + [[A, B]_*, \Phi(C)]$$

is an additive $*$-derivation.

Motivated by the above results, we prove that if $\mathcal{A}$ is a prime $*$-algebra then the map $\Phi : \mathcal{A} \to \mathcal{A}$ which satisfies

$$\Phi[[A, B], C]_\dagger = [[\Phi(A), B], C]_\dagger + [[A, \Phi(B)], C]_\dagger + [[A, B], \Phi(C)]_\dagger$$

is additive $*$-derivation, where $[A, B] = AB + BA$ and $[A, B]_\dagger = A^*B + B^*A$ for all $A, B \in \mathcal{A}$.

We say that $\mathcal{A}$ is prime, that is, for $A, B \in \mathcal{A}$ if $A\mathcal{A}B = \{0\}$, then $A = 0$ or $B = 0$.

## 2. Main Results

Our main theorem is as follows:

**Theorem 2.1.** *Let $\mathcal{A}$ be a prime $*$-algebra. Let $\Phi : \mathcal{A} \to \mathcal{A}$ satisfies in*

$$\Phi[[A, B], C]_\dagger = [[\Phi(A), B], C]_\dagger + [[A, \Phi(B)], C]_\dagger + [[A, B], \Phi(C)]_\dagger$$

*where $[A, B] = AB + BA$ and $[A, B]_\dagger = A^*B + B^*A$ for all $A, B \in \mathcal{A}$. Then, $\Phi$ is additive $*$-derivation.*

**Proof.** Let $P_1$ be a nontrivial projection in $\mathcal{A}$ and $P_2 = I_{\mathcal{A}} - P_1$. Denote $\mathcal{A}_{ij} = P_i\mathcal{A}P_j,\ i, j = 1, 2$, then $\mathcal{A} = \sum_{i,j=1}^{2} \mathcal{A}_{ij}$. For every $A \in \mathcal{A}$ we may write $A = A_{11} + A_{12} + A_{21} + A_{22}$. In all that follow, when we write $A_{ij}$, it indicates that $A_{ij} \in \mathcal{A}_{ij}$. For showing additivity of $\Phi$ on $\mathcal{A}$, we use above partition of $\mathcal{A}$ and give some claims that prove $\Phi$ is additive on each $\mathcal{A}_{ij},\ i, j = 1, 2$.

We prove the above theorem by several claims.

**Claim 1.** *We show that $\Phi(0) = 0$.*

By assuming $A = B = C = 0$ then we obtain the desired result.

**Claim 2.** *For all $A_{ij} \in \mathcal{A}_{ij}$ ($i, j \in \{1,2\}$ such that $i \neq j$) we have*

$$\Phi(A_{ii} + A_{ij}) = \Phi(A_{ii}) + \Phi(A_{ij}).$$

We show that

$$T = \Phi(A_{ii} + A_{ij}) - (\Phi(A_{ii}) + \Phi(A_{ij})) = 0.$$

For any $X_{lk}$ such that $l \neq k \in \{1,2\}$ and $i \neq l$ we have

$$\begin{aligned}
&\left[\left[\Phi\left(\frac{I}{2}\right), (A_{ii} + A_{ij})\right], X_{lk}\right]_{\dagger} + \left[\left[\frac{I}{2}, \Phi(A_{ii} + A_{ij})\right], X_{lk}\right]_{\dagger} + \left[\left[\frac{I}{2}, A_{ii} + A_{ij}\right], \Phi(X_{lk})\right]_{\dagger} \\
&= \Phi\left[\left[\frac{I}{2}, A_{ii} + A_{ij}\right], X_{lk}\right]_{\dagger} \\
&= \Phi\left[\left[\frac{I}{2}, A_{ii}\right], X_{lk}\right]_{\dagger} + \Phi\left[\left[\frac{I}{2}, A_{ij}\right], X_{lk}\right]_{\dagger} \\
&= \left[\left[\Phi\left(\frac{I}{2}\right), A_{ii} + A_{ij}\right], X_{lk}\right]_{\dagger} + \left[\left[\frac{I}{2}, \Phi(A_{ii}) + \Phi(A_{ij})\right], X_{lk}\right]_{\dagger} \\
&+ \left[\left[\frac{I}{2}, A_{ii} + A_{ij}\right], \Phi(X_{lk})\right]_{\dagger}.
\end{aligned}$$

So $\left[\left[\frac{I}{2}, T\right], X_{lk}\right]_{\dagger} = 0$ then we have

$$\left[\left[\frac{I}{2}, T_{11} + T_{12} + T_{21} + T_{22}\right], X_{lk}\right]_{\dagger}.$$

Since $\mathcal{A}$ is prime, from the above equation, we have $T_{11} = 0$ for $l = 1$ and $T_{22} = 0$ for $l = 2$.

Similarly by applying $iX_{lk}$ in the above equation and the primeness of $\mathcal{A}$ we have $T_{12} = 0$ for $l = 1$ and $T_{21} = 0$ for $l = 2$.

Now we show that $T_{11} = T_{12} = 0$ for $l = 2$ and $T_{22} = T_{21} = 0$ for $l = 1$.
For $l \neq k \in \{1,2\}$ we have

$$\begin{aligned}
\Phi\left[\left[\frac{I}{2}, \left(\left[\frac{I}{2}, A_{ij} + A_{ii}\right], X_{lk}\right]_{\dagger}\right)\right], P_l\right]_{\dagger} &= \left[\left[\Phi\left(\frac{I}{2}\right), \left(\left[\left[\frac{I}{2}, A_{ii} + A_{ij}\right], X_{lk}\right]_{\dagger}\right)\right], P_l\right]_{\dagger} \\
&+ \left[\left[\frac{I}{2}, \Phi\left(\left[\left[\frac{I}{2}, A_{ii} + A_{ij}\right], X_{lk}\right]_{\dagger}\right)\right], P_l\right]_{\dagger} \\
&+ \left[\left[\frac{I}{2}, \left(\left[\left[\frac{I}{2}, A_{ii} + A_{ij}\right], X_{lk}\right]_{\dagger}\right)\right], \Phi(P_l)\right]_{\dagger} \\
&= \left[\left[\Phi\left(\frac{I}{2}\right), \left(\left[\left[\frac{I}{2}, A_{ii} + A_{ij}\right], X_{lk}\right]_{\dagger}\right)\right], P_l\right]_{\dagger} \\
&+ \left[\left[\frac{I}{2}, (\left[\left[\Phi\left(\frac{I}{2}\right), A_{ii} + A_{ij}\right], X_{lk}\right]_{\dagger})\right], P_l\right]_{\dagger} \\
&+ \left[\left[\frac{I}{2}, \left(\left[\left[\frac{I}{2}, \Phi(A_{ii} + A_{ij})\right], X_{lk}\right]_{\dagger}\right], P_l\right]_{\dagger} \\
&+ \left[\left[\frac{I}{2}, \left(\left[\left[\frac{I}{2}, A_{ij} + A_{jj}\right], \Phi(X_{lk})\right]_{\dagger}\right)\right], P_l\right]_{\dagger} \\
&+ \left[\left[\frac{I}{2}, \left(\left[\left[\frac{I}{2}, A_{ii} + A_{jj}\right], X_{lk}\right]_{\dagger}\right)\right], \Phi(P_l)\right]_{\dagger}.
\end{aligned} \tag{1}$$

It is easy to check that

$$\left[\left[\frac{I}{2}, \left(\left[\left[\frac{I}{2}, A_{ij}\right], X_{lk}\right]_{\dagger}\right)\right], P_l\right] = 0.$$

So we have

$$
\begin{aligned}
&\Phi\left[\left[\frac{I}{2},\left(\left[\left[\frac{I}{2},A_{ij}+A_{ii}\right],X_{lk}\right]_{\dagger}\right]\right),P_l\right]_{\dagger}\\
&=\Phi\left[\left[\frac{I}{2},\left(\left[\left[\frac{I}{2},A_{ij}\right],X_{lk}\right]_{\dagger}\right]\right),P_l\right]_{\dagger}+\Phi\left[\left[\frac{I}{2},\left(\left[\left[\frac{I}{2},A_{ii}\right],X_{lk}\right]_{\dagger}\right]\right)P_l\right]_{\dagger}\\
&=\left[\left[\Phi\left(\frac{I}{2}\right),\left(\left[\left[\frac{I}{2},A_{ij}\right],X_{lk}\right]_{\dagger}\right]\right),P_l\right]_{\dagger}+\left[\left[\frac{I}{2},\Phi\left(\left[\left[\frac{I}{2},A_{ij}\right],X_{lk}\right]_{\dagger}\right)\right],P_l\right]_{\dagger}\\
&+\left[\left[\frac{I}{2},\left(\left[\left[\frac{I}{2},A_{ij}\right],X_{lk}\right]_{\dagger}\right)\right],\Phi(P_l)\right]_{\dagger}+\left[\Phi\left(\frac{I}{2}\right),\left(\left[\left[\frac{I}{2},A_{ii}\right],X_{lk}\right]_{\dagger}\right),P_l\right]\\
&+\left[\left[\frac{I}{2},\Phi\left(\left[\left[\frac{I}{2},A_{ii}\right],X_{lk}\right]_{\dagger}\right)\right],P_l\right]_{\dagger}+\left[\left[\frac{I}{2},\left(\left[\left[\frac{I}{2},A_{ii}\right],X_{lk}\right]_{\dagger}\right)\right],\Phi(P_l)\right]_{\dagger}\\
&=\left[\left[\Phi\left(\frac{I}{2}\right),\left(\left[\left[\frac{I}{2},A_{ij}\right],X_{lk}\right]_{\dagger}\right]\right),P_l\right]_{\dagger}+\left[\left[\frac{I}{2},\left(\left[\left[\Phi\left(\frac{I}{2}\right),A_{ij}\right],X_{lk}\right]_{\dagger}\right)\right],P_l\right]_{\dagger}\\
&+\left[\left[\frac{I}{2},\left(\left[\left[\frac{I}{2},\Phi(A_{ij})\right],X_{lk}\right]_{\dagger}\right)\right],P_l\right]_{\dagger}+\left[\left[\frac{I}{2},\left(\left[\left[\frac{I}{2},A_{ij}\right],\Phi(X_{lk})\right]_{\dagger}\right)\right],P_l\right]_{\dagger}\\
&+\left[\left[\frac{I}{2},\left(\left[\left[\frac{I}{2},A_{ij}\right],X_{lk}\right]_{\dagger}\right)\right],\Phi(P_l)\right]_{\dagger}+\left[\left[\Phi\left(\frac{I}{2}\right),\left(\left[\left[\frac{I}{2},A_{ii}\right],X_{lk}\right]_{\dagger}\right]\right),P_l\right]_{\dagger}\\
&+\left[\left[\frac{I}{2},\left(\left[\left[\Phi\left(\frac{I}{2}\right),A_{ii}\right],X_{lk}\right]_{\dagger}\right)\right],P_l\right]_{\dagger}+\left[\left[\frac{I}{2},\left(\left[\left[\frac{I}{2},\Phi(A_{ii})\right],X_{lk}\right]_{\dagger}\right)\right],P_l\right]_{\dagger}\\
&+\left[\left[\frac{I}{2},\left(\left[\left[\frac{I}{2},A_{ii}\right],\Phi(X_{lk})\right]_{\dagger}\right)\right],P_l\right]_{\dagger}+\left[\left[\frac{I}{2},\left(\left[\left[\frac{I}{2},A_{ii}\right],X_{lk}\right]_{\dagger}\right)\right],\Phi(P_l)\right]_{\dagger}\\
&=\left[\left[\Phi\left(\frac{I}{2}\right),\left(\left[\left[\frac{I}{2},A_{ii}+A_{ij}\right],X_{lk}\right]_{\dagger}\right)\right],P_l\right]_{\dagger}\\
&+\left[\left[\frac{I}{2},\left(\left[\left[\Phi\left(\frac{I}{2}\right),A_{ii}+A_{ij}\right],X_{lk}\right]_{\dagger}\right)\right],P_l\right]_{\dagger}\\
&+\left[\left[\frac{I}{2},\left(\left[\left[\frac{I}{2},\Phi(A_{ii})+\Phi(A_{ij})\right],X_{lk}\right]_{\dagger}\right],P_l\right]_{\dagger}\\
&+\left[\left[\frac{I}{2},\left(\left[\left[\frac{I}{2},A_{ij}+A_{ii}\right],\Phi(X_{lk})\right]_{\dagger}\right)\right],P_l\right]_{\dagger}\\
&+\left[\left[\frac{I}{2},\left(\left[\left[\frac{I}{2},A_{ii}+A_{ij}\right],X_{lk}\right]_{\dagger}\right)\right],\Phi(P_l)\right]_{\dagger}. \qquad (2)
\end{aligned}
$$

From (1) and (2) we have

$$\left[\left[\frac{I}{2},\left(\left[\left[\frac{I}{2},\Phi(A_{ii}+A_{ij})\right],X_{lk}\right]_{\dagger}\right)\right],P_l\right]_{\dagger}=\left[\left[\frac{I}{2},\left(\left[\left[\frac{I}{2},\Phi(A_{ii})+\Phi(A_{ij})\right],X_{lk}\right]_{\dagger}\right)\right],P_l\right]_{\dagger}.$$

So $\left[\left[\frac{I}{2},\left(\left[\left[\frac{I}{2},T\right],X_{lk}\right]_{\dagger}\right)\right],P_l\right]_{\dagger}=0$. Therefore

$$\left[\left[\frac{I}{2},\left(\left[\left[\frac{I}{2},T_{11}+T_{12}+T_{21}+T_{22}\right],X_{lk}\right]_{\dagger}\right)\right],P_l\right]_{\dagger}=0.$$

From the above equation and primeness of $\mathcal{A}$ we have $T_{22}=T_{21}=0$ for $l=1$ and $T_{11}=T_{12}=0$ for $l=2$.

**Claim 3.** *For each* $A_{ij}\in\mathcal{A}_{ij}$ $(i,j\in\{1,2\})$, *we have*

$$\Phi\left(\left(\sum_{i,j=1}^{2}A_{ij}\right)+A_{ii}\right)=\left(\sum_{i,j=1}^{2}\Phi(A_{ij})\right)+\Phi(A_{ii}).$$

We show that

$$T=\Phi\left(\left(\sum_{i,j=1}^{2}A_{ij}\right)+A_{ii}\right)-\left(\left(\sum_{i,j=1}^{2}\Phi(A_{ij})\right)+\Phi(A_{ii})\right)=0.$$

For $i \neq j$ and $l \neq k \in \{1,2\}$, from Claim 2, we have

$$\left[\left[\Phi\left(\frac{I}{2}\right), \left(\sum_{i,j=1}^{2} A_{ij}\right) + A_{ii}\right], X_{lk}\right]_{\dagger} + \left[\left[\frac{I}{2}, \Phi\left(\left(\sum_{i,j=1}^{2} A_{ij}\right) + A_{ii}\right)\right], X_{lk}\right]_{\dagger}$$
$$+ \left[\left[\frac{I}{2}, \left(\sum_{i,j=1}^{2} A_{ij}\right) + A_{ii}\right], \Phi(X_{lk})\right]_{\dagger} = \Phi\left[\left[\frac{I}{2}, \left(\sum_{i,j=1}^{2} A_{ij}\right) + A_{ii}\right], (X_{lk})\right]_{\dagger}$$
$$= \left[\left[\Phi\left(\frac{I}{2}\right), \left(\sum_{i,j=1}^{2} A_{ij}\right) + A_{ii}\right], X_{lk}\right] + \left[\left[\frac{I}{2}, \left(\sum_{i,j=1}^{2} \Phi\left(A_{ij}\right)\right) + \Phi(A_{ii})\right], X_{lk}\right]_{\dagger}$$
$$+ \left[\left[\frac{I}{2}, \left(\sum_{i,j=1}^{2} A_{ij}\right) + A_{ii}\right], \Phi(X_{lk})\right]_{\dagger}.$$

So $\left[\left[\frac{I}{2}, T\right], X_{lk}\right]_{\dagger} = 0$. Therefore

$$T_{11} = T_{12} = T_{21} = T_{22} = 0.$$

**Claim 4.** *For each $A_{ij} \in \mathcal{A}_{ij}$ ($i, j \in \{1,2\}$) we have*

$$\Phi\left(\sum_{i,j=1}^{2} A_{ij}\right) = \sum_{i,j=1}^{2} \Phi(A_{ij}).$$

Let $T = \Phi\left(\sum_{i,j=1}^{2} A_{ij}\right) - \sum_{i,j=1}^{2} \Phi(A_{ij})$. We prove that $T = 0$. For $l \neq k$ and $l, k \in \{1,2\}$, from Claim 3, we have

$$\left[\left[\Phi\left(\frac{I}{2}\right), \sum_{i,j=1}^{2} A_{ij}\right], X_{lk}\right]_{\dagger} + \left[\left[\frac{I}{2}, \Phi\left(\sum_{i,j=1}^{2} A_{ij}\right)\right], X_{lk}\right]_{\dagger} + \left[\left[\frac{I}{2}, \sum_{i,j=1}^{2} A_{ij}\right], \Phi(X_{lk})\right]_{\dagger}$$
$$= \Phi\left(\left[\left[\frac{I}{2}, \sum_{i,j=1}^{2} \mathcal{A}_{ij}\right], X_{lk}\right]_{\dagger}\right)$$
$$= \Phi\left[\left[\frac{I}{2}, \left(\sum_{i,j=1}^{2} A_{ij}\right) + A_{11}\right], X_{lk}\right]_{\dagger} + \Phi\left[\left[\frac{I}{2}, A_{22}\right], X_{lk}\right]_{\dagger}$$
$$= \Phi\left[\left[\frac{I}{2}, \sum_{i,j=1}^{2} A_{ij}\right], X_{lk}\right]_{\dagger}$$
$$= \left[\left[\Phi\left(\frac{I}{2}\right), \sum_{i,j=1}^{2} A_{ij}\right], X_{lk}\right]_{\dagger} + \left[\left[\frac{I}{2}, \sum_{i,j=1}^{2} \Phi(A_{ij})\right], X_{lk}\right]_{\dagger} + \left[\left[\frac{I}{2}, \sum_{i,j=1}^{2} A_{ij}\right], \Phi(X_{lk})\right]_{\dagger}.$$

Hence $\left[\left[\frac{I}{2}, T\right], X_{lk}\right]_{\dagger} = 0$ then $\left[\left[\frac{I}{2}, \sum_{i,j=1}^{2} T_{ij}\right], X_{lk}\right]_{\dagger} = 0$ therefore

$$T_{11} = T_{12} = T_{21} = T_{22} = 0.$$

**Claim 5.** *For each $A_{ij}, B_{ij} \in \mathcal{A}_{ii}$ we have*

$$\Phi(A_{ij} + B_{ij}) + \Phi(A_{ij}^* + B_{ij}^*) = \Phi(A_{ij}) + \Phi(A_{ij}^*) + \Phi(B_{ij}) + \Phi(B_{ij}^*).$$

From Claim 4 we have

$$\begin{aligned}
\Phi(A_{ij}+B_{ij})+\Phi(A_{ij}^*+B_{ij}^*) &= \Phi\left[\left[\frac{I}{2},P_i+A_{ij}^*\right],P_j+B_{ij}\right]_\dagger \\
&= \left[\left[\Phi\left(\frac{I}{2}\right),P_i+A_{ij}^*\right],P_j+B_{ij}\right]_\dagger + \left[\left[\frac{I}{2},\Phi(P_i+A_{ij}^*)\right],P_j+B_{ij}\right]_\dagger \\
&+ \left[\left[\frac{I}{2},P_i+A_{ij}^*\right],\Phi(P_j+B_{ij})\right]_\dagger \\
&= \left[\left[\Phi\left(\frac{I}{2}\right),P_i+A_{ij}^*\right],P_j+B_{ij}\right]_\dagger + \left[\left[\frac{I}{2},\Phi(P_i)+\Phi(A_{ij}^*)\right],P_j+B_{ij}\right]_\dagger \\
&+ \left[\left[\frac{I}{2},P_i+A_{ij}^*\right],\Phi(P_j)+\Phi(B_{ij})\right]_\dagger \\
&= \Phi\left[\left[\frac{I}{2},A_{ij}^*\right],P_j\right]_\dagger + \Phi\left[\left[\frac{I}{2},P_i\right],B_{ij}\right]_\dagger \\
&= \Phi(A_{ij})+\Phi(A_{ij}^*)+\Phi(B_{ij})+\Phi(B_{ij}^*)
\end{aligned}$$

**Claim 6.** *For each $A_{ii}, B_{ii} \in \mathcal{A}_{ii}$ ($i \in \{1,2\}$) we have*

$$\Phi(A_{ii}+B_{ii}) = \Phi(A_{ii})+\Phi(B_{ii}).$$

We show that $T = \Phi(A_{ii}+B_{ii}) - \Phi(A_{ii}) + \Phi(B_{ii}) = 0$. We can check that

$$\begin{aligned}
&\left[\left[\Phi\left(\frac{I}{2}\right),A_{ii}+B_{ii}\right],P_j\right]_\dagger + \left[\left[\frac{I}{2},\Phi(A_{ii}+B_{ii})\right],P_j\right]_\dagger + \left[\left[\frac{I}{2},A_{ii}+B_{ii}\right],\Phi(P_j)\right]_\dagger \\
&= \Phi\left[\left[\frac{I}{2},A_{ii}\right],P_j\right]_\dagger + \Phi\left[\left[\frac{I}{2},B_{ii}\right],P_j\right]_\dagger \\
&= \left[\left[\Phi\left(\frac{I}{2}\right),A_{ii}\right],P_j\right]_\dagger + \left[\left[\frac{I}{2},\Phi(A_{ii})\right],P_j\right]_\dagger \\
&+ \left[\left[\frac{I}{2},A_{ii}\right],\Phi(P_j)\right]_\dagger \\
&+ \left[\left[\Phi\left(\frac{I}{2}\right),B_{ii}\right],P_j\right]_\dagger + \left[\left[\frac{I}{2},\Phi(B_{ii})\right],P_j\right]_\dagger \\
&+ \left[\left[\frac{I}{2},B_{ii}\right],\Phi(P_j)\right]_\dagger \\
&= \left[\left[\Phi\left(\frac{I}{2}\right),A_{ii}+B_{ii}\right],P_j\right]_\dagger + \left[\left[\frac{I}{2},\Phi(A_{ii})+\Phi(B_{ii})\right],P_j\right]_\dagger \\
&+ \left[\left[\frac{I}{2},A_{ii}+B_{ii}\right],\Phi(P_j)\right]_\dagger .
\end{aligned}$$

So $\left[\left[\frac{I}{2},T\right],P_j\right]_\dagger = 0$ therefore $T_{ij} = T_{ji} = T_{jj} = 0$.

On the other hand, from Claims 4 and 5, we have

$$\begin{aligned}
&\left[\left[\Phi\left(\frac{I}{2}\right), A_{ii}+B_{ii}\right], X_{ij}\right]_{\dagger} + \left[\left[\frac{I}{2}, \Phi(A_{ii}+B_{ii})\right], X_{ij}\right]_{\dagger} + \left[\left[\frac{I}{2}, A_{ii}+B_{ii}\right], \Phi(X_{ij})\right]_{\dagger}\\
&= \Phi\left[\left[\frac{I}{2}, A_{ii}+B_{ii}\right], X_{ij}\right]_{\dagger}\\
&= \Phi(A_{ii}^{*}X_{ij} + B_{ii}^{*}X_{ij}) + \Phi(X_{ij}^{*}A_{ii} + X_{ij}^{*}B_{ii})\\
&= \Phi(A_{ii}^{*}X_{ij}) + \Phi(B_{ii}^{*}X_{ij}) + \Phi(X_{ij}^{*}A_{ii}) + \Phi(X_{ij}^{*}B_{ii})\\
&= \Phi(A_{ii}^{*}X_{ij} + X_{ij}^{*}A_{ii}) + \Phi(B_{ii}^{*}X_{ij} + X_{ij}^{*}B_{ii})\\
&= \Phi\left[\left[\frac{I}{2}, A_{ii}\right], X_{ij}\right]_{\dagger}) + \Phi\left[\left[\frac{I}{2}, B_{ii}\right], X_{ij}\right]_{\dagger}\\
&= \left[\left[\Phi\left(\frac{I}{2}\right), A_{ii}+B_{ii}\right], X_{ij}\right]_{\dagger} + \left[\left[\frac{I}{2}, \Phi(A_{ii})+\Phi(B_{ii})\right], X_{ij}\right]_{\dagger}\\
&+ \left[\left[\frac{I}{2}, A_{ii}+B_{ii}\right], \Phi(X_{ij})\right]_{\dagger}.
\end{aligned}$$

So $[[\frac{1}{2}, T], X_{ij}]_{\dagger} = 0$ then $T_{ii} = 0$. Hence, the additivity of $\Phi$ comes from the above claims. In the rest of this paper, we show that $\Phi$ is $*$-derivation.

**Claim 7.** *We show that* $\Phi(I) = \Phi(iI) = 0$.

$$\Phi([[I, I], I]_{\dagger}) = [[\Phi(I), I], I]_{\dagger} + [[I, \Phi(I)], I]_{\dagger} + [[I, I], \Phi(I)]_{\dagger}.$$

So $\Phi(I) = \frac{3}{2}(\Phi(I) + \Phi(I)^{*})$. We say that $\Phi(I)$ is self-adjoint. It follows that $\Phi(I) = \frac{3}{2}(\Phi(I) + \Phi(I))$ therefore $\Phi(I) = 0$.
Similarly,

$$\Phi([[I, I], iI]_{\dagger} = [[\Phi(I), I], iI]_{\dagger} + [[I, \Phi(I)], iI]_{\dagger} + [[I, I], \Phi(iI)]_{\dagger}.$$

It follows that

$$\Phi(iI)^{*} = -\Phi(iI). \tag{3}$$

So,

$$\begin{aligned}
\Phi([[I, iI], iI]_{\dagger} = {} & [[\Phi(I), iI], iI]_{\dagger} + [[I, \Phi(iI)], iI]_{\dagger}\\
& + [[I, iI], \Phi(iI)]_{\dagger}.
\end{aligned}$$

It follows that

$$\Phi(iI) = i(\Phi(iI)^{*} - \Phi(iI)). \tag{4}$$

Finally, from (3) and (4), we have $\Phi(iI) = 0$

**Claim 8.** $\Phi$ *preserves star.*

It is easy to check that

$$\Phi\left[\left[\frac{I}{2}, I\right], A\right]_{\dagger} = \left[\left[\frac{I}{2}, I\right], \Phi(A)\right]_{\dagger}$$

equivalently,

$$\Phi(A + A^{*}) = \Phi(A) + \Phi(A)^{*}.$$

So $\Phi(A)^{*} = \Phi(A^{*})$.

**Claim 9.** $\Phi(iA) = i\Phi(A)$

By an easy computation, we can write

$$\Phi([[I, iA], I]_\dagger = \Phi([[I, -A], iI]_\dagger$$

then

$$[[I, \Phi(iA)], I]_\dagger = ([[I, \Phi(-A)], iI]_\dagger.$$

So

$$\Phi(iA)^* + \Phi(iA) = i\Phi(-A)^* - i\Phi(-A). \tag{5}$$

Also

$$\Phi[[I, iA], iI]_\dagger = \Phi[[-I, -A], I]_\dagger$$

it yields

$$[[I, \Phi(iA)], iI]_\dagger = ([[-I, \Phi(-A)], iI]_\dagger.$$

So

$$-\Phi(iA)^* + \Phi(iA) = i\Phi(A) + i\Phi(A)^*. \tag{6}$$

Therefore, from (5) and (6), we have $\Phi(iA) = i\Phi(A)$

**Claim 10.** *We prove that* $\Phi$ *is derivation.*

For every $A, B \in \mathcal{A}$ we have

$$\begin{aligned}\Phi(AB + B^*A^*) &= \Phi\left[\left[\frac{I}{2}, A^*\right], B\right] \\ &= \left[\left[\frac{I}{2}, \Phi(A^*)\right], B\right] + \left[\left[\frac{I}{2}, A^*\right], \Phi(B)\right] \\ &= \Phi(A^*)^*B + \Phi(B)^*A^* + B^*\Phi(A^*) + A\Phi(B).\end{aligned}$$

On the other hand, since $\Phi$ preserves star, we obtain

$$\Phi(AB + B^*A^*) = \Phi(A)B + A\Phi(B) + B^*\Phi(A^*) + \Phi(B)^*A^*. \tag{7}$$

So, from (7), we have

$$\begin{aligned}\Phi(i(AB - B^*A^*) &= \Phi(A(iB) + (iB)^*A^*) \\ &= \Phi(A)(iB) + A\Phi(iB) + (iB)^*\Phi(A^*) + \Phi(iB)^*A^*.\end{aligned}$$

Therefore, from Claim 9 it follows that

$$\Phi(AB - B^*A^*) = \Phi(A)B + A\Phi(B) - B^*\Phi(A^*) - \Phi(B^*)A^*. \tag{8}$$

By adding equations (7) and (8), we have

$$\Phi(AB) = \Phi(A)B + A\Phi(B).$$

This completes the proof.

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